



Entropy and the Spectral Action

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Received: 19 September 2018 / Accepted: 26 September 2018

Published online: 6 February 2019 – © The Author(s) 2019

Abstract: We compute the information theoretic von Neumann entropy of the state associated to the fermionic second quantization of a spectral triple. We show that this entropy is given by the spectral action of the spectral triple for a specific universal function. The main result of our paper is the surprising relation between this function and the Riemann zeta function. It manifests itself in particular by the values of the coefficients $c(d)$ by which it multiplies the d dimensional terms in the heat expansion of the spectral triple. We find that $c(d)$ is the product of the Riemann xi function evaluated at $-d$ by an elementary expression. In particular $c(4)$ is a rational multiple of $\zeta(5)$ and $c(2)$ a rational multiple of $\zeta(3)$. The functional equation gives a duality between the coefficients in positive dimension, which govern the high energy expansion, and the coefficients in negative dimension, exchanging even dimension with odd dimension.

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1. Introduction

The spectral action principle [4] has so far been developed for arbitrary test functions of the form $\chi(D^2/\Lambda^2)$ of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The spectral action defined, for a fixed choice of χ , as

$$\text{Tr}(\chi(D^2/\Lambda^2))$$

is an additive functional of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, *i.e.* when evaluated on the direct sum of two spectral triples it behaves additively. This additivity is, at the conceptual level, a key property of the spectral action. Motivated by the formalism of Quantum field theory, one gets another natural way to obtain an additive functional of spectral triples: given $(\mathcal{A}, \mathcal{H}, D)$ one performs the fermionic second quantization using the Clifford algebra of the underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$ and evaluating the von Neumann information theoretic entropy of the unique state satisfying the KMS condition at inverse temperature β with respect to the time evolution of the Clifford algebra induced by the operator D . The second quantization transforms direct sums into tensor products and moreover the von Neumann entropy of the tensor product of two states is the sum of the respective entropies. Thus one indeed obtains an additive functional of spectral triples in this way, which we simply call the “entropy” $S(\mathcal{A}, \mathcal{H}, D)$ of the spectral triple. After recalling the KMS condition in Sect. 2.1 and the fermionic second quantization in Sect. 2.2 we show in Sect. 3 that the entropy functional is the spectral action for a specific universal test function $\chi(x) = h(\sqrt{x})$. The main result of this paper is the surprising intimate relation, explored in Sect. 4, between the function h and the Riemann zeta function. This is explained at the conceptual level to try to understand why it appears.

The content of this paper is purely mathematical. As far as physical applications are concerned, using the role of the spectral action as a framework to unify all fundamental interactions, this work indicates in particular the need to develop the spectral approach in connection with second quantization.

2. Spectral Triples and Second Quantization

We use the operator algebra formalism of C^* -dynamical systems and KMS condition to pass from the first-quantized or “one-particle” level of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ to the second-quantized level. The Hilbert space \mathcal{H} is used to construct the complexified Clifford algebra $C := \text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ of its underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$ when considered as a Euclidean space. The operator D is used as the generator of a one-parameter group $\sigma_t \in \text{Aut}(C)$ of automorphisms of the Clifford algebra. The algebra \mathcal{A} manifests itself through the semigroup of inner fluctuations [5] which labels deformations D' of the operator D . The action of the semigroup of inner fluctuations continues to make sense at the second-quantized level and gives rise to deformations $\sigma'_t \in \text{Aut}(C)$ of the above one-parameter group of automorphisms. We concentrate here on the meaning of the spectral action, and thus take the C^* -dynamical system (\mathcal{C}, σ_t) as our starting point, keeping in mind that the results will automatically apply to the deformations (\mathcal{C}, σ'_t) .

2.1. KMS and the dynamical system $(\text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}), \sigma_t = \text{Cliff}(e^{itD}))$. We first briefly recall the KMS condition for C^* -dynamical systems, that is, a C^* -algebra \mathcal{C} together with a one-parameter group of automorphisms $\sigma_t \in \text{Aut}(\mathcal{C}), t \in \mathbb{R}$.

Definition 2.1. Let (\mathcal{C}, σ_t) be a C^* -dynamical system. For a given $0 < \beta < \infty$, a state φ on the unital C^* -algebra \mathcal{C} satisfies the KMS condition at inverse temperature β if for all $a, b \in \mathcal{C}$, there exists a function $F_{a,b}(z)$ which is holomorphic on the strip

$$I_{\beta} = \{z \in \mathbb{C} \mid 0 < \Im(z) < \beta\}, \tag{1}$$

continuous on the boundary ∂I_β and bounded, with the property that for all $t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a). \tag{2}$$

In short the KMS condition at inverse temperature β means that one has the formal equality

$$\varphi(a\sigma_t(b))|_{t=i\beta} = \varphi(ba), \quad \forall a, b \in \mathcal{C}. \tag{3}$$

What matters in the context of the present paper is the existence and uniqueness of KMS states on a matrix algebra $M_n(\mathbb{C})$ and we give the short proof for convenience. A one-parameter group of automorphisms $\sigma_t \in \text{Aut}(M_n(\mathbb{C}))$ is always associated to a self-adjoint $H = H^* \in M_n(\mathbb{C})$ by

$$\sigma_t(A) = e^{itH} A e^{-itH}, \quad \forall t \in \mathbb{R}, \quad A \in M_n(\mathbb{C}).$$

Given a state ψ on the matrix algebra $M_n(\mathbb{C})$ there exists a unique density matrix $\rho \geq 0$ such that

$$\psi(T) = \text{Tr}(\rho T), \quad \forall T \in M_n(\mathbb{C}).$$

By uniqueness of the trace it follows that

$$\psi(AB) = \psi(BA), \quad \forall A, B \Rightarrow \rho = \frac{1}{n} \text{id}.$$

A state which is KMS_β for $\sigma_t \in \text{Aut}(M_n(\mathbb{C}))$ is invariant. In fact

$$\psi(e^{-\beta H} B e^{\beta H}) = \psi(B), \quad \forall B \Rightarrow e^{\beta H} \rho e^{-\beta H} = \rho \Rightarrow \rho e^{\beta H} = e^{\beta H} \rho.$$

It follows using (3) that if ψ is KMS_β then, with $B' = e^{-\beta H} B$

$$\psi(BA) = \psi(Ae^{-\beta H} B e^{\beta H}) \Rightarrow \text{Tr}(\rho e^{\beta H} B' A) = \text{Tr}(\rho A B' e^{\beta H})$$

so that $\rho e^{\beta H} = e^{\beta H} \rho$ defines a trace and hence is a scalar multiple of id . This shows that $\rho = Z e^{-\beta H}$ for $Z = 1/\text{Tr}(e^{-\beta H})$ and gives the uniqueness of the KMS_β state. The same formula gives the existence. For completeness, we include a proof of the following result on KMS-states on Clifford algebras (cf. [2, Prop. 5.2.23]).

Proposition 2.2. *Let \mathcal{H} be a complex Hilbert space, D a self-adjoint operator in \mathcal{H} with compact resolvent. Let $C := \text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ be the complexified Clifford algebra of the underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$ and $\sigma_t \in \text{Aut}(C)$ be the one-parameter group of automorphisms associated to $\exp(itD) \in \text{Aut}(\mathcal{H}_{\mathbb{R}})$. Then for any $\beta > 0$ there exists a unique KMS_β state ψ_β on the C^* -dynamical system (C, σ_t) .*

Proof. One applies the existence and uniqueness of KMS states on a matrix algebra to the subalgebra of $C = \text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ associated to the subspace corresponding to a finite dimensional spectral projection of D . This is enough to prove the uniqueness of the KMS_β state. The existence also follows since the existence part for matrix algebras gives a coherent system of states which define a state on the inductive limit of the C^* -algebras. \square

Proposition 2.3. *Let $\mathcal{H}, D, C := \text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}), \sigma_t \in \text{Aut}(C)$ and ψ_β be as in Proposition 2.2. Then if the operator $\exp(-\beta|D|)$ is of trace class, the state ψ_β is of type I and the associated irreducible representation is given by the fermionic second quantization associated to the complex structure $I := i \text{sign } D$ on $\mathcal{H}_{\mathbb{R}}$.*

The proof of this proposition will be given in Sect. 2.2, Proposition 2.6 after recalling some terminology.

2.2. Fermionic second quantization. In this section we recall the procedure of (fermionic) second quantization. We refer to [1, 3] and [7, Sect. 5.3 and 6.1] for excellent expositions on fermionic second quantization.

Consider the real Euclidean vector space $V := \mathcal{H}_{\mathbb{R}}$ that underlies the complex Hilbert space \mathcal{H} . Our first goal is to find irreducible representations of the Clifford algebra associated to the real Euclidean vector space $\mathcal{H}_{\mathbb{R}}$ underlying \mathcal{H} and for this it turns out that a crucial role is played by complex structures. We let I be an orthogonal complex structure on V , which is not necessarily the one coming from \mathcal{H} . Then we may regard V as a complex vector space when we define i to act as I . The resulting complex Hilbert space will be denoted by V_I .

A representation of the complexified Clifford algebra $\text{Cliff}_{\mathbb{C}}(V)$ is given on the Fock space $\bigwedge V_I$ that is built on V_I by the usual formula

$$\begin{aligned} \gamma_I : \text{Cliff}_{\mathbb{C}}(V) &\rightarrow \mathcal{L}(\bigwedge V_I) \\ v &\mapsto a_I^*(v) + a_I(v); \quad (v \in V). \end{aligned}$$

Here the *creation operators* $a_I^*(v)$ depend \mathbb{C} -linearly on $v \in V_I$ and are given by exterior multiplication by v while the *annihilation operator* $a_I(v)$ is its adjoint. We choose a unit vector $\Omega_I \in \bigwedge^0 V_I$ and call it the *vacuum vector*. It is annihilated by $a_I(v)$ for all $v \in V$. The following is well-known.

Lemma 2.4. *The above representation γ_I of the complexified Clifford algebra $\text{Cliff}_{\mathbb{C}}(V)$ on Fock space $\bigwedge V_I$ is irreducible.*

Proof. We may assume that V is a inductive limit of finite-dimensional Hilbert spaces, and, accordingly, that $\text{Cliff}_{\mathbb{C}}(V)$ is the C^* -algebraic inductive limit of finite-dimensional Clifford algebras. Without loss of generality we may thus assume that $\dim V < \infty$ so that $\text{Cliff}_{\mathbb{C}}(V)$ are simple matrix algebras. We invoke Schur’s Lemma to conclude that γ_I is irreducible if and only if every operator $T : V \rightarrow V$ commuting with all $\gamma_I(v)$ ($v \in V^{\mathbb{C}}$) is a scalar.

For any v , one has, using the \mathbb{C} -linearity of $a_I^*(v)$ and \mathbb{C} -anti-linearity of $a_I(v)$

$$a_I^*(v) = \frac{1}{2} (\gamma_I(v) - i\gamma_I(Iv)), \quad a_I(v) = \frac{1}{2} (\gamma_I(v) + i\gamma_I(Iv)).$$

Hence, any T that commutes with $\gamma_I(v)$ for all v commutes with $a_I^*(v)$ and $a_I(v)$. From this it follows that

$$a_I(v)(T\Omega_I) = T(a_I(v)\Omega_I) = 0$$

so that $T\Omega_I \in \Lambda^0(V_I)$. In other words, $T\Omega_I = t\Omega_I$ for some $t \in \mathbb{C}$. Moreover,

$$\begin{aligned} T(v_1 \wedge \cdots \wedge v_k) &= T(a_I^*(v_1) \cdots a_I^*(v_k)\Omega) \\ &= (a_I^*(v_1) \cdots a_I^*(v_k))(T\Omega) = t(v_1 \wedge \cdots \wedge v_k), \end{aligned}$$

so that $T = t \cdot \text{id}$. \square

Any orthogonal operator $T : V \rightarrow V$ induces an automorphism of $\text{Cliff}_{\mathbb{C}}(V)$ by sending $\gamma_I(v) \rightarrow \gamma_I(Tv)$. In some cases this automorphism can be lifted to the Fock space $\bigwedge V_I$, for instance, if U is a unitary operator on V_I . Then, if $\bigwedge U$ is the unitary operator in the Fock space such that on simple tensors

$$\bigwedge U(v_1 \wedge \cdots \wedge v_n) := U(v_1) \wedge \cdots \wedge U(v_n),$$

one has the covariance

$$\bigwedge U \circ a_I^*(v) \circ \bigwedge U^* = a_I^*(Uv).$$

We thus get the equality

$$(\bigwedge U)\gamma_I(v)(\bigwedge U^*) = \gamma_I(Uv).$$

Remark 2.5. More generally speaking, the *Shale–Stinespring Theorem* [9] says that the automorphism on $\text{Cliff}_{\mathbb{C}}(V)$ defined by an orthogonal operator $T : V \rightarrow V$ is implementable by a unitary operator on Fock space $\bigwedge V_I$ if and only if $T + ITI$ is Hilbert–Schmidt.

Suppose now that we are given a (complex) Hilbert space \mathcal{H} and a self-adjoint operator D in \mathcal{H} with compact resolvent. Again, let $V = \mathcal{H}_{\mathbb{R}}$ denote the underlying real vector space. Suppose that we take the natural complex structure on V so that $V_I = \mathcal{H}$. Then the above construction gives us an irreducible representation γ of the canonical anti-commutation relations (CAR) algebra on $\bigwedge \mathcal{H}$ but, from a physical point of view this representation is not the right one to consider. In fact, Dirac realized in [6] that in order to avoid unwanted negative energy solutions to his Dirac equation, one has to fill up (what is now called) the *Dirac sea*. Mathematically speaking, this construction is nicely captured by choosing another irreducible representation, corresponding to a different complex structure on \mathcal{H} . Let us describe it in some detail.

If E_{\pm} are the spectral projections of D corresponding to the positive and negative eigenspaces of D in \mathcal{H} , let us define the following complex structure:

$$I = i(E_+ - E_-).$$

In other words, $I = iF$ where $F = D|D|^{-1}$ is the sign^1 of D . In view of the previous section, this gives rise to another irreducible representation γ_I of $\text{Cliff}_{\mathbb{C}}(V)$ in Fock space, where the key difference with respect to the original Fock space representation γ is that i acts as $-i$ on $E_-(\mathcal{H})$. In other words, the operator D can be considered to act as $|D|$ which now only has positive eigenvalues. More precisely,

Proposition 2.6. (i) *The one-parameter group $\sigma_t \in \text{Aut}(C)$ is implemented in the (physical) Fock representation by the one-parameter unitary group $W(t) = \bigwedge \exp(it|D|)$, i.e. one has*

$$\gamma_I(\sigma_t(A)) = \bigwedge (e^{it|D|})\gamma_I(A) \bigwedge (e^{-it|D|}), \quad \forall A \in \text{Cliff}_{\mathbb{C}}(V). \tag{4}$$

(ii) *If $\exp(-\beta|D|)$ is of trace class the state ψ_{β} is of type I and is given by*

$$\psi_{\beta}(A) = \frac{1}{Z} \text{Tr} \left(\bigwedge \exp(-\beta|D|)\gamma_I(A) \right), \quad \forall A \in \text{Cliff}_{\mathbb{C}}(V) \tag{5}$$

where the normalization factor Z is finite.

Proof. The associated KMS_{β} state is obtained after normalization from the density matrix $W(i\beta)$ obtained by analytic continuation. Thus it coincides here with the operator

$$\rho = \bigwedge \exp(-\beta|D|).$$

Since $T = \exp(-\beta|D|)$ is positive and of trace class we get that $\rho = \bigwedge T$ is also positive and of trace class (with trace given by the determinant of $1 + T$). Thus $Z < \infty$. The state ψ_{β} is implemented by a density matrix in an irreducible representation and is thus of type I. \square

¹ We take the convention that the sign of 0 is 1.

3. von Neumann Information Theoretic Entropy

We start by briefly recalling von Neumann's notion of entropy. Consider a density matrix ρ on a Hilbert space \mathcal{H} , i.e. a positive trace-class operator with normalized trace. It induces a state ϕ on any C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ by setting $\phi(\cdot) = \text{Tr}(\rho \cdot)$. The entropy of this state ϕ is then defined to be

$$S(\phi) := -\text{Tr}(\rho \log \rho).$$

For composite systems $\phi_1 \otimes \phi_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ one finds the following important additivity property for entropy

$$S(\phi_1 \otimes \phi_2) = S(\phi_1) + S(\phi_2).$$

Let us start with a basic example of entropy that will play a crucial role in what follows.

Lemma 3.1. *Let $x > 0$, the entropy of the partition of the unit interval in two intervals with ratio of size x , is given by*

$$\mathcal{E}(x) := \log(x+1) - \frac{x \log(x)}{x+1}.$$

Proof. The sizes of the intervals are $\frac{1}{x+1}$ and $\frac{x}{x+1}$. One has

$$-\frac{\log\left(\frac{1}{x+1}\right)}{x+1} - \frac{x \log\left(\frac{x}{x+1}\right)}{x+1} = \frac{x+1}{x+1} \log(x+1) - \frac{x \log(x)}{x+1} = \mathcal{E}(x).$$

□

Corollary 3.2. *One has $\mathcal{E}(x) = \mathcal{E}(1/x)$ for any $x > 0$.*

Proof. The obtained partitions are isomorphic. □

The following result gives an expression for the entropy of density matrices that arise as a second-quantized operator (Fig. 1).

Lemma 3.3. *Let $T \in \mathcal{L}^1(\mathcal{H})^+$ be a positive trace class operator and ϕ the state associated to $\bigwedge T$ then*

$$S(\phi) = \text{Tr}(\mathcal{E}(T)). \tag{6}$$

Proof. Let first $T = T_1 \oplus T_2$ be an orthogonal decomposition. Let us show that the associated states fulfill $\phi = \phi_1 \otimes \phi_2$. One has with $\rho_I = \bigwedge T_I$ the equality $\rho = \rho_1 \otimes \rho_2$ by the compatibility of the wedge functor with direct sums. Then

$$\text{Tr}(\rho_1 \otimes \rho_2) = \text{Tr}(\rho_1)\text{Tr}(\rho_2) \Rightarrow \phi = \phi_1 \otimes \phi_2.$$

Next the entropy functional fulfills

$$S(\phi_1 \otimes \phi_2) = S(\phi_1) + S(\phi_2).$$

This shows that the functional $S(\phi)$ is additive for direct sum decompositions. It also applies to infinite sums and one can thus consider only the one dimensional case. In this case the state associated to the operator T of multiplication by x corresponds to $\bigwedge T$ whose spectrum is $\{1, x\}$ and hence has entropy given by the function $\mathcal{E}(x)$ of Lemma 3.1. □

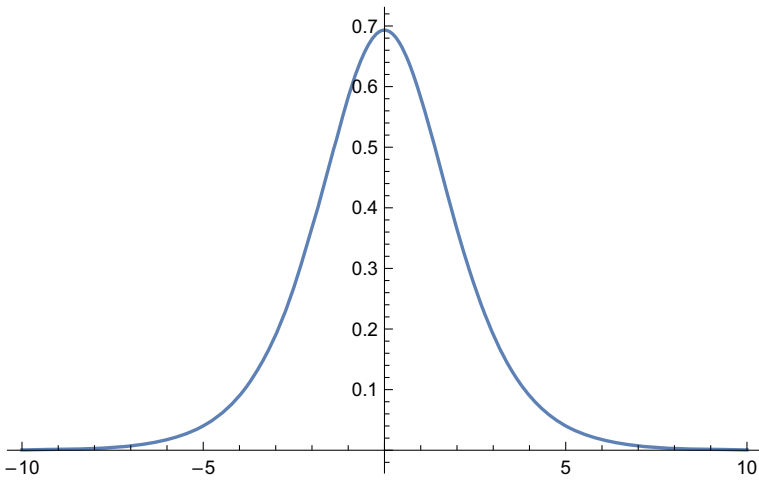


Fig. 1. Graph of the function h

Theorem 3.4. *Let \mathcal{H} , D , $C := \text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$, $\sigma_t \in \text{Aut}(C)$ and ψ_{β} be as in Proposition 2.2. Then if the operator $\exp(-\beta|D|)$ is of trace class, the state ψ_{β} is of type I and its von Neumann entropy is equal to the spectral action $\text{Tr}(h(\beta D))$ for the spectral function $h(x) := \mathcal{E}(e^{-x})$.*

Proof. The first statement follows from Proposition 2.6. The statement about the entropy follows from Lemma 3.3 together with the fact that $x \mapsto \mathcal{E}(e^{-x})$ is an even function (cf. Corollary 3.2). \square

4. The Function h and Riemann’s ξ Function

Theorem 3.4 provides a very specific choice of test function for the spectral action and in this section we shall analyze the surprising relation between this specific function and the Riemann ξ function. Thus consider

$$\begin{aligned} h(x) &:= \mathcal{E}(e^{-x}) = \frac{x}{1 + e^x} + \log(1 + e^{-x}) \\ &= \log(2) - \frac{x^2}{8} + \frac{x^4}{64} - \frac{x^6}{576} + \frac{17x^8}{92160} - \frac{31x^{10}}{1612800} + O(x^{11}). \end{aligned}$$

Lemma 4.1. *The function $h(x)$ is an even positive function of the variable $x \in \mathbb{R}$. Its derivative is*

$$h'(x) = -\frac{x}{4 \cosh^2\left(\frac{x}{2}\right)}.$$

Lemma 4.2. *One has, for $x \geq 0$,*

$$\sum_{\mathbb{Z}} \frac{(2\pi n + \pi)^2 - x}{((2\pi n + \pi)^2 + x)^2} = \frac{1}{4 \cosh^2\left(\frac{\sqrt{x}}{2}\right)}.$$

Proof. The simplest periodic function written as a convergent Eisenstein series is

$$\sum_{\mathbb{Z}} \frac{1}{(\pi n + x)^2} = \frac{1}{(\sin x)^2}. \tag{7}$$

One has

$$\frac{(2\pi n + \pi)^2 + u^2}{((2\pi n + \pi)^2 - u^2)^2} = \frac{1}{2(-2\pi n + u - \pi)^2} + \frac{1}{2(2\pi n + u + \pi)^2}.$$

Thus if we take $u = i\sqrt{x}$ we obtain

$$\sum_{\mathbb{Z}} \frac{(2\pi n + \pi)^2 - x}{((2\pi n + \pi)^2 + x)^2} = \sum_{\mathbb{Z}} \frac{1}{(2\pi n + u + \pi)^2} = \frac{1}{4} \sum_{\mathbb{Z}} \frac{1}{(\pi n + u/2 + \pi/2)^2}$$

which, using (7), gives the required equality since $\sin(u/2 + \pi/2) = \cos(u/2) = \cosh \sqrt{x}/2$. \square

Next one has

$$\int_0^\infty \left(2\pi^2(2n + 1)^2t - 1\right) e^{\pi^2(-2n+1)^2t-tx} dt = \frac{(2\pi n + \pi)^2 - x}{((2\pi n + \pi)^2 + x)^2}.$$

We want to apply the Fubini theorem to justify the formula

$$\frac{1}{4 \cosh^2\left(\frac{\sqrt{x}}{2}\right)} = \int_0^\infty g(t)e^{-tx} dt, \quad g(t) := \sum_{\mathbb{Z}} \left(2\pi^2(2n + 1)^2t - 1\right) e^{\pi^2(-2n+1)^2t}. \tag{8}$$

It suffices to show that the double sum of absolute values is convergent, *i.e.*

$$\int_0^\infty \sum_{\mathbb{Z}} |2\pi^2(2n + 1)^2t - 1| e^{\pi^2(-2n+1)^2t-tx} dt < \infty.$$

One has $ve^{-v} \leq e^{-v/2}$ and $e^{-v} \leq e^{-v/2}$ for $v > 0$ so that the above integral is bounded by

$$2 \int_0^\infty \sum_{\mathbb{Z}} e^{\pi^2(-2n+1)^2t/2-tx} dt$$

which is finite since for t small $\sum_{\mathbb{Z}} e^{\pi^2(-2n+1)^2t/2} = O\left(\frac{1}{\sqrt{t}}\right)$ while it decays exponentially for large t . This justifies the formula (8) but we now need a better estimate of the function $g(t)$ when $t \rightarrow 0$.

Lemma 4.3. *One has*

$$g(t) = \frac{-1}{4\sqrt{\pi}t^{3/2}} \sum_{\mathbb{Z}} (-1)^n n^2 q^{n^2}, \quad q = e^{-\frac{1}{4t}}. \tag{9}$$

Proof. We apply the Poisson summation formula, for the Fourier transform

$$F(f)(\omega) := \int_{-\infty}^{\infty} f(u)e^{2i\pi u\omega} dt$$

and the function

$$f(u) := \left(2\pi^2(2u + 1)^2t - 1\right) e^{\pi^2(-(2u+1)^2)t}.$$

One has

$$F(f)(\omega) = -\frac{\omega^2 e^{-i\pi\omega} e^{-\frac{\omega^2}{4t}}}{4\sqrt{\pi}t^{3/2}}.$$

Thus (9) follows from (8) and the Poisson summation formula $\sum_{\mathbb{Z}} f(n) = \sum_{\mathbb{Z}} F(f)(m)$. □

This formula (9) together with the lack of constant term in the function $\sum_{\mathbb{Z}} (-1)^n n^2 q^{n^2} = q \partial_q \theta_4(0, q)$ show that the function $\tilde{g}(t) := g(t)/2t$ is of rapid decay near $t = 0$ and in particular is integrable.

Proposition 4.4. *One has*

$$h(x) = \int_0^{\infty} e^{-tx^2} \tilde{g}(t) dt. \tag{10}$$

Proof. Let $k(x)$ be the right hand side of (10). One has, using (8),

$$\partial_x k(x) = -2x \int_0^{\infty} e^{-tx^2} t \tilde{g}(t) dt = -x \int_0^{\infty} g(t) e^{-tx^2} dt = \frac{-x}{4 \cosh^2\left(\frac{x}{2}\right)} = \partial_x h(x).$$

Thus the two functions h, k are equal because they both tend to zero when $x \rightarrow \infty$. □

Note that in terms of theta functions one has by Jacobi triple product formula

$$\theta_4(0, q) = \sum_{\mathbb{Z}} (-1)^n q^{n^2} = \prod_1^{\infty} (1 - q^{2m})(1 - q^{2m-1})^2.$$

This formula gives us a good control of the sign of the function $\sum_{\mathbb{Z}} (-1)^n n^2 q^{n^2} = q \partial_q \theta_4(0, q)$. Indeed one has $0 < q < 1$ so that all terms in the product are positive. Moreover the logarithmic derivative gives

$$q \partial_q \theta_4(0, q) = \theta_4(0, q) \sum_1^{\infty} \left(-\frac{2mq^{2m}}{1 - q^{2m}} - \frac{2(2m - 1)q^{2m-1}}{1 - q^{2m-1}} \right) \leq 0.$$

Thus by (9) we get that both $g(t)$ and $\tilde{g}(t)$ are positive functions.

We now compute the moments of the function h which are relevant in dimension 4, they are given by

Lemma 4.5. (i) *One has*

$$2 \int_0^\infty h(x)x dx = \frac{9 \zeta(3)}{2}, \quad 2 \int_0^\infty h(x)x^3 dx = \frac{225 \zeta(5)}{4}. \tag{11}$$

(ii) *More generally*

$$\int_0^\infty h(x)x^\alpha dx = \frac{1 - 2^{-\alpha-1}}{\alpha + 1} \Gamma(\alpha + 3) \zeta(\alpha + 2). \tag{12}$$

Proof. One has for $\alpha > 0$, integrating by parts,

$$\alpha \int_0^\infty h(x)x^{\alpha-1} dx = \int_0^\infty \frac{x^{\alpha+1}}{4 \cosh^2\left(\frac{x}{2}\right)} dx. \tag{13}$$

Moreover by Lemma 4.2 one has

$$\frac{1}{4 \cosh^2\left(\frac{x}{2}\right)} = 2 \sum_{n=0}^\infty \frac{(2\pi n + \pi)^2 - x^2}{((2\pi n + \pi)^2 + x^2)^2}.$$

Consider the function

$$f(x) := \frac{x^2 - 1}{(x^2 + 1)^2}.$$

One has

$$x^{-2} f\left(\frac{2\pi n + \pi}{x}\right) = \frac{(2\pi n + \pi)^2 - x^2}{((2\pi n + \pi)^2 + x^2)^2}$$

while for $\Re(y) \in (-1, 1)$

$$\int_0^\infty f(a)a^y da = \frac{\pi y}{2 \cos\left(\frac{\pi y}{2}\right)},$$

so that

$$\int_0^\infty x^{-2} f\left(\frac{2\pi n + \pi}{x}\right) x^{\alpha+1} dx = \int_0^\infty f\left(\frac{2\pi n + \pi}{x}\right) x^{\alpha-1} dx$$

and changing variables by $x' = \frac{2\pi n + \pi}{x}$ gives for $\Re(\alpha + 1) \in (-1, 1)$

$$\begin{aligned} \int_0^\infty x^{-2} f\left(\frac{2\pi n + \pi}{x}\right) x^{\alpha+1} dx &= (2\pi n + \pi)^\alpha \int_0^\infty f(x')x'^{-\alpha-1} dx \\ &= (2\pi n + \pi)^\alpha \frac{-\pi(1 + \alpha)}{2 \cos\left(\frac{-\pi(1+\alpha)y}{2}\right)}. \end{aligned}$$

Thus for $\Re(\alpha) \in (-2, -1)$ we get the equality

$$\sum_{n=0}^\infty \int_0^\infty x^{-2} f\left(\frac{2\pi n + \pi}{x}\right) x^{\alpha+1} dx = \frac{-\pi(1 + \alpha)}{2 \cos\left(\frac{-\pi(1+\alpha)}{2}\right)} \pi^\alpha \sum_{n=0}^\infty (2n + 1)^\alpha.$$

One has

$$\sum_{n=0}^{\infty} (2n + 1)^\alpha = 2^\alpha (2^{-\alpha} - 1) \zeta(-\alpha)$$

and

$$\frac{-\pi(1 + \alpha)}{2 \cos\left(\frac{-\pi(1+\alpha)}{2}\right)} \pi^\alpha = \frac{\pi^{1+\alpha}(1 + \alpha)}{2 \sin\left(\frac{\pi\alpha}{2}\right)}.$$

One has the functional equation in the form

$$\zeta(1 - s) = \frac{2}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$$

and it gives for $s = \alpha + 1$

$$\frac{\pi^{1+\alpha}(1 + \alpha)}{2 \sin\left(\frac{\pi\alpha}{2}\right)} 2^\alpha (2^{-\alpha} - 1) \zeta(-\alpha) = \frac{1}{2} (1 - 2^{-\alpha}) \Gamma(\alpha + 2) \zeta(\alpha + 1)$$

and by analytic continuation one gets (in concordance with [8, Formula 3.527(3)])

$$\begin{aligned} \int_0^\infty \frac{x^{\alpha+1}}{4 \cosh^2\left(\frac{x}{2}\right)} dx &= 2 \sum_{n=0}^\infty \int_0^\infty x^{-2} f\left(\frac{2\pi n + \pi}{x}\right) x^{\alpha+1} dx \\ &= (1 - 2^{-\alpha}) \Gamma(\alpha + 2) \zeta(\alpha + 1). \end{aligned}$$

Thus using (13) one gets (12). \square

This gives us for the coefficient of t^{-a} in the heat expansion the term

$$\frac{1}{\Gamma(a)} \int_0^\infty h(v^{\frac{1}{2}}) v^{a-1} dv = \frac{2}{\Gamma(a)} \int_0^\infty h(u) u^{2a-1} du = \frac{(1 - 2^{-2a}) \Gamma(2a + 2) \zeta(2a + 1)}{a \Gamma(a)}. \tag{14}$$

We need to check several properties of the right hand side. First it should define an everywhere holomorphic function of $a \in \mathbb{C}$ because as the coefficient of t^{-a} in the heat expansion it is also given, using (10) by

$$\int_0^\infty t^{-a} \tilde{g}(t) dt$$

which is an entire function of a . Now $\zeta(z)$ has a pole at $z = 1$ but the corresponding pole of $\zeta(2a + 1)$ is cancelled by the following term which is 0 at $a = 0$:

$$\frac{(1 - 2^{-2a})}{a \Gamma(a)}.$$

The other potential poles come from the term $\Gamma(2a + 2)$ which has a pole when $2a + 2 = -m$ is a negative integer. There are two cases: if $m = 2b$ is even, then the term $a \Gamma(a) = \Gamma(a + 1)$ also has a pole at the negative integer $-b$. If $m = 2b + 1$ is odd, then the term $a \Gamma(a) = \Gamma(a + 1)$ does not have a pole at $-b - \frac{1}{2}$ but the term $\zeta(2a + 1)$

vanishes for $2a + 2 = -m = -2b - 1$ since $2a + 1 = -2b - 2$ is a strictly negative even number.

Next we should check that when a is a strictly negative integer the formula (14) should give a rational number which agrees up to sign and division by $n!$ with the Taylor expansion of $h(\sqrt{x})$ at $x = 0$. The rationality follows from the rationality of zeta at odd negative integers.

The expansion of $h(\sqrt{x})$ at $x = 0$ is

$$h(\sqrt{x}) = \log(2) - \frac{x}{8} + \frac{x^2}{64} - \frac{x^3}{576} + \frac{17x^4}{92160} - \frac{31x^5}{1612800} + O(x^6).$$

The terms $k(a) := \frac{(1-2^{-2a})\Gamma(2a+2)}{a\Gamma(a)}$ evaluated at negative integers are of the form

$$k(-1) = -\frac{3}{2}, k(-2) = \frac{15}{4}, k(-3) = -\frac{21}{8}, k(-4) = \frac{17}{16}, k(-5) = -\frac{341}{1120}$$

while the values of zeta at odd negative integers gives

$$\zeta(-1) = -\frac{1}{12}, \zeta(-3) = \frac{1}{120}, \zeta(-5) = -\frac{1}{252}, \zeta(-7) = \frac{1}{240}, \zeta(-9) = -\frac{1}{132}$$

and one checks that the Taylor expansion of $h(\sqrt{x})$ at $x = 0$ is given by

$$h(\sqrt{x}) = \log(2) + \sum_1^\infty (-1)^n k(-n) \zeta(-2n + 1) x^n / n!.$$

To consider the odd case when a is a negative half integer we use the functional equation to rewrite the terms in (14) as follows

$$\begin{aligned} \frac{\Gamma(2a + 2)\zeta(2a + 1)}{a\Gamma(a)} &= (2a + 1) \frac{\Gamma(2a + 1)\zeta(2a + 1)}{\Gamma(a + 1)} = \frac{(2a + 1)\zeta(-2a)(2\pi)^{2a+1}}{2 \cos(\pi(a + \frac{1}{2}))\Gamma(a + 1)} \\ &= (2a + 1)\Gamma(-a)\zeta(-2a)(2\pi)^{2a+1}/(2\pi) \end{aligned}$$

using the formula of complements in the form

$$\Gamma(-a)\Gamma(a + 1) = \frac{\pi}{\cos(\pi(a + \frac{1}{2}))}.$$

Moreover in terms of Riemann’s ξ function,

$$\xi(s) := \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

one has

$$(2a + 1)\Gamma(-a)\zeta(-2a)(2\pi)^{2a+1}/(2\pi) = 2^{2a}\pi^a\xi(-2a)/a$$

and we get

$$\frac{(1 - 2^{-2a})\Gamma(2a + 2)\zeta(2a + 1)}{a\Gamma(a)} = \frac{2^{2a} - 1}{a}\pi^a\xi(-2a). \tag{15}$$

We thus get, summarizing the above computations, and changing $a \rightarrow -a$,

Proposition 4.6. *The expansion of the entropy functional in powers of beta near $\beta = 0$ is obtained from the heat expansion by setting $t = \beta^2$ and by multiplying the term in t^a by the following expression*

$$\gamma(a) = \frac{1 - 2^{-2a}}{a} \pi^{-a} \xi(2a).$$

As a corollary we double check that this is an entire function of the variable $a \in \mathbb{C}$ and that when a is a negative integer $-n$ the result does involve the value of zeta at the odd positive integer $2n + 1$. This shows that these odd zeta values show up in the even dimensional case for the coefficients of negative powers of t and in the odd dimensional case for the coefficients of positive powers of t .

$\gamma(-2) = \frac{225\zeta(5)}{4}$
$\gamma\left(-\frac{3}{2}\right) = \frac{14\pi^{7/2}}{45}$
$\gamma(-1) = \frac{9\zeta(3)}{2}$
$\gamma\left(-\frac{1}{2}\right) = \frac{\pi^{3/2}}{3}$
$\gamma(0) = \log(2)$
$\gamma\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}}$
$\gamma(1) = \frac{1}{8}$
$\gamma\left(\frac{3}{2}\right) = \frac{7\zeta(3)}{8\pi^{5/2}}$
$\gamma(2) = \frac{1}{32}$
$\gamma\left(\frac{5}{2}\right) = \frac{93\zeta(5)}{32\pi^{9/2}}$

What this table of values also shows is that the functional equation gives a duality between the coefficients of the high energy expansion in even dimension with the coefficients of the low energy expansion in the odd dimensional case.

Acknowledgements The work of A. H. C is supported in part by the National Science Foundation Grant No. Phys-1518371. A. H. C would also like to thank Institute for Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen for hospitality where part of this work was done. The involved research has partly been enabled by a Radboud Excellence Professorship awarded to Prof. Chamseddine. WvS would like to thank Sijbrand de Jong for stimulating discussions on entropy and the spectral action. WvS thanks IHÉS for hospitality during a visit in early 2018, as well as NWO for support via VIDI-Grant 016.133.326.

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A. Appendix

We recall briefly in this appendix the relation of the constants appearing in the heat expansion and the spectral action. Consider a test function $\chi(u)$ which is given as a simple superposition of exponentials as a Laplace transform of the form

$$\chi(u) = \int_0^\infty e^{-su} g(s) ds, \tag{16}$$

where the function $g(s)$ is of rapid decay near 0 and ∞ . Let Δ be a positive self-adjoint operator

$$\chi(t\Delta) = \int_0^\infty e^{-st\Delta} g(s) ds. \tag{17}$$

An asymptotic expansion of the form $\text{Trace}(\exp(-t\Delta)) \sim \sum a_\alpha t^\alpha$ then yields

$$\text{Trace}(\chi(t\Delta)) \sim \sum a_\alpha t^\alpha \int_0^\infty s^\alpha g(s) ds. \tag{18}$$

For $\alpha < 0$ one has,

$$s^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-sv} v^{-\alpha-1} dv$$

and

$$\int_0^\infty s^\alpha g(s) ds = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \chi(v) v^{-\alpha-1} dv.$$

Moreover, for $\alpha = 0$ one has

$$\int_0^\infty g(s) ds = \chi(0).$$

Thus, one gets

$$\begin{aligned} \text{Trace}(\chi(t\Delta)) &\sim \sum_{\alpha < 0} a_\alpha t^\alpha \frac{1}{\Gamma(-\alpha)} \int_0^\infty \chi(v) v^{-\alpha-1} dv \\ &\quad + a_0 \chi(0) + \dots \end{aligned} \tag{19}$$

For $\alpha = n$ a positive integer, using $(\partial_u)^n(e^{-su}) = (-1)^n s^n e^{-su}$, one has²

$$\begin{aligned} \int_0^\infty s^\alpha g(s) ds &= (-1)^n \left(\int_0^\infty (\partial_u)^n(e^{-su}) g(s) ds \right)_{u=0} \\ &= (-1)^n (\partial_u)^n \chi(u)|_{u=0} = (-1)^n \chi^{(n)}(0). \end{aligned}$$

² Note that in the odd-dimensional case α takes half-integral values and this formula does not apply.

In dimension 4 the values of negative α which are relevant are -1 and -2 , the coefficients are

$$\left(\int_0^\infty \chi(v) v dv \right) t^{-2} a_{-2} + \left(\int_0^\infty \chi(v) dv \right) t^{-1} a_{-1} + \chi(0) a_0 + \dots$$

and they can be written in terms of $f(u) = \chi(u^2)$ using

$$\int_0^\infty \chi(v) v^{-\alpha-1} dv = 2 \int_0^\infty f(u) u^{-2\alpha-1} du.$$

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Communicated by Y. Kawahigashi