The Distribution of Superconductivity Near a Magnetic Barrier

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Abstract: We consider the Ginzburg–Landau functional, defined on a two-dimensional simply connected domain with smooth boundary, in the situation when the applied magnetic field is piecewise constant with a jump discontinuity along a smooth curve. In the regime of large Ginzburg–Landau parameter and strong magnetic field, we study the concentration of the minimizing configurations along this discontinuity by computing the energy of the minimizers and their weak limit in the sense of distributions.

1. Introduction

1.1. Motivation. The Ginzburg–Landau theory, introduced in [LG50], is a phenomenological macroscopic model describing the response of a superconducting sample to an external magnetic field, when the sample is close to its critical temperature T_c . The phenomenological quantities associated with a superconductor are the order parameter ψ and the magnetic potential $\mathbf{A}$, where $|\psi|^2$ measures the density of the superconducting Cooper pairs and $\text{curl } \mathbf{A}$ represents the induced magnetic field in the sample.

In this paper, the superconducting sample is an infinite cylindrical domain subjected to a magnetic field with a direction parallel to the axis of the cylinder. For this specific geometry, it is enough to consider the horizontal cross section of the sample, $\Omega \subset \mathbb{R}^2$. The phenomenological configuration $(\psi, \mathbf{A})$ is then defined on the domain Ω .

The study of the Ginzburg–Landau model in the case of a *uniform* or a *smooth* non-uniform applied magnetic field has been the focus of much attention in the literature. We refer to the two monographs [FH10, SS07] for the uniform magnetic field case. Smooth magnetic fields are the subject of the papers [Att15a, Att15b, HK15, LP99, PK02]. Given the current interest in magnetic steps for various physical systems, we focus on the case where the applied magnetic field is a step function, which is not covered in the aforementioned papers.

Nonhomogeneous magnetic fields have been the focus of great amount of research. Current fabrication techniques allow the creation of such magnetic fields [FLBP94,

[STH+94,GGD+97], something that opens new paths in quantum physics and possible applications [RP98, JBY+97, MJR97]. Indeed, these magnetic fields appear in models involved in nanophysics such as in quantum transport in 2DEG (bidimensional electron gas) (see [PM93, RP00] and references therein) and in the Ginzburg–Landau model in superconductivity [SJST69]. More recently, piecewise constant magnetic fields are considered in the analysis of transport properties in graphene [GDMH+08, ORK+08]. Such magnetic fields are interesting because they induce snake states, carriers of edge currents flowing in the interface separating the distinct values of the magnetic field—the *magnetic barrier* (for instance see [HPRS16, HS15, DHS14, HS08, RP00, PM93]). While such edge currents have been discussed for linear problems in earlier works, the main contribution of this manuscript lies in establishing their existence in the context of the non-linear Ginzburg–Landau functional in superconductivity, by examining the presence of superconductivity along the magnetic barrier. Our configuration is illustrated in Fig. 1.

In an earlier contribution [AK16], we explored the influence of a step magnetic field on the distribution of bulk superconductivity, which highlighted the regime where an edge current might occur near the magnetic barrier. In this contribution, we will demonstrate the existence of such a current by providing examples where superconductivity concentrates at the interface separating the distinct values of the magnetic field.

1.2. The functional and the mathematical set-up. We assume that the domain Ω is open in $\mathbb{R}^2$, bounded, and simply connected. The Ginzburg–Landau (GL) free energy is given by the functional

$$\begin{aligned} \mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) = & \int_{\Omega} \left(|(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 \right. \\ & \left. + \frac{\kappa^2}{2} |\psi|^4 \right) dx + \kappa^2 H^2 \int_{\Omega} |\operatorname{curl} \mathbf{A} - B_0|^2 dx, \end{aligned} \quad (1.1)$$

with $\psi \in H^1(\Omega; \mathbb{C})$ and $\mathbf{A} = (A_1, A_2) \in H^1(\Omega; \mathbb{R}^2)$. Here, $\kappa > 0$ is a large GL parameter, the function $B_0 : \Omega \rightarrow [-1, 1]$ is the profile of the applied magnetic field, and $H > 0$ is the intensity of this applied magnetic field.

The parameter κ depends on the temperature and the type of the material. It is a physical characteristic scale of the sample, the inverse of the penetration depth, and it measures the size of vortex cores (which is proportional to κ^{-1} , in some typical situations dependent on the strength of the applied magnetic field). Vortex cores are narrow regions in the sample, which corresponds to κ being a large parameter. That is the main reason behind our analysis of the asymptotic regime $\kappa \rightarrow +\infty$, following many early contributions addressing this asymptotic regime (see e.g. [SS07]). We work under the following assumptions on the domain Ω and the magnetic field B_0 , which are quite generic as revealed from the illustration in Fig. 2.

Assumption 1.1. (1) $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are two disjoint open sets.

(2) Ω_1 and Ω_2 have a finite number of connected components.

(3) $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise smooth with (possibly) a finite number of corners.

(4) $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the union of a finite number of disjoint simple smooth curves $\{\Gamma_k\}_{k \in \mathcal{K}}$; we will refer to Γ as the *magnetic barrier*.

(5) $\Omega = (\Omega_1 \cup \Omega_2 \cup \Gamma)^\circ$ and $\partial\Omega$ is *smooth*.

(6) $\Gamma \cap \partial\Omega$ is either empty or finite.

Fig. 1. Schematic representation of the set Ω subjected to a step magnetic field B_0 , with the magnetic barrier Γ

Fig. 2. Schematic representations of the set Ω

- (7) For any $k \in \mathcal{K}$, if Γ_k intersects $\partial\Omega$ then the intersection is at two distinct points. This intersection is transversal, i.e. $T_{\partial\Omega} \times T_{\Gamma_k} \neq 0$ at the intersection point, where $T_{\partial\Omega}$ and T_{Γ_k} are respectively unit tangent vectors of $\partial\Omega$ and Γ_k .

Assumption 1.2. $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, where $a \in [-1, 1) \setminus \{0\}$ is a given constant.

The ground state of the superconductor describes its behaviour at equilibrium. It is obtained by minimizing the GL functional in (1.1) with respect to $(\psi, \mathbf{A})$. The corresponding energy is called the ground state energy, denoted by $E_{\text{g.st}}(\kappa, H)$, where

$$E_{\text{g.st}}(\kappa, H) = \inf \left\{ \mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \right\}.$$

One may restrict the minimization of the GL functional to the space $H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ where

$$H_{\text{div}}^1(\Omega) = \left\{ \mathbf{A} \in H^1(\Omega; \mathbb{R}^2) : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \mathbf{A} \cdot \nu_{\partial\Omega} = 0 \text{ on } \partial\Omega \right\}. \quad (1.2)$$

Indeed, the functional in (1.1) enjoys the property of *gauge invariance*.¹ Consequently, the ground state energy can be written as follows (see [FH10, Appendix D])

$$E_{\text{g.st}}(\kappa, H) = \inf \left\{ \mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega) \right\}. \quad (1.3)$$

This restriction allows us to make profit from some well-known regularity properties of vector fields in $H_{\text{div}}^1(\Omega)$ (see [AK16, Appendix B]).

¹ It does not change under the transformation $(\psi, \mathbf{A}) \mapsto (e^{i\varphi\kappa} \psi, \mathbf{A} + \nabla\varphi)$, for any (say smooth) function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$. The physically meaningful quantities are the gauge invariant ones, such as $|\psi|$ and $\operatorname{curl} \mathbf{A}$.

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ of $\mathcal{E}_{\kappa, H}$ are weak solutions of the following GL equations:

$$\begin{cases} (\nabla - i\kappa H \mathbf{A})^2 \psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\ -\nabla^\perp(\text{curl } \mathbf{A} - B_0) = \frac{1}{\kappa H} \text{Im}(\overline{\psi}(\nabla - i\kappa H \mathbf{A})\psi) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H \mathbf{A})\psi = 0 & \text{on } \partial\Omega, \\ \text{curl } \mathbf{A} = B_0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Here,

$$(\nabla - i\kappa H \mathbf{A})^2 \psi = \Delta \psi - i\kappa H(\text{div } \mathbf{A})\psi - 2i\kappa H \mathbf{A} \cdot \nabla \psi - \kappa^2 H^2 |\mathbf{A}|^2 \psi$$

and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ is the Hodge gradient.

1.3. Some earlier results for uniform magnetic fields. The value of the ground state energy $E_{\text{g.st}}(\kappa, H)$ depends on κ and H in a non-trivial fashion. The physical explanation is that a superconductor undergoes phase transitions as the intensity of the applied magnetic field varies.

To illustrate the dependence on the intensity of the applied magnetic field, we assume that κ is large and $H = b\kappa$, for some fixed parameter $b > 0$. Such magnetic field strengths are considered in many papers (for instance see [AH07, LP99, Pan02, SS03]).

Assuming that the applied magnetic field is uniform, which corresponds to taking $B_0 = 1$ in (1.1), the following scenario takes place. If $b > \Theta_0^{-1}$, where $\Theta_0 \approx 0.59$ is a universal constant defined in (2.5) below, the only minimizers of the GL functional are the trivial states $(0, \hat{\mathbf{F}})$, where $\text{curl } \hat{\mathbf{F}} = 1$ (see [GP99, LP99]). This corresponds in Physics to the destruction of superconductivity when the sample is submitted to a large external magnetic field, and occurs when the intensity H crosses a specific threshold value, the so-called *third critical field*, denoted by H_{C_3} .

Another well-known critical field is the *second critical field* H_{C_2} , which is much harder to define. When $H < H_{C_2}$, superconductivity is uniformly distributed in the interior of the sample (see [SS03]). This is the bulk superconductivity regime. When $H_{C_2} < H < H_{C_3}$, the surface superconductivity regime occurs: superconductivity disappears from the interior and is localized in a thin layer near the boundary of the sample (see [AH07, HFPS11, Pan02, CR14]). The transition from surface to bulk superconductivity takes place when H varies around the critical value κ , which we informally take as the definition of H_{C_2} (see [FK11]).

One more critical field left is H_{C_1} . It marks the transition from the pure superconducting phase to the phase with vortices. We refer to [SS07] for its definition.

1.4. Expected behaviour under magnetic steps. Let us return to the case where the magnetic field is a step function as in Assumption 1.2. At some stage, the expected behaviour of the superconductor in question deviates from the one submitted to a uniform magnetic field. Recently, this case was considered in [AK16] and the following was obtained. Suppose that $H = b\kappa$ and κ is large. If $b < 1/|a|$ then bulk superconductivity persists; if $b > 1/|a|$ then superconductivity disappears in the bulk of Ω_1 and Ω_2 , and may nucleate in thin layers near $\Gamma \cup \partial\Omega$ (see Assumption 1.1 and Fig. 1). The present contribution affirms the presence of superconductivity in the vicinity of Γ when b is greater than, but close to the value $1/|a|$, for some negative values of a . The precise

statements are given in Theorems 1.7 and 1.11 below.

The aforementioned behaviour of the superconductor in presence of magnetic steps is consistent with the existing literature about the electron motion near the magnetic barrier at which the strength and/or the sign of the magnetic fields change (for instance see [HPRS16, HS15, DHS14, RP00, Iwa85]). Particularly, the case where $a \in [-1, 0)$ is called the *trapping magnetic step* (see [HPRS16]), where the discontinuous magnetic field may create supercurrents (*snake orbits*) flowing along the discontinuity edge. On the other hand, such supercurrents do not seem detectable in the case when $a \in (0, 1)$, which is called the *non-trapping magnetic step*. However, the approach was generally spectral where some properties of relevant linear models were analysed (see [HPRS16, HS15, Iwa85, RP00]), and no estimates for the non-linear GL energy in (1.1) were established in these cases.

The contribution of this paper together with [AK16] provide such estimates. Particularly in the case when $a \in [-1, 0)$ and $b > 1/|a|$, Theorems 1.7 and 1.11 below establish global and local asymptotic estimates for the ground state energy $E_{\text{g.st}}(\kappa, H)$, and the L^4 -norm of the minimizing order parameter. These theorems assert the nucleation of superconductivity near the magnetic barrier Γ (and the surface $\partial\Omega$) when b crosses the threshold value $1/|a|$.

1.5. Main results. Our results are valid under the following additional assumption.

Assumption 1.3. The parameter H depends on κ in the following manner

$$H = b\kappa, \quad (1.5)$$

where b is a fixed parameter satisfying

$$b > \frac{1}{|a|}, \quad a \in [-1, 1) \setminus \{0\}.$$

Remark 1.4. Our study does not cover the potentially interesting case $a = 0$, which deserves to be studied independently in a future work. This case, referred to as *magnetic wall*, was considered in [RP00, HPRS16].

Remark 1.5. Even though the case $a \in (0, 1)$ is included in Assumption 1.3, it will not be central in our study (the reader may notice this in the majority of our theorems statements). The reason is that, our main concern is to analyse the interesting phenomenon happening when bulk superconductivity is only restricted to a narrow neighbourhood of the magnetic edge Γ , and this only occurs when the values of the two magnetic fields *interacting* near Γ are of *opposite signs*, that is when $a \in [-1, 0)$. This can be seen through the trivial cases in Section 3.2, and is consistent with the aforementioned literature findings (non-trapping magnetic steps). Moreover, the case $b < 1/|a|$ is treated previously in [AK16] and corresponds to the bulk regime.

The statements of the main theorems involve two non-decreasing continuous functions

$$\epsilon_a : [|a|^{-1}, +\infty) \rightarrow (-\infty, 0] \quad \text{and} \quad E_{\text{surf}} : [1, +\infty) \rightarrow (-\infty, 0],$$

respectively defined in (3.5) and (6.27) below. The energy E_{surf} has been studied in many papers (for instance see [CR14, FKP13, FK11, HFPS11, AH07, Pan02]). We will refer to E_{surf} as the *surface energy*. The function ϵ_a is constructed in this paper, and we will refer to it as the *barrier energy*.

Remark 1.6. It is worthy of mention that $\epsilon_a(b)$ vanishes if and only if

- $a \in (0, 1)$; or
- $a \in [-1, 0)$ and $b \geq 1/\beta_a$, where β_a is defined in (2.11) below and satisfies $\beta_a \in (0, |a|)$ (see Theorem 2.6).

The surface energy $E_{\text{surf}}(b)$ vanishes if and only if $b \geq \Theta_0^{-1}$, where Θ_0 is the constant defined in (2.5).

The main contribution of this paper is summarized in Theorems 1.7 and 1.11 below.

Theorem 1.7 [Global asymptotics] *For all $a \in [-1, 1) \setminus \{0\}$ and $b > 1/|a|$, the ground state energy $E_{\text{g.st}}(\kappa, H)$ in (1.3) satisfies, when $H = b\kappa$,*

$$E_{\text{g.st}}(\kappa, H) = E_a^{\text{L}}(b)\kappa + o(\kappa) \quad (\kappa \rightarrow +\infty), \quad (1.6)$$

where

$$E_a^{\text{L}}(b) = b^{-1/2} \left(|\Gamma| \epsilon_a(b) + |\partial\Omega_1 \cap \partial\Omega| E_{\text{surf}}(b) + |\partial\Omega_2 \cap \partial\Omega| |a|^{-1/2} E_{\text{surf}}(b|a|) \right).$$

Furthermore, every minimizer $(\psi, \mathbf{A})_{\kappa, H} \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ of the functional in (1.1) satisfies

$$\int_{\Omega} |\psi|^4 dx = -2E_a^{\text{L}}(b)\kappa^{-1} + o(\kappa^{-1}) \quad (\kappa \rightarrow +\infty). \quad (1.7)$$

Remark 1.8. In the asymptotics displayed in Theorem 1.7, the term $|\Gamma|b^{-1/2}\epsilon_a(b)$ corresponds to the energy contribution of the magnetic barrier. The rest of the terms indicate the energy contributions of the surface of the sample. In light of Remark 1.6, the critical value $b = \beta_a^{-1}$ marks the transition between the superconducting and normal states along Γ .

Remark 1.9. The edge Γ creates vertices in the case where $\Gamma \cap \partial\Omega \neq \emptyset$ (see Fig. 2) which may have non-trivial energy contributions hidden in the remainder term in (1.6). This case alters the breakdown of superconductivity too and shares some similarities with corner domains [BNF07, CG17, HK18, Ass].

Remark 1.10. Theorem 1.7 does not cover the case when the intensity of the magnetic field satisfies $b = 1/|a|$. However, we expect that some additional bulk terms will contribute to the estimate of the energy in this case, by analogy with [FK11].

Our next result, Theorem 1.11 below, describes the local behaviour of the minimizing order parameter ψ , thereby enhancing the statement in Theorem 1.7. We define the following distribution in $\mathbb{R}^2$,

$$C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathcal{T}^b(\varphi),$$

where

$$\begin{aligned} \mathcal{T}^b(\varphi) = & -2b^{-\frac{1}{2}} \left(\epsilon_a(b) \int_{\Gamma} \varphi ds_{\Gamma} + E_{\text{surf}}(b) \int_{\partial\Omega_1 \cap \partial\Omega} \varphi ds \right. \\ & \left. + |a|^{-\frac{1}{2}} E_{\text{surf}}(b|a|) \int_{\partial\Omega_2 \cap \partial\Omega} \varphi ds \right). \end{aligned}$$

Here ds_{Γ} and ds denote the arc-length measures on Γ and $\partial\Omega$ respectively.

Fig. 3. Superconductivity distribution in the set Ω subjected to a magnetic field B_0 , in the regime where $a = -1$, $H = b\kappa$, and $|a|^{-1} < b < \Theta_0^{-1}$. The white regions are in a normal state, while the grey regions carry superconductivity

Theorem 1.11 [Local asymptotics] *For all $a \in [-1, 1] \setminus \{0\}$ and $b > 1/|a|$, if $(\psi, \mathbf{A})_{\kappa, H} \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a minimizer of the functional in (1.1) for $H = b\kappa$, then, as $\kappa \rightarrow +\infty$,*

$$\kappa T_{\kappa}^b \rightharpoonup T^b \text{ in } \mathcal{D}'(\mathbb{R}^2),$$

where T_{κ}^b is the distribution in $\mathbb{R}^2$ defined as follows

$$C_c^{\infty}(\mathbb{R}^2) \ni \varphi \mapsto T_{\kappa}^b(\varphi) = \int_{\Omega} |\psi|^4 \varphi \, dx,$$

and the convergence of T_{κ}^b to T^b is understood in the following sense

$$\forall \varphi \in C_c^{\infty}(\mathbb{R}^2), \quad \lim_{\kappa \rightarrow +\infty} \kappa T_{\kappa}^b(\varphi) = T^b(\varphi). \quad (1.8)$$

Similarly as in [CR16b], we expect that the second correction term in the asymptotics in (1.8) will depend on the surface geometry of Γ and $\partial\Omega$, and will require a restrictive assumption on the way the support of the test function φ meets the edges Γ and $\partial\Omega$.

Discussion of the main results.

We will discuss the results in Theorems 1.7 and 1.11, in the interesting case where the magnetic barrier Γ intersects the boundary of Ω . Hence, we will assume that $\partial\Omega_j \cap \partial\Omega \neq \emptyset$ for $j \in \{1, 2\}$. When this condition is violated, the discussion below can be adjusted easily.

The following observations mainly rely on Remark 1.6 and the order of the values $|a|\Theta_0$, Θ_0 , β_a , and $|a|$.

- For $a = -1$, we have $\beta_a = \Theta_0 < |a|$ [see (2.26)]. Consequently, in light of Remark 1.6:
 - If $1 < b < \Theta_0^{-1}$, then the surface of the sample carries superconductivity and the entire bulk is in a normal state *except* for the region near the magnetic barrier (see Fig. 3). Moreover, the energy contributions of the magnetic barrier and the surface of the sample are of the same order and described by the surface energy, since in this case $\epsilon_a(b) = E_{\text{surf}}(b)$, see Remark 3.12. This behaviour is remarkably distinct from the case of a uniform applied magnetic field.
 - If $b \geq \Theta_0^{-1}$, then all the aforementioned energy contributions vanish, $E_a^L(b) = 0$.

Fig. 4. Superconductivity distribution in the set Ω subjected to a magnetic field B_0 , in the regime where $a \in (-\Theta_0, 0)$, $H = b\kappa$, and respectively $|a|^{-1} < b < \beta_a^{-1}$ and $\beta_a^{-1} \leq b < |a|^{-1}\Theta_0^{-1}$. The white regions are in a normal state, while the grey regions carry superconductivity

Fig. 5. Variation of β_a with respect to a , for $a \in [-1, 1] \setminus \{0\}$

- For $a \in (-1, 0)$, comparing the values β_a , Θ_0 and $|a|$ is more subtle. In (2.18), (2.23) and Theorem 2.6 below, we show that

$$\forall a \in (-\Theta_0, 0), \quad |a|\Theta_0 < \beta_a < |a| < \Theta_0. \quad (1.9)$$

Moreover, numerical results about the variation of β_a with respect to a show that β_a is strictly decreasing for $a \in [-1, 0)$ (see Fig. 5).² Having $\beta_{-1} = \Theta_0$ [see (2.26)], this suggests that $\beta_a < \Theta_0$ for $a \in (-1, 0)$. However, such a result is not rigorously established yet.

With (1.9) in hand, Theorem 1.11 and Remark 1.6 indicate the following behaviour for $a \in (-\Theta_0, 0)$ and $b > |a|^{-1}$:

- The part of the sample's surface near $\partial\Omega_1 \cap \partial\Omega$ does not carry superconductivity.
- If $|a|^{-1} < b < \beta_a^{-1}$, then *surface superconductivity* is confined to the *part* of the surface near $\partial\Omega_2 \cap \partial\Omega$. At the same time, superconductivity is observed along the magnetic barrier Γ (see Fig. 4). This behaviour is interesting for two

² The graph in Fig. 5 was obtained after a numerical computation done by Virginie Bonnaillie-Noël.

reasons. Firstly, it demonstrates the existence of the *edge current* along the magnetic barrier, which is consistent with physics (see [HPRS16]). Secondly, it marks a distinct behaviour from the one known for uniform applied magnetic fields, in which case the whole surface carries superconductivity evenly (see for instance [HK17,FKP13,Pan02]).

- If $\beta_a^{-1} \leq b < |a|^{-1}\Theta_0^{-1}$, then superconductivity only survives along $\partial\Omega_2 \cap \partial\Omega$ (see Fig. 4). Our results then display the strength of the applied magnetic field responsible for the breakdown of the edge current along the barrier.
- If $b \geq |a|^{-1}\Theta_0^{-1}$, then all energy contributions in Theorem 1.7 disappear.
- For $a \in (0, 1)$, $\beta_a = a$ [see (2.19)]. When $b > a^{-1}$, Theorem 1.11 reveals the absence of superconductivity along the magnetic barrier. As for the distribution of superconductivity along the surface of the sample, we distinguish between two regimes: *Regime 1*, $a \in (0, \Theta_0]$ The part of the boundary, $\partial\Omega_1 \cap \partial\Omega$, does not carry superconductivity. It remains to inspect the energy contribution of $\partial\Omega_2 \cap \partial\Omega$. In that respect:
 - If $a^{-1} < b < a^{-1}\Theta_0^{-1}$, then superconductivity exists along $\partial\Omega_2 \cap \partial\Omega$.
 - If $b \geq a^{-1}\Theta_0^{-1}$, then superconductivity disappears along $\partial\Omega_2 \cap \partial\Omega$.
- Regime 2*, $a \in (\Theta_0, 1)$ We observe the following :
 - If $a^{-1} < b < \Theta_0^{-1}$, then the entire surface of the sample is in a superconducting state, though the superconductivity distribution is not uniform.
 - If $\Theta_0^{-1} \leq b < a^{-1}\Theta_0^{-1}$, then only $\partial\Omega_2 \cap \partial\Omega$ carries superconductivity.
 - If $b \geq a^{-1}\Theta_0^{-1}$, then all the energy contributions in Theorem 1.7 vanish.

1.6. Notation.

- The letter C denotes a positive constant whose value may change from one formula to another. Unless otherwise stated, the constant C depends on the value of a and the domain Ω , and is independent of κ and H .
- Let $a(\kappa)$ and $b(\kappa)$ be two positive functions. We write $a(\kappa) \approx b(\kappa)$, if there exist constants κ_0 , C_1 and C_2 such that for all $\kappa \geq \kappa_0$, $C_1 a(\kappa) \leq b(\kappa) \leq C_2 a(\kappa)$.
- The quantity $o(1)$ indicates a function of κ , defined by universal quantities, the domain Ω , given functions, etc., and such that $|o(1)| \rightarrow 0$ as $\kappa \rightarrow +\infty$. Any expression $o(1)$ is independent of the minimizer $(\psi, \mathbf{A})$ of (1.1). Similarly, $\mathcal{O}(1)$ indicates a function of κ , absolutely bounded by a constant independent of the minimizers of (1.1).
- Let $n \in \mathbb{N}$, $p \in \mathbb{N}$, $N \in \mathbb{N}$, $\alpha \in (0, 1)$, $K \subset \mathbb{R}^N$ be an open set. We use the following Hölder space

$$C^{n,\alpha}(\overline{K}) = \left\{ f \in C^n(\overline{K}) \mid \sup_{x \neq y \in K} \frac{|D^n f(x) - D^n f(y)|}{|x - y|^\alpha} < +\infty \right\}.$$

- Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval. We use the space

$$B^n(I) = \{u \in L^2(I) : x^i D^j u \in L^2(I), \forall i, j \in \mathbb{N} \text{ s.t. } i + j \leq n\}. \quad (1.10)$$

1.7. Heuristics of the proofs. In this section, we present our approach in an informal way, not organized according to the order of appearance of various effective models

in the paper, but following a scheme highlighting some important links between these models.

We are mainly interested in examining the behaviour of the minimizer of the GL energy in (1.1) near the magnetic barrier Γ . Working under Assumption 1.3, one can use the (Agmon) decay estimates established in [AK16] (see Theorem 2.4) to neglect the bulk energy contribution and restrict the study near the edge Γ and the boundary $\partial\Omega$.

As the applied magnetic field behaves uniformly near $\partial\Omega \setminus \Gamma$, the study of surface superconductivity is the same as that in the case of uniform fields, frequently encountered in the literature. Therefore in Sect. 6.2, the reader is referred to the existing literature.

The rest of the paper mainly focuses on the study of superconductivity in a tubular neighbourhood of Γ . In Sect. 6, we decompose this neighbourhood into small cells, each of size $\mathcal{O}(\kappa^{-3/2})$, in order to establish the local asymptotics of the minimizer as well as the corresponding energy estimates as $\kappa \rightarrow +\infty$. This decomposition aims to reveal the existence of superconductivity in each of these small patches, in a certain regime of the applied magnetic field (i.e. for certain values of the parameter b , as in Assumption 1.3).

Using Frenet coordinates, cut-off functions, a suitable gauge transformation allowing to replace the induced magnetic field $\mathbf{A}$ by the applied magnetic field $\mathbf{F}$ ($\text{curl } \mathbf{F} = B_0$, see Lemma 2.2), together with a rescaling argument (Sects. 4–6), we may reduce the study of the GL energy in (1.1) into that of the 2D-effective energy $\mathcal{G}_{a,b,R}$ defined on the strip $S_R = (-R/2, R/2) \times (-\infty, +\infty)$, for $R > 1$ (Sect. 3)

$$\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left(b |(\nabla - i\sigma \mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx,$$

with Dirichlet boundary conditions imposed on u , where $\sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a\mathbb{1}_{\mathbb{R}_-}(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Here, x_1 and x_2 are respectively the tangential and the normal coordinates with respect to the magnetic edge. We also consider the ground state energy

$$\mathfrak{g}_a(b, R) = \inf_u \mathcal{G}_{a,b,R}(u).$$

Hence, we launch an investigation of the new energy model, $\mathcal{G}_{a,b,R}$, with a step magnetic field. It is standard to begin by exploring the linear part of this energy, which leads us to the following linear magnetic Schrödinger operator defined in the plane (Sect. 2.4)

$$\mathcal{L}_a = -(\nabla - i\sigma \mathbf{A}_0)^2,$$

where $\mathbf{A}_0(x) = (-x_2, 0)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The ground state energy corresponding to this operator is denoted by β_a . One can easily see that the non-triviality of the energy $\mathcal{G}_{a,b,R}$ minimizer (that is when $\mathfrak{g}_a(b, R) \neq 0$) is equivalent to $1/|a| < b < 1/\beta_a$ (under Assumption 1.3). Therefore, to ensure the non-emptiness of the interval $(1/|a|, 1/\beta_a)$, thus the non-triviality of our study, we shall compare the values $|a|$ and β_a .

In order to get the aforementioned comparison (of $|a|$ and β_a), we use partial Fourier transform to perform a new reduction, this time of the 2D-operator $\mathcal{L}_a$ to a 1D-effective operator in $\mathbb{R}$, $\mathfrak{h}_a[\xi]$, parametrized by $\xi \in \mathbb{R}$ (Sect. 2.4):

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{dt^2} + V_a(\xi, t),$$

with the effective potential

$$V_a(\xi, t) = \begin{cases} (\xi + at)^2 & (t < 0), \\ (\xi + t)^2 & (t > 0), \end{cases}$$

and with a lowest eigenvalue denoted by $\mu_a(\xi)$. The ground state energy β_a satisfies

$$\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi).$$

Next, we provide information about this infimum by collecting some spectral properties of the operator $\mathfrak{h}_a[\xi]$. This 1D-operator has already been considered in the literature, and some spectral information was established experimentally and rigorously in earlier works (for instance see [HPRS16, HS15, DHS14, RP00, Iwa85]). However, the approach in the aforementioned references was rather complicated, since all energy levels were examined. In addition, some of the spectral results we need in our study were not explicitly stated in these references. Therefore, for the sake of clarity and since we are only interested in the lowest eigenvalue, we opt to use a direct approach to provide such results (see Sect. 2.4). Moreover, our results slightly improve those of the aforementioned works (see Theorem 2.6). Our proofs call some spectral data of well-known effective models in the half-line (Sect. 2.3).

From Sect. 2.4, we collect the following useful properties:

- $\beta_a = a$, for $a \in (0, 1)$,
- $\beta_{-1} = \Theta_0$,
- $|a|\Theta_0 < \beta_a < |a|$, for $a \in [-1, 0)$.

Here, Θ_0 is the value in (2.5). Now, the comparison of β_a and a is in hand and a consequence of this is the following observation:

$$\mathfrak{g}_a(b, R) = 0, \quad \text{for } a \in (0, 1) \text{ and } b > \frac{1}{a}$$

while

$$\mathfrak{g}_a(b, R) < 0, \quad \text{for } a \in [-1, 0) \text{ and } \frac{1}{|a|} < b < \frac{1}{\beta_a}.$$

We highlight the contribution of Theorem 2.6 in obtaining the latter property. This gives us the desired information about the values of a and b for which our study is non-trivial. Subsequently, we neglect the case $a \in (0, 1)$ and proceed under the more restrictive assumption

$$a \in [-1, 0), \quad \frac{1}{|a|} < b < \frac{1}{\beta_a}.$$

The main results about the reduced energy $\mathcal{G}_{a,b,R}$ are stated in Theorem 3.1. In particular, this theorem introduces the limiting energy $\mathfrak{e}_a(b)$ appearing in our main theorems (Theorems 1.7 and 1.11):

$$\mathfrak{e}_a(b) = \lim_{R \rightarrow +\infty} \frac{\mathfrak{g}_a(b, R)}{R}.$$

In addition, the bounds in the last item of this theorem are important to control the error terms arising while establishing the energy and minimizer estimates in Sect. 6. The proof of Theorem 3.1 occupies Sect. 3. It relies on the approach in [Pan02, FKP13] in the case of uniform fields, with some additional technical difficulties caused by the discontinuity of our magnetic field. For instance, we step carefully while establishing some regularity properties needed in proving the existence of $\mathcal{G}_{a,b,R}$ minimizer (see Lemmas B.3–B.6).

Finally, inspired by the recent work of Correggi–Rougerie [CR14] studying the surface superconductivity in the case of constant fields (more precisely by their energy lower bound proof), we interestingly prove that the 2D-limiting energy $\epsilon_a(b)$ is nothing but a one dimensional energy, $E_{a,b}^{\text{ID}}$, defined in Sect. 3.6. This reduction serves in providing a more explicit definition of the energy $\epsilon_a(b)$ and suggests that the profile of the minimizing order parameter ψ near the edge is as follows (up to a gauge transformation):

$$\psi \approx f_0(b\kappa t)e^{i\tilde{\xi}_0 b\kappa s} \quad (1.11)$$

where $(f_0, \tilde{\xi}_0)$ is a minimizing couple of the energy $\mathcal{E}_{a,b,\xi}^{\text{ID}}$ defined in (3.16), s is the tangential distance along Γ and t is the normal distance to Γ . Such a profile suggests that the supercurrent along the edge Γ , $j = \text{Im}(\bar{\psi}(\nabla - i\kappa H\mathbf{A})\psi)$, behaves to leading order as $b\kappa\tilde{\xi}_0 f_0(0)^2\tau$, with τ being a unit tangent vector along the edge Γ .

The rigorous derivation of (1.11) is not given in the present paper, but we expect that the analysis in this paper paves the way to a future investigation of the profile of ψ displayed in (1.11). In that respect, a special attention is required due to the non-homogeneity of the order parameter ψ as revealed in Theorem 1.11; indeed ψ seems to have different profiles along Γ and the parts of $\partial\Omega$.

One remarkable aspect of our proofs is that we have not used the *a priori* elliptic L^∞ -estimate $\|(\nabla - i\kappa H\mathbf{A})\psi\|_\infty \leq C\kappa$. Such estimate is not known to hold in our case of discontinuous magnetic field B_0 . Instead, we used the easy energy estimate $\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leq C\kappa$ and the regularity of the curl-div system (see Theorem 2.3). This also spares us the complex derivation of the L^∞ -estimate (see [FH10, Chapter 11]).

1.8. Organization of the paper. Section 2 presents some preliminaries, particularly, *a priori* estimates, exponential decay results, and a linear 2D-operator with a step magnetic field. Theorem 2.6 is an improvement of a result in [HPRS16]. Section 3 introduces the 2D-reduced GL energy along with the barrier energy $\epsilon_a(\cdot)$. In Sect. 4, we present the Frenet coordinates defined in a tubular neighbourhood of the curve Γ . These coordinates are frequently used in the context of surface superconductivity (see [FH10, Appendix F]). In Sect. 5, we introduce a reference energy that describes the local behaviour of the full GL energy in (1.1). Section 6 is devoted for the analysis of the local behaviour of the minimizing order parameter near the edge Γ . Also, we recall well-known results about the local behaviour of the order parameter near the surface $\partial\Omega$. Finally, collecting all the estimates established in Sect. 6, we complete the proof of our main theorems (Theorems 1.7 and 1.11 above).

2. Preliminaries

2.1. *A priori estimates.* We present some celebrated estimates needed in the sequel to control the various errors arising while estimating the energy in (1.1).

Proposition 2.1. *If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ is a weak solution of (1.4), then*

$$\|\psi\|_{L^\infty(\Omega)} \leq 1.$$

The proof of Proposition 2.1 can be found in [FH10, Proposition 10.3.1].

Recall the magnetic field B_0 introduced in Assumption 1.2. In the next lemma, we will fix the gauge for the magnetic potential generating B_0 (see [AK16, Lemma A.1])

Lemma 2.2. *Suppose that the conditions in Assumptions 1.1 and 1.2 hold. There exists a unique vector field $\mathbf{F} \in H_{\text{div}}^1(\Omega)$ such that*

$$\text{curl } \mathbf{F} = B_0.$$

Furthermore, $\mathbf{F}$ is in $C^\infty(\Omega_i)$, $i = 1, 2$.

We collect below some useful estimates whose proofs are given in [AK16, Theorem 4.2].

Theorem 2.3. *Let $\alpha \in (0, 1)$ be a constant. Suppose that the conditions in Assumptions 1.1 and 1.2 hold. There exists a constant $C > 0$ (dependent on b) such that if (1.5) is satisfied and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a solution of (1.4), then*

- (1) $\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \leq C\kappa.$
- (2) $\|\text{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \frac{C}{\kappa}.$
- (3) $\mathbf{A} - \mathbf{F} \in H^2(\Omega)$ and $\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq \frac{C}{\kappa}.$
- (4) $\mathbf{A} - \mathbf{F} \in C^{0,\alpha}(\overline{\Omega})$ and $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leq \frac{C}{\kappa}.$

2.2. Exponential decay of the order parameter. The following theorem displays a regime for the intensity of the applied magnetic field where the order parameter and the GL energy are exponentially small in the bulk of the domains Ω_1 and Ω_2 .

Theorem 2.4. *Given $a \in [-1, 1] \setminus \{0\}$ and $b > 1/|a|$, there exist constants $\kappa_0 > 0$, $C > 0$, and $\alpha_0 > 0$ such that, if*

$$\kappa \geq \kappa_0, \quad \kappa_0 \kappa^{-1} \leq \ell < 1, \quad \text{and } (\psi, \mathbf{A}) \text{ is a solution of (1.4) for } H = b\kappa,$$

then

$$\int_{\Omega_j \cap \{\text{dist}(x, \partial\Omega_j) \geq \ell\}} \left(|\psi|^2 + (\kappa H)^{-1} |(\nabla - i\kappa H\mathbf{A})\psi|^2 \right) dx \leq C\kappa^{-1} e^{-\alpha_0 \kappa \ell},$$

for $j \in \{1, 2\}$.

Remark 2.5. In the proof of Theorem 2.4 below, we see that $\alpha_0 \rightarrow 0$ when $b \rightarrow (1/|a|)_+$.

Proof of Theorem 2.4. The proof is a consequence of the Agmon-type estimates established in [AK16, Theorems 1.5 & 7.3]; indeed, for a fixed $b > 1/|a|$, there exist $\kappa_0, C > 0$ such that, for $\kappa \geq \kappa_0$ and $H = b\kappa$,

$$\begin{aligned} & \int_{\Omega_j \cap \{\text{dist}(x, \partial\Omega_j) \geq \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \frac{1}{\kappa H} |(\nabla - i\kappa H\mathbf{A})\psi|^2 \right) \exp \left(2\varepsilon \sqrt{\kappa H} \text{dist}(x, \partial\Omega_j) \right) dx \\ & \leq C \int_{\Omega_j \cap \{\text{dist}(x, \partial\Omega_j) \leq \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 dx, \end{aligned} \tag{2.1}$$

for $j \in \{1, 2\}$ and $0 < \varepsilon < b - 1/|a|$. We modify the choice of κ_0 so that $\kappa_0 \geq 1/\sqrt{b}$. That way, for $\kappa \geq \kappa_0$ and $\kappa_0 \kappa^{-1} \leq \ell < 1$, we get $\ell \geq 1/\sqrt{\kappa H}$. Using (2.1), one can easily verify the claim of Theorem 2.4, with $\alpha_0 = \alpha_0(b) = 2\varepsilon\sqrt{b}$. $\square$

2.3. *A family of Sturm–Liouville operators on $L^2(\mathbb{R}_+)$.* In this section, we will briefly present some spectral properties of the self-adjoint realization on $L^2(\mathbb{R}_+)$ of the Sturm–Liouville operator:

$$H[\gamma, \xi] = -\frac{d^2}{dt^2} + (t - \xi)^2, \quad (2.2)$$

defined over the domain

$$\text{Dom}(H[\gamma, \xi]) = \{u \in B^2(\mathbb{R}_+) : u'(0) = \gamma u(0)\},$$

where $B^2(\mathbb{R}_+)$ is the space introduced in (1.10), and ξ and γ are two real parameters.

Denote by $\mu(\gamma, \xi)$ the lowest eigenvalue of the operator $H[\gamma, \xi]$

$$\mu(\gamma, \xi) = \inf \text{sp}(H[\gamma, \xi]). \quad (2.3)$$

For all $\gamma \in \mathbb{R}$, we define

$$\Theta(\gamma) = \inf_{\xi \in \mathbb{R}} \mu(\gamma, \xi). \quad (2.4)$$

The particular case where $\gamma = 0$ corresponds to the *Neumann* realization, and we use the following notation,

$$H^N[\xi] = H[0, \xi], \quad \mu^N(\xi) = \mu(0, \xi), \quad \Theta_0 = \Theta(0). \quad (2.5)$$

For all $\gamma \in \mathbb{R}$, there exists a unique minimum $\xi(\gamma)$ for the function $\xi \mapsto \mu(\gamma, \xi)$. Furthermore (see [Kac06, Sect. 2.3])

$$\xi(\gamma) = \sqrt{\Theta(\gamma) + \gamma^2} \quad (2.6)$$

and

$$\forall \gamma \geq 0, \quad 0 < \Theta(\gamma) < 1. \quad (2.7)$$

2.4. *An operator with a step magnetic field.* Let $a \in [-1, 1) \setminus \{0\}$. We consider the magnetic potential $\mathbf{A}_0$ defined by

$$\mathbf{A}_0(x) = (-x_2, 0) \quad \left(x = (x_1, x_2) \in \mathbb{R}^2 \right), \quad (2.8)$$

which satisfies $\text{curl } \mathbf{A}_0 = 1$. We define the step function σ as follows. For $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a \mathbb{1}_{\mathbb{R}_-}(x_2). \quad (2.9)$$

We introduce the self-adjoint *magnetic Hamiltonian*

$$\mathcal{L}_a = -(\nabla - i\sigma \mathbf{A}_0)^2, \quad (2.10)$$

with domain of definition

$$\text{Dom}(\mathcal{L}_a) = \{u \in L^2(\mathbb{R}^2) : (\nabla - i\sigma \mathbf{A}_0)^j u \in L^2(\mathbb{R}^2), \text{ for } j \in \{1, 2\}\}.$$

The ground state energy of the operator $\mathcal{L}_a$ is denoted by

$$\beta_a = \inf \operatorname{sp}(\mathcal{L}_a). \quad (2.11)$$

Since the Hamiltonian $\mathcal{L}_a$ is invariant with respect to translations in the x_1 -direction then, by using the partial Fourier transform with respect to the x_1 -variable, we can reduce $\mathcal{L}_a$ to a family of Shrödinger operators on $L^2(\mathbb{R})$, $\mathfrak{h}_a[\xi]$, parametrized by $\xi \in \mathbb{R}$ and called *fiber operators* (see [HPRS16, HS15]). The operator $\mathfrak{h}_a[\xi]$ is defined by

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{dt^2} + V_a(\xi, t), \quad (2.12)$$

with

$$V_a(\xi, t) = \begin{cases} (\xi + at)^2, & t < 0, \\ (\xi + t)^2, & t > 0. \end{cases} \quad (2.13)$$

The domain of $\mathfrak{h}_a[\xi]$ is given by

$$\operatorname{Dom}(\mathfrak{h}_a[\xi]) = \left\{ u \in B^1(\mathbb{R}) : \left(-\frac{d^2}{dt^2} + V_a(\xi, t) \right) u \in L^2(\mathbb{R}), u'(0_+) = u'(0_-) \right\}.$$

The quadratic form associated to $\mathfrak{h}_a[\xi]$ is

$$q_a[\xi](u) = \int_{\mathbb{R}} (|u'(t)|^2 + V_a(\xi, t)|u(t)|^2) dt, \quad (2.14)$$

defined on the form domain

$$\operatorname{Dom}(q_a[\xi]) = B^1(\mathbb{R}). \quad (2.15)$$

The spectra of the operators $\mathcal{L}_a$ and $\mathfrak{h}_a[\xi]$ are linked together as follows (see [FH10, Sect. 4.3])

$$\operatorname{sp}(\mathcal{L}_a) = \overline{\bigcup_{\xi \in \mathbb{R}} \operatorname{sp}(\mathfrak{h}_a[\xi])}. \quad (2.16)$$

We introduce the lowest eigenvalue of the fiber operator $\mathfrak{h}_a[\xi]$,

$$\mu_a(\xi) = \inf \operatorname{sp}(\mathfrak{h}_a[\xi]) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{q_a[\xi](u)}{\|u\|_{L^2(\mathbb{R})}^2}. \quad (2.17)$$

Consequently, for all $a \in [-1, 1] \setminus \{0\}$, we may express the ground state energy in (2.11) by

$$\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi). \quad (2.18)$$

Below, we collect some properties of the eigenvalue $\mu_a(\xi)$.

The case $0 < a < 1$ This case is studied in [HS15, Iwa85]. The eigenvalue $\mu_a(\xi)$ is simple and is a decreasing function of ξ . The monotonicity of $\mu_a(\cdot)$ and its asymptotics in Proposition A.4 imply that

$$a < \mu_a(\xi) < 1 \quad (\xi \in \mathbb{R}),$$

and that β_a , introduced in (2.11), satisfies

$$\beta_a = a. \quad (2.19)$$

The case $a = -1$ This case is studied in [HPRS16]. Using symmetry arguments, $\mu_{-1}(\xi)$ is simple and satisfies

$$\mu_{-1}(\xi) = \mu^N(-\xi), \quad (2.20)$$

where $\mu^N(\cdot)$ is introduced in (2.5). By (2.5)–(2.7),

$$0 < \min_{\xi \in \mathbb{R}} \mu_{-1}(\xi) = \mu_{-1}(\xi_{-1}) = \Theta_0 < 1, \quad (2.21)$$

where $\xi_{-1} = -\xi(0) = -\sqrt{\Theta_0}$, Θ_0 and $\xi(0)$ are respectively introduced in (2.5) and (2.7).

The case $-1 < a < 0$ See also [HPRS16] for the study of this case. The eigenvalue $\mu_a(\xi)$ is simple, and there exists $\xi_a < 0$ satisfying³

$$|a| \geq \mu_a(\xi_a) = \min_{\xi \in \mathbb{R}} \mu_a(\xi). \quad (2.22)$$

Moreover, we have (see Proposition A.6)

$$|a|\Theta_0 < \min_{\xi \in \mathbb{R}} \mu_a(\xi). \quad (2.23)$$

Combining the foregoing discussion in the case $a \in [-1, 0)$, we get that β_a , introduced in (2.11), satisfies

$$|a|\Theta_0 \leq \beta_a \leq |a|, \quad (2.24)$$

$$\beta_a = \mu_a(\xi_a) \text{ with } \xi_a < 0. \quad (2.25)$$

In particular,

$$\beta_{-1} = (\xi_{-1})^2 = \Theta_0. \quad (2.26)$$

In the next theorem, we will use a direct approach, different from the one in [HPRS16], to establish the existence of a global minimum ξ_a in the case when $a \in (-1, 0)$ and to prove that $\beta_a < |a|$. Theorem 2.6 slightly improves the estimates in [HPRS16], since it provides an upper bound of β_a stronger than $|a|$. This theorem is necessary to validate the hypothesis $1/|a| < 1/\beta_a$ in (3.7), under which we work in Sect. 3. Indeed, it guarantees the existence of a non-empty b -parameter region where the minimizer of the reduced GL energy $\mathcal{G}_{a,b,R}$, introduced in Sect. 3, is non-trivial, which is key in the study of this energy.

Theorem 2.6. *For all $a \in (-1, 0)$, there exists $\xi < 0$ such that $\mu_a(\xi)$, the lowest eigenvalue of the operator $\mathfrak{h}_a[\xi]$, satisfies*

$$\mu_a(\xi) < |a|\Theta \left(\sqrt{\frac{1}{2|a|(1-|a|)}} \right) < |a|,$$

where $\Theta(\cdot)$ is defined in (2.4). Consequently, the function $\xi \mapsto \mu_a(\xi)$ admits a global minimum satisfying

$$\min_{\xi \in \mathbb{R}} \mu_a(\xi) < |a|.$$

³ ξ_a was not explicitly proven to be negative in [HPRS16]. For the convenience of the reader, we show that $\xi_a < 0$ in Proposition A.7.

Proof. The proof is inspired by [Kac07]. For all $\gamma \in \mathbb{R}$, let $\Theta(\gamma)$ and $\xi(\gamma)$ be the quantities introduced in (2.4) and (2.6) respectively (such that $\Theta(\gamma) = \mu(\gamma, \xi(\gamma))$). Denote by $\varphi_\gamma = \varphi_{\gamma, \xi(\gamma)}$ the positive L^2 -normalized eigenfunction of the operator in (2.2) with eigenvalue $\Theta(\gamma)$. Define the function

$$u(t) = \begin{cases} \varphi_\gamma(0) \exp(-mt), & t \geq 0, \\ \varphi_\gamma(-\sqrt{|a|}t), & t < 0, \end{cases} \quad (2.27)$$

where γ and m are two positive constants to be fixed later. One can check that $u \in \text{Dom}(q_a[\xi])$, hence by the min-max principle, for all $\xi \in \mathbb{R}$,

$$\mu_a(\xi) \leq \frac{q_a[\xi](u)}{\|u\|_{L^2(\mathbb{R})}^2}. \quad (2.28)$$

Pick $\xi \in \mathbb{R}$. We will choose ξ precisely later. The quadratic form $q_a[\xi](u)$ defined in (2.14) can be decomposed as follows:

$$q_a[\xi](u) = q_a^{(1)}[\xi](u) + q_a^{(2)}[\xi](u),$$

where

$$q_a^{(1)}[\xi](u) = \int_0^{+\infty} (|u'(t)|^2 + |(t + \xi)u(t)|^2) dt,$$

and

$$q_a^{(2)}[\xi](u) = \int_{-\infty}^0 (|u'(t)|^2 + |(at + \xi)u(t)|^2) dt.$$

A simple computation gives

$$q_a^{(1)}[\xi](u) = \left(\frac{m}{2} + \frac{\xi^2}{2m} + \frac{\xi}{2m^2} + \frac{1}{4m^3} \right) |\varphi_\gamma(0)|^2. \quad (2.29)$$

On the other hand, for $t < 0$, we do the change of variable $y = -\sqrt{|a|}t$, which in turn yields

$$q_a^{(2)}[\xi](u) = \sqrt{|a|} \int_0^{+\infty} \left(|\varphi'_\gamma(y)|^2 + \left| \left(y + \frac{\xi}{\sqrt{|a|}} \right) \varphi_\gamma(y) \right|^2 \right) dy.$$

Now we select $\xi = -\sqrt{|a|}\xi(\gamma)$ [see (2.6)]. That way we get

$$q_a^{(2)}[\xi](u) = \sqrt{|a|} (\Theta(\gamma) - \gamma |\varphi_\gamma(0)|^2). \quad (2.30)$$

The definition of the function u in (2.27) yields

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt = \frac{|\varphi_\gamma(0)|^2}{2m} + \frac{1}{\sqrt{|a|}}. \quad (2.31)$$

Combining the results in (2.29)–(2.31) and using (2.7), we rewrite (2.28) as follows

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\Theta(\gamma) + \left(\frac{m}{2} - \sqrt{|a|}\gamma + \frac{|a|(\Theta(\gamma)+\gamma^2)}{2m} - \frac{\sqrt{|a|(\Theta(\gamma)+\gamma^2)}}{2m^2} + \frac{1}{4m^3}\right)|\varphi_\gamma(0)|^2}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$

Since $0 < \Theta(\gamma) < 1$ for $\gamma > 0$,

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\Theta(\gamma) + \left(\frac{m}{2} - \sqrt{|a|}\gamma + \frac{|a|(1+\gamma^2)}{2m} - \frac{\sqrt{|a|}\gamma}{2m^2} + \frac{1}{4m^3}\right)|\varphi_\gamma(0)|^2}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$

Now we choose $\gamma = \sqrt{1/(2|a|(1-|a|))}$ and $m = \sqrt{|a|}\gamma$. Using again the fact that $\Theta(\gamma) < 1$, we obtain

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\Theta(\gamma)}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}} < |a|\Theta(\gamma) < |a|. \quad (2.32)$$

The existence of the global minimum is now standard (it is a consequence of Proposition A.4 in the appendix). $\square$

Remark 2.7. Collecting the foregoing results in (2.19)–(2.23) and Theorem 2.6, we deduce the following facts regarding the bottom of the spectrum of the operator $\mathcal{L}_a$ introduced in (2.10).

- (1) For all $a \in (0, 1)$, $\beta_a = a$.
- (2) For all $a \in [-1, 0)$, $|a|\Theta_0 \leq \beta_a < |a|$, and there exist $\xi_a < 0$ and a L^2 -normalized function $\phi_a \in B^2(\mathbb{R})$ such that

$$\mathfrak{h}_a[\xi_a]\phi_a = \beta_a\phi_a \text{ in } \mathbb{R}, \quad (2.33)$$

where $\mathfrak{h}_a[\cdot]$ is introduced in (2.12).

3. Reduced Ginzburg–Landau Energy

3.1. The functional and the main result. Assume that $a \in [-1, 1) \setminus \{0\}$ is fixed, σ is the step function defined in (2.9) and $\mathbf{A}_0$ is the magnetic potential defined in (2.8). For every $R > 1$, consider the strip

$$S_R = (-R/2, R/2) \times (-\infty, +\infty). \quad (3.1)$$

We introduce the space

$$\mathcal{D}_R = \left\{ u \in L^2(S_R) : (\nabla - i\sigma\mathbf{A}_0)u \in L^2(S_R), u\left(x_1 = \pm\frac{R}{2}, x_2\right) = 0 \right\}. \quad (3.2)$$

Note that the boundary condition in the domain $\mathcal{D}_R$ is meant in the trace sense. For $b > 0$, we define the following Ginzburg–Landau energy on $\mathcal{D}_R$ by

$$\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left(b|(\nabla - i\sigma\mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx, \quad (3.3)$$

along with the ground state energy

$$\mathfrak{g}_a(b, R) = \inf_{u \in \mathcal{D}_R} \mathcal{G}_{a,b,R}(u). \quad (3.4)$$

Our objective is to prove

Theorem 3.1. *Assume that $a \in [-1, 1) \setminus \{0\}$, $b \geq 1/|a|$, $R > 1$, $\mathfrak{g}_a(b, R)$ is the ground state energy in (3.4), and β_a is defined in (2.11).*

The following holds:

- (1) $\mathfrak{g}_a(b, R) \leq 0$.
- (2) If $a \in (0, 1)$, then $\mathfrak{g}_a(b, R) = 0$.
- (3) If $a \in [-1, 0)$, then there exists a constant $\epsilon_a(b) \leq 0$ such that

$$\lim_{R \rightarrow +\infty} \frac{\mathfrak{g}_a(b, R)}{R} = \epsilon_a(b). \quad (3.5)$$

Furthermore, $\epsilon_a(b) = 0$ if and only if $b \geq 1/\beta_a$.

- (4) For all $a \in [-1, 0)$, the function $[1/|a|, +\infty) \ni b \mapsto \epsilon_a(b)$ is monotone non-decreasing and continuous.
- (5) For all $a \in [-1, 0)$, there exists $C > 0$ such that

$$\forall R \geq 4, \quad \epsilon_a(b) \leq \frac{\mathfrak{g}_a(b, R)}{R} \leq \epsilon_a(b) + \frac{C}{R^{\frac{1}{3}}}. \quad (3.6)$$

The proof of Theorem 3.1, along with other properties of $\epsilon_a(b)$, will occupy the rest of this section.

3.2. The trivial case. We start by handling the trivial situation where the ground state energy vanishes:

Lemma 3.2. *If $a \in [-1, 1) \setminus \{0\}$ and $b \geq 1/\beta_a$, then for all $R > 1$, $\mathfrak{g}_a(b, R) = 0$.*

Remark 3.3.

- (1) Under the assumptions in Lemma 3.2, the function $u = 0 \in \mathcal{D}_R$ is the unique minimizer of the functional in (3.3).
- (2) When $a \in (0, 1)$, $\beta_a = a$ by Remark 2.7, hence Lemma 3.2 yields that $\mathfrak{g}_a(b, R) = 0$ for all $b \geq 1/a$ and $R > 1$.

Proof of Lemma 3.2. We have the obvious upper bound $\mathfrak{g}_a(b, R) \leq \mathcal{G}_{a,b,R}(0) = 0$. For the lower bound, pick an arbitrary function $u \in \mathcal{D}_R$ and extend it by zero on $\mathbb{R}^2$. Using the min–max principle, we get

$$\mathcal{G}_{a,b,R}(u) \geq b\beta_a \int_{S_R} |u|^2 dx + \int_{S_R} \left(-|u|^2 + \frac{1}{2}|u|^4 \right) dx \geq 0 \text{ since } b \geq \frac{1}{\beta_a}. \quad \square$$

3.3. Existence of minimizers. Now we handle the following case (which is complementary to the one in Lemma 3.2)

$$-1 \leq a < 0 \quad \text{and} \quad \frac{1}{|a|} \leq b < \frac{1}{\beta_a}, \quad (3.7)$$

where β_a is the lowest eigenvalue introduced in (2.11). Under the hypothesis in (3.7), we can prove the existence of a non-trivial minimizer of the functional in (3.3) along with decay estimates at infinity.

Proposition 3.4. *Assume that (3.7) holds. For all $R > 1$, there exists a function $\varphi_{a,b,R} \in \mathcal{D}_R$ such that $\varphi_{a,b,R} \not\equiv 0$ for R large enough,*

$$\mathcal{G}_{a,b,R}(\varphi_{a,b,R}) = \mathfrak{g}_a(b, R), \quad \text{and} \quad \|\varphi_{a,b,R}\|_{L^\infty(S_R)} \leq 1. \quad (3.8)$$

Here $\mathcal{G}_{a,b,R}$ is the functional in (3.3) and $\mathfrak{g}_a(b, R)$ is the ground state energy in (3.4).

Furthermore, there exists a universal constant $C > 0$ such that, for all $R > 1$, the function $\varphi_{a,b,R}$ satisfies

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left(|(\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \quad (3.9)$$

and

$$\int_{S_R} \left(b |(\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR. \quad (3.10)$$

The proof of Proposition 3.4 relies on the approach in [FKP13, Theorem 3.6] and [Pan02]. It can be described in a heuristic manner as follows. The unboundedness of the set S_R makes the existence of the minimizer $\varphi_{a,b,R}$ in (3.8) non-trivial. To overcome this issue, we consider a reduced Ginzburg–Landau energy $\mathcal{G}_{a,b,R,m}$ defined on the bounded set $S_{R,m} = (-R/2, R/2) \times (-m, m)$, and we establish some decay estimates of its minimizer $\varphi_{a,b,R,m}$. Later, using a limiting argument on $\mathcal{G}_{a,b,R,m}$ and $\varphi_{a,b,R,m}$ for large values of m , we obtain the existence of the minimizer $\varphi_{a,b,R}$ together with the properties in Proposition 3.4. The details are given in Appendix B for the convenience of the reader.

3.4. The limit energy. In this section, we will prove the existence of the limit energy $\epsilon_a(b)$, defined as the limit of $\mathfrak{g}_a(b, R)/R$ as $R \rightarrow +\infty$. After that, we will study, when the parameter a is fixed, some properties of the function $b \mapsto \epsilon_a(b)$.

In the sequel, we assume that a, b, R are constants such that $R \geq 1$ and (3.7) holds. The next lemma displays some simple, yet very important, property of the energy. This property is mainly needed in Theorem 3.1 to establish an upper bound of the limit energy $\epsilon_a(b)$.

Lemma 3.5. *Let $n \in \mathbb{N}$. Consider the ground state energy $\mathfrak{g}_a(b, R)$ defined in (3.4), then*

$$\mathfrak{g}_a(b, nR) \leq n\mathfrak{g}_a(b, R).$$

Proof. Lemma 3.5 follows from the translation invariance of the integrand of $\mathcal{G}_{a,b,R}$ with respect to the variable x_1 and the Dirichlet boundary conditions, where $\mathcal{G}_{a,b,R}$ is defined in (3.3). $\square$

Our next result easily follows from the property of monotonicity with respect to the domain.

Lemma 3.6. *The function $R \mapsto g_a(b, R)$ defined in (3.4) is monotone non-increasing.*

The existence of the limit of $g_a(b, R)/R$ as $R \rightarrow +\infty$ will be derived from a well-known abstract result (see [FK13, Lemma 2.2]). To apply this abstract result, we need some estimates on the energy $g_a(b, R)$, that we give in Lemmas 3.7 and 3.8 below.

Lemma 3.7. *Let $g_a(b, R)$ be the ground state energy in (3.4). There exist positive constants C_1 , C_2 , and C_3 dependent only on a and b such that*

$$-C_1 R \leq \frac{g_a(b, R)}{1 - b\beta_a} \leq -C_2 R + \frac{C_3}{R}. \quad (3.11)$$

Proof. Upper bound. Let $\theta \in C_c^\infty(\mathbb{R})$ be a smooth cut-off function satisfying

$$\text{supp } \theta \subset \left(-\frac{1}{2}, \frac{1}{2}\right), \quad 0 \leq \theta \leq 1, \quad \theta = 1 \text{ in } \left[-\frac{1}{4}, \frac{1}{4}\right],$$

and let $\theta_R(x) = \theta(x/R)$. Recall the function $\phi_a \not\equiv 0$ defined in (2.33), we define in $\mathbb{R}^2$ the functions

$$\psi_a(x_1, x_2) = e^{i\xi_a x_1} \phi_a(x_2) \quad \text{and} \quad v(x_1, x_2) = \theta_R(x_1) \psi_a(x_1, x_2). \quad (3.12)$$

The function ψ_a satisfies $-(\nabla - i\sigma \mathbf{A}_0)^2 \psi_a = \beta_a \psi_a$. Multiplying this equation by $\overline{\psi_a} \theta_R^2$ and integrating by parts yield

$$\begin{aligned} \beta_a \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 dx &= \int_{S_R} \theta_R^2(x_1) |(\nabla - i\sigma \mathbf{A}_0) \psi_a(x)|^2 dx \\ &\quad + 2 \int_{S_R} \theta_R(x_1) \theta_R'(x_1) \overline{\psi_a}(x) (\nabla - i\sigma \mathbf{A}_0) \psi_a(x) dx. \end{aligned}$$

Taking the real part of each side of the equation above, we get

$$\begin{aligned} \beta_a \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 dx &= \int_{S_R} \theta_R^2(x_1) |(\nabla - i\sigma \mathbf{A}_0) \psi_a(x)|^2 dx \\ &\quad + 2 \text{Re} \int_{S_R} \theta_R(x_1) \theta_R'(x_1) \overline{\psi_a}(x) (\nabla - i\sigma \mathbf{A}_0) \psi_a(x) dx \\ &= \int_{S_R} |(\nabla - i\sigma \mathbf{A}_0) v|^2 dx - \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 dx \\ &= \int_{S_R} |(\nabla - i\sigma \mathbf{A}_0) v|^2 dx - \int_{S_R} \theta_R^2(x_1) |\phi_a(x_2)|^2 dx. \end{aligned}$$

Hence, using $\|\phi_a\|_{L^2(\mathbb{R})} = 1$ and the properties of θ_R in (3.12), we obtain

$$\int_{S_R} |(\nabla - i\sigma \mathbf{A}_0) v|^2 dx \leq \beta_a \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 dx + \frac{C}{R}.$$

Consequently, for $t = \sqrt{(1 - b\beta_a)/\nu_a}$ and $\nu_a = \int_{\mathbb{R}} |\phi_a(x_2)|^4 dx_2$, we get

$$\begin{aligned} \mathfrak{g}_a(b, R) &\leq \mathcal{G}_{a,b,R}(tv) \\ &\leq t^2 R \left((b\beta_a - 1) + \frac{t^2}{2} \int_{\mathbb{R}} |\phi_a(x_2)|^4 dx_2 \right) + Cb \frac{t^2}{R} \\ &\leq (1 - b\beta_a) \left(-C_2 R + \frac{C_3}{R} \right), \end{aligned}$$

where $C_2 = (1/2)t^2$ and $C_3 = Cb/\nu_a$.

Lower bound. Let $\varphi = \varphi_{a,b,R}$ be the minimizer in Proposition 3.4. It follows from the min–max principle that

$$\mathfrak{g}_a(b, R) = \mathcal{G}_{a,b,R}(\varphi) \geq (b\beta_a - 1) \int_{S_R} |\varphi|^2 dx.$$

By (3.10), $\int_{S_R} |\varphi|^2 dx \leq CbR$, where $C > 0$ is some constant. Hence, choosing $C_1 = C/\beta_a$ establishes the desired lower bound. $\square$

Lemma 3.8. *There exists a universal constant C such that, for all $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, the ground state energy $\mathfrak{g}_a(b, R)$ defined in (3.4) satisfies, for $R > 1$,*

$$\frac{\mathfrak{g}_a(b, n^2 R)}{n^2 R} \geq \frac{\mathfrak{g}_a(b, (1 + \alpha)^2 R)}{(1 + \alpha)^2 R} - Cb^2 \left(\alpha + \frac{1}{\alpha^2 R} \right). \quad (3.13)$$

Proof. Let $n \geq 1$ be a natural number, $\alpha \in (0, 1)$ and consider the family of strips

$$S_j = \left(-n^2 - 1 - \alpha + (2j - 1) \left(1 + \frac{\alpha}{2} \right), -n^2 - 1 + (2j + 1) \left(1 + \frac{\alpha}{2} \right) \right) \times \mathbb{R}, \quad (j \in \mathbb{Z}).$$

Notice that the width of S_j is $2(1 + \alpha)$, and the overlapping occurs only between two adjacent strips (S_j and S_{j-1} , for any j). There exists a universal constant $\tilde{C} > 0$ and a partition of unity $(\chi_j)_{j \in \mathbb{Z}}$ of $\mathbb{R}^2$ such that

$$\sum_j |\chi_j|^2 = 1, \quad \text{supp } \chi_j \subset S_j, \quad 0 \leq \chi_j \leq 1, \quad |\nabla \chi_j| \leq \frac{\tilde{C}}{\alpha},$$

and

$$\chi_j = 1 \text{ in } \left(-n^2 - 1 + (2j - 1) \left(1 + \frac{\alpha}{2} \right), -n^2 - 1 - \alpha + (2j + 1) \left(1 + \frac{\alpha}{2} \right) \right).$$

Since the overlapping is between a finite number of strips, one may further write

$$\sum_j |\chi_j|^2 = 1, \quad 0 \leq \chi_j \leq 1, \quad \sum_j |\nabla \chi_j|^2 \leq \frac{C}{\alpha^2}, \quad \text{supp } \chi_j \subset S_j,$$

where C is some universal constant. Define

$$\chi_{R,j}(x) = \chi_j(2x/R),$$

$(\chi_{R,j})$ is then a new partition of unity satisfying

$$\sum_j |\chi_{R,j}|^2 = 1, \quad 0 \leq \chi_{R,j} \leq 1, \quad \sum_j |\nabla \chi_{R,j}|^2 \leq \frac{C}{\alpha^2 R^2}, \quad \text{supp } \chi_{R,j} \subset S_{R,j}, \quad (3.14)$$

where $S_{R,j} = \{xR/2 : x \in S_j\}$. The family of strips $(S_{R,j})_{j \in \{1,2,\dots,n^2\}}$ yields a covering of $S_{n^2R} = (-n^2R/2, n^2R/2) \times \mathbb{R}$ by n^2 strips, each of width $(1+\alpha)R$. Let $\varphi_{a,b,n^2R} \in \mathcal{D}_{n^2R}$ be the minimizer in Proposition 3.4. We decompose the energy associated to φ_{a,b,n^2R} as follows

$$\begin{aligned} \mathfrak{g}_a(b, n^2R) &\geq \sum_{j=1}^{n^2} \left(\mathcal{G}_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) - b \left\| (\nabla \chi_{R,j}) \varphi_{a,b,n^2R} \right\|_{L^2(S_{n^2R})}^2 \right) \\ &\geq \sum_{j=1}^{n^2} \mathcal{G}_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) - C \frac{b^2 n^2}{\alpha^2 R}. \end{aligned}$$

The first inequality above follows from the celebrated *IMS localization formula* (see [CFKS09, Theorem 3.2]), while the second comes from (3.10) and the properties of $(\chi_{R,j})$ in (3.14). Notice that $\chi_{R,j} \varphi_{a,b,n^2R}$ is supported in an infinite strip of width $(1+\alpha)R$. By energy translation invariance along the x_1 -direction, we have

$$\mathcal{G}_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) \geq \mathfrak{g}_a(b, (1+\alpha)R).$$

As a consequence,

$$\mathfrak{g}_a(b, n^2R) \geq n^2 \mathfrak{g}_a(b, (1+\alpha)R) - C \frac{b^2 n^2}{\alpha^2 R}.$$

For $R \geq 1$, dividing both sides by n^2R and using the monotonicity of $R \mapsto \mathfrak{g}_a(b, R)$, we get

$$\begin{aligned} \frac{\mathfrak{g}_a(b, n^2R)}{n^2R} &\geq \frac{\mathfrak{g}_a(b, (1+\alpha)R)}{R} - \frac{Cb^2}{\alpha^2 R^2} \\ &\geq \frac{\mathfrak{g}_a(b, (1+\alpha)^2R)}{(1+\alpha)^2R} - Cb^2 \left(\alpha + \frac{1}{\alpha^2 R^2} \right) \\ &\geq \frac{\mathfrak{g}_a(b, (1+\alpha)^2R)}{(1+\alpha)^2R} - Cb^2 \left(\alpha + \frac{1}{\alpha^2 R} \right). \quad \square \end{aligned}$$

3.5. Proof of Theorem 3.1. Here we will verify all the statements appearing in Theorem 3.1. Noticing that $\mathcal{G}_{a,b,R}(0) = 0$, we get Item (1). The second item is already proven in Lemma 3.2.

Defining $\mathfrak{e}_a(b) = 0$ for $b \geq 1/\beta_a$, the items (3) and (5) hold trivially since $\mathfrak{g}_a(b, R) = 0$ in this case. We handle now the case where $1/|a| \leq b < 1/\beta_a$. Define in $\mathbb{R}$ the two functions $d_{a,b}(l) = \mathfrak{g}_a(b, l^2)$ and $f_{a,b}(l) = d_{a,b}(l)/l^2$. Using Lemmas 3.6–3.8, we see that the functions $d_{a,b}(l)$ and $f_{a,b}(l)$ satisfy the following properties:

- $d_{a,b}(\cdot)$ is non-positive, monotone non-increasing, and $f_{a,b}(\cdot)$ is bounded.

- For $l \geq 1$, $f_{a,b}(nl) \geq f_{a,b}((1+\alpha)l) - C\left(\alpha + \frac{1}{\alpha^2 l^2}\right)$, where $C > 0$ is a constant dependent on b and independent from l , n and α .

Then, by [FK13, Lemma 2.2], the following limit exists

$$\lim_{R \rightarrow +\infty} \frac{\mathfrak{g}_a(b, R)}{R} = \lim_{l \rightarrow +\infty} f_{a,b}(l) = \mathfrak{e}_a(b),$$

and for $R \geq 4$

$$\frac{\mathfrak{g}_a(b, R)}{R} \leq \mathfrak{e}_a(b) + \frac{2C}{R^{\frac{1}{3}}}.$$

Moreover, for every integer $n \geq 1$, Lemma 3.5 asserts that,

$$\mathfrak{g}_a(b, nR) \leq n \mathfrak{g}_a(b, R).$$

Dividing both sides by nR and taking $n \rightarrow +\infty$ yields $\mathfrak{e}_a(b) \leq \mathfrak{g}_a(b, R)/R$.

By Lemma 3.7, $\mathfrak{e}_a(b) < 0$; that the function $\mathfrak{e}_a(\cdot)$ is monotone non-decreasing follows from the monotonicity of the function $b \mapsto \mathfrak{g}_a(b, R)$; the continuity of the function $\mathfrak{e}_a(\cdot)$ follows from the estimates in (3.10) and the bounds in (3.6) (see [FKP13, Theorem 3.13]).

3.6. An effective one-dimensional energy. Assume that $a \in [-1, 1) \setminus \{0\}$ and $b > 0$. For all $\xi \in \mathbb{R}$, consider the functional $\mathcal{E}_{a,b,\xi}^{\text{ID}}$ defined over the space $B^1(\mathbb{R})$

$$\mathcal{E}_{a,b,\xi}^{\text{ID}}(f) = \int_{\mathbb{R}} \left(b|f'(t)|^2 + bV_a(\xi, t)|f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) dt, \quad (3.15)$$

where $V_a(\xi, t)$ is introduced in (2.13). Let

$$E_{a,b}^{\text{ID}}(\xi) = \inf_{f \in B^1(\mathbb{R})} \mathcal{E}_{a,b,\xi}^{\text{ID}}(f). \quad (3.16)$$

We would like to find a relationship between the 2D-energy in (3.4) and the 1D-energy in (3.16) for some specific value of ξ . The existing results on the Ginzburg–Landau functional with a uniform magnetic field suggest that we should select ξ so as to minimize the function $\xi \mapsto E_{a,b}^{\text{ID}}(\xi)$, see [AH07, CR14, Pan02].

In light of Remark 3.3, we will assume that a and b satisfy

$$a \in [-1, 0) \quad \text{and} \quad b \geq \frac{1}{|a|}. \quad (3.17)$$

We can list some elementary properties of the functional $\mathcal{E}_{a,b,\xi}^{\text{ID}}$ in (3.15):

Proposition 3.9. *Let $a \in [-1, 0)$ and $b \geq 1/|a|$.*

- (1) *The functional $\mathcal{E}_{a,b,\xi}^{\text{ID}}$ has a non-trivial minimizer in $B^1(\mathbb{R})$ if and only if $1/|a| \leq b < 1/\mu_a(\xi)$. Furthermore, one can find a positive minimizer $f_{a,b,\xi}$, dependent on a and b , such that any minimizer has the form $cf_{a,b,\xi}$ where $c \in \mathbb{C}$ and $|c| = 1$.*
- (2) *Any minimizer f of $\mathcal{E}_{a,b,\xi}^{\text{ID}}$ satisfies $\|f\|_{\infty} \leq 1$ and the equation:*

$$-f''(t) + V_a(\xi, t)f(t) = \frac{1}{b}(1 - |f(t)|^2)f(t), \quad \text{for } t \in \mathbb{R}.$$

(3) For $1/|a| < b < 1/\beta_a$, there exists $\tilde{\xi}_0$, dependent on a and b , such that

$$E_{a,b}^{1D}(\tilde{\xi}_0) = \inf_{\xi \in \mathbb{R}} E_{a,b}^{1D}(\xi).$$

(4) (Feynman–Hellmann)

$$\int_{-\infty}^0 (at + \tilde{\xi}_0) |f_{\tilde{\xi}_0}(t)|^2 dt + \int_0^{+\infty} (t + \tilde{\xi}_0) |f_{\tilde{\xi}_0}(t)|^2 dt = 0.$$

(5) Any minimizer f of $\mathcal{E}_{a,b,\xi}^{1D}$ satisfies

$$E_{a,b}^{1D}(\xi) = -\frac{1}{2} \int_{\mathbb{R}} f^4(x_2) dx_2.$$

Remark 3.10. Guided by the numerical computations of [HPRS16, Sect. 1.3], we expect that:

- the global minimum β_a , defined in (2.18), is attained at a *unique* point ξ_a ;
- ξ_a is the unique critical point of the function $\xi \mapsto \mu_a(\xi)$.

However, such results have not been analytically proven yet.

The proof of Proposition 3.9 may be derived as done in [FH10, Sect. 14.2] devoted to the analysis of the following 1D-functional

$$\mathcal{E}_{b,\xi}^{1D}(f) = \int_0^{+\infty} \left(b|f'(t)|^2 + b(t+\xi)^2|f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) dt,$$

defined over the space $B^1(\mathbb{R}_+)$. We introduce the energies

$$E_b^{1D}(\xi) = \inf_{f \in B^1(\mathbb{R}_+)} \mathcal{E}_{b,\xi}^{1D}(f),$$

and

$$E_b^{1D} = \inf_{\xi \in \mathbb{R}} E_b^{1D}(\xi). \quad (3.18)$$

The ground state energy in (3.18) plays a crucial role in the study of surface superconductivity under the presence of a uniform magnetic field (see e.g. [AH07, FH10, HFPS11, CR14]). Let $E_{\text{g.st}}^{\text{unif}}(\kappa, H)$ be the ground state energy of the functional in (1.1) for $B_0 = 1$. Assuming that $H = b\kappa$ and $1 < b < \Theta_0^{-1}$, Θ_0 is the constant in (2.5), then as $\kappa \rightarrow +\infty$,

$$E_{\text{g.st}}^{\text{unif}}(\kappa, H) = |\partial\Omega| \kappa b^{-\frac{1}{2}} E_b^{1D} + \mathcal{O}(1), \quad (3.19)$$

where the remainder term $\mathcal{O}(1)$ depends on the geometry and is explicitly computed in [CR16a, CR16b, CDR17].

That has been conjectured by Pan [Pan02], then proven by Almog–Helffer and Helffer–Fournais–Persson [AH07, HFPS11] under a restrictive assumption on b , using a spectral approach. In the whole regime $b \in (1, \Theta_0^{-1})$, the upper bound part in (3.19) easily holds (see [FH10, Sect. 14.4.2]), while the matching lower bound is more difficult to obtain and has been finally proven by Correggi–Rougerie [CR14]. The proof of Correggi–Rougerie, based on the non-negativity of a certain *cost function*, was markedly different from the spectral approach of [AH07, HFPS11].

Going back to our step magnetic field problem and the one dimensional energy in (3.15), we prove the following theorem.

Theorem 3.11. Assume that $-1 \leq a < 0$ and $1/|a| < b < 1/\beta_a$, where β_a is defined in (2.11). Then, the energy $\mathfrak{e}_a(b)$ introduced in (3.5) satisfies

$$\mathfrak{e}_a(b) = E_{a,b}^{\text{1D}},$$

where

$$E_{a,b}^{\text{1D}} = \inf_{\xi \in \mathbb{R}} E_{a,b}^{\text{1D}}(\xi), \quad (3.20)$$

and $E_{a,b}^{\text{1D}}(\cdot)$ is defined in (3.16).

Remark 3.12. By a symmetry argument, Theorem 3.11 trivially holds in the case $a = -1$, namely

$$\mathfrak{e}_{-1}(b) = E_{-1,b}^{\text{1D}} = E_b^{\text{1D}}.$$

To prove Theorem 3.11, we will adopt the method of [CR14], which relies on remarkable identities, including an energy splitting [LM99], along with the non-positivity of a certain *potential function* and the non-negativity of another *cost function*.

We propose the *potential* and *cost* functions of our problem. These are defined as follows,

$$F_0(t) = \begin{cases} 2 \int_t^0 (a\eta + \tilde{\xi}_0) f_0^2(\eta) d\eta, & t \leq 0, \\ 2 \int_0^t (\eta + \tilde{\xi}_0) f_0^2(\eta) d\eta, & t > 0, \end{cases} \quad (3.21)$$

and

$$K_0(t) = f_0^2(t) + F_0(t), \quad \text{for } t \in \mathbb{R}, \quad (3.22)$$

where $\tilde{\xi}_0$ and $f_0 = f_{a,b,\tilde{\xi}_0}$ are introduced in Proposition 3.9. We recall the set S_R in (3.1) and the energy $\mathcal{G}_{a,b,R}$ in (3.3) defined over the space $\mathcal{D}_R$ in (3.2). Let $u \in C_0^\infty(S_R)$ (note that this space is dense in $\mathcal{D}_R$ with respect to the norm $\|(\nabla - i\sigma \mathbf{A}_0)u\|_{L^2(S_R)} + \|u\|_{L^2(S_R)}$). Since $f_0 > 0$ on $\mathbb{R}$ (see Proposition 3.9), we may introduce the function v via the relation

$$u(x_1, x_2) = e^{i\tilde{\xi}_0 x_1} f_0(x_2) v(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2. \quad (3.23)$$

Lemma 3.13. It holds

$$\mathcal{G}_{a,b,R}(u) = R E_{a,b}^{\text{1D}} + \mathcal{E}_0(v),$$

where

$$\begin{aligned} \mathcal{E}_0(v) &= \int_{S_R} b f_0^2(x_2) \left(|\partial_{x_2} v|^2 + |\partial_{x_1} v|^2 + 2(\sigma x_2 + \tilde{\xi}_0)(i v, \partial_{x_1} v) \right. \\ &\quad \left. + \frac{1}{2b} f_0^2(x_2) (1 - |v|^2)^2 \right) dx, \\ (i v, \partial_{x_1} v) &= \frac{i}{2} (v \partial_{x_1} \bar{v} - \bar{v} \partial_{x_1} v), \end{aligned} \quad (3.24)$$

and σ is defined in (2.9).

Proof. Note that

$$\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left(b|\partial_{x_2}u|^2 + b|(\partial_{x_1} + i\sigma x_2)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx. \quad (3.25)$$

We will compute each term of $\mathcal{G}_{a,b,R}(u)$ apart. Starting with

$$\int_{S_R} |\partial_{x_2}u|^2 dx = \int_{S_R} \left(|\partial_{x_2}f_0|^2|v|^2 + f_0^2|\partial_{x_2}v|^2 \right) dx + \int_{S_R} f_0\partial_{x_2}f_0\partial_{x_2}|v|^2 dx. \quad (3.26)$$

An integration by parts yields

$$\begin{aligned} \int_{S_R} f_0\partial_{x_2}f_0\partial_{x_2}|v|^2 dx &= - \int_{S_R} |\partial_{x_2}f_0|^2|v|^2 dx \\ &\quad - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^0 f_0\partial_{x_2}^2f_0|v|^2 dx - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_0^{+\infty} f_0\partial_{x_2}^2f_0|v|^2 dx, \end{aligned} \quad (3.27)$$

since the functions f_0 and f'_0 vanish at $\pm\infty$. Plugging (3.27) in (3.26) and using the second item of Proposition 3.9, we find

$$\int_{S_R} |\partial_{x_2}u|^2 dx = \int_{S_R} \left(f_0^2|\partial_{x_2}v|^2 + f_0|v|^2 \left(-(\sigma x_2 + \tilde{\xi}_0)^2 f_0 + \frac{1}{b}f_0(1 - f_0^2) \right) \right) dx. \quad (3.28)$$

Next, we compute the second term of $\mathcal{G}_{a,b,R}(u)$

$$\begin{aligned} \int_{S_R} |(\partial_{x_1} + i\sigma x_2)u|^2 dx &= \int_{S_R} f_0^2 \left(|\partial_{x_1}v|^2 + (\sigma x_2 + \tilde{\xi}_0)^2|v|^2 \right. \\ &\quad \left. + 2(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1}v) \right) dx. \end{aligned} \quad (3.29)$$

Moreover, by Proposition 3.9 we have

$$E_{a,b}^{\text{ID}} = \mathcal{E}_{a,b,\tilde{\xi}_0}^{\text{ID}}(f_0) = -\frac{1}{2} \int_{\mathbb{R}} f_0^4(x_2) dx_2. \quad (3.30)$$

We put (3.28)–(3.30) in (3.25) to complete the proof. $\square$

Lemma 3.14. *Let F_0 and K_0 be the functions defined respectively in (3.21) and (3.22). If $F_0 \leq 0$ and $F_0(\pm\infty) := \lim_{t \rightarrow \pm\infty} F_0(t) = 0$, then*

$$\mathcal{G}_{a,b,R}(u) \geq RE_{a,b}^{\text{ID}} + \mathcal{E}_1(v),$$

where

$$\mathcal{E}_1(v) = \int_{S_R} bK_0(x_2) \left(|\partial_{x_2}v|^2 + |\partial_{x_1}v|^2 \right) dx + \frac{1}{2} \int_{S_R} f_0^4(x_2)(1 - |v|^2)^2 dx.$$

Proof. Note that

$$F_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0)f_0^2(t), & t < 0, \\ 2(t + \tilde{\xi}_0)f_0^2(t), & t > 0. \end{cases}$$

Since $F_0(0) = 0$ and $F_0(\pm\infty) = 0$, we can handle the next term through an integration by parts:

$$\begin{aligned} & 2 \int_{S_R} f_0^2(x_2)(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1} v) dx \\ &= - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^0 (iv, \partial_{x_1} v) \partial_{x_2} F_0 dx + \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_0^{+\infty} (iv, \partial_{x_1} v) \partial_{x_2} F_0 dx \\ &= \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^0 F_0 \partial_{x_2} (iv, \partial_{x_1} v) dx - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_0^{+\infty} F_0 \partial_{x_2} (iv, \partial_{x_1} v) dx. \end{aligned} \quad (3.31)$$

Now we handle the integral involving the term in (3.24). An integration by parts yields

$$\begin{aligned} \int_{-\frac{R}{2}}^{\frac{R}{2}} \partial_{x_2} (iv, \partial_{x_1} v) dx_1 &= i \int_{-\frac{R}{2}}^{\frac{R}{2}} \left(\partial_{x_2} v \partial_{x_1} \bar{v} - \partial_{x_2} \bar{v} \partial_{x_1} v \right) dx_1 + \frac{i}{2} \left[v \partial_{x_2} \bar{v} - \bar{v} \partial_{x_2} v \right]_{-\frac{R}{2}}^{\frac{R}{2}} \\ &= i \int_{-\frac{R}{2}}^{\frac{R}{2}} \left(\partial_{x_2} v \partial_{x_1} \bar{v} - \partial_{x_2} \bar{v} \partial_{x_1} v \right) dx_1, \end{aligned} \quad (3.32)$$

since $u = 0$ (and consequently v) for $x_1 = \pm R/2$.

We plug (3.32) into (3.31) and we use Cauchy's inequality to get

$$\begin{aligned} 2 \int_{S_R} f_0^2(x_2)(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1} v) dx &\geq -2 \int_{S_R} |F_0| |\partial_{x_1} v| |\partial_{x_2} v| dx \\ &\geq \int_{S_R} F_0 \left(|\partial_{x_1} v|^2 + |\partial_{x_2} v|^2 \right) dx, \end{aligned}$$

since $F_0 \leq 0$. This completes the proof in light of Lemma 3.13 and the definition of the function K_0 . $\square$

Now, looking at the expression of $\mathcal{E}_1(v)$ in Lemma 3.14, we obtain

Lemma 3.15. *If $K_0 \geq 0$ then*

$$\mathcal{G}_{a,b,R}(u) \geq RE_{a,b}^{\text{ID}}.$$

Thus, if $F_0 \leq 0$, $F_0(\pm\infty) = 0$ and $K_0 \geq 0$, then we get the lower bound

$$\frac{\mathcal{G}_{a,b,R}(u)}{R} \geq E_{a,b}^{\text{ID}}. \quad (3.33)$$

Our next task is to verify these conditions.

We have the following Feynman–Hellmann equation (see Proposition 3.9):

$$\int_{-\infty}^0 (at + \tilde{\xi}_0) f_0^2(t) dt + \int_0^{+\infty} (t + \tilde{\xi}_0) f_0^2(t) dt = 0, \quad (3.34)$$

which can be expressed as follows

$$F_0(-\infty) + F_0(+\infty) = 0. \quad (3.35)$$

Regarding the function K_0 , we get immediately from (3.22),

$$K_0(\pm\infty) = F_0(\pm\infty). \quad (3.36)$$

If we manage to prove that $F_0(\pm\infty) = 0$, then by the same argument in [CR14, Lemma 3.2 and Proposition 3.4], we may prove that $F_0 \leq 0$ and $K_0 \geq 0$.

Such information is known in the particular case $a = -1$, thanks to symmetry considerations and [CR14]; indeed

$$\int_{-\infty}^0 (at + \tilde{\xi}_0) f_0^2(t) dt = \int_0^{+\infty} (t + \tilde{\xi}_0) f_0^2(t) dt = 0. \quad (3.37)$$

In the asymmetric case when $a \in (-1, 0)$, one needs to work a little bit more for obtaining (3.37).

The next lemma will be useful for establishing that $F_0(\pm\infty) = 0$.

Lemma 3.16 (Alternative expression of F_0). *It holds*

$$F_0(t) = \begin{cases} \frac{1}{|a|} \left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b} f_0^2(t) + \frac{1}{2b} f_0^4(t) \right), & t \leq 0, \\ -f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b} f_0^2(t) + \frac{1}{2b} f_0^4(t), & t > 0. \end{cases}$$

Proof. For $t \leq 0$ and $a < 0$, we have

$$\begin{aligned} F_0(t) &= 2 \int_t^0 (a\eta + \tilde{\xi}_0) f_0^2(\eta) d\eta \\ &= \frac{1}{|a|} \int_0^t \partial_\eta (a\eta + \tilde{\xi}_0)^2 f_0^2(\eta) d\eta \\ &= \frac{1}{|a|} \left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b} f_0^2(t) + \frac{1}{2b} f_0^4(t) \right) + R_1 \end{aligned}$$

where

$$R_1 = \frac{1}{|a|} \left(f_0'^2(0) - \tilde{\xi}_0^2 f_0^2(0) + \frac{1}{b} f_0^2(0) - \frac{1}{2b} f_0^4(0) \right).$$

Similarly, one proves for $t > 0$ that

$$F_0(t) = -f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b} f_0^2(t) + \frac{1}{2b} f_0^4(t) + R_2,$$

where

$$R_2 = f_0'^2(0) - \tilde{\xi}_0^2 f_0^2(0) + \frac{1}{b} f_0^2(0) - \frac{1}{2b} f_0^4(0).$$

Now, we use the Feynman–Hellmann equation in (3.35) and the vanishing of f_0 and f_0' at ∞ to get

$$R_1 + R_2 = F_0(-\infty) + F_0(+\infty) = 0.$$

Since $R_1 = 1/|a|R_2$, we conclude that $R_1 = R_2 = 0$. $\square$

Lemma 3.17. *Let F_0 be the potential function defined in (3.21). It holds*

$$F_0(t) \leq 0 \text{ for all } t \in \mathbb{R}, \text{ and } F_0(\pm\infty) = 0.$$

Proof. From the definition of F_0 , we have $F_0(0) = 0$. In addition, the alternative expression of F_0 in Lemma 3.16 and the decay and vanishing of f_0 and f'_0 at ∞ imply that

$$F_0(-\infty) = \lim_{t \rightarrow -\infty} \frac{1}{|a|} \left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b} f_0^2(t) + \frac{1}{2b} f_0^4(t) \right) = 0,$$

and similarly that $F_0(+\infty) = 0$. Next, we will study the variations of F_0 . Recall the derivative of F_0

$$F_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0) f_0^2(t), & t < 0, \\ 2(t + \tilde{\xi}_0) f_0^2(t), & t > 0. \end{cases}$$

We know that $f_0 > 0$ on $\mathbb{R}$. Hence, assuming that $\tilde{\xi}_0 \geq 0$ yields that $F_0'(t) > 0$ for all $t > 0$, which contradicts the fact that $F_0(0) = F_0(+\infty) = 0$. This proves that $\tilde{\xi}_0 < 0$. Consequently, we find that $F_0' < 0$ in a right-neighbourhood of 0, and $F_0' > 0$ in a left-neighbourhood of 0. Since $F_0(0) = 0$, we find that $F_0 \leq 0$ in a neighbourhood of 0.

On the other hand, $F_0'(t) = 0$ iff $t = -\tilde{\xi}_0 > 0$ or $t = -\tilde{\xi}_0/a < 0$. Having the additional properties $F_0(0) = 0$ and $F_0(\pm\infty) = 0$, we get that $F_0 \leq 0$ in $\mathbb{R}$. $\square$

Remark 3.18. Along the proof of Lemma 3.17, we proved that any $\tilde{\xi}_0$ minimizing $E_{a,b}^{1D}(\cdot)$ satisfies $\tilde{\xi}_0 < 0$.

Now, we are ready to prove the non-negativity of the cost function K_0 .

Lemma 3.19. *Let K_0 be the cost function defined in (3.22). It holds*

$$K_0(t) \geq 0 \text{ for all } t \in \mathbb{R}.$$

Proof. Lemma 3.17 and (3.36) simply imply that $K_0(\pm\infty) = 0$. Consequently if K_0 becomes negative at some point t , this definitely means the existence of a global minimum at some point t_0 in $\mathbb{R}^*$, since $K_0(0) > 0$. We have then $K_0(t_0) < 0$ and $K_0'(t_0) = 0$, where

$$K_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0) f_0^2(t) + 2f_0(t) f_0'(t), & t < 0, \\ 2(t + \tilde{\xi}_0) f_0^2(t) + 2f_0(t) f_0'(t), & t > 0. \end{cases}$$

Since $K_0'(t_0) = 0$ and $f_0(t_0) > 0$, we get that

$$f_0'(t_0) = \begin{cases} (at_0 + \tilde{\xi}_0) f_0(t_0), & \text{if } t_0 < 0, \\ -(t_0 + \tilde{\xi}_0) f_0(t_0), & \text{if } t_0 > 0. \end{cases} \quad (3.38)$$

On the other hand, we may use the alternative expression of F_0 in Lemma 3.16 to write the function K_0 in the following form

$$K_0(t) = \begin{cases} \left(1 - \frac{1}{|a|b}\right) f_0^2(t) - \frac{1}{|a|} f_0'^2(t) + \frac{1}{|a|} (at + \tilde{\xi}_0)^2 f_0^2(t) + \frac{1}{2|a|b} f_0^4(t), & t \leq 0, \\ \left(1 - \frac{1}{b}\right) f_0^2(t) - f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) + \frac{1}{2b} f_0^4(t), & t > 0. \end{cases} \quad (3.39)$$

Plug (3.38) into (3.39) to get

$$K_0(t_0) = \begin{cases} \left(1 - \frac{1}{|a|b}\right) f_0^2(t_0) + \frac{1}{2|a|b} f_0^4(t_0), & t_0 < 0, \\ \left(1 - \frac{1}{b}\right) f_0^2(t_0) + \frac{1}{2b} f_0^4(t_0), & t_0 > 0. \end{cases}$$

Since $a \in [-1, 0)$, $b > 1/|a|$ and $f_0 > 0$ everywhere in $\mathbb{R}$, then obviously $K_0(t_0) > 0$ which is the desired contradiction. $\square$

Collecting the aforementioned lemmas, we can now prove Theorem 3.11.

Proof of Theorem 3.11. The upper bound $\epsilon_a(b) \leq E_{a,b}^{\text{1D}}$ follows by using the trial function

$$u(x_1, x_2) = \chi_R(x_1) f_0(x_2) e^{i\tilde{\xi}_0 x_1},$$

and passing to the limit $R \rightarrow +\infty$. Here, $\tilde{\xi}_0$ and $f_0 = f_{a,b,\tilde{\xi}_0}$ are introduced in Proposition 3.9, and χ_R is a smooth cut-off function supported in S_R and satisfying $\chi_R(x_1) \in (0, 1)$ for $x_1 \in (-R/2, R/2)$, and $\chi_R = 1$ in $(-R/2 + 1, R/2 - 1)$.

The lower bound $\epsilon_a(b) \geq E_{a,b}^{\text{1D}}$ is a consequence of (3.33) after passing to the limit $R \rightarrow +\infty$. $\square$

4. The Frenet Coordinates

In this section, we assume that the set Γ consists of a single simple smooth curve that may intersect the boundary of Ω transversely in two points. In the general case, Γ consists of a finite number of such (disjoint) curves. By working on each component separately, we reduce to the simple case above.

To study the energy contribution along Γ , we will use the *Frenet coordinates* which are valid in a tubular neighbourhood of Γ . For more details regarding these coordinates, see e.g. [FH10, Appendix F]. We will list the basic properties of these coordinates here.

Let $(-|\Gamma|/2, |\Gamma|/2) \ni s \mapsto M(s) \in \Gamma$ (respectively $[-|\Gamma|/2, |\Gamma|/2] \ni s \mapsto M(s) \in \Gamma$) be the arc-length parametrization of Γ , when $\Gamma \cap \partial\Omega = \emptyset$ (respectively when $\Gamma \cap \partial\Omega \neq \emptyset$). The vector

$$T(s) = M'(s) \tag{4.1}$$

is the unit tangent vector to Γ at the point $M(s)$. Let $\nu(s)$ be the unit normal of Γ at the point $M(s)$ pointed toward Ω_1 . The orientation of the parametrization M is displayed as follows

$$\det(T(s), \nu(s)) = 1.$$

The curvature k_r of Γ is defined by

$$T'(s) = k_r(s) \nu(s).$$

For $t_0 > 0$, we define

$$S(t_0) = \begin{cases} \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2}\right) \times (-t_0, t_0), & \text{if } \Gamma \cap \partial\Omega = \emptyset, \\ \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2}\right) \times (-t_0, t_0), & \text{if } \Gamma \cap \partial\Omega \neq \emptyset, \end{cases} \tag{4.2}$$

and the transformation

$$\Phi : S(t_0) \ni (s, t) \longmapsto M(s) + t\nu(s) \in \mathbb{R}^2. \quad (4.3)$$

For a sufficiently small t_0 , Φ is a diffeomorphism from $S(t_0)$ to $\Gamma(t_0)$, where

$$\Gamma(t_0) := \text{Im}\Phi. \quad (4.4)$$

The Jacobian of Φ is

$$\alpha(s, t) = \det(D\Phi) = 1 - tk_r(s).$$

The inverse of Φ , Φ^{-1} , defines a system of coordinates for the tubular neighbourhood $\Gamma(t_0)$ of Γ ,

$$\Phi^{-1}(x) = (s(x), t(x)).$$

To each function $u \in H_0^1(\Gamma(t_0))$, we associate the function $\tilde{u} \in H^1(S(t_0))$ as follows

$$\tilde{u}(s, t) = u(\Phi(s, t)). \quad (4.5)$$

We also associate to any vector field $A = (A_1, A_2) \in H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^2)$, the vector field

$$\tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in H^1(S(t_0)),$$

where

$$\tilde{A}_1(s, t) = \alpha(s, t)A(\Phi(s, t)) \cdot T(s) \quad \text{and} \quad \tilde{A}_2(s, t) = A(\Phi(s, t)) \cdot \nu(s). \quad (4.6)$$

Then we have the following change of variable formulae:

$$\int_{\Gamma(t_0)} |(\nabla - iA)u|^2 dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} \left(\alpha^{-2} |(\partial_s - i\tilde{A}_1)\tilde{u}|^2 + |(\partial_t - i\tilde{A}_2)\tilde{u}|^2 \right) \alpha ds dt$$

and

$$\int_{\Gamma(t_0)} |u(x)|^2 dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} |\tilde{u}|^2 \alpha ds dt. \quad (4.7)$$

We define

$$\tilde{B}(s, t) = B(\Phi(s, t)), \quad \text{for all } (s, t) \in S(t_0).$$

Note that

$$\text{curl } \tilde{A}(s, t) = \partial_s \tilde{A}_2(s, t) - \partial_t \tilde{A}_1(s, t) = (1 - tk_r(s)) \tilde{B}(s, t). \quad (4.8)$$

The following lemma presents a special gauge transformation, that will allow us to express a given vector field in a canonical manner.

Lemma 4.1. *Let $a \in [-1, 1] \setminus \{0\}$ and $[s_0, s_1] \subset (-|\Gamma|/2, |\Gamma|/2)$ such that $\Phi((s_0, s_1) \times (-t_0, t_0)) \subset \Omega$. If A is a vector field in $H^1(\Omega, \mathbb{R}^2)$ with $\text{curl } A = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, then there exists a function $\omega_{s_0, s_1} \in H^2((s_0, s_1) \times (-t_0, t_0))$ such that the vector field $\tilde{A}_{new} := \tilde{A} - \nabla_{(s,t)} \omega_{s_0, s_1}$ satisfies on $(s_0, s_1) \times (-t_0, t_0)$*

$$\left(\tilde{A}_{new}\right)_1(s, t) = \begin{cases} -(t - \frac{t^2}{2}k_r(s)), & \text{if } t > 0; \\ -a(t - \frac{t^2}{2}k_r(s)), & \text{if } t < 0; \end{cases} \quad \left(\tilde{A}_{new}\right)_2(s, t) = 0. \quad (4.9)$$

Proof. For $(s, t) \in (s_0, s_1) \times (-t_0, t_0)$, let $\omega_{s_0, s_1}(s, t) = \int_0^t \tilde{A}_2(s, t') dt' + \int_{s_0}^s \tilde{A}_1(s', 0) ds'$.

Obviously, $\left(\tilde{A}_{new}\right)_2(s, t) = 0$ and

$$\begin{aligned} \left(\tilde{A}_{new}\right)_1(s, t) &= \int_0^t \left(\partial_t \tilde{A}_1(s, t') - \partial_s \tilde{A}_2(s, t')\right) dt' \\ &= - \int_0^t (1 - t'k_r(s)) \tilde{B}(s, t') dt' \quad (\text{by (4.8)}), \end{aligned}$$

which is the desired result since $\tilde{B}(s, t) = \begin{cases} 1, & \text{if } t > 0 \\ a, & \text{if } t < 0 \end{cases}$. $\square$

5. A Local Effective Energy

In this section, we will introduce a ‘local version’ of the Ginzburg–Landau functional in (1.1). For this local functional, we will be able to write precise estimates of the ground state energy, which in turn will prove useful in estimating the ground state energy of the full functional in (1.1).

Select a positive number t_0 sufficiently small so that the Frenet coordinates of Sect. 4 are valid in the tubular neighbourhood $\Gamma(t_0)$ defined in (4.4). Let $0 < c_1 < c_2$ be fixed constants and ℓ be a parameter that is allowed to vary in such a manner that

$$c_1 \kappa^{-\frac{3}{4}} < \ell < c_2 \kappa^{-\frac{3}{4}}. \quad (5.1)$$

We will refer to (5.1) by writing $\ell \approx \kappa^{-3/4}$. Let $s_0 \in \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2}\right)$. After performing a linear change of variable, we may assume that $s_0 = 0$ (for simplicity). For large values of κ , consider the neighbourhood of s_0

$$\mathcal{V}(\ell) = \left\{ (s, t) \in \Phi^{-1}(\Gamma(t_0)) : -\frac{\ell}{2} < s < \frac{\ell}{2}, -\ell < t < \ell \right\}. \quad (5.2)$$

Let $\tilde{F}$ be the magnetic potential defined in $\mathcal{V}(\ell)$ by

$$\tilde{F}(s, t) = \left(\tilde{F}_1(s, t), 0\right) = \left(-\sigma\left(t - \frac{t^2}{2}k_r(s)\right), 0\right), \quad (5.3)$$

where $\sigma = \sigma(s, t)$ was defined in (2.9). Consider the domain

$$\mathcal{D}_\ell = \left\{ u \in H_0^1(\mathcal{V}(\ell)) \cap L^\infty(\mathcal{V}(\ell)) : \|u\|_\infty \leq 1 \right\}. \quad (5.4)$$

For $u \in \mathcal{D}_\ell$, we define the (local) energy

$$\mathfrak{G}(u; \mathcal{V}(\ell)) = \int_{\mathcal{V}(\ell)} \left(\mathfrak{a}^{-2} |(\partial_s - i\kappa H \tilde{F}_1)u|^2 + |\partial_t u|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right) \mathfrak{a} \, ds \, dt, \quad (5.5)$$

where $\mathfrak{a}(s, t) = 1 - tk_r(s)$. Now we introduce the following ground state energy

$$\mathfrak{G}_0 = \inf_{u \in \mathcal{D}_\ell} \mathfrak{G}(u; \mathcal{V}(\ell)). \quad (5.6)$$

Using standard variational methods, one can prove the existence of a minimizer u_0 of $\mathfrak{G}$.

Our aim is to write matching upper and lower bounds for $\mathfrak{G}_0$, as $\kappa \rightarrow +\infty$, in the regime

$$H = b\kappa, \quad a \in [-1, 0) \quad \text{and} \quad b \geq \frac{1}{|a|}. \quad (5.7)$$

5.1. Lower bound of $\mathfrak{G}_0$.

Lemma 5.1. *Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on a and b such that, if $\kappa \geq \kappa_0$ and ℓ is as in (5.1), then*

$$\mathfrak{G}_0 \geq b^{-\frac{1}{2}} \kappa \ell \mathfrak{e}_a(b) - C, \quad (5.8)$$

where $\mathfrak{G}_0$ and $\mathfrak{e}_a(b)$ are defined in (5.6) and (3.5) respectively.

Proof. Notice that $\mathfrak{a}(s, t)$ is bounded in the set $\mathcal{V}(\ell)$ as follows

$$1 - C\ell \leq \mathfrak{a}(s, t) \leq 1 + C\ell. \quad (5.9)$$

Consequently

$$\mathfrak{G}(u; \mathcal{V}(\ell)) \geq (1 - C\ell) \mathcal{J}(u) - C\kappa^2 \ell \int_{\mathcal{V}(\ell)} |u|^2 \, ds \, dt, \quad (5.10)$$

where

$$\mathcal{J}(u) = \int_{\mathcal{V}(\ell)} \left(|(\partial_s - i\kappa H \tilde{F}_1)u|^2 + |\partial_t u|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right) \, ds \, dt. \quad (5.11)$$

We apply Cauchy's inequality and the uniform bound of u to get

$$\mathcal{J}(u) \geq (1 - \kappa^{-\frac{1}{2}}) \mathcal{T}(u) - C \left(\kappa^{\frac{3}{2}} \ell^2 + \kappa^{\frac{5}{2}} H^2 \ell^6 \right), \quad (5.12)$$

where

$$\mathcal{T}(u) = \int_{\mathcal{V}(\ell)} \left(|(\partial_s + i\sigma\kappa H t)u|^2 + |\partial_t u|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right) \, ds \, dt.$$

We introduce the parameters $R = \sqrt{\kappa H} \ell$, $\gamma = \sqrt{\kappa H} s$, $\tau = \sqrt{\kappa H} t$, and define the re-scaled function

$$\check{u}(\gamma, \tau) = \begin{cases} u(s, t) & \text{if } (\gamma, \tau) \in (-R/2, R/2) \times (-R, R), \\ 0 & \text{otherwise.} \end{cases}$$

In the new scale, we may write

$$\mathcal{T}(u) = \int_{-R/2}^{R/2} \int_{-\infty}^{+\infty} \left[|(\partial_\gamma + i\sigma\tau)\check{u}|^2 + |\partial_\tau \check{u}|^2 - \frac{1}{b} |\check{u}|^2 + \frac{1}{2b} |\check{u}|^4 \right] d\gamma d\tau = \frac{1}{b} \mathcal{G}_{a,b,R}(\check{u}),$$

where $\mathcal{G}_{a,b,R}$ is the functional in (3.3), and $\check{u} \in \mathcal{D}_R$ the domain in (3.2) (since $u \in \mathcal{D}_\ell$). Invoking Theorem 3.1, we conclude that

$$\mathcal{T}(u) \geq \frac{1}{b} R \mathfrak{e}_a(b). \quad (5.13)$$

We plug the estimates (5.12) and (5.13) in (5.10), then we use $\mathfrak{e}_a(b) \leq 0$ and the assumptions on κ and ℓ to complete the proof of Lemma 5.1. $\square$

5.2. Upper bound of $\mathfrak{G}_0$.

Lemma 5.2. *Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on a and b such that, if $\kappa \geq \kappa_0$ and ℓ is as in (5.1), then*

$$\mathfrak{G}_0 \leq b^{-\frac{1}{2}} \kappa \ell \mathfrak{e}_a(b) + C \kappa^{\frac{1}{8}}, \quad (5.14)$$

where $\mathfrak{G}_0$ and $\mathfrak{e}_a(b)$ are defined in (5.6) and (3.5) respectively.

Proof. For $R = \ell \sqrt{\kappa H}$, consider $\varphi = \varphi_{a,b,R}$ the minimizer of $\mathcal{G}_{a,b,R}$ defined in (3.8). We define the function u in $\mathcal{D}_\ell$ as follows

$$u(s, t) = \chi \left(\frac{t}{\ell} \right) \varphi \left(s \sqrt{\kappa H}, t \sqrt{\kappa H} \right), \quad (5.15)$$

where χ is a smooth cut-off function satisfying

$$0 \leq \chi \leq 1 \text{ in } \mathbb{R}, \quad \chi = 1 \text{ in } \left[-\frac{1}{2}, \frac{1}{2} \right] \quad \text{and} \quad \text{supp } \chi \subset (-1, 1).$$

Next, we define the following function (with the re-scaled variables)

$$v(\gamma, \tau) = u(s, t) \quad \left((\gamma, \tau) \in (-R/2, R/2) \times (-R, R) \right),$$

with $\gamma = \sqrt{\kappa H} s$, $\tau = \sqrt{\kappa H} t$. Using (5.9) and (3.10), we get

$$\mathfrak{G}(u) \leq (1 + C\ell) \mathcal{J}(u) + C\kappa^2 \ell \int_{\mathcal{V}(\ell)} |u|^2 ds dt \leq (1 + C\ell) \mathcal{K}(v) + C\kappa^2 \ell^3, \quad (5.16)$$

where $\mathcal{J}(u)$ was defined in (5.11),

$$\mathcal{K}(v) = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^R \left[\left| \left(\partial_\gamma + i\sigma \left(\tau - \epsilon \frac{\tau^2}{2} k_r \left(\frac{\gamma}{\epsilon} \right) \right) \right) v \right|^2 + |\partial_\tau v|^2 - \frac{1}{b} |v|^2 + \frac{1}{2b} |v|^4 \right] d\gamma d\tau,$$

and $\epsilon = 1/\sqrt{\kappa H}$.

Let $\chi_R(\tau) = \chi(\tau/R) = \chi(t/\ell)$. We will estimate now each term of $\mathcal{K}(v)$ apart, using mainly the estimates on the minimizer φ in (3.10) and the properties of the function χ_R . We start with the following two estimates that result from Cauchy–Schwarz inequality,

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^R |\partial_\tau v|^2 d\gamma d\tau \leq (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\partial_\tau \varphi|^2 d\gamma d\tau + C\kappa^{-\frac{3}{4}} \ell^{-1},$$

and

$$\begin{aligned} & \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^R \left| \left(\partial_\gamma + i\sigma \left(\tau - \epsilon \frac{\tau^2}{2} k_r \left(\frac{s}{\epsilon} \right) \right) \right) v \right|^2 d\gamma d\tau \\ & \leq (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |(\partial_\gamma + i\sigma\tau)\varphi|^2 d\gamma d\tau + C\kappa^{\frac{13}{4}} \ell^5. \end{aligned} \quad (5.17)$$

Next, we may select R_0 sufficiently large so that, for all $R \geq R_0$,

$$|\tau| \geq \frac{R}{2} \implies \frac{|\tau|}{\ln^2 |\tau|} \geq R^{\frac{1}{2}}. \quad (5.18)$$

The decay of φ in (3.9), and (5.18) yield

$$\begin{aligned} \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^R |v|^2 d\gamma d\tau &= \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau + \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} (\chi_R^2(\tau) - 1) |\varphi|^2 d\gamma d\tau \\ &\geq \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{|\tau| \geq R/2} |\varphi|^2 d\gamma d\tau \\ &\geq \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau - C\kappa^{\frac{1}{2}} \ell^{\frac{1}{2}}. \end{aligned}$$

Finally, we write the obvious inequality

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^R |v|^4 d\gamma d\tau \leq \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^4 d\gamma d\tau.$$

Gathering the foregoing estimates, we get

$$\mathcal{K}(v) \leq \frac{(1 + \kappa^{-\frac{1}{4}})}{b} \mathcal{G}_{a,b,R}(\varphi) + C\kappa^{\frac{1}{8}}. \quad (5.19)$$

Invoking Theorem 3.1, we implement (5.19) into (5.16) to get the desired upper bound. $\square$

6. Local Estimates

6.1. Superconductivity near the magnetic barrier. The aim of this section is to study the concentration of the minimizers $(\psi, \mathbf{A})$ of the functional in (1.1) near the set Γ that separates the values of the applied magnetic field (see Assumptions 1.1 and 1.2). This will be displayed by the local estimates of the Ginzburg–Landau energy and the L^4 -norm of the Ginzburg–Landau parameter in Theorem 6.1.

We will introduce the necessary notations and assumptions. Starting with the local energy of the configuration $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$, in any open set $D \subset \Omega$ as follows

$$\begin{aligned} \mathcal{E}_0(\psi, \mathbf{A}; D) &= \int_D \left(|(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{1}{2} \kappa^2 |\psi|^4 \right) dx, \\ \mathcal{E}(\psi, \mathbf{A}; D) &= \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_{\Omega} |\text{curl } \mathbf{A} - B_0|^2 dx. \end{aligned} \quad (6.1)$$

Choose $t_0 > 0$ sufficiently small. For all $x \in \Gamma(t_0)$, define the point $p(x) \in \Gamma$ as follows

$$\text{dist}(x, p(x)) = \text{dist}(x, \Gamma).$$

Let $\ell \approx \kappa^{-3/4}$ satisfy (5.1) (for some fixed choice of the constants c_1 and c_2). For κ sufficiently large (hence ℓ sufficiently small), let $x_0 \in \Gamma \setminus \partial\Omega$ be chosen so that

$$\text{dist}(x_0, \partial\Omega) > 2\ell. \quad (6.2)$$

Consider the following neighbourhood of x_0 ,

$$\mathcal{N}_{x_0}(\ell) = \left\{ x \in \mathbb{R}^2 : \text{dist}_{\Gamma}(x_0, p(x)) < \frac{\ell}{2}, \text{dist}_{\Omega}(x, p(x)) < \ell \right\}. \quad (6.3)$$

Thanks to (6.2), we have $\overline{\mathcal{N}_{x_0}(\ell)} \subset \Omega$. As a consequence of the assumption in (6.2), all the estimates that we will write will hold uniformly with respect to the point x_0 .

We assume that $a \in [-1, 0)$ and $b > 0$ are fixed and satisfy

$$b > \frac{1}{|a|}. \quad (6.4)$$

When (6.4) holds, we are able to use the exponential decay of the Ginzburg–Landau parameter away from the set Γ and the surface $\partial\Omega$ (see Theorem 2.4).

Theorem 6.1. *Let $a \in [-1, 0)$ and $b > 1/|a|$. There exists $\kappa_0 > 0$ and a function $\tau : [\kappa_0, +\infty) \rightarrow (0, +\infty)$ such that $\lim_{\kappa \rightarrow +\infty} \tau(\kappa) = 0$ and the following is true. For $\kappa \geq \kappa_0$, $H = b\kappa$ and $\ell \approx \kappa^{-3/4}$ as in (5.1), for any $x_0 \in \Gamma$ satisfying (6.2), every minimizer $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ of the functional in (1.1) satisfies*

$$\left| \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) - b^{-\frac{1}{2}} \kappa \ell \epsilon_a(b) \right| \leq \kappa \ell \tau(\kappa), \quad (6.5)$$

and

$$\left| \frac{1}{\ell} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx + 2b^{-\frac{1}{2}} \kappa^{-1} \epsilon_a(b) \right| \leq \kappa^{-1} \tau(\kappa), \quad (6.6)$$

where $\mathcal{N}_{x_0}(\cdot)$ is the set in (6.3), $\mathcal{E}_0$ is the local energy in (6.1), and $\epsilon_a(b)$ is the limiting energy in (3.5).

Furthermore, the function τ is independent of the point $x_0 \in \Gamma$.

The proof of Theorem 6.1 follows by combining the results of Proposition 6.3 and Proposition 6.4 below, which are derived along the lines of [HK17, Sect. 4] in the study of local surface superconductivity.

Part of the proof of Theorem 6.1 is based on the following remark. After performing a translation, we may assume that the Frenet coordinates of x_0 are $(s = 0, t = 0)$ (see Sect. 4). Recall the local Ginzburg–Landau energy $\mathcal{E}_0$ introduced in (6.1). Let $\mathbf{F}$ be the vector field introduced in Lemma 2.2. We have the following relation

$$\mathcal{E}_0(u, \mathbf{F}; \mathcal{N}_{x_0}(\ell)) = \mathfrak{G}(\tilde{v}; \mathcal{V}(\ell)), \quad (6.7)$$

where $\mathfrak{G}$ is defined in (5.5), $u \in H_0^1(\mathcal{N}_{x_0}(\ell))$, $\tilde{v} = e^{-i\kappa H\omega} \tilde{u}$, $\tilde{u}$ is the function associated to u by the transformation Φ^{-1} [see (4.5)], and $\omega = \omega_{-\ell, \ell}$ is the gauge function defined in Lemma 4.1.

6.1.1. Lower bound of the local energy. We start by establishing a lower bound for the local energy $\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell))$ for an arbitrary function $u \in H_0^1(\mathcal{N}_{x_0}(\ell))$ satisfying $|u| \leq 1$. We will work under the assumptions made in this section, notably, we assume that (6.4) holds, and $\ell \approx \kappa^{-3/4}$ [see (5.1)], and in the regime where $H = b\kappa$.

Proposition 6.2. *There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for $\kappa \geq \kappa_0$ and for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If*

- $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a solution of (1.4)
- $u \in H_0^1(\mathcal{N}_{x_0}(\ell))$ satisfies $|u| \leq 1$

then

$$\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq b^{-\frac{1}{2}} \kappa \ell \epsilon_a(b) - C,$$

where $\mathcal{N}_{x_0}(\cdot)$ is the neighbourhood in (6.3), $\mathcal{E}_0$ is the functional in (6.1), and $\epsilon_a(b)$ is the limiting energy in (3.5).

Proof. Let $\alpha \in (0, 1)$ and $\mathbf{F}$ be the vector field introduced in Lemma 2.2. We define the function $\phi_{x_0}(x) = (\mathbf{A}(x_0) - \mathbf{F}(x_0)) \cdot x$. As a consequence of the fourth item in Theorem 2.3, we get the following useful approximation of the vector potential $\mathbf{A}$

$$|\mathbf{A}(x) - \nabla \phi_{x_0}(x) - \mathbf{F}(x)| \leq \frac{C}{\kappa} \ell^\alpha, \quad \text{for } x \in \mathcal{N}_{x_0}(\ell). \quad (6.8)$$

We choose $\alpha = 2/3$ in (6.8). Let $h = e^{-i\kappa H\phi_{x_0}} u$. Using (6.8), Cauchy's inequality, and the uniform bound $|h| \leq 1$, we may write

$$\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{1}{2}}) \mathcal{E}_0(h, \mathbf{F}; \mathcal{N}_{x_0}(\ell)) - C \left(\kappa^{\frac{3}{2}} \ell^2 + \kappa^{\frac{5}{2}} \ell^{\frac{10}{3}} \right).$$

Now, define the function $\tilde{v} = e^{-i\kappa H\omega} \tilde{h}$ on $\Phi^{-1}(\mathcal{N}_{x_0}(\ell))$, where $\tilde{h}$ is the function associated to h by the transformation Φ^{-1} [see (4.5)], and $\omega = \omega_{s_0, s_1}$ is the function introduced in Lemma 4.1 with $s_0 = -\ell$ and $s_1 = \ell$. We may use the relation in (6.7) to write

$$\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{1}{2}}) \mathfrak{G}(\tilde{v}; \mathcal{V}(\ell)) - C \left(\kappa^{\frac{3}{2}} \ell^2 + \kappa^{\frac{5}{2}} \ell^{\frac{10}{3}} \right).$$

Finally, the lower bound in Lemma 5.1, together with the inequality $\epsilon_a(b) \leq 0$, yield the claim of the inequality. $\square$

6.1.2. Sharp upper bound on the L^4 -norm. We will derive a lower bound of the local energy $\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell))$ and an upper bound of the L^4 -norm of ψ , valid for any critical point $(\psi, \mathbf{A})$ of the functional in (1.1). Again, we remind the reader that we assume that (6.4) holds, $\ell \approx \kappa^{-3/4}$ [see (5.1)] and $H = b\kappa$.

Proposition 6.3. *There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a critical point of the functional in (1.1) for $\kappa \geq \kappa_0$, then*

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq b^{-\frac{1}{2}} \kappa \ell \mathfrak{e}_a(b) - C\kappa^{\frac{3}{16}}, \quad (6.9)$$

and

$$\frac{1}{\ell} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx \leq -2b^{-\frac{1}{2}} \kappa^{-1} \mathfrak{e}_a(b) + C\kappa^{-\frac{17}{16}}. \quad (6.10)$$

Here $\mathcal{N}_{x_0}(\cdot)$, $\mathcal{E}_0$, and $\mathfrak{e}_a(b)$ are respectively defined in (6.3), (6.1), and (3.5).

Proof. In the sequel, $\gamma = \kappa^{-3/16}$ and κ is sufficiently large so that $\gamma \in (0, 1)$. We denote by $\hat{\ell} = (1 + \gamma)\ell$.

Consider a smooth function f satisfying

$$\begin{aligned} f &= 1 \text{ in } \mathcal{N}_{x_0}(\ell), \quad f = 0 \text{ in } \mathcal{N}_{x_0}(\hat{\ell})^c, \\ 0 &\leq f \leq 1, \quad |\nabla f| \leq C\gamma^{-1}\ell^{-1} \text{ and } |\Delta f| \leq C\gamma^{-2}\ell^{-2} \text{ in } \Omega. \end{aligned} \quad (6.11)$$

Proof of (6.9). We use the following simple identity (see [KN16, p. 2871])

$$\begin{aligned} \int_{\mathcal{N}_{x_0}(\hat{\ell})} |(\nabla - i\kappa H\mathbf{A})f\psi|^2 dx &= \int_{\mathcal{N}_{x_0}(\hat{\ell})} |f(\nabla - i\kappa H\mathbf{A})\psi|^2 dx \\ &\quad - \int_{\mathcal{N}_{x_0}(\hat{\ell})} f\Delta f |\psi|^2 dx. \end{aligned} \quad (6.12)$$

Having in hand (6.12), $|\psi| \leq 1$ and $|\text{supp}(\Delta f)| \leq C\gamma\ell^2$, we can write

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} |(\nabla - i\kappa H\mathbf{A})f\psi|^2 dx \leq \int_{\mathcal{N}_{x_0}(\hat{\ell})} |f(\nabla - i\kappa H\mathbf{A})\psi|^2 dx + C\gamma^{-1}.$$

On the other hand, we write

$$\begin{aligned} \int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 |\psi|^2 dx &= \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\psi|^2 dx - \int_{\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) \leq \gamma\ell\}} (1 - f^2) |\psi|^2 dx \\ &\quad - \int_{\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}} (1 - f^2) |\psi|^2 dx. \end{aligned} \quad (6.13)$$

Recall that $\gamma = \kappa^{-3/16}$, then $\gamma\ell \gg \kappa^{-1}$ which, together with (6.4), allow us to use the exponential decay of $|\psi|^2$ in $\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}$ (see Theorem 2.4). Consequently, the integral over $\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}$ in (6.13) is exponentially small when $\kappa \rightarrow +\infty$; in addition, we have

$$\left| \text{supp}(1 - f^2) \cap \mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) \leq \gamma\ell\} \right| = \mathcal{O}(\gamma^2 \ell^2),$$

this yields

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 |\psi|^2 dx \geq \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\psi|^2 dx - C\gamma^2 \ell^2.$$

Hence,

$$\mathcal{E}_0(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) \leq \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) + C\kappa^{\frac{3}{16}}. \quad (6.14)$$

The fact that $f\psi \in H_0^1(\mathcal{N}_{x_0}(\hat{\ell}))$ and $|f\psi| \leq 1$ allows us to use the lower bound result established in Proposition 6.2, for $u = f\psi$. This yields together with (6.14)

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) \geq b^{-\frac{1}{2}} \kappa \hat{\ell} \epsilon_a(b) - C\kappa^{\frac{3}{16}}. \quad (6.15)$$

This completes the proof of (6.9), but with $\hat{\ell}$ appearing instead of ℓ . However, this is not harmful, as we could start the argument with $\tilde{\ell} = (1 + \gamma)^{-1} \ell$ in place of ℓ and then modify $\hat{\ell}$ accordingly; in this case we would get $\hat{\ell} = (1 + \gamma) \tilde{\ell} = \ell$ as required.

Proof of (6.10). In light of (1.4), we get using integration by parts (see [FK11, (6.2)])

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} \left(|(\nabla - i\kappa H \mathbf{A}) f\psi|^2 - |\nabla f|^2 |\psi|^2 \right) dx = \kappa^2 \int_{\mathcal{N}_{x_0}(\hat{\ell})} \left(|\psi|^2 - |\psi|^4 \right) f^2 dx.$$

Consequently,

$$\mathcal{E}_0(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) = \kappa^2 \int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 \left(-1 + \frac{1}{2} f^2 \right) |\psi|^4 dx + \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\nabla f|^2 |\psi|^2 dx. \quad (6.16)$$

Since $f = 1$ in $\mathcal{N}_{x_0}(\ell)$ and $-1 + 1/2 f^2 \leq -1/2$ in $\mathcal{N}_{x_0}(\hat{\ell})$, we get

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 \left(-1 + \frac{1}{2} f^2 \right) |\psi|^4 dx \leq -\frac{1}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx.$$

We use the previous inequality, (6.16) and the estimate $|\text{supp } |\nabla f|| \leq C\gamma\ell^2$ to obtain

$$\mathcal{E}_0(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) \leq -\frac{\kappa^2}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx + C\kappa^{\frac{3}{16}}. \quad (6.17)$$

Finally we plug the lower bound in Proposition 6.2 into (6.17). $\square$

6.1.3. Sharp lower bound on the L^4 -norm. Complementary to Proposition 6.3, we will prove Proposition 6.4 below, whose conclusion holds for minimizing configurations only. We continue working under the assumption that (6.4) holds, $\ell \approx \kappa^{-3/4}$ [see (5.1)] and $H = b\kappa$.

Proposition 6.4. *There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a minimizer of the functional in (1.1) for $\kappa \geq \kappa_0$, then*

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \leq b^{-\frac{1}{2}} \kappa \ell \mathfrak{e}_a(b) + C \kappa^{\frac{3}{16}}, \quad (6.18)$$

and

$$\frac{1}{\ell} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx \geq -2b^{-\frac{1}{2}} \kappa^{-1} \mathfrak{e}_a(b) - C \kappa^{-\frac{17}{16}}. \quad (6.19)$$

Here $\mathcal{N}_{x_0}(\cdot)$, $\mathcal{E}_0$, and $\mathfrak{e}_a(b)$ are respectively defined in (6.3), (6.1), and (3.5).

Proof. The proof is divided into five steps.

Step 1. Construction of a test function and decomposition of the energy. The construction of the test function is inspired from that by Sandier and Serfaty, in their study of bulk superconductivity in [SS03]. For $\gamma = \kappa^{-3/16}$ and $\hat{\ell} = (1 + \gamma)\ell$, we define the function

$$u(x) = \mathbb{1}_{\mathcal{N}_{x_0}(\hat{\ell})}(x) e^{i\kappa H \phi_{x_0}(x)} v_0(x) + \eta(x) \psi(x), \quad (6.20)$$

where $v_0(x) = \left(e^{i\kappa H \omega} u_0 \right) \circ \Phi^{-1}(x)$ for $x \in \mathcal{N}_{x_0}(\hat{\ell})$, ϕ_{x_0} is the gauge function introduced in (6.8), $\omega = \omega_{s_0, s_1}$ is the function introduced in Lemma 4.1 for $s_0 = -\hat{\ell}$ and $s_1 = \hat{\ell}$, Φ is the coordinate transformation in (4.3), u_0 is a minimizer of the functional $\mathfrak{G}(\cdot, \mathcal{V}(\hat{\ell}))$ defined in (5.5), and η is a smooth function satisfying

$$\begin{aligned} \eta &= 0 \text{ in } \mathcal{N}_{x_0}(\hat{\ell}), \quad \eta = 1 \text{ in } \mathcal{N}_{x_0}((1 + 2\gamma)\ell)^{\mathbb{G}}, \\ 0 \leq \eta &\leq 1, \quad |\nabla \eta| \leq C \gamma^{-1} \ell^{-1}, \quad \text{and } |\Delta \eta| \leq C \gamma^{-2} \ell^{-2} \text{ in } \Omega. \end{aligned} \quad (6.21)$$

Recall the energies defined in (1.1) and (6.1). Let us write the obvious decomposition

$$\mathcal{E}_0(\cdot, \mathbf{A}; \Omega) = \mathcal{E}_0(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) + \mathcal{E}_0(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\mathbb{G}}).$$

Adding the magnetic energy term $\kappa^2 H^2 \|\text{curl } \mathbf{A} - B_0\|_{L^2(\Omega)}^2$ on both sides, we obtain the following identity,

$$\mathcal{E}_{\kappa, H}(\cdot, \mathbf{A}) = \mathcal{E}_0(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})) + \mathcal{E}(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\mathbb{G}}),$$

since the same magnetic energy term is present in both energies $\mathcal{E}_{\kappa, H}(\cdot, \mathbf{A})$ and $\mathcal{E}(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\mathbb{G}})$. We denote by

$$\begin{aligned} \mathcal{E}_1(\cdot, \mathbf{A}) &= \mathcal{E}_0(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})), \quad \mathcal{E}_2(\cdot, \mathbf{A}) = \mathcal{E}_0(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\tilde{\ell}) \setminus \mathcal{N}_{x_0}(\hat{\ell})), \quad \mathcal{E}_3(\cdot, \mathbf{A}) \\ &= \mathcal{E}(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\tilde{\ell})^{\mathbb{G}}), \end{aligned}$$

where $\tilde{\ell} = (1 + 2\gamma)\ell$. Hence, we get the following decomposition of the functional in (1.1),

$$\mathcal{E}_{\kappa, H}(\cdot, \mathbf{A}) = \mathcal{E}_1(\cdot, \mathbf{A}) + \mathcal{E}_2(\cdot, \mathbf{A}) + \mathcal{E}_3(\cdot, \mathbf{A}).$$

Step 2. Estimating $\mathcal{E}_1(u, \mathbf{A})$. Using (6.8) for $\alpha = 2/3$, $|v_0| \leq 1$ and the Cauchy-Schwarz inequality, we write

$$\mathcal{E}_1(u, \mathbf{A}) \leq (1 + \kappa^{-\frac{1}{2}}) \mathcal{E}_0(v_0, \mathbf{F}; \mathcal{N}_{x_0}(\hat{\ell})) + C. \quad (6.22)$$

But by (6.7), we have $\mathcal{E}_0(v_0, \mathbf{F}; \mathcal{N}_{x_0}(\hat{\ell})) = \mathfrak{G}(u_0, \mathcal{V}(\hat{\ell}))$. Hence, Lemma 5.2 and (6.22) imply

$$\mathcal{E}_1(u, \mathbf{A}) \leq b^{-\frac{1}{2}} \kappa \hat{\ell} \mathfrak{e}_a(b) + C \kappa^{\frac{1}{8}}. \quad (6.23)$$

Step 3. Estimating $\mathcal{E}_2(u, \mathbf{A})$. Notice that $u = \eta \psi$ with $0 \leq \eta \leq 1$ in $\mathcal{N}_{x_0}(\tilde{\ell}) \setminus \mathcal{N}_{x_0}(\hat{\ell})$. Then, we do a straightforward computation, similar to the one done in the proof of (6.14), replacing f by η and $\mathcal{N}_{x_0}(\hat{\ell})$ by $\mathcal{N}_{x_0}(\tilde{\ell}) \setminus \mathcal{N}_{x_0}(\hat{\ell})$. This gives the following relation between $\mathcal{E}_2(u, \mathbf{A})$ and $\mathcal{E}_2(\psi, \mathbf{A})$

$$\mathcal{E}_2(u, \mathbf{A}) \leq \mathcal{E}_2(\psi, \mathbf{A}) + C \kappa^{\frac{3}{16}}. \quad (6.24)$$

Step 4. Estimating $\mathcal{E}_1(\psi, \mathbf{A})$. Since $(\psi, \mathbf{A})$ is a minimizer of the functional $\mathcal{E}_{\kappa, H}$ defined in (1.1), we write $\mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) \leq \mathcal{E}_{\kappa, H}(u, \mathbf{A})$. Noticing that $\mathcal{E}_3(u, \mathbf{A}) = \mathcal{E}_3(\psi, \mathbf{A})$, we get

$$\mathcal{E}_1(\psi, \mathbf{A}) + \mathcal{E}_2(\psi, \mathbf{A}) \leq \mathcal{E}_1(u, \mathbf{A}) + \mathcal{E}_2(u, \mathbf{A}).$$

We plug (6.23) and (6.24) into the previous inequality to get

$$\mathcal{E}_1(\psi, \mathbf{A}) \leq b^{-\frac{1}{2}} \kappa \hat{\ell} \mathfrak{e}_a(b) + C \kappa^{\frac{3}{16}}. \quad (6.25)$$

Recalling that $\mathcal{E}_1(\psi, \mathbf{A}) = \mathcal{E}_1(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell}))$, we see that (6.25) is nothing but (6.18) with $\hat{\ell}$ appearing instead of ℓ .

Step 5. Lower bound of the L^4 -norm of ψ . Consider the function f defined in (6.11). We use the properties of this function, mainly that $f = 1$ in $\mathcal{N}_{x_0}(\ell)$ and $0 \leq f \leq 1$ in Ω , to obtain

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 \left(-1 + \frac{1}{2} f^2 \right) |\psi|^4 dx \geq -\frac{1}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx - \int_{\mathcal{N}_{x_0}(\hat{\ell}) \setminus \mathcal{N}_{x_0}(\ell)} |\psi|^4 dx.$$

Following an argument similar to the one for (6.13), we divide the set $\mathcal{N}_{x_0}(\hat{\ell}) \setminus \mathcal{N}_{x_0}(\ell)$ into the two sets $(\mathcal{N}_{x_0}(\hat{\ell}) \setminus \mathcal{N}_{x_0}(\ell)) \cap \{\text{dist}(x, \Gamma) \leq \gamma \ell\}$ and $(\mathcal{N}_{x_0}(\hat{\ell}) \setminus \mathcal{N}_{x_0}(\ell)) \cap \{\text{dist}(x, \Gamma) > \gamma \ell\}$, and we use this time the exponential decay of $|\psi|^4$, deduced from Theorem 2.4, to get

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 \left(-1 + \frac{1}{2} f^2 \right) |\psi|^4 dx \geq -\frac{1}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx - C \kappa^{-\frac{15}{8}}. \quad (6.26)$$

Inserting (6.26) into (6.16) gives

$$\mathcal{E}_1(f\psi, \mathbf{A}) \geq -\frac{\kappa^2}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 dx - C \kappa^{\frac{1}{8}}.$$

The previous inequality together with (6.14) and (6.25) establish the lower bound in (6.19). $\square$

6.1.4. *Proof of Theorem 6.1.* Gather results in Propositions 6.3 and 6.4.

6.2. *Surface superconductivity.* In this section, we are concerned in the local behaviour of the sample near the boundary of Ω , under the assumption

$$b > \frac{1}{|a|}, \quad a \in [-1, 0).$$

The analysis of superconductivity near $\partial\Omega$ in our case of a step magnetic field (B_0 satisfying 1.2) is essentially the same as that in the uniform field case, since B_0 is constant in each of Ω_1 and Ω_2 . Thereby, the results presented in this section are well-known in the literature since the celebrated work of Saint-James and de Gennes [SJG63]. We refer to [CG17, CR16a, CR16b, CR14, FKP13, FK11, HFPS11, AH07, FH05, Pan02, LP99] for rigorous results in general 2D and 3D samples subjected to a constant magnetic field, and to [NSG+09] for recent experimental results. Particularly, local surface estimates were recently established in [HK17], when $B_0 \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$. We will adapt these results to our discontinuous magnetic field (see Theorem 6.5 below).

The statement of Theorem 6.5 involves the surface energy E_{surf} , that we introduce in the next section.

6.2.1. *The surface energy function.* Let $b \geq 1$ and Θ_0 be the value defined in (2.5). When $b \in (1, \Theta_0^{-1})$, the surface energy has been described by the 1D-energy, E_b^{1D} , introduced in (3.18) (see [CR14, CR16a, CR16b, CDR17, AH07, HFPS11] and references therein).

This same energy was introduced earlier in the literature via a 2D-reduced Ginzburg–Landau functional defined in what follows. Let $R > 1$. We consider

$$\mathcal{W}(U_R) \ni \phi \mapsto \mathcal{E}_{b,R}(\phi) = \int_{U_R} \left(b |(\nabla - i\mathbf{A}_0)\phi|^2 - |\phi|^2 + \frac{1}{2}|\phi|^4 \right) d\gamma d\tau,$$

where $(\gamma, \tau) \in \mathbb{R}^2$, $\mathbf{A}_0(\gamma, \tau) = (-\tau, 0)$, $U_R = (-R/2, R/2) \times (0, +\infty)$, and

$$\mathcal{W}(U_R) = \left\{ u \in L^2(U_R) : (\nabla - i\mathbf{A}_0)u \in L^2(U_R), u(\pm R, \cdot) = 0 \right\}.$$

The boundary condition in the domain $\mathcal{W}(U_R)$ is meant in the trace sense. Let $d(b, R)$ be the ground state energy defined by

$$d(b, R) = \inf_{\phi \in \mathcal{W}(U_R)} \mathcal{E}_{b,R}(\phi).$$

Pan proved in [Pan02] the existence of a non-decreasing continuous function $E_{\text{surf}} : [1, \Theta_0^{-1}] \rightarrow (-\infty, 0]$ such that

$$E_{\text{surf}}(b) = \lim_{R \rightarrow +\infty} \frac{d(b, R)}{R}. \quad (6.27)$$

Later, it was proven that (see e.g. [CR14])

$$E_{\text{surf}}(b) = E_b^{\text{1D}}, \text{ for } b \in (1, \Theta_0^{-1}).$$

One important property of the function $E_{\text{surf}}(\cdot)$ is (see [FH05])

$$E_{\text{surf}}(\Theta_0^{-1}) = 0 \text{ and } E_{\text{surf}}(b) < 0, \text{ for all } b \in \left[1, \Theta_0^{-1}\right). \quad (6.28)$$

This property allows us to extend the function $E_{\text{surf}}(\cdot)$ continuously to $[1, +\infty)$, by setting it to zero on $[\Theta_0^{-1}, +\infty)$. This extension of the surface energy is still denoted by E_{surf} for simplicity.

6.2.2. Local surface superconductivity. Let $t_0 > 0$ and $j \in \{1, 2\}$. We define the following set

$$\Omega_j(t_0) = \{x \in \Omega_j : \text{dist}(x, \partial\Omega_j \cap \partial\Omega) < t_0\}. \quad (6.29)$$

Assume that t_0 is sufficiently small, then for any $x \in \Omega_j(t_0)$, there exists a unique point $p(x) \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\text{dist}(x, \partial\Omega_j \cap \partial\Omega) = \text{dist}(x, p(x)).$$

Let $\ell \approx \kappa^{-3/4}$ be the parameter in (5.1). Assume that κ is sufficiently large and choose $x_0 \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\text{dist}(x_0, \Gamma) > 2\ell. \quad (6.30)$$

We introduce the following neighbourhood of x_0

$$\mathcal{N}_{x_0}^j(\ell) = \left\{x \in \Omega_j : \text{dist}_{\partial\Omega}(x_0, p(x)) < \frac{\ell}{2}, \text{dist}_{\Omega}(x, p(x)) < \ell\right\}. \quad (6.31)$$

The assumption on x_0 in (6.30) guarantees that $\overline{\mathcal{N}_{x_0}^j(\ell)} \subset \Omega_j$. Consequently, the estimates in Theorem 6.5 below hold uniformly with respect to the point x_0 .

Recall the magnetic field B_0 defined in Assumption 1.2 ($B_0 = \mathbb{1}_{\Omega_1} + a \mathbb{1}_{\Omega_2}$).

Theorem 6.5. *Let $a \in [-1, 0)$ and $b > 1/|a|$. There exists $\kappa_0 > 0$ and a function $\check{\epsilon} : [\kappa_0, +\infty) \rightarrow (0, +\infty)$ such that $\lim_{\kappa \rightarrow +\infty} \check{\epsilon}(\kappa) = 0$ and the following is true. For $\kappa \geq \kappa_0$, $H = b\kappa$, ℓ as in (5.1), $j \in \{1, 2\}$, $x_0 \in \partial\Omega_j \cap \partial\Omega$ satisfying (6.30), and every minimizer $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ of the functional in (1.1), we have*

$$\left| \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}^j(\ell)) - b^{-\frac{1}{2}} |B_0(x_0)|^{-\frac{1}{2}} \kappa \ell E_{\text{surf}}(b|B_0(x_0)|) \right| \leq \kappa \ell \check{\epsilon}(\kappa),$$

and

$$\left| \frac{1}{\ell} \int_{\mathcal{N}_{x_0}^j(\ell)} |\psi|^4 dx + 2b^{-\frac{1}{2}} |B_0(x_0)|^{-\frac{1}{2}} \kappa^{-1} E_{\text{surf}}(b|B_0(x_0)|) \right| \leq \kappa^{-1} \check{\epsilon}(\kappa),$$

where $\mathcal{N}_{x_0}^j(\cdot)$ is the set in (6.31), and $\mathcal{E}_0$ is the local energy in (6.1).

Furthermore, the function $\check{\epsilon}$ is independent of the point x_0 .

The estimates in Theorem 6.5 are established in [HK17], when the function B_0 is smooth. Since B_0 is constant in the neighbourhood $\mathcal{N}_{x_0}^j(\ell)$, the proof in [HK17] still holds in our case.

6.3. Proof of main results. In this section, we work under the conditions of Theorems 1.7 and 1.11. We will gather the results of the two previous sections to establish the two aforementioned theorems.

Proof of Theorem 1.11. We will decompose the sample Ω into the sets $\Gamma^*(\ell)$, $\Omega_1^*(\ell)$, $\Omega_2^*(\ell)$, $\Omega_{\text{bulk}}(\ell)$ and $T(\ell)$ introduced below in this section (see Fig. 6), and we will

Fig. 6. Schematic representation of the sets $\Omega_{\text{bulk}}(\ell)$, $T(\ell)$, $\mathcal{N}_{x_i}(\ell)$, $\mathcal{N}_{y_j}(\ell)$ and $\mathcal{N}_{z_k}(\ell)$, where $\ell \approx \kappa^{-3/4}$. In the regime where $a \in [-1, 0)$, $H = b\kappa$ and $1/|a| < b$, the white region is in the normal state, while the other regions may carry superconductivity

analyse the behaviour of the minimizer in each of these sets. We assume ℓ to be the parameter in (5.1) which satisfies $\ell \approx \kappa^{-3/4}$.

In a neighbourhood of the magnetic barrier We start by introducing the set $\Gamma^* = \Gamma^*(\ell)$ which covers almost all of the set Γ . Recall the assumption that Γ consists of a finite collection of simple disjoint smooth curves that may intersect $\partial\Omega$ transversely. For the simplicity of the exposition, we will focus on the particular case of a single curve intersecting $\partial\Omega$ at two points. The construction below may be adjusted to cover the general case by considering every single component of Γ separately. We may select two constants $\ell_0 \in (0, 1)$ and $c > 2$, and for all $\ell \in (0, \ell_0)$, a collection of pairwise distinct points $(x_i)_{i=1}^N \subset \Gamma$ such that,

$$\begin{aligned} (x_i)_{i=1}^N &\subset \{x \in \Gamma : \text{dist}(x, \partial\Omega) > c\ell\}, \\ \forall i \in \{1, \dots, N-1\}, \quad \text{dist}_\Gamma(x_i, x_{i+1}) &= \ell, \end{aligned}$$

and

$$\{x \in \Omega : \text{dist}(x, \Gamma) < \ell, \text{dist}(x, \partial\Omega) > c\ell\} \subset \Gamma^*(\ell) := \left(\bigcup_{i=1}^N \overline{\mathcal{N}_{x_i}(\ell)} \right)^\circ, \quad (6.32)$$

where $\mathcal{N}_{x_i}(\ell)$ is the set introduced in (6.3). The family $(\mathcal{N}_{x_i}(\ell))_{1 \leq i \leq N}$ consists of pairwise disjoint sets. The number N depends on ℓ as follows

$$|\Gamma|\ell^{-1} - \mathcal{O}(1) \leq N \leq |\Gamma|\ell^{-1}, \quad (\ell \rightarrow 0). \quad (6.33)$$

In a neighbourhood of the boundary Now, we define the two sets $\Omega_1^* = \Omega_1^*(\ell)$ and $\Omega_2^* = \Omega_2^*(\ell)$ which cover almost all of the set $\partial\Omega$. In a similar fashion to the definition of $\Gamma^*(\ell)$, we fix $\ell_0 \in (0, 1)$ and $c > 2$ and we select two collections of points

$$(y_j)_{j=1}^{N_1} \subset \{x \in \partial\Omega_1 : \text{dist}(x, \Gamma) > c\ell\} \text{ and } (z_k)_{k=1}^{N_2} \subset \{x \in \partial\Omega_2 : \text{dist}(x, \Gamma) > c\ell\},$$

such that

$$\text{dist}_{\partial\Omega_1}(y_j, y_{j+1}) = \ell \quad \text{and} \quad \text{dist}_{\partial\Omega_2}(z_k, z_{k+1}) = \ell,$$

for $1 \leq j \leq N_1 - 1$ and $1 \leq k \leq N_2 - 1$. Furthermore,

$$\{x \in \Omega : \text{dist}(x, \partial\Omega_1) < \ell, \text{dist}(x, \Gamma) > c\ell\} \subset \Omega_1^*(\ell) := \left(\bigcup_{j=1}^{N_1} \overline{\mathcal{N}_{y_j}(\ell)} \right)^\circ, \quad (6.34)$$

$$\{x \in \Omega : \text{dist}(x, \partial\Omega_2) < \ell, \text{dist}(x, \Gamma) > c\ell\} \subset \Omega_2^*(\ell) := \left(\bigcup_{k=1}^{N_2} \overline{\mathcal{N}_{z_k}(\ell)} \right)^\circ, \quad (6.35)$$

where $\mathcal{N}_{y_j}^1(\ell)$ and $\mathcal{N}_{z_k}^2(\ell)$ are defined in (6.31). The numbers N_1 and N_2 depend on ℓ as follows

$$|\partial\Omega_1|\ell^{-1} - \mathcal{O}(1) \leq N_1 \leq |\partial\Omega_1|\ell^{-1} \quad \text{and} \quad |\partial\Omega_2|\ell^{-1} - \mathcal{O}(1) \leq N_2 \leq |\partial\Omega_2|\ell^{-1}, \quad (\ell \rightarrow 0).$$

The bulk set Next, we introduce the set $\Omega_{\text{bulk}} = \Omega_{\text{bulk}}(\ell)$ representing the bulk of the sample:

$$\Omega_{\text{bulk}}(\ell) = \{x \in \Omega : \text{dist}(x, \partial\Omega_1 \cup \partial\Omega_2) > \ell\}. \quad (6.36)$$

In a neighbourhood of the T -zone We finally introduce the remaining set in the decomposition of Ω , the neighbourhood $T = T(\ell)$ of $\Gamma \cap \partial\Omega$

$$T(\ell) := \Omega \setminus \overline{\left(\bigcup_{j=1}^2 \Omega_j^*(\ell) \cup \Gamma^*(\ell) \cup \Omega_{\text{bulk}}(\ell) \right)}.$$

The definition of the sets Γ^* , Ω_1^* , Ω_2^* and Ω_{bulk} in (6.32), (6.34), (6.35) and (6.36) ensures that $|T| = \mathcal{O}(\ell^2)$ as $\ell \rightarrow 0$.

Behaviour of the minimizer Now, we are ready to prove the convergence of $|\psi|^4$ in the sense of distributions, claimed in Theorem 1.11.

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We have

$$\begin{aligned} \kappa T_\kappa^b(\varphi) &= \kappa \int_{\Omega_{\text{bulk}}} |\psi|^4 \varphi \, dx + \kappa \int_T |\psi|^4 \varphi \, dx + \kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx \\ &\quad + \kappa \int_{\Omega_1^*} |\psi|^4 \varphi \, dx + \kappa \int_{\Omega_2^*} |\psi|^4 \varphi \, dx. \end{aligned} \quad (6.37)$$

We will estimate each of these right hand side integrals. Starting with

$$\left| \kappa \int_{\Omega_{\text{bulk}}} |\psi|^4 \varphi \, dx \right| \leq \kappa \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega_{\text{bulk}}} |\psi|^4 \, dx = o(1), \quad (6.38)$$

by the exponential decay of ψ in Theorem 2.4.

Secondly, since $\|\psi\|_{L^\infty(\Omega)} \leq 1$ (see Proposition 2.1), $|T| = \mathcal{O}(\ell^2)$ as $\ell \rightarrow 0$ and by (5.1), we get

$$\left| \kappa \int_T |\psi|^4 \varphi \, dx \right| \leq C\kappa\ell^2 = o(1), \quad (6.39)$$

C is a constant independent of κ .

Next, we have [see (6.32)]

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi|^4 \varphi \, dx. \quad (6.40)$$

For $i \in \{1, \dots, N\}$, let p_i and q_i be two points of Γ such that

$$\varphi(p_i) = \max_{x \in \mathcal{N}_{x_i}(\ell) \cap \Gamma} \varphi(x) \quad \text{and} \quad \varphi(q_i) = \min_{x \in \mathcal{N}_{x_i}(\ell) \cap \Gamma} \varphi(x).$$

We may write

$$\begin{aligned} \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi|^4 \varphi \, dx &= \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx \\ &\quad + \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 (\varphi(x) - \varphi(p_i)) \, dx. \end{aligned} \quad (6.41)$$

We estimate $|\varphi(x) - \varphi(p_i)|$ in $\mathcal{N}_{x_i}(\ell)$ by the mean value theorem. Using the size of $\mathcal{N}_{x_i}(\ell)$ and the bound $\|\psi\|_{L^\infty(\Omega)} \leq 1$, we get

$$\left| \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 (\varphi(x) - \varphi(p_i)) \, dx \right| \leq C \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 |x - p_i| \, dx \leq C \ell^3,$$

for some C independent of κ . Hence, by (5.1) and (6.33)

$$\kappa \sum_{i=1}^N \left| \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 (\varphi(x) - \varphi(p_i)) \, dx \right| \leq C N \kappa \ell^3 = o(1). \quad (6.42)$$

On the other hand, using the uniform bounds in (6.10) and (6.19), we get

$$\left| \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx + 2b^{-\frac{1}{2}} \epsilon_a(b) \sum_{i=1}^N \ell \varphi(p_i) \right| \leq C \kappa^{-\frac{1}{16}} \sum_{i=1}^N \ell |\varphi(p_i)|,$$

where C is a constant independent of κ . We use further that $\sum_{i=1}^N \ell |\varphi(p_i)| \leq \|\varphi\|_\infty N \ell$ and $N \ell = \mathcal{O}(1)$ by (6.33). We get that

$$\left| \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx + 2b^{-\frac{1}{2}} \epsilon_a(b) \sum_{i=1}^N \ell \varphi(p_i) \right| \leq \tilde{C} \kappa^{-1/16}, \quad (6.43)$$

where $\tilde{C}$ is a new constant independent of κ . Combining (6.40)–(6.43) yields

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx \geq -2b^{-\frac{1}{2}} \epsilon_a(b) \sum_{i=1}^N \ell \varphi(p_i) + o(1) \geq -2b^{-\frac{1}{2}} \epsilon_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_\Gamma + o(1), \quad (6.44)$$

since our choice of the points (p_i) is such that the term $\sum_{i=1}^N \ell \varphi(p_i)$ is an upper Riemann sum of the function $\varphi(x)$ on the set $\Gamma^* \cap \Gamma$. Similarly, using $\sum_{i=1}^N \ell \varphi(q_i)$ the lower Riemann sum of the function $\varphi(x)$ on the set $\Gamma^* \cap \Gamma$, we get

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx \leq -2b^{-\frac{1}{2}} \epsilon_a(b) \sum_{i=1}^N \ell \varphi(q_i) + o(1) \leq -2b^{-\frac{1}{2}} \epsilon_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_{\Gamma} + o(1). \quad (6.45)$$

We combine (6.44) and (6.45) to obtain

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = -2b^{-\frac{1}{2}} \epsilon_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_{\Gamma} + o(1).$$

But by (6.33)

$$\left| \int_{\Gamma \setminus (\Gamma^* \cap \Gamma)} \varphi \, ds_{\Gamma} \right| \leq \|\varphi\|_{L^\infty(\Omega)} |\Gamma \setminus (\Gamma^* \cap \Gamma)| \leq C\ell = o(1).$$

Hence,

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = -2b^{-\frac{1}{2}} \epsilon_a(b) \int_{\Gamma} \varphi \, ds_{\Gamma} + o(1). \quad (6.46)$$

One can proceed similarly to prove that

$$\begin{aligned} \kappa \int_{\Omega_1^*} |\psi|^4 \varphi \, dx &= -2b^{-\frac{1}{2}} E_{\text{surf}}(b) \int_{\partial\Omega_1 \cap \partial\Omega} \varphi \, ds + o(1) \\ \text{and } \kappa \int_{\Omega_2^*} |\psi|^4 \varphi \, dx &= -2|a|^{-\frac{1}{2}} b^{-\frac{1}{2}} E_{\text{surf}}(b|a|) \int_{\partial\Omega_2 \cap \partial\Omega} \varphi \, ds + o(1). \end{aligned} \quad (6.47)$$

Gathering pieces in (6.38), (6.39), (6.46) and (6.47), we establish Theorem 1.11.

Proof of Theorem 1.7 We apply Theorem 1.11 for $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi = 1$ in a neighbourhood of Ω to get (1.7).

Multiplying both sides of the first equation in (1.4) by $\overline{\psi}$ then integrating by parts give

$$E_{\text{g.st}}(\kappa, H) = \mathcal{E}(\psi, \mathbf{A}; \Omega) \geq \mathcal{E}_0(\psi, \mathbf{A}; \Omega) = -\frac{1}{2} \kappa^2 \int_{\Omega} |\psi|^4 \, dx, \quad (6.48)$$

where $\mathcal{E}(\psi, \mathbf{A}; \cdot)$ and $\mathcal{E}_0(\psi, \mathbf{A}; \cdot)$ are the energies in (6.1). Using (6.48) and (1.7), we get the lower bound of $E_{\text{g.st}}(\kappa, H)$ in (1.6).

The upper bound of $E_{\text{g.st}}(\kappa, H)$ can be derived by the help of a suitable trial configuration. We are still considering the parameter ℓ as in (5.1). Let $\mathbf{F}$ be the magnetic potential introduced in Lemma 2.2. We define the function $h_{\Gamma} \in H^1(\Omega; \mathbb{C}) \cap H_0^1(\Gamma^*(\ell))$

$$h_{\Gamma}(x) = \sum_{i=1}^N \mathbb{1}_{\mathcal{N}_{x_i}(\ell)}(x) v_i(x),$$

where $\Gamma^*(\ell)$ and $\mathcal{N}_{x_i}(\ell)$ are respectively the sets in (6.32) and (6.3), $v_i(x) = \left(e^{i\kappa H\omega} u_i\right) \circ \Phi^{-1}(x)$, $\omega = \omega_{-\ell, \ell}$ is the gauge function in Lemma 4.1, Φ is the coordinate transformation in (4.3), u_i is defined by $u_i(s, t) = u_0(s - s_i, t)$ for $(s_i, t_i) = \Phi^{-1}(x_i)$, and u_0 is the minimizer of $\mathfrak{G}(\cdot, \mathcal{V}(\ell))$ defined in (5.5). From the definition of v_i , we derive the following [see (6.7)]

$$\mathcal{E}_0(v_i, \mathbf{F}; \mathcal{N}_{x_i}(\ell)) = \mathfrak{G}(u_0, \mathcal{V}(\ell)) .$$

The previous identity together with Lemma 5.2, (6.33) and $(\ell \approx \kappa^{-3/4})$ give

$$\mathcal{E}_0(h_\Gamma, \mathbf{F}; \Omega) = \sum_{i=1}^N \mathcal{E}_0(v_i, \mathbf{F}; \mathcal{N}_{x_i}(\ell)) \leq |\Gamma| b^{-1/2} \kappa \mathfrak{e}_a(b) + o(\kappa) , \quad (\kappa \rightarrow +\infty) . \quad (6.49)$$

Similarly, for $j \in \{1, 2\}$, using the results of Theorem 6.5, one may define a function $h_j \in H^1(\Omega; \mathbb{C}) \cap H_0^1(\Omega_j^*(\ell))$ satisfying

$$\begin{aligned} \mathcal{E}_0(h_1, \mathbf{F}; \Omega_1^*(\ell)) &\leq |\partial\Omega_1 \cap \partial\Omega| b^{-\frac{1}{2}} \kappa E_{\text{surf}}(b) + o(\kappa) , \\ \mathcal{E}_0(h_2, \mathbf{F}; \Omega_2^*(\ell)) &\leq |\partial\Omega_2 \cap \partial\Omega| b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} \kappa E_{\text{surf}}(b|a|) + o(\kappa) , \end{aligned} \quad (6.50)$$

where $\Omega_j^*(\ell)$ is defined in (6.34) and (6.35). Now, we define the trial function

$$h(x) = \mathbb{1}_{\Gamma^*(\ell)}(x) h_\Gamma(x) + \mathbb{1}_{\Omega_1^*(\ell)}(x) h_1(x) + \mathbb{1}_{\Omega_2^*(\ell)}(x) h_2(x) .$$

Noticing that $E_{\text{g.st}}(\kappa, H) \leq \mathcal{E}(h, \mathbf{F}; \Omega) = \mathcal{E}_0(h, \mathbf{F}; \Omega)$ [see (6.1)], we gather the results in (6.49) and (6.50) to derive the upper bound in (1.6).

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Appendix A. Some Spectral Properties of Fiber Operators

A.1. Harmonic oscillators on the semi-axis. Let $\xi \in \mathbb{R}$. Besides the Robin and Neumann realizations of the harmonic oscillator, we introduce the Dirichlet realization of

$$H^D[\xi] = -\frac{d^2}{dt^2} + (t - \xi)^2 , \quad (A.1)$$

with domain

$$\text{Dom}(H^D[\xi]) = \{u \in B^2(\mathbb{R}_+) : u(0) = 0\} ,$$

and lowest eigenvalue

$$\mu^D(\xi) = \inf \operatorname{sp}(H^D[\xi]). \quad (\text{A.2})$$

The perturbation theory [Kat66] ensures that the functions

$$\xi \mapsto \mu^D(\xi), \quad \xi \mapsto \mu^N(\xi), \quad \text{and} \quad \xi \mapsto \mu(\gamma, \xi)$$

are analytic, where $\mu(\gamma, \xi)$ and $\mu^N(\xi)$ are respectively defined in (2.3) and (2.5). We list the following well-known spectral properties (for instance see [DH93, RS72, Kac06]):

Proposition A.1. *The function $\xi \mapsto \mu^D(\xi)$ introduced in (A.2) satisfies*

$$\lim_{\xi \rightarrow -\infty} \mu^D(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mu^D(\xi) = 1.$$

For all $\gamma \in \mathbb{R}$, the function $\xi \mapsto \mu(\gamma, \xi)$ introduced in (2.3) satisfies

$$\lim_{\xi \rightarrow -\infty} \mu(\gamma, \xi) = +\infty \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mu(\gamma, \xi) = 1.$$

A.2. Spectral properties of the operator $\mathfrak{h}_a[\xi]$. Let $a \in [-1, 1) \setminus \{0\}$ and $\xi \in \mathbb{R}$. Recall the operator $\mathfrak{h}_a[\xi]$ introduced in (2.13) and its associated quadratic form $q_a[\xi]$ defined in (2.14). The embedding of the domain of $q_a[\xi]$ is compact in $L^2(\mathbb{R})$, hence the spectrum of $\mathfrak{h}_a[\xi]$ is an increasing sequence of eigenvalues converging to $+\infty$. The lowest eigenvalue is denoted by $\mu_a(\xi)$.

The result in the following proposition may be derived similarly as done in [FH10, Sect. 3.2.1]:

Proposition A.2. *The lowest eigenvalue $\mu_a(\xi)$ of $\mathfrak{h}_a[\xi]$ is simple. Furthermore, there exists a positive eigenfunction $g_{a,\xi}$ normalized with respect to the norm $\|\cdot\|_{L^2(\mathbb{R})}$. $g_{a,\xi}$ is the unique function satisfying such properties.*

The functions $\xi \mapsto \mu_a(\xi)$ and $\xi \mapsto g_{a,\xi}$ are in C^∞ by the perturbation theory (see [FH10, Theorem C.2.2]).

The bounds in Lemma A.3 are useful for establishing Proposition A.4, which is crucial in our study of the eigenvalue $\mu_a(\xi)$ (see Sect. 2.4).

Lemma A.3. *Let $a \in [-1, 1) \setminus \{0\}$ and $\xi \in \mathbb{R}$. It holds*

- If $a \in (0, 1)$, then

$$\min \left(\mu^N(-\xi), a\mu^N \left(\frac{\xi}{\sqrt{a}} \right) \right) \leq \mu_a(\xi) \leq \min \left(\mu^D(-\xi), a\mu^D \left(\frac{\xi}{\sqrt{a}} \right) \right).$$

- If $a \in [-1, 0)$, then

$$\min \left(\mu^N(-\xi), |a|\mu^N \left(-\frac{\xi}{\sqrt{|a|}} \right) \right) \leq \mu_a(\xi) \leq \min \left(\mu^D(-\xi), |a|\mu^D \left(-\frac{\xi}{\sqrt{|a|}} \right) \right). \quad (\text{A.3})$$

Proof. We will prove the lemma in the case $a \in (-1, 0)$. The proof follows similarly in the case $a \in (0, 1)$.

We start by establishing the upper bound in (A.3). Let $\xi \in \mathbb{R}$. Consider $u = u_{-\xi}^D$ the normalized eigenfunction of the operator $H^D[-\xi]$ defined in (A.1), corresponding to the lowest eigenvalue $\mu^D(-\xi)$. Then

$$\mu^D(-\xi) = \int_0^{+\infty} \left(|u'(t)|^2 + (t + \xi)^2 |u(t)|^2 \right) dt.$$

Noticing that $u \in H_0^1(\mathbb{R}_+)$, we extend it by zero on $\mathbb{R}_-$ (the extension is still denoted by u for simplicity). Hence, we have $q_a[\xi](u) = \mu^D(-\xi)$, where $q_a[\xi]$ is the quadratic form in (2.14). Using the min-max principle, we get

$$\mu_a[\xi] \leq \frac{q_a[\xi](u)}{\|u\|_{L^2(\mathbb{R})}^2} = \mu^D(-\xi).$$

Similarly, using $v = v_{-\xi/\sqrt{|a|}}^D$ the normalized eigenfunction of $H^D[-\xi/\sqrt{|a|}]$ corresponding to the lowest eigenvalue $\mu^D(-\xi/\sqrt{|a|})$, we can prove that

$$\mu^D\left(\frac{-\xi}{\sqrt{|a|}}\right) = \int_0^{+\infty} \left(|v'(t)|^2 + \left(t + \frac{\xi}{\sqrt{|a|}}\right)^2 |v(t)|^2 \right) dt \geq \frac{1}{|a|} \mu_a(\xi),$$

by the min-max principle, after employing the change of variable $x = -t/\sqrt{|a|}$ and extending the resulting function by 0 on $\mathbb{R}_+$.

Next, we establish the lower bound in (A.3). We consider $g = g_{a,\xi}$ the normalized eigenfunction of the operator $\mathfrak{h}_a[\xi]$ corresponding to the lowest eigenvalue $\mu_a(\xi)$ (see Proposition A.2). We have

$$\mu_a(\xi) = \int_{-\infty}^0 \left(|g'(t)|^2 + (at + \xi)^2 |g(t)|^2 \right) dt + \int_0^{+\infty} \left(|g'(t)|^2 + (t + \xi)^2 |g(t)|^2 \right) dt. \quad (\text{A.4})$$

Using the min-max principle, we write a lower bound for each integral appearing in (A.4). Indeed,

$$\int_0^{+\infty} \left(|g'(t)|^2 + (t + \xi)^2 |g(t)|^2 \right) dt \geq \mu^N(-\xi) \int_0^{+\infty} |g(t)|^2 dt, \quad (\text{A.5})$$

and

$$\int_{-\infty}^0 \left(|g'(t)|^2 + (at + \xi)^2 |g(t)|^2 \right) dt \geq |a| \mu^N\left(\frac{-\xi}{\sqrt{|a|}}\right) \int_{-\infty}^0 |g(t)|^2 dt. \quad (\text{A.6})$$

Note that, for obtaining (A.6), we performed the change of variable $x = -\sqrt{|a|}t$ which yielded

$$\begin{aligned} \int_{-\infty}^0 \left(|g'(t)|^2 + (at + \xi)^2 |g(t)|^2 \right) dt &= \sqrt{|a|} \int_0^{+\infty} \left(|w'(x)|^2 + \left(x + \frac{\xi}{\sqrt{|a|}}\right)^2 |w(x)|^2 \right) dx, \\ \int_{-\infty}^0 |g(t)|^2 dt &= \frac{1}{\sqrt{|a|}} \int_0^{+\infty} |w(x)|^2 dx, \end{aligned}$$

where $w(x) = g(-x/\sqrt{|a|})$.

Combining (A.4), (A.5) and (A.6), and using the normalization of g , we obtain the desired lower bound. $\square$

Proposition A.4. *Let $a \in [-1, 1) \setminus \{0\}$. We have*

- For $a \in (0, 1)$,

$$\lim_{\xi \rightarrow -\infty} \mu_a(\xi) = 1 \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mu_a(\xi) = a.$$

- For $a \in [-1, 0)$,

$$\lim_{\xi \rightarrow -\infty} \mu_a(\xi) = |a| \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mu_a(\xi) = +\infty.$$

Proof. It is sufficient to apply Proposition A.1 and Lemma A.3. $\square$

Proposition A.5 ([HS15]). *For any $a \in [-1, 1) \setminus \{0\}$ and $\xi \in \mathbb{R}$ we have*

$$\partial_\xi \mu_a(\xi) = \left(1 - \frac{1}{a}\right) \left(g'_{a,\xi}(0)^2 + (\mu_a(\xi) - \xi^2) g_{a,\xi}(0)^2\right), \quad (\text{A.7})$$

where $g_{a,\xi}$ is the eigenfunction in Proposition A.2.

Proof. (Feynman–Hellmann) For simplicity, we write μ, g and $\mathfrak{h}$ respectively for $\mu_a(\xi)$, $g_{a,\xi}$, and $\mathfrak{h}_a[\xi]$. Differentiating with respect to ξ and integrating by parts in

$$(\mathfrak{h} - \mu)g = 0 \quad (\text{A.8})$$

we get

$$\langle (\partial_\xi \mathfrak{h} - \partial_\xi \mu)g, g \rangle + \langle (\mathfrak{h} - \mu)\partial_\xi g, g \rangle = 0.$$

Hence using

$$\langle (\mathfrak{h} - \mu)\partial_\xi g, g \rangle = \langle \partial_\xi g, (\mathfrak{h} - \mu)g \rangle = 0,$$

and recalling that g is normalized, we obtain

$$\partial_\xi \mu = \langle \partial_\xi \mathfrak{h} g, g \rangle = 2 \int_{-\infty}^0 (\xi + at) g^2(t) dt + 2 \int_0^{+\infty} (\xi + t) g^2(t) dt. \quad (\text{A.9})$$

Integrating by parts the right hand side of (A.9), and using (A.8) establish the result. $\square$

Proposition A.6. *Let $a \in (-1, 0)$ and $\beta_a = \min_{\xi \in \mathbb{R}} \mu_a(\xi)$. We have*

$$|a|\Theta_0 < \beta_a,$$

where Θ_0 is the value in (2.5).

Proof. Let ξ_a be such that $\beta_a = \mu_a(\xi_a)$ (see [HPRS16]). We use the lower bound proof of Lemma A.3, with $g = g_{a,\xi_a}$ the positive normalized eigenfunction of the operator $\mathfrak{h}_a[\xi_a]$ corresponding to $\mu_a(\xi_a)$ (see Proposition A.2). We get

$$\mu_a(\xi_a) \geq |a|\Theta_0 \int_{-\infty}^0 g^2(t) dt + \Theta_0 \int_0^{+\infty} g^2(t) dt. \quad (\text{A.10})$$

Since g is normalized and positive, and $|a|\Theta_0 < \Theta_0$ for $a \in (-1, 0)$, the proof is completed. $\square$

Proposition A.7. *Let $a \in (-1, 0)$. If $\xi_a \in \mathbb{R}$ satisfies $\mu_a(\xi_a) = \min_{\xi \in \mathbb{R}} \mu_a(\xi)$, then $\xi_a < 0$.*

Proof. Suppose that $\xi_a \geq 0$. Let g_{a,ξ_a} be the positive normalized eigenfunction of the operator $\mathfrak{h}_a[\xi_a]$ corresponding to the lowest eigenvalue $\mu_a(\xi_a)$ (see Proposition A.2).

- If $\xi_a > 0$, then since $a < 0$, one sees that $q_a[0](g_{a,\xi_a}) < q_a[\xi_a](g_{a,\xi_a})$, where $q_a[\cdot]$ is the form in (2.14); consequently, the min-max principle gives $\mu_a(0) < \mu_a(\xi_a)$. This contradicts the definition of ξ_a .
- If $\xi_a = 0$, then by Proposition A.5,

$$0 = \partial_{\xi} \mu_a(\xi_a) = \left(1 - \frac{1}{a}\right) \left(g'_{a,0}(0)^2 + \mu_a(0) g_{a,0}(0)^2\right)^2 > 0,$$

since $a \in (-1, 0)$, g is positive and, by Proposition A.6, $\mu_a(0) > |a|\Theta_0 > 0$. $\square$

Appendix B. Decay Estimates for the 2D-Effective Model

The aim of this appendix is to prove Proposition 3.4. Recall that we work under (3.7), namely,

$$-1 \leq a < 0 \quad \text{and} \quad \frac{1}{|a|} \leq b < \frac{1}{\beta_a},$$

where β_a is the lowest eigenvalue introduced in (2.11).

For every $m \in \mathbb{N}$ and $R > 1$, we introduce the set $S_{R,m} = (-R/2, R/2) \times (-m, m)$ and the functional

$$\mathcal{G}_{a,b,R,m}(u) = \int_{S_{R,m}} \left(b |(\nabla - i\sigma \mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx \quad (\text{B.1})$$

defined over the space

$$\mathcal{D}_{R,m} = \left\{ u \in L^2(S_{R,m}) : (\nabla - i\sigma \mathbf{A}_0)u \in L^2(S_{R,m}), \right. \\ \left. u\left(x_1 = \pm \frac{R}{2}, \cdot\right) = u\left(\cdot, x_2 = \pm m\right) = 0 \right\}. \quad (\text{B.2})$$

Here σ was defined in (2.9). Now we define the ground state energy

$$\mathfrak{g}_a(b, R, m) = \inf_{u \in \mathcal{D}_{R,m}} \mathcal{G}_{a,b,R,m}(u). \quad (\text{B.3})$$

Lemma B.1. *There exists a universal constant $C > 0$, and for all $R > 1$, $m \geq 1$, there exists a function $\varphi_{a,b,R,m} \in \mathcal{D}_{R,m}$ satisfying,*

$$\|\varphi_{a,b,R,m}\|_{L^\infty(S_{R,m})} \leq 1, \quad (\text{B.4})$$

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left(|(\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R,m}|^2 + |\varphi_{a,b,R,m}|^2 \right) dx \leq CbR, \quad (\text{B.5})$$

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_{a,b,R,m}|^4 dx \leq C b^2 R, \quad (\text{B.6})$$

and

$$\mathcal{G}_{a,b,R,m}(\varphi_{a,b,R,m}) = \mathfrak{g}_a(b, R, m). \quad (\text{B.7})$$

Here $\mathcal{G}_{a,b,R,m}$ is the functional introduced in (B.1) and $\mathfrak{g}_a(b, R, m)$ is the ground state energy introduced in (B.3).

Proof. The boundedness and the regularity of the domain $S_{R,m}$ guarantee the existence of a minimizer $\varphi_m := \varphi_{a,b,R,m}$ of $\mathcal{G}_{a,b,R,m}$ in $\mathcal{D}_{R,m}$, satisfying

$$-b(\nabla - i\sigma \mathbf{A}_0)^2 \varphi_m = (1 - |\varphi_m|^2) \varphi_m \quad \text{in } S_{R,m}, \quad (\text{B.8})$$

see e.g. [FH10, Chapter 11]. Furthermore, Proposition 10.3.1 in [FH10] ensures that

$$\|\varphi_m\|_{L^\infty(S_{R,m})} \leq 1.$$

Next, select $\chi \in C^\infty(\mathbb{R})$ such that $\chi(x_2) = 0$ if $|x_2| \leq 1$, and $\chi(x_2) = |x_2|^2 / \ln |x_2|$ if $|x_2| \geq 4$. Consequently, the function χ satisfies

$$0 < |\chi'(x_2)| < \frac{3\sqrt{|x_2|}}{2 \ln |x_2|} \quad \text{for all } |x_2| \geq 4.$$

Multiply (B.8) by $\chi^2 \overline{\varphi_m}$ and integrate by parts,

$$\int_{S_{R,m}} \left(b |(\nabla - i\sigma \mathbf{A}_0) \chi \varphi_m|^2 - \chi^2 |\varphi_m|^2 + \chi^2 |\varphi_m|^4 \right) dx = b \int_{S_{R,m}} \chi'^2 |\varphi_m|^2 dx. \quad (\text{B.9})$$

Since the function $x \mapsto \chi(x_2) \varphi_m(x)$ is supported in $S_{R,m} \cap \{|x_2| \geq 1\}$ where $\text{curl}(\sigma \mathbf{A}_0) = \sigma$, we can apply the spectral inequality in [FH10, Lemma 1.4.1] to get, under the assumption $1/|a| \leq b < 1/\beta_a$,

$$b \int_{S_{R,m}} |(\nabla - i\sigma \mathbf{A}_0) \chi \varphi_m|^2 dx \geq b \int_{S_{R,m}} |\sigma| \chi^2 |\varphi_m|^2 dx \geq \int_{S_{R,m}} \chi^2 |\varphi_m|^2 dx. \quad (\text{B.10})$$

It follows from (B.9) and (B.10)

$$\begin{aligned} \int_{S_{R,m}} \chi^2(x_2) |\varphi_m|^4 dx &\leq b \int_{S_{R,m}} \chi'^2(x_2) |\varphi_m|^2 dx \\ &\leq b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \chi'^2(x_2) |\varphi_m|^2 dx + b \int_{S_{R,m} \cap \{|x_2| < 4\}} \chi'^2(x_2) |\varphi_m|^2 dx \\ &\leq C b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |\varphi_m|^2 dx + C b R. \end{aligned} \quad (\text{B.11})$$

Using Hölder's inequality,

$$\begin{aligned}
 & \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |\varphi_m|^2 dx \\
 & \leq \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{1}{|x_2| (\ln |x_2|)^2} dx \right)^{\frac{1}{2}} \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 dx \right)^{\frac{1}{2}} \\
 & \leq CR^{\frac{1}{2}} \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{|(\ln |x_2|)^2} |\varphi_m|^4 dx \right)^{\frac{1}{2}}. \tag{B.12}
 \end{aligned}$$

Now, using Cauchy–Schwarz inequality together with (B.11) and (B.12), we obtain

$$\begin{aligned}
 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 dx & \leq \int_{S_{R,m}} \chi^2(x_2) |\varphi_m|^4 dx \\
 & \leq CR^{\frac{1}{2}} b \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{|(\ln |x_2|)^2} |\varphi_m|^4 dx \right)^{\frac{1}{2}} + CbR \\
 & \leq Cb^2R + CbR. \tag{B.13}
 \end{aligned}$$

Consequently, under the assumption $1 \leq 1/|a| \leq b < 1/\beta_a$, we get (B.6). Inserting (B.6) into (B.12), we get

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |\varphi_m|^2 dx \leq CbR. \tag{B.14}$$

We still need to establish

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 dx \leq CbR. \tag{B.15}$$

To that end, we select $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x_2) = 0$ if $|x_2| \leq 1$, and $\eta(x_2) = \sqrt{|x_2|/\ln |x_2|}$ if $|x_2| \geq 4$. Multiplying the equation in (B.8) by $\eta^2 \overline{\varphi_m}$ and integrating over $S_{R,m}$, we get

$$\begin{aligned}
 & b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} |(\nabla - i\sigma \mathbf{A}_0)\eta(x_2)\varphi_m|^2 dx \\
 & = \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \left(\eta^2(x_2) |\varphi_m|^2 - \eta^2(x_2) |\varphi_m|^4 + b\eta'^2(x_2) |\varphi_m|^2 \right) dx. \tag{B.16}
 \end{aligned}$$

It is easy to check by a straightforward computation, and using Cauchy's inequality, that

$$\begin{aligned}
 & \int_S \left(\eta^2(x_2) |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 \right) dx \\
 & \leq \int_S \left(|(\nabla - i\sigma \mathbf{A}_0)\eta(x_2)\varphi_m|^2 - \eta'^2(x_2) |\varphi_m|^2 \right) dx \\
 & \quad + 2 |Re \langle \varphi_m \eta'(x_2), \eta(x_2)(\nabla - i\sigma \mathbf{A}_0)\varphi_m \rangle| \\
 & \leq \int_S \left(|(\nabla - i\sigma \mathbf{A}_0)\eta(x_2)\varphi_m|^2 + \frac{1}{2} \eta^2(x_2) |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 + \eta'^2(x_2) |\varphi_m|^2 \right) dx.
 \end{aligned}$$

where $S = S_{R,m} \cap \{|x_2| \geq 4\}$. Hence,

$$\begin{aligned} & \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2) |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 dx \\ & \leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \left(|(\nabla - i\sigma \mathbf{A}_0)\eta(x_2)\varphi_m|^2 + \eta'^2(x_2)|\varphi_m|^2 \right) dx. \end{aligned} \quad (\text{B.17})$$

Combining (B.16) and (B.17), we get

$$\begin{aligned} b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2) |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 dx & \leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2) |\varphi_m|^2 dx \\ & + 4b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta'^2(x_2) |\varphi_m|^2 dx. \end{aligned} \quad (\text{B.18})$$

The definition of η yields that, in $S_{R,m} \cap \{|x_2| \geq 4\}$, $\eta^2 = |x_2|/(\ln |x_2|)^2$, and $\eta'^2 \leq 4\eta^2$. Hence, (B.14) and (B.18) imply (B.15). $\square$

Corollary B.2. *There exists a universal constant $C > 0$ such that the minimizer $\varphi_{a,b,R,m}$ in Lemma B.1 satisfies, for all $R > 1$, $m \geq 1$,*

$$\int_{S_{R,m}} b \left| (\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R,m} \right|^2 + |\varphi_{a,b,R,m}|^2 dx \leq CbR. \quad (\text{B.19})$$

Proof. For the sake of brevity, we will write φ_m for $\varphi_{a,b,R,m}$. Using (B.14) and the fact that $|x_2|/(\ln |x_2|)^2 \geq 1$ for $|x_2| \geq 4$, we get

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} |\varphi_m|^2 dx \leq CbR.$$

On the other hand, using $\|\varphi_m\|_\infty \leq 1$ and $b > 1$ we get

$$\int_{S_{R,m} \cap \{|x_2| < 4\}} |\varphi_m|^2 dx \leq CbR.$$

Next, since φ_m satisfies

$$-b(\nabla - i\sigma \mathbf{A}_0)^2 \varphi_m = (1 - |\varphi_m|^2)\varphi_m \quad \text{in } S_{R,m},$$

a simple integration by parts over $S_{R,m}$ yields

$$\int_{S_{R,m}} b |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 dx = \int_{S_{R,m}} |\varphi_m|^2 dx - \int_{S_{R,m}} |\varphi_m|^4 dx \leq CbR. \quad \square$$

Now, we will investigate the regularity of the minimizer $\varphi_{a,b,R,m}$ in Lemma B.1. We have to be careful at this point since the magnetic field is a step function and therefore has singularities. As a byproduct, we will extract a convergent subsequence of $(\varphi_{a,b,R,m})_{m \geq 1}$.

We will use the following terminology. Let $\Omega \subset \mathbb{R}^2$ be an open set. If $(u_m)_{m \geq 1}$ is a sequence in $H^k(\Omega)$, then by saying that (u_m) is bounded/convergent in $H_{\text{loc}}^k(\Omega)$, we mean that it is bounded/convergent in $H^k(K)$, for every $K \subset \Omega$ open and relatively compact. A similar terminology applies for boundedness/convergence in $C_{\text{loc}}^{k,\alpha}(\Omega)$: A sequence $(u_m)_{m \geq 1}$ is bounded/convergent in $C_{\text{loc}}^{k,\alpha}(\Omega)$ if it is bounded/convergent in $C^{k,\alpha}(\overline{K})$, for every $K \subset \Omega$ open and relatively compact.

Lemma B.3. Assume that (3.7) holds. Let $R > 1$ and $\alpha \in (0, 1)$ be fixed. The sequence $(\varphi_{a,b,R,m})_{m \geq 1}$ defined by Lemma B.1 is bounded in $H_{\text{loc}}^3(S_R)$ and consequently in $C_{\text{loc}}^{1,\alpha}(S_R)$.

Proof. For simplicity, we will write $\varphi_m = \varphi_{a,b,R,m}$. The proof is split into three steps.

Step 1. We first prove the boundedness of (φ_m) in $H_{\text{loc}}^2(S_R)$. Using (B.8) we may write

$$\Delta \varphi_m = \frac{1}{b} (|\varphi_m|^2 - 1) \varphi_m + 2i\sigma \mathbf{A}_0 \cdot \nabla \varphi_m + |\sigma|^2 |\mathbf{A}_0|^2 \varphi_m. \quad (\text{B.20})$$

Let $K \subset S_R$ be open and relatively compact. Choose an open and bounded set $\tilde{K}$ such that $\bar{K} \subset \tilde{K} \subset S_R$. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $\tilde{K} \subset S_{R,m}$ and by Cauchy's inequality,

$$\int_{\tilde{K}} |\nabla \varphi_m|^2 dx \leq 2 \int_{\tilde{K}} |(\nabla - i\sigma \mathbf{A}_0) \varphi_m|^2 dx + 2 \int_{\tilde{K}} |\sigma|^2 |\mathbf{A}_0|^2 |\varphi_m|^2 dx.$$

Using $|\varphi_m| \leq 1$, the decay estimate in (B.19) and the boundedness of σ and $\mathbf{A}_0$ in $\tilde{K}$, we get a constant $C = C(\tilde{K}, R)$ such that

$$\int_{\tilde{K}} |\nabla \varphi_m|^2 dx \leq C, \quad \text{and} \quad \int_{\tilde{K}} |\Delta \varphi_m|^2 dx \leq C,$$

in light of (B.20). By the interior elliptic estimates (see for instance [FH10, Sect. E.4.1]), we get that $\varphi_m \in H^2(K)$ and

$$\|\varphi_m\|_{H^2(K)} \leq C \left(\|\Delta \varphi_m\|_{L^2(\tilde{K})} + \|\varphi_m\|_{L^2(\tilde{K})} \right) \leq \tilde{C}, \quad (\text{B.21})$$

where $\tilde{C}$ is a constant independent from m . This proves that $(\varphi_m)_{m \geq 1}$ is bounded in $H_{\text{loc}}^2(S_R)$.

Step 2. Here we will improve the result in Step 1 and prove that $(\varphi_m)_{m \geq 1}$ is bounded in $H_{\text{loc}}^3(S_R)$. It is enough to prove that the sequence $(\nabla \varphi_m)_{m \geq 1}$ is bounded in $H_{\text{loc}}^2(S_R)$.

Let $\varsigma_m = \partial_{x_2} \varphi_m$. We will prove that $(\Delta \varsigma_m)_{m \geq 1}$ is bounded in $L_{\text{loc}}^2(S_R)$. Recall that, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\mathbf{A}_0(x) = (-x_2, 0) \quad \text{and} \quad \sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a \mathbb{1}_{\mathbb{R}_-}(x_2),$$

hence,

$$(\sigma \mathbf{A}_0)(x) = (-x_2 \mathbb{1}_{\mathbb{R}_+}(x_2) - ax_2 \mathbb{1}_{\mathbb{R}_-}(x_2), 0), \quad (\text{B.22})$$

$$(\sigma^2 |\mathbf{A}_0|^2)(x) = x_2^2 \mathbb{1}_{\mathbb{R}_+}(x_2) + a^2 x_2^2 \mathbb{1}_{\mathbb{R}_-}(x_2). \quad (\text{B.23})$$

Obviously, the functions in (B.22) and (B.23) admit respectively the following weak partial derivatives

$$\partial_{x_2} (\sigma \mathbf{A}_0)(x) = (-\mathbb{1}_{\mathbb{R}_+}(x_2) - a \mathbb{1}_{\mathbb{R}_-}(x_2), 0) = (-\sigma(x), 0) \quad (\text{B.24})$$

$$\partial_{x_2} (\sigma^2 |\mathbf{A}_0|^2)(x) = 2x_2 \mathbb{1}_{\mathbb{R}_+}(x_2) + 2a^2 x_2 \mathbb{1}_{\mathbb{R}_-}(x_2) = 2x_2 \sigma^2(x). \quad (\text{B.25})$$

A straightforward computation using (B.20), (B.24) and (B.25) yields

$$\begin{aligned}\Delta \zeta_m &= \partial_{x_2} \Delta \varphi_m \\ &= \frac{1}{b} \varphi_m^2 \partial_{x_2} \overline{\varphi_m} + \frac{1}{b} |\varphi_m|^2 \partial_{x_2} \varphi_m - 2i\sigma x_2 \partial_{x_2} \partial_{x_1} \varphi_m \\ &\quad - 2i\sigma \partial_{x_1} \varphi_m + \sigma^2 x_2^2 \partial_{x_2} \varphi_m + 2\sigma^2 x_2 \varphi_m ,\end{aligned}$$

in the sense of weak derivatives. By Step 1, the sequence (φ_m) is bounded in $H_{\text{loc}}^2(S_R)$. Consequently, since $|\varphi_m| \leq 1$, it is clear that $(\Delta \zeta_m)_{m \geq 1}$ is bounded in $L_{\text{loc}}^2(S_R)$. By the interior elliptic estimates, we get that $(\zeta_m = \partial_{x_2} \varphi_m)_{m \geq 1}$ is bounded in $H_{\text{loc}}^2(S_R)$.

In a similar fashion, we prove that $(\partial_{x_1} \varphi_m)_{m \geq 1}$ is bounded in $H_{\text{loc}}^2(S_R)$.

Step 3. Finally, for every relatively compact open set $K \subset \Omega$, the space $H^3(K)$ is embedded in $C^{1,\alpha}(\overline{K})$. Consequently, (φ_m) is bounded in $C_{\text{loc}}^{1,\alpha}(S_R)$. $\square$

Lemma B.4. Assume that $R > 1$ and that (3.7) holds. Let $(\varphi_{a,b,R,m})_{m \geq 1}$ be the sequence defined in Lemma B.1. There exist a function $\varphi_{a,b,R} \in H_{\text{loc}}^3(S_R)$ and a subsequence, denoted by $(\varphi_{a,b,R,m})_{m \geq 1}$, such that

$$\varphi_{a,b,R,m} \longrightarrow \varphi_{a,b,R} \text{ in } H_{\text{loc}}^2(S_R) \text{ and } \varphi_{a,b,R,m} \longrightarrow \varphi_{a,b,R} \text{ in } C_{\text{loc}}^{0,\alpha}(S_R) \quad (\alpha \in (0, 1)).$$

Furthermore, for all $\alpha \in (0, 1)$, $\varphi_{a,b,R} \in C_{\text{loc}}^{1,\alpha}(S_R)$.

Proof. We continue writing φ_m for $\varphi_{a,b,R,m}$. Let $K \subset S_R$ be open and relatively compact. By Lemma B.3, $(\varphi_m)_{m \geq 1}$ is bounded in $H^3(K)$, hence it has a weakly convergent subsequence by the Banach–Alaoglu theorem. By the compact embedding of $H^3(K)$ in $H^2(K)$, and of $H^2(K)$ in $C^{0,\alpha}(\overline{K})$, we may extract a subsequence, that we denote by (φ_m) , such that it is strongly convergent in $H^2(K)$ and $C^{0,\alpha}(\overline{K})$. The subsequence in Lemma B.4 and its limit are then constructed via the standard Cantor’s diagonal process. $\square$

Lemma B.5. Let $R > 1$ and $\varphi_{a,b,R}$ be the function defined by Lemma B.4. The following statements hold:

$$\varphi_{a,b,R} \in \mathcal{D}_R, \quad |\varphi_{a,b,R}| \leq 1 \text{ in } S_R, \quad (\text{B.26})$$

$$-b(\nabla - i\sigma \mathbf{A}_0)^2 \varphi_{a,b,R} = (1 - |\varphi_{a,b,R}|^2) \varphi_{a,b,R} \quad \text{in } S_R, \quad (\text{B.27})$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left(|(\nabla - i\sigma \mathbf{A}_0) \varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \quad (\text{B.28})$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_{a,b,R}|^4 dx \leq Cb^2R, \quad (\text{B.29})$$

$$\int_{S_R} \left(b|(\nabla - i\sigma \mathbf{A}_0) \varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \quad (\text{B.30})$$

where $C > 0$ is a universal constant, and $\mathcal{D}_R$ is the space introduced in (3.2).

Proof. Let $(\varphi_{a,b,R,m})$ be the subsequence in Lemma B.4. Again, we will use (φ_m) and φ for $(\varphi_{a,b,R,m})$ and $\varphi_{a,b,R}$ respectively.

By Lemma B.1, the inequality $|\varphi_m| \leq 1$ holds for all m . The inequality $|\varphi| \leq 1$ then follows from the uniform convergence of (φ_m) stated in Lemma B.4. By the convergence of (φ_m) in $H_{\text{loc}}^2(S_R)$ and $C_{\text{loc}}^{0,\alpha}(S_R)$, we get (B.27) from

$$-b(\nabla - i\sigma \mathbf{A}_0)^2 \varphi_m = (1 - |\varphi_m|^2) \varphi_m.$$

Now we prove that $\varphi \in \mathcal{D}_R$. Pick an arbitrary integer $m_0 \geq 1$. For all $m \geq m_0$, $S_{R,m_0} \subset S_{R,m}$. Thus using the decay of φ_m in (B.19) we have

$$\int_{S_{R,m_0}} |\varphi_m|^2 dx \leq \int_{S_{R,m}} |\varphi_m|^2 dx \leq CbR.$$

The uniform convergence of (φ_m) to φ gives us

$$\int_{S_{R,m_0}} |\varphi|^2 dx = \lim_{m \rightarrow +\infty} \int_{S_{R,m_0}} |\varphi_m|^2 dx \leq CbR.$$

Taking $m_0 \rightarrow +\infty$, we write by the monotone convergence theorem,

$$\int_{S_R} |\varphi|^2 dx \leq CbR.$$

This proves that $\varphi \in L^2(S_R)$. Next we will prove that $(\nabla - i\sigma \mathbf{A}_0)\varphi \in L^2(S_R)$. In light of the convergence of (φ_m) in $H_{\text{loc}}^1(S_R)$, we can refine the subsequence (φ_m) so that

$$(\nabla - i\sigma \mathbf{A}_0)\varphi_m \rightarrow (\nabla - i\sigma \mathbf{A}_0)\varphi \text{ a.e.}$$

Furthermore, by Lemma B.3, (φ_m) is bounded in $C_{\text{loc}}^1(S_R)$, hence in $C^1(S_{R,m_0})$, for all $m_0 \geq 1$. Using the dominated convergence theorem and the estimate in (B.19), we may write, for all $m_0 \geq 1$,

$$\int_{S_{R,m_0}} |(\nabla - i\sigma \mathbf{A}_0)\varphi|^2 dx = \lim_{m \rightarrow +\infty} \int_{S_{R,m_0}} |(\nabla - i\sigma \mathbf{A}_0)\varphi_m|^2 dx \leq CR.$$

Sending m_0 to $+\infty$ and using the monotone convergence theorem, we get

$$\int_{S_R} |(\nabla - i\sigma \mathbf{A}_0)\varphi|^2 dx \leq CR.$$

Thus, we have proven that $\varphi, (\nabla - i\sigma \mathbf{A}_0)\varphi \in L^2(S_R)$. It remains to prove that φ satisfies the boundary condition

$$\varphi\left(x_1 = \pm \frac{R}{2}, x_2\right) = 0, \quad \text{for all } x_2 \in \mathbb{R}.$$

To see this, let $x_2 \in \mathbb{R}$. There exists m_0 such that $x_2 \in (-m_0, m_0)$. By the convergence of (φ_m) to φ in $C^{0,\alpha}(\overline{S_{R,m_0}})$, we get

$$\varphi\left(x_1 = \pm \frac{R}{2}, x_2\right) = \lim_{m \rightarrow +\infty} \varphi_m\left(x_1 = \pm \frac{R}{2}, x_2\right) = 0.$$

Finally, we may use similar limiting arguments to pass from the decay estimates of φ_m in (B.5) and (B.6) to the decay estimates of φ in (B.28) and (B.29). $\square$

Now, we are ready to establish the existence of a minimizer of the Ginzburg–Landau energy $\mathcal{G}(a, b, R)$ defined in the unbounded set S_R .

Lemma B.6. *Let $R > 1$. The function $\varphi_{a,b,R} \in \mathcal{D}_R$ defined in Lemma B.4 is a minimizer of $\mathcal{G}_{a,b,R}$, that is*

$$\mathcal{G}_{a,b,R}(\varphi_{a,b,R}) = \mathfrak{g}_a(b, R).$$

Here $\mathcal{G}_{a,b,R}$ is the functional introduced in (3.3) and $\mathfrak{g}_a(b, R)$ is the ground state energy defined in (3.4).

Proof. The proof is divided into three steps.

Step 1. (Convergence of the ground state energy). Let $\mathfrak{g}_a(b, R, m)$ and $\mathfrak{g}_a(b, R)$ be the energies defined in (B.3) and (3.4) respectively. In this step, we will prove that

$$\lim_{m \rightarrow +\infty} \mathfrak{g}_a(b, R, m) = \mathfrak{g}_a(b, R). \quad (\text{B.31})$$

Let $u \in \mathcal{D}_{R,m}$. We can extend u by 0 to a function $\tilde{u} \in \mathcal{D}_R$. As an immediate consequence, we get $\mathfrak{g}_a(b, R, m) \geq \mathfrak{g}_a(b, R)$, for all $m \geq 1$. Thus, $\liminf_{m \rightarrow +\infty} \mathfrak{g}_a(b, R, m) \geq \mathfrak{g}_a(b, R)$. Next, we will prove that

$$\limsup_{m \rightarrow +\infty} \mathfrak{g}_a(b, R, m) \leq \mathfrak{g}_a(b, R). \quad (\text{B.32})$$

Consider $(\varphi_n) \subset \mathcal{D}_R$ a minimizing sequence of $\mathcal{G}_{a,b,R}$, that is $\mathfrak{g}_a(b, R) = \lim_{n \rightarrow +\infty} \mathcal{G}_{a,b,R}(\varphi_n)$.

Let $\vartheta \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \leq \vartheta \leq 1 \text{ in } \mathbb{R}, \quad \text{supp } \vartheta \subset (-1, 1), \quad \vartheta = 1 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Consider the re-scaled function $\vartheta_m(x_2) = \vartheta(x_2/m)$. The function $\vartheta_m(x_2)\varphi_n(x)$ restricted to $S_{R,m}$ belongs to $\mathcal{D}_{R,m}$ and consequently

$$\mathfrak{g}_a(b, R, m) \leq \mathcal{G}_{a,b,R}(\vartheta_m \varphi_n). \quad (\text{B.33})$$

By Cauchy's inequality, for all $\epsilon \in (0, 1)$

$$|(\nabla - i\sigma \mathbf{A}_0)\vartheta_m \varphi_n|^2 \leq (1 + \epsilon) |\vartheta_m(\nabla - i\sigma \mathbf{A}_0)\varphi_n|^2 + 2\epsilon^{-1} |\nabla \vartheta_m|^2 |\varphi_n|^2.$$

Thus, using the definition of the ground state energy $\mathfrak{g}_a(b, R, m)$ and the functional $\mathcal{G}_{a,b,R}$ in (B.3) and (3.3) respectively, we obtain

$$\begin{aligned} \mathfrak{g}_a(b, R, m) &\leq (1 + \epsilon) \mathcal{G}_{a,b,R}(\varphi_n) + \frac{2b\epsilon^{-1}}{m^2} \|\vartheta'\|_{L^\infty(\mathbb{R})}^2 \int_{S_R} |\varphi_n|^2 dx \\ &\quad + \int_{S_R} (1 - \vartheta_m^2 + \epsilon) |\varphi_n|^2 dx. \end{aligned} \quad (\text{B.34})$$

Introducing $\limsup_{m \rightarrow +\infty}$ on both sides of (B.34), and using the dominated convergence theorem, we get

$$\limsup_{m \rightarrow +\infty} \mathfrak{g}_a(b, R, m) \leq (1 + \epsilon) \mathcal{G}_{a,b,R}(\varphi_n) + \epsilon \int_{S_R} |\varphi_n|^2 dx.$$

Taking the successive limits $\epsilon \rightarrow 0_+$ and $n \rightarrow +\infty$, we get (B.32).

Step 2. (The L^4 -norm of the limit function). Let $(\varphi_m = \varphi_{a,b,R,m})$ be the sequence in Lemma B.4 which converges to the function $\varphi = \varphi_{a,b,R}$. We would like to verify that the limit function φ is a minimizer of the functional $\mathcal{G}_{a,b,R}$. To that end, we will prove first that

$$\lim_{m \rightarrow +\infty} \int_{S_{R,m}} |\varphi_m|^4 dx = \int_{S_R} |\varphi|^4 dx. \quad (\text{B.35})$$

We begin by proving that

$$\liminf_{m \rightarrow +\infty} \int_{S_{R,m}} |\varphi_m|^4 dx \geq \int_{S_R} |\varphi|^4 dx. \quad (\text{B.36})$$

Pick a fixed integer $m_0 \geq 1$. Since $S_{R,m} \supset S_{R,m_0}$ for all $m \geq m_0$, the following inequality holds

$$\int_{S_{R,m}} |\varphi_m|^4 dx \geq \int_{S_{R,m_0}} |\varphi_m|^4 dx. \quad (\text{B.37})$$

In addition, having in hand the uniform convergence of φ_m to φ on the compact set S_{R,m_0} , we get as $m \rightarrow \infty$

$$\int_{S_{R,m_0}} |\varphi_m|^4 dx \rightarrow \int_{S_{R,m_0}} |\varphi|^4 dx. \quad (\text{B.38})$$

We introduce $\liminf_{m \rightarrow +\infty}$ on both sides of (B.37), and we use (B.38) to get

$$\liminf_{m \rightarrow +\infty} \int_{S_{R,m}} |\varphi_m|^4 dx \geq \int_{S_{R,m_0}} |\varphi|^4 dx.$$

This is true for every integer $m_0 \geq 1$. Consequently (B.36) simply follows by applying the monotone convergence theorem.

Next, we prove that

$$\limsup_{m \rightarrow +\infty} \int_{S_{R,m}} |\varphi_m|^4 dx \leq \int_{S_R} |\varphi|^4 dx. \quad (\text{B.39})$$

Let C be the universal constant in (B.6), $\epsilon > 0$ be fixed, and $R > 1$ be arbitrary. We select an integer $m_0 \geq 1$ such that

$$\frac{Cb^2R}{m_0} < \epsilon. \quad (\text{B.40})$$

In light of (B.38), there exists $m_1 \geq m_0$ such that

$$\forall m \geq m_1, \quad \left| \int_{S_{R,m_0}} |\varphi_m|^4 dx - \int_{S_{R,m_0}} |\varphi|^4 dx \right| \leq \epsilon.$$

Noticing that $\int_{S_{R,m_0}} |\varphi|^4 dx \leq \int_{S_R} |\varphi|^4 dx$, we may write for all $m \geq m_1$

$$\int_{S_{R,m_0}} |\varphi_m|^4 dx \leq \int_{S_R} |\varphi|^4 dx + \epsilon. \quad (\text{B.41})$$

On the other hand, for $|x_2| \geq m_0 \geq 1$ we have $m_0 \leq \frac{|x_2|^3}{(\ln |x_2|)^2}$. Thus, the estimate in (B.6) yields for all $m \geq m_0$,

$$\int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 dx \leq \underbrace{\frac{Cb^2R}{m_0}}_{\text{by (B.40)}} < \epsilon. \quad (\text{B.42})$$

Combining (B.41) and (B.42), we get for all $m \geq m_1 \geq m_0$

$$\int_{S_{R,m}} |\varphi_m|^4 dx = \int_{S_{R,m_0}} |\varphi_m|^4 dx + \int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 dx \leq \int_{S_R} |\varphi|^4 dx + 2\epsilon.$$

Taking the successive limits $m \rightarrow +\infty$ and $\epsilon \rightarrow 0_+$, we get (B.39).

Step 3. (The limit function is a minimizer). The convergence in (B.35) is crucial in establishing that φ is a minimizer of $\mathcal{G}_{a,b,R}$. In light of Eq. (B.8), an integration by parts yields, for all $m \geq 1$,

$$\mathfrak{g}_a(b, R, m) = -\frac{1}{2} \int_{S_{R,m}} |\varphi_m|^4 dx.$$

We take $m \rightarrow +\infty$, and we use the results in (B.31) and (B.35). We get

$$\mathfrak{g}_a(b, R) = -\frac{1}{2} \int_{S_R} |\varphi|^4 dx. \quad (\text{B.43})$$

By Lemma B.5, $\varphi \in \mathcal{D}_R$ and satisfies (B.27), so after integrating by parts, we get

$$\mathcal{G}_{a,b,R}(\varphi) = -\frac{1}{2} \int_{S_R} |\varphi|^4 dx. \quad (\text{B.44})$$

Comparing (B.43) and (B.44) yields that $\mathcal{G}_{a,b,R}(\varphi) = \mathfrak{g}_a(b, R)$. $\square$

Proof of Proposition 3.4. All the properties stated in Proposition 3.4 (except the non-triviality of the minimizer) are simply a convenient collection in one place of already proven facts in Lemmas B.5 and B.6. With these properties in hand, the non-triviality of $\varphi_{a,b,R}$ follows from Lemma 3.7.

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