Communications in
Mathematical
Physics

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Analyticity of Nekrasov Partition Functions

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Abstract: We prove that the K-theoretic Nekrasov instanton partition functions have a positive radius of convergence in the instanton counting parameter and are holomorphic functions of the Coulomb parameters in a suitable domain. We discuss the implications for the AGT correspondence and the analyticity of the norm of Gaiotto states for the deformed Virasoro algebra. The proof is based on random matrix techniques and relies on an integral representation of the partition function, due to Moore, Nekrasov, and Shatashvili, which we also prove.

Contents

1. Introduction

The purpose of this paper is to study the analytic properties of the Nekrasov instanton partition function. In $\mathcal{N} = 2$ supersymmetric gauge theory in four and five dimensions, Nekrasov's instanton partition function [18] plays the role of a basic building block. In the physical interpretation, the instanton partition function is the non-perturbative contribution of instantons to a gauge theory in an Ω-background. Mathematically it is the generating function of integrals of torus equivariant cohomology classes (K-theory classes in the five dimensional theory) on the moduli space of framed torsion free sheaves with fixed rank on the complex projective plane. The torus is a product of a two-dimensional torus acting on the projective plane and a torus acting on the framing. The parameters ϵ_1, ϵ_2 of the Ω-background are the equivariant parameters of the two-dimensional torus in the mathematical description and serve as an infrared regulator. One early success of the theory [18,19] was a microscopic justification of the Seiberg–Witten formula [22] for the prepotential of the low energy effective theory, which arises in the limit $\epsilon_1, \epsilon_2 \to 0$. The Nekrasov instanton partition function is a power series in a complex variable $\mathfrak{q}$ parametrizing the strength of the interaction in gauge theory. The coefficient of $\mathfrak{q}^n$ is the contribution of instanton number n to the partition function. The simplest case of a Nekrasov partition function appears in the pure $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with gauge group $U(r)$ on $\mathbb{R}^4 \times S^1$. It is given as a sum over r-tuples $\vec{Y} = (Y_i)_{i=1}^r$ of Young diagrams of total size $|\vec{Y}|$:

$$
Z(\epsilon_1, \epsilon_2, a, \mathfrak{q}, \lambda) = \sum_{\vec{Y}} \mathfrak{q}^{|\vec{Y}|} \prod_{\alpha,\beta=1}^r \prod_{b \in Y_\alpha} \frac{(\lambda/2)^2}{\sinh\left(\frac{\lambda}{2} E_{\alpha\beta}(b)\right) \sinh\left(\frac{\lambda}{2}(\epsilon_1 + \epsilon_2 - E_{\alpha\beta}(b))\right)},
$$
$$
E_{\alpha\beta}(b) = a_\alpha - a_\beta - l_{Y_\beta}(b)\epsilon_1 + (a_{Y_\alpha}(b) + 1)\epsilon_2.
$$

Here λ is the circumference of the circle S^1 and $a = (a_1, \ldots, a_r)$ belongs to the Lie algebra of $U(r)$ and parametrizes boundary conditions of scalar fields in the $\mathcal{N} = 2$ vector multiplet. The arm length $a_Y(b)$ of a box b is the number of boxes in the Young diagram Y to the right of b; the leg length $l_Y(b)$ is the (possibly negative) number of boxes in Y below b. The four dimensional theory on $\mathbb{R}^4$ arises in the limit $\lambda \to 0$ and amounts to replacing $\sinh(\lambda x)/\lambda$ by x and $\mathfrak{q} = \Lambda^{2r}$ is related to the dynamical mass scale Λ of the gauge theory. In the mathematical description, we view $\mathbb{R}^4$ as $\mathbb{C}^2$ with its action of $(\mathbb{C}^\times)^2$ and embed it in $\mathbb{CP}^2$ by adding a line at infinity $\ell_\infty \cong \mathbb{CP}^1$. Let $\mathcal{M}(r, n)$ be the moduli space of torsion free sheaves on $\mathbb{CP}^2$ of rank r and second Chern class n with a framing, i.e., a trivialization on ℓ_∞. It is a smooth algebraic variety of dimension $2nr$. The action of the group $(\mathbb{C}^\times)^2$ lifts to an action on $\mathcal{M}(r, n)$. Moreover GL_r, and in particular its Cartan torus $T = (\mathbb{C}^\times)^r$ acts on $\mathcal{M}(r, n)$ by changing the framing. Therefore we have an action of $\tilde{T} = (\mathbb{C}^\times)^2 \times T$ on $\mathcal{M}(r, n)$. For a semisimple $\tilde{T}$-module V with finite dimensional weight spaces V_χ, we denote by $\mathrm{ch}(V) = \sum_\chi e^\chi \dim V_\chi$ its formal character in the completed group ring of the weight lattice. It is a formal series in the weight variables q_1, q_2 for the $(\mathbb{C}^\times)^2$-action and $u_1, \ldots, u_r$ for the T-action. Then

$$
Z(\epsilon_1, \epsilon_2, a, \mathfrak{q}, \lambda) = \sum_{n=0}^{\infty} z^n \sum_{i=1}^{2nr} (-1)^i \mathrm{ch}\, H^i(\mathcal{M}(r, n), \mathcal{O}), \quad z = \mathfrak{q}\lambda^{2r} e^{-\lambda(\epsilon_1+\epsilon_2)r/2}, \tag{1}
$$

is the pushforward of the $\tilde{T}$-equivariant K-theory class of the structure sheaf by the map to a point, with the identification

$$q_1 = e^{-\lambda \epsilon_1}. \quad q_2 = e^{-\lambda \epsilon_2}, \quad u_\alpha = e^{-\lambda a_\alpha}, \quad \alpha = 1, \ldots, r.$$

The combinatorial formula above is obtained by localization to fixed points. There are many variants and generalizations of the Nekrasov partition functions. In this paper we focus on the case where one includes matter fields in the gauge theory, which corresponds to replacing the structure sheaf by more general K-theory classes in the mathematical description.

The absolute value squared $|Z|^2$ of the Nekrasov partition function, in the limit $\lambda \to 0$, also appears in the integrand of Pestun's formula for gauge theory on the round sphere S^4 [21] and in its extension to S^4 with an ellipsoid metric parametrized by $\epsilon_1, \epsilon_2 > 0$ [11]. For $\lambda > 0$ this formula extends to $S^4 \times S^1$ and other compact manifolds, see [20] for a review. For these reasons it is important to understand the convergence properties of the formal power series defining the partition function. In this paper, we prove that the power series has a positive radius of convergence if $\lambda > 0$ and in a suitable range of parameters. For example, in the case of pure gauge theory, we show that if $\lambda > 0$ and $\epsilon_1, \epsilon_2 > 0$, the series (1) converges for all z in the unit disk to an analytic function of the Coulomb parameters a_i in a neighbourhood of the imaginary axis, see Theorem 5 for the general result. Note that for this range of parameters a direct estimation of the sum over partitions is problematic because of small denominators. In fact, the individual terms are not defined if $\epsilon_1/\epsilon_2 \in \pi\mathbb{Q}$ and have arbitrarily small denominators otherwise. The situation is different in the case $\epsilon_1 < 0 < \epsilon_2$ (which we do not consider) where a positive radius of convergence may be obtained by direct estimates on individual terms for generic a_i. Our proof applies to the case $\epsilon_1, \epsilon_2 > 0$ and also $\epsilon_1 = \bar{\epsilon}_2$ not imaginary, and relies on an integral representation of the coefficients of Z which resembles a unitary random matrix integral. This integral representation is actually the form in which this type of partition functions first appeared in the literature, as a regularized volume of instanton moduli spaces, see [14, , Section 6]. We estimate it using methods of random matrix theory and an explicit formula from representation homology [4]. The combinatorial formula for Z above is the sum of residues at certain poles of the integrand. As the choice of integration cycle is essential for the estimate, we carefully prove that the integral formula with the correct integration cycle is equal to the combinatorial formula. This requires showing that certain apparent residues actually cancel out. The limit $\lambda \to 0$ to the four dimensional theory is subtle and appears to require a different approach; we hope to return to this problem in the future.

Another reason for the interest in the radius of convergence of the power series Z comes from conformal field theory in 2 dimensions through the Alday–Gaiotto–Tachikawa (AGT) correspondence [2]. According to this correspondence, which was verified in a number of cases, the instanton partition functions of $\mathcal{N} = 2$ supersymmetric gauge theories with suitable matter fields are equal, up to a known scalar factor, to conformal blocks of W-algebras. For example, for $r = 2$ and $\lambda = 0$ they are related to the four point conformal blocks of the Virasoro algebra is defined as a matrix element of products of certain intertwining operators (primary fields). The parameters are the central charge $c = 13 + 6b^2 + 6b^{-2}$ with $\epsilon_1 = b = \epsilon_2^{-1}$ and the highest weights of four Virasoro representations, related to masses of matter fields. A priori the four point conformal block is a formal power series in the cross ratio $\mathfrak{q}$ of the four points. Except in the special cases of degenerate representations, occurring in minimal models at $c < 1$, where they are solutions of differential equations of hypergeometric type, giving full control on the

radius of convergence and analytic continuation, representation theory does not seem to give information on the convergence of the power series. This is particularly relevant for the unitary representations of the Virasoro algebra with $c > 1$, arising in Liouville theory. We treat both the weakly coupled Liouville theory with $c \geq 25$ and $\epsilon_1, \epsilon_2 > 0$ and the strongly coupled range $1 < c \leq 25$, with $\epsilon_1 = \bar{\epsilon}_2$ on the unit circle. The limiting case $c = 1$ ($\epsilon_1 = i = -\epsilon_2$) was considered for $\lambda = 0$ in [12]. The case $\epsilon_1 = -\epsilon_2$ for $\lambda \neq 0$ was recently considered in [5]. In cases where the AGT correspondence is understood, our estimates on the gauge theory side imply the convergence of conformal blocks. We make this explicit in the mathematically well-understood case of the norm of Gaiotto states for the q-deformed Virasoro algebra. Gaiotto states [9] are Whittaker vectors in completions of Verma modules of W-algebras and their deformations. In the case of the Virasoro algebra a Gaiotto state is a formal power series in the eigenvalue of the generator L_1. The squared norm of a Gaiotto state can be understood as suitable limits of a conformal block and corresponds via the AGT correspondence to the Nekrasov partition function of the pure Yang–Mills theory. As an application of our result we prove that for a suitable range of parameters the squared norm of the Gaiotto state for the two-parameter deformation of the Virasoro algebra is a holomorphic function of the eigenvalue with a convergence radius that converges to infinity in the $\lambda \to 0$ limit.

This agrees with the findings of Its et al. [12] and Bershtein et al. [5]. They consider the norm of the Gaiotto state in the case $\lambda = 0$ and $\lambda > 0$, respectively. Moreover, they require $\epsilon_1 + \epsilon_2 = 0$, which is different from our setting. In their respective setups, the Plancherel measure on partitions appears and allows for a direct estimate, which proves analyticity of the norm of the Gaiotto state on the whole complex plane.

The paper is organized as follows: in Sect. 2 we review the mathematical definition of the Nekrasov partition function with matter fields by introducing $M(r, n)$ as a special case of a Nakajima quiver variety via the ADHM construction and explain the localization formula leading to the combinatorial expression for Z. We then introduce the integral representation of Z and compute the large n behaviour of the coefficient of q^n using methods of random matrix theory and estimate the radius of convergence. In Sect. 3 we discuss how our method can be generalized to more general gauge theory, including the $\mathcal{N} = 2^*$ theory. In Sect. 4 we present the application to the norm of Gaiotto states for the deformed Virasoro algebra. In Sect. 5 we discuss some of the problems that are left open. The Appendix contains the proof of the integral representation of the partition function.

2. Nekrasov Partition Functions

In this section, we define the K-theoretic Nekrasov partition function as discussed in [16]. It is defined as a formal power series. Our aim is to show that for certain parameter ranges, the series converges.

2.1. A Nakajima quiver variety. We follow the exposition in [15,16]. Fix a positive integer r. The Nekrasov partition function is a generating functions for certain K-theory classes computed from a sequence of so-called Nakajima quiver varieties $M(r, n)$, where $n = 0, 1, 2, \ldots$. They are constructed as follows: Set $V = \mathbb{C}^n$ and $W = \mathbb{C}^r$. We define

$$M(r, n) = \Big\{ (B_1, B_2, i, j) \in \mathrm{End}(V) \times \mathrm{End}(V) \times \mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W) :$$

$$[B_1, B_2] + i j = 0 \text{ and there does not exist a proper subspace } S \subset V$$

$$\text{such that } B_1(S), B_2(S) \subset S \text{ and } i(W) \subset S \Big\} \Big/ GL(V),$$

where an element $g \in GL(V)$ acts on (B_1, B_2, i, j) via

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

The space $M(r, n)$ is a nonsingular algebraic variety of dimension $2nr$.

Let $T \subset GL_r(\mathbb{C})$ be the maximal torus consisting of diagonal matrices. The group $\tilde{T} = T \times \mathbb{C}^* \times \mathbb{C}^*$ acts on $M(r, n)$. This action is induced[1] by

$$(B_1, B_2, i, j) \cdot (t_1, t_2, e_1, \dots, e_r) = (t_1 B_1, t_2 B_2, ie, t_1 t_2 e^{-1} j),$$

where $(t_1, t_2, e_1, \dots, e_r) \in \tilde{T}$ and e is the diagonal matrix with entries $e_1, \dots, e_r$.

We want to consider $\tilde{T}$-equivariant K-theory on $M(r, n)$. By the localization principle, we first want a description of the fixed points of the $\tilde{T}$-action. In order to describe the fixed points of this action, we use partitions. First we fix our conventions: We write $l(Y) = l$ for the length of the partition $Y = (Y(1), \dots, Y(l))$ and $|Y|$ for its size $Y(1) + \dots + Y(l)$. We use the English convention to draw the Young diagram corresponding to the partitions. For example the partition $Y = (5, 3, 2)$ of size 10 has length $l(Y) = 3$ and its Young diagram is given by
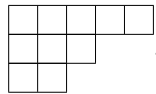
In the Young diagram, the row index x increases as we go south and the column index y increases as we go east. In the following we identify the partition Y with its Young diagram. In particular, we write $Y = \{(x, y) : 1 \le x \le l(Y), 1 \le y \le Y(x)\}$.

The fixed point set $M(r, n)^{\tilde{T}}$ is a discrete set whose points $I_{\vec{Y}}$ are indexed by r-tuples $\vec{Y} = (Y_1, \dots, Y_r)$ of partitions with total size $|\vec{Y}| := |Y_1| + \dots + |Y_r| = n$. We denote the corresponding inclusion maps as follows:

$$\iota : M(r, n)^{\tilde{T}} \to M(r, n), \quad \iota_{\vec{Y}} : \{I_{\vec{Y}}\} \to M(r, n), \quad |\vec{Y}| = n.$$

2.2. Notations for K-theory. Let t_i be the $\tilde{T}$-character given by $(t_1, t_2, e_1, \dots, e_r) \mapsto t_i$. Let e_α be the $\tilde{T}$-character given by $(t_1, t_2, e_1, \dots, e_r) \mapsto e_\alpha$. All the $\tilde{T}$-equivariant K-theory groups $K^{\tilde{T}}(-)$ are modules for the ring

$$R(\tilde{T}) := K^{\tilde{T}}(pt) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, e_1^{\pm 1}, \dots, e_r^{\pm 1}].$$

Let $\mathcal{R} \cong \mathbb{Q}(t_1, t_2, e_1, \dots, e_r)$ denote its quotient field. The maps ι and $\iota_{\vec{Y}}$ all define pushforwards in K-theory. The Thomason localization theorem in K-theory says that

$$\iota_* : K^{\tilde{T}}(M(r, n)^{\tilde{T}}) \to K^{\tilde{T}}(M(r, n))$$

becomes an isomorphism after localization, i.e. tensoring with $\mathcal{R}$. We write

$$K_{\mathrm{loc}}^{\tilde{T}}(-) = K^{\tilde{T}}(-) \otimes_{R(\tilde{T})} \mathcal{R}.$$

---

[1] Note our convention differs from the one used in [15] by $e_\alpha \mapsto e_\alpha^{-1}$.

2.3. Tangent spaces at fixed points. We will need a description of the tangent space $T_{\vec{Y}} M(r, n)$ at a fixed point $I_{\vec{Y}}$ as a $\tilde{T}$-module. It is given by [15, Theorem 2.11]

$$T_{\vec{Y}} M(r, n) = \sum_{\alpha, \beta = 1}^{r} N_{\alpha, \beta}(t_1, t_2) \in K^{\tilde{T}}(pt),$$

$$N_{\alpha, \beta}(t_1, t_2) = e_\alpha e_\beta^{-1} \left(\sum_{s \in Y_\alpha} t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha}(s)+1} + \sum_{t \in Y_\beta} t_1^{l_{Y_\alpha}(t)+1} t_2^{-a_{Y_\beta}(t)} \right). \qquad (2)$$

Here we use the arm length $a_Y(s) = Y(x) - y$ and leg length $l_Y(s) = Y^T(y) - x$ of a box $s = (x, y) \in \mathbb{Z}^2$ with respect to the partition Y. The symbol Y^T denotes the transpose of the partition Y. Note that box s does not have to belong to the partition Y, hence arm length and leg length can be negative.

2.4. The tautological bundle. In order to include theories with massive matter, we introduce formal parameters $\vec{b} = (b_1, \ldots, b_s)$. The tautological bundle $\mathcal{V}$ is a vector bundle over $M(r, n)$ of rank n whose fiber is V, see e.g. [17]. Let $c_0, \ldots, c_n \in K_{\mathrm{loc}}^{\tilde{T}}(M(r, n))$ denote its Chern classes. Consider the polynomial $b(z) = (z - b_1) \cdots (z - b_s)$. Construct the element

$$\kappa_n(b_1, \ldots, b_s) := b(z_1) \cdots b(z_n)|_{\sigma_j = c_j} \in K_{\mathrm{loc}}^{\tilde{T}}(M(r, n))[b_1, \ldots, b_s]$$

by expressing the symmetric polynomial $b(z_1) \cdots b(z_n)$ as a polynomial in the elementary symmetric functions $\sigma_0, \ldots, \sigma_n$ and replacing each σ_j by the Chern class c_j.

2.5. K-theoretic Nekrasov partition function. We now define the Nekrasov partition function arising in the K-theory of the moduli space of instantons. Let $(\epsilon_1, \epsilon_2, a_1, \ldots, a_r)$ be coordinates on the Lie algebra of $\tilde{T}$ and λ be a parameter such that $t_j = e^{\lambda \epsilon_j}$ and $e_\alpha = e^{\lambda a_\alpha}$. Write $b_m = e^{\lambda w_m}$. The Nekrasov partition function is defined as a formal power series

$$Z(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}; \mathfrak{q}, \lambda) = \sum_{n \geq 0} \left(\mathfrak{q} \lambda^{2r-s} e^{-r\lambda(\epsilon_1 + \epsilon_2)/2} \right)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}; \lambda), \qquad (3)$$

with coefficients

$$Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}; \lambda) = \sum_{i=0}^{2nr} (-1)^i \, \mathrm{ch} \, H^i(M(r, n), \kappa_n(b_1, \ldots, b_s))$$

$$= \sum_{|\vec{Y}| = n} (\iota_*)^{-1} \Big(\kappa_n(b_1, \ldots, b_s) \Big) \in \mathcal{R}[b_1, \ldots, b_s].$$

Here $\sum_{|\vec{Y}| = n}$ denotes the summation map

$$K_{\mathrm{loc}}^{\tilde{T}}(M(r, n)^{\tilde{T}})[b_1, \ldots, b_s] = \bigoplus_{|\vec{Y}| = n} \mathcal{R}[b_1, \ldots, b_s] \to \mathcal{R}[b_1, \ldots, b_s].$$

The fixed point theorem says that after localization we have

$$(\iota_*)^{-1} = \sum_{|\vec{Y}|=n} \frac{\iota_{\vec{Y}}^*}{\Lambda_{-1} T_{\vec{Y}} M(r, n)}, \tag{4}$$

where Λ_{-1} denotes the alternating sum of exterior powers. Combining Eq. (4) with the description (2) of the tangent space at a fixed point to compute $\Lambda_{-1} T_{\vec{Y}} M(r, n)$, one gets

$$Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}; \lambda) = \sum_{|\vec{Y}|=n} \frac{\iota_{\vec{Y}}^*\left(\kappa_n(b_1, \ldots, b_s)\right)}{\prod_{\alpha,\beta=1}^r n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}; \lambda)}, \tag{5}$$

where

$$n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}; \lambda) = \prod_{s \in Y_\alpha} \left(1 - e^{-\lambda\left(-l_{Y_\beta}(s)\,\epsilon_1 + (a_{Y_\alpha}(s)+1)\,\epsilon_2 + a_\alpha - a_\beta\right)}\right)$$

$$\prod_{t \in Y_\beta} \left(1 - e^{-\lambda\left((l_{Y_\alpha}(t)+1)\,\epsilon_1 - a_{Y_\beta}(t)\,\epsilon_2 + a_\alpha - a_\beta\right)}\right).$$

For $s = 0$, i.e. $b(z) = 1$, we have $\kappa_n(\emptyset) = \mathcal{O}$, the K-theory class corresponding to the structure sheaf on $M(r, n)$. This corresponds to pure Yang Mills theory and the numerator in Eq. (5) equals one. For general $\vec{b} = (b_1, \ldots, b_s)$ and a fixed point $I_{\vec{Y}}$, we have [17, equation (2.27)]

$$\iota_{\vec{Y}}^*\left(\kappa_n(b_1, \ldots, b_s)\right) = \prod_{\alpha=1}^r \prod_{(x,y)\in Y_\alpha} \prod_{m=1}^s (e^{-\lambda(a_\alpha + (x-1)\,\epsilon_1 + (y-1)\,\epsilon_2)} - b_m). \tag{6}$$

Our aim is to estimate the general form of the coefficients (5) in order to prove convergence of the K-theoretic partition function. To do so, we use an integral representation for the coefficients $Z_n(\epsilon_1, \epsilon_2, \vec{a}, \vec{w}; \lambda)$.

2.6. Integral representation and estimate. In this section we define an integral representation for the coefficients of the Nekrasov partition function. We will use this representation to estimate the coefficients.

2.6.1. Definition of the integral. Let q_1 and q_2 be a pair of complex numbers in the open unit disk. Assume that either $q_1 = \overline{q_2}$ or $q_1, q_2 \in (0, 1)$. Note that in either case $q_1 q_2 = |q_1 q_2| \in (0, 1)$. Later these numbers will be identified with exponential functions of ϵ_1, ϵ_2.

Fix $r \geq 1$. Let $\vec{u} = (u_1, \ldots, u_r)$ be a vector of complex numbers such that

$$|q_i| \max_{\alpha=1,\ldots,r} |u_\alpha| < \min_{\alpha=1,\ldots,r} |u_\alpha|, \quad \forall\, i = 1, 2.$$

Let $\vec{p} = (p_1, \ldots, p_s)$ be another vector of complex numbers. The condition on $|u_\alpha|$ ensures that we can pick $\rho > 0$ with

$$|u_\alpha| < \rho < |q_i|^{-1} |u_\alpha| \quad \forall\, \alpha = 1, \ldots, r, \forall\, i = 1, 2. \tag{7}$$

Let $C_\rho \subset \mathbb{C}$ be the set of complex numbers of modulus ρ. We define

$$\mathcal{Z}_n(\vec{u}; \vec{p}) = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n$$
$$\int_{C_\rho^n} \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \prod_{j=1}^n \prod_{m=1}^s (z_j - p_m) \, \mathcal{I}(z_1, \ldots, z_n; \vec{u}), \qquad (8)$$

where the integrand contains the symmetric function

$$\mathcal{I}(z_1, \ldots, z_n; \vec{u}) = \prod_{j=1}^n \prod_{\alpha=1}^r \frac{-u_\alpha z_j}{(z_j - u_\alpha)(q_1 q_2 z_j - u_\alpha)} \prod_{1 \le j \ne k \le n} \frac{(z_j - z_k)(z_j - q_1 q_2 z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)}.$$
$$(9)$$

In the following, we evaluate the coefficients $\mathcal{Z}_n(\vec{u}; \vec{p})$ using residue calculus and estimate them. We then identify them with the coefficients Z_n of the K-theoretic partition function up to a change of variables in Sect. 2.8.

2.6.2. Evaluation of the integral. We evaluate the integral in $\mathcal{Z}_n(\vec{u}; \vec{p})$ using residue calculus. The residues are indexed by r-tuples $\vec{Y}$ of partitions Y_α with total size $|\vec{Y}| = |Y_1| + \cdots + |Y_r| = n$. We again identify a partition with its Young diagram. For a box $s = (x, y) \in Y_\alpha$ we define

$$z_s^\alpha = z_{x,y}^\alpha = u_\alpha \, q_1^{x-1} q_2^{y-1}. \qquad (10)$$

Theorem 1. *Under the assumption*

$$u_\alpha u_\beta^{-1} \ne q_1^x q_2^y, \qquad \forall x, y \in \{-n, \ldots, n\}, \quad \forall \alpha \ne \beta \in \{1, \ldots, r\},$$
$$q_1^x \ne q_2^{y+1}, \quad q_1^{x+1} \ne q_2^y, \qquad \forall x, y \in \{0, \ldots, n-1\}, \qquad (11)$$

the value of $\mathcal{Z}_n(\vec{u}; \vec{p})$ is given as a sum over r-tuples of partitions of total size n in two equivalent ways:

$$\mathcal{Z}_n(\vec{u}; \vec{p}) = \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^r \prod_{s \in Y_\alpha} \prod_{m=1}^s (z_s^\alpha - p_m)}{N_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u})},$$
$$N_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u}) = \prod_{s \in Y_\alpha} \left(1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(s)+1} q_2^{-a_{Y_\beta}(s)} \right) \prod_{t \in Y_\beta} \left(1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(t)} q_2^{a_{Y_\alpha}(t)+1} \right).$$
$$(12)$$

Alternatively,

$$\mathcal{Z}_n(\vec{u}; \vec{p}) = \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^r \prod_{s \in Y_\alpha} \prod_{m=1}^s (z_s^\alpha - p_m)}{M_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u})},$$
$$M_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u}) = \prod_{s \in Y_\alpha} \left(1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(s)} q_2^{a_{Y_\alpha}(s)+1} \right) \prod_{t \in Y_\beta} \left(1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(t)+1} q_2^{-a_{Y_\beta}(t)} \right).$$
$$(13)$$

The proof is of this theorem is technical. In the literature, several arguments for the validity of this or similar formulae have been given, see [8,10,17,18,24]. They are based on taking the iterated residues at $z_{x,y}^\alpha$, $(x, y) \in Y_\alpha$, $\alpha = 1, \ldots, r$. However, the integrand also has further poles and it is a nontrivial fact, that we prove in this paper, that the residues at those poles cancel. The proof of Theorem 1 is postponed to Appendix A.

Remark 1. The assumption (11) is necessary to ensure that all terms $\mathcal{Z}_{\vec{Y}}(\vec{u}; \vec{p})$ are well-defined. If it is violated, some residues might not be simple residues anymore and consequently some $\mathcal{Z}_{\vec{Y}}(\vec{u}; \vec{p})$ might be infinite. However, their sum $\mathcal{Z}_n(\vec{u}; \vec{p})$ is still well-defined, as the integral in Eq. (8) is.

2.6.3. Estimate for the integral. In this section we apply potential theory to estimate the coefficients $\mathcal{Z}_n(\vec{u}; \vec{p})$ in the integral form given by Eq. (8) in the limit of large n.

Theorem 2. *We have*

$$\limsup_{n \to \infty} |\mathcal{Z}_n(\vec{u}; \vec{p})|^{\frac{1}{n}} \le \prod_{m=1}^s \max\{|p_m|, |u_1|, \ldots, |u_r|\}.$$

We prove this by comparing the growth of $\mathcal{Z}_n(\vec{u}; \vec{p})$ to the growth of the coefficients

$$a_n = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \int_{C_1^n} \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \prod_{j \ne k} \frac{(z_j - z_k)(z_j - q_1 q_2 z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)}. \tag{14}$$

We also want to introduce the language of potential theory. Let $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$ be the torus. Define $f : \mathbb{T} \to \mathbb{R} \cup \{\infty\}$ by

$$f(\theta) := -\log \frac{|e^{i\theta} - 1||e^{i\theta} - q_1 q_2|}{|e^{i\theta} - q_1||e^{i\theta} - q_2|}. \tag{15}$$

For each $n \in \mathbb{N}$, we define a probability measure on $\mathbb{T}^n$ by

$$P_n(\theta) d\theta = \frac{1}{Z_n} e^{-\sum_{j \ne k} f(\theta_k - \theta_j)} d\theta, \quad d\theta = d\theta_1 \cdots d\theta_n,$$

where $Z_n = \int_{\mathbb{T}^n} d\theta\, e^{-\sum_{j \ne k} f(\theta_k - \theta_j)}$. Denote the associated expectation functionals by $\mathbb{E}_n[-]$. By changing variables $z_j = \rho e^{i\theta_j}$ in Eq. (8) and $z_j = e^{i\theta_j}$ in Eq. (14), we get

$$\mathcal{Z}_n(\vec{u}; \vec{p}) = a_n\, \mathbb{E}_n \left[\prod_{j=1}^n g(\rho, \theta_j; \vec{u}; \vec{p}) \right],$$

where

$$g(\rho, \theta; \vec{u}; \vec{p}) = \prod_{m=1}^s (\rho e^{i\theta} - p_m) \prod_{\alpha=1}^r \frac{-u_\alpha \rho e^{i\theta}}{(\rho e^{i\theta} - u_\alpha)(q_1 q_2 \rho e^{i\theta} - u_\alpha)}.$$

We estimate $|\mathcal{Z}_n(\vec{u}; \vec{p})|$ by taking the absolute value inside. We arrive at

$$\left| \mathcal{Z}_n(\vec{u}; \vec{p}) \right|^{\frac{1}{n}} \le |a_n|^{\frac{1}{n}} \exp \left(\frac{1}{n} \log \mathbb{E}_n \left[e^{\sum_j \log |g(\rho, \theta_j; \vec{u}; p)|} \right] \right). \tag{16}$$

We estimate both factors on the right hand side separately. The first one is related to a known power series:

Lemma 1. *We have*

$$\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 1.$$

Proof. From [4] we know that

$$\sum_{n \geq 0} a_n z^n = \exp\left(\sum_{n \geq 1} \frac{1 - q_1^n q_2^n}{(1 - q_1^n)(1 - q_2^n)} \frac{z^n}{n}\right),$$

as a formal power series. Since all coefficients are positive in this expansion and $q_i^n \to 0$ as $n \to \infty$ we see that the radius of convergence equals 1. $\square$

These estimates already give a non-quantitative convergence result.

Lemma 2. *Assume that $\lambda > 0$, q_1, q_2 in the unit disk, both real or complex conjugate to each other. Then on each compact subset of the domain*

$$\{(\vec{u}, \vec{p}) \in \mathbb{C}^{r+s} \mid |q_i| \max_\alpha |u_\alpha| < \min_\alpha |u_\alpha|, \forall i = 1, 2\},$$

the power series $\sum_{n=0}^{\infty} \mathcal{Z}_n(\vec{u}; \vec{p})z^n$ converges uniformly with a positive radius of convergence.

To prove this lemma notice that on such compact subsets $|g(\rho, \theta; \vec{u}; \vec{p})|$ is uniformly bounded on the integration circle C_ρ for proper choice of ρ. Thus the coefficients $\mathcal{Z}_n(\vec{u}; \vec{p})$ are bounded by const^n.

To get a quantitative estimate of the radius of convergence we need to work harder. We have the following result:

Theorem 3. *Let h be a continuous, real-valued function on the torus $\mathbb{T}$. We have*

$$\frac{1}{n} \log \mathbb{E}_n \left[e^{\sum_j h(\theta_j)} \right] \to \frac{1}{2\pi} \int_\mathbb{T} h(\theta) d\theta \quad (n \to \infty). \tag{17}$$

The proof of this theorem uses ideas from potential theory and is postponed to Sect. 2.7. It applies to a more general class of functions f defining the probability measure, as we discuss in Sect. 3.2. Now we can apply Lemma 1 and Theorem 3 in Eq. (16) to estimate

$$\limsup_{n \to \infty} |\mathcal{Z}_n(\vec{u}; \vec{p})|^{\frac{1}{n}} \leq \exp\left(\frac{1}{2\pi} \int_\mathbb{T} \log |g(\rho, \theta; \vec{u}; \vec{p})| d\theta\right).$$

Using that condition (7) says $\rho^{-1}|u_\alpha| < 1$, but $\rho^{-1}|q_1 q_2|^{-1}|u_\alpha| > 1$, formula 18 from below implies

$$\frac{1}{2\pi} \int_\mathbb{T} \log |g(\rho, \theta; \vec{u}; p)| d\theta = \sum_{m=1}^{s} \max\{\log |p_m|, \log \rho\}.$$

We obtain

$$\limsup_{n \to \infty} |\mathcal{Z}_n(\vec{u}; \vec{p})|^{\frac{1}{n}} \leq \prod_{m=1}^{s} \max\{|p_m|, \rho\}.$$

According to condition (7), the lower bound for ρ is given by $\max_\alpha\{|u_\alpha|\}$. We let ρ tend to this bound, completing the proof of Theorem 2.

2.7. Potential theory. In this section we prove Theorem 3 using techniques adapted from [13].

2.7.1. Setup of potential theory. Let $M(\mathbb{T})$ be the set of all Borel probability measures on $\mathbb{T}$ and $M_0(\mathbb{T})$ be the subset of all such measures which in addition do not contain point masses.

The function $f : \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ defined in Eq. (15) is continuous, bounded from below and has a single pole at $\theta = 0$. Set

$$I[\mu] = \iint_{\theta \neq \phi} f(\theta - \phi) d\mu(\theta) d\mu(\phi),$$

where $\mu \in M(\mathbb{T})$. Since f is bounded from below, I is bounded from below, too. Define

$$I_0 := \inf_{\mu \in M_0(\mathbb{T})} I[\mu] > -\infty.$$

The aim of this section is to prove

Theorem 4. *The normalized Lebesgue measure is the unique measure μ with $I[\mu] = I_0$. Moreover $I_0 = 0$.*

Define the function $g_\sigma(\theta) := \frac{1}{2} \log(1 + \sigma^2 - 2\sigma \cos \theta) = \log |e^{i\theta} - \sigma|$ for $\sigma > 0$. Its Fourier coefficients

$$c_k(g_\sigma) = \frac{1}{2\pi} \int_{\mathbb{T}} g_\sigma(\theta) e^{-ik\theta} d\theta = \begin{cases} -\frac{1}{2|k|} \min\{\sigma^{|k|}, \sigma^{-|k|}\}, & \text{if } k \neq 0, \\ \max\{0, \log \sigma\}, & \text{otherwise,} \end{cases} \quad (18)$$

are known. For $\sigma \neq 1$ see for example [7]. For the case $\sigma = 1$, note that g_σ is uniformly bounded from above and

$$g_\sigma(\theta) = \log(1 - 2\sigma + \sigma^2 + 2\sigma - 2\sigma \cos(\theta)) \geq \log \sigma + g_1(\theta).$$

By dominated convergence, it suffices to show that g_1 is integrable. The only pole is at $\theta = 0$. By changing variables to $x = 2 - 2\cos \theta$ we have to consider $\log x \frac{1}{\sqrt{x}}$ which is integrable as one can see from integration by parts.

We write $q_j = |q_j| e^{i\tau_j}$. We have

$$c_k(f) = -c_k(g_1) - c_k(g_{|q_1 q_2|}) + e^{-ik\tau_1} c_k(g_{|q_1|}) + e^{-ik\tau_2} c_k(g_{|q_2|}).$$

Since $|q_i| < 1$, we obtain $c_0(f) = 0$, so in particular

Lemma 3. *I vanishes for the normalized Lebesgue measure.*

For $k \neq 0$ we get

$$c_k(f) = \frac{1}{2|k|}(1 + |q_1 q_2|^k - e^{-ik\tau_1}|q_1|^{|k|} - e^{-ik\tau_2}|q_2|^{|k|}).$$

If $q_1 = \overline{q_2}$, this is bounded from below by $\frac{1}{2|k|}(1 - |q_1 q_2|^{|k|/2})^2 > 0$. If $q_1, q_2 \in (0, 1)$ it equals $\frac{1}{2|k|}(1 - |q_1|^{|k|})(1 - |q_2|^{|k|}) > 0$. In either case we obtain

Lemma 4. *The Fourier coefficients $c_k(f)$ of $f \in L^1(\mathbb{T})$ satisfy $c_k(f) > 0$ if $k \neq 0$ and $c_0(f) = 0$. The Fourier series of f converges everywhere except at $\theta = 0$.*

We use this to prove

Lemma 5. *We have $I_0 = 0$.*

Proof. Since the normalized Lebesgue measure is in $M_0(\mathbb{T})$ we have $I_0 \leq 0$. On the other hand, for any $\mu \in M_0(\mathbb{T})$, we have

$$
\begin{aligned}
I[\mu] &= \iint f(\theta - \phi) d\mu(\theta) d\mu(\phi) \\
&= \sum_{k \neq 0} c_k(f) \iint e^{i(\theta - \phi)} d\mu(\theta) d\mu(\phi) \\
&= \sum_{k \neq 0} c_k(f) |c_k(\mu)|^2 \geq 0.
\end{aligned}
$$

Firstly, we have dropped the condition $\theta \neq \phi$ using the fact that μ does not contain point masses. Then we have applied Tonelli's theorem using the fact that $c_k(f) e^{i(\theta - \phi)}$ is bounded from below. $\square$

Lemma 6. *The normalized Lebesgue measure is the unique measure $\mu \in M_0(\mathbb{T})$ for which $I_0 = I[\mu]$.*

Proof. Let $\mu, \nu \in M_0(\mathbb{T})$ with $I[\mu] = I[\nu] = I_0$. Since $I[\mu]$ and $I[\nu]$ are both finite,

$$
\begin{aligned}
I[\mu - \nu] &= \iint f(\theta - \phi) d\mu(\theta) d\mu(\phi) + \iint f(\theta - \phi) d\nu(\theta) d\nu(\phi) \\
&\quad - \iint f(\theta - \phi) d\mu(\theta) d\nu(\phi) - \iint f(\theta - \phi) d\nu(\theta) d\mu(\phi) \in [-\infty, \infty)
\end{aligned}
\tag{19}
$$

is well-defined. Using Tonelli's theorem for each summand, we get

$$
I[\mu - \nu] = \sum_{k \neq 0} c_k(f) |c_k(\mu) - c_k(\nu)|^2 \geq 0.
\tag{20}
$$

In particular, $I[\mu - \nu]$ and all four terms in its expansion (19) are finite.

For $t \in [0, 1]$ we have $\nu + t(\mu - \nu) \in M_0(\mathbb{T})$ and hence

$$
\begin{aligned}
0 = I_0 &\leq I[\nu + t(\mu - \nu)] \\
&= I[\nu] + t\left(\iint f(\theta - \phi) d\nu(\theta) d(\mu - \nu)(\phi) \right. \\
&\quad \left. + \iint f(\theta - \phi) d(\mu - \nu)(\theta) d\nu(\phi) \right) + t^2 I[\mu - \nu],
\end{aligned}
$$

where all terms are finite. The right hand side is a polynomial in t which is nonnegative for $t \in [0, 1]$ and vanishes at $t = 0$ and $t = 1$. We obtain $I[\mu - \nu] \leq 0$. From Eq. (20) we get $c_k(\mu) = c_k(\nu)$ for all $k \neq 0$, since $c_k(f) > 0$ for $k \neq 0$. We obtain $\mu = \nu$ since $c_0(\mu) = c_0(\nu)$ trivially because μ and ν are probability measures. $\square$

The proof of Theorem 4 is complete.

2.7.2. Application of potential theory. To a point $\theta \in \mathbb{T}^n$ we associate the probability measure

$$\delta_\theta = \frac{1}{n} \sum_{j=1}^{n} \delta_{\theta_j},$$

where on the right hand side we have a convex combination of ordinary Dirac measures.

Set $\mathbb{T}_0^n = \{\theta \in \mathbb{T}^n : \theta_j \neq \theta_k (j \neq k)\}$. For $\theta \in \mathbb{T}_0^n$ we have $\theta_j \neq \theta_k \Leftrightarrow j \neq k$ and thus

$$n^2 I[\delta_\theta] = \sum_{j \neq k} f(\theta_j - \theta_k).$$

The intuition behind Theorem 3 is the following. The limit behavior for large n of the quantity

$$\frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}] = \frac{1}{n} \log \frac{1}{Z_n} \int_{\mathbb{T}^n} d\theta e^{\sum_j h(\theta_j)} e^{-n^2 I[\delta_\theta]}$$

will be dominated by such $\theta \in \mathbb{T}^n$, for which $I[\delta_\theta]$ is close to $I_0 = 0$. This is the content of Lemma 7. Those δ_θ will then for large n equidistribute to approximate the normalized Lebesgue measure yielding $\frac{1}{2\pi} \int_{\mathbb{T}} h(\theta) d\theta$. This will be the content of Lemma 8 and the discussion afterwards.

We use the notion of weak convergence for measures: We say a sequence of measures (μ_n) in $M(\mathbb{T})$ converges to a measure $\mu \in M(\mathbb{T})$ iff for all continuous and bounded functions g on $\mathbb{T}$ we have

$$\int_{\mathbb{T}} g d\mu_n \rightarrow \int_{\mathbb{T}} g d\mu.$$

It is well-known that the space $M(\mathbb{T})$ with this notion of convergence is sequentially compact.

For $\eta > 0$ define

$$A_{n,\eta} = \{\theta \in \mathbb{T}_0^n : I_0 \leq I[\delta_\theta] \leq I_0 + \eta\} = \{\theta \in \mathbb{T}^n : \sum_{j \neq k} f(\theta_k - \theta_j) \leq \eta n^2\}.$$

This set is compact. Denote by $P_n[S]$ the measure of a measurable set $S \subset \mathbb{T}^n$ with respect to the probability measure P_n. We have

Lemma 7. $0 \leq P_n[\mathbb{T}^n \setminus A_{n,\eta}] \leq e^{-\eta n^2}$.

Proof. We have

$$\int_{\mathbb{T}^n \setminus A_{n,\eta}} d\theta e^{-\sum_{j \neq k} f(\theta_k - \theta_j)} \leq \int_{\mathbb{T}^n \setminus A_{n,\eta}} d\theta e^{-\eta n^2} \leq (2\pi)^n e^{-\eta n^2}.$$

Let μ denote the normalized Lebesgue measure. By the Jensen inequality we have

$$\frac{Z_n}{(2\pi)^n} = \int_{\mathbb{T}_0^n} \prod_j d\mu(\theta_j) e^{-\sum_{j \neq k} f(\theta_k - \theta_j)}$$

$$\geq \exp\left(\int_{\mathbb{T}_0^n} \prod_j d\mu(\theta_j)\left(-\sum_{j\neq k} f(\theta_k - \theta_j)\right)\right).$$

Now

$$-\sum_{j\neq k}\int_{\mathbb{T}_0^n}\prod_{j'} d\mu(\theta_{j'}) f(\theta_k - \theta_j) = -n(n-1)\iint f(\theta - \phi) d\mu(\theta) d\mu(\phi)$$

$$= -n(n-1)I[\mu] = 0.$$

$\square$

Moreover, we have

Lemma 8. *If the measures $v_{n,\eta}$ are Dirac measures supported at $\tau^{n,\eta} \in A_{n,\eta}$ and $v_{n_k,\eta} \to v_\eta$ is a convergent subsequence, we have $v_\eta \in M_0(\mathbb{T})$ and $I[v_\eta] \leq \eta$. If $v_{\eta_k} \to v$ is a convergent sequence with $\eta_k \to 0$, the limit v has to be the normalized Lebesgue measure.*

Proof. Let $L \in \mathbb{R}$ and separate the diagonal part:

$$\eta \geq I[v_{n,\eta}] = \iint_{\theta\neq\phi} f(\theta - \phi) dv_{n,\eta}(\theta) dv_{n,\eta}(\phi)$$

$$\geq \iint_{\theta\neq\phi} \min\{f(\theta - \phi), L\} dv_{n,\eta}(\theta) dv_{n,\eta}(\phi)$$

$$= \frac{1}{n^2} \sum_{j\neq k} \min\{f(\tau_j^{n,\eta} - \tau_k^{n,\eta}), L\}$$

$$= \iint \min\{f(\theta - \phi), L\} dv_{n,\eta}(\theta) dv_{n,\eta}(\phi) - \frac{L}{n}.$$

Let $\epsilon > 0$. Using the Weierstrass approximation theorem, pick a polynomial $p(\theta, \phi)$ which uniformly approximates the last integrand up to an error of ϵ. For $n = n_k$ we get

$$\eta \geq \iint p(\theta, \phi) dv_{n_k,\eta}(\theta) dv_{n_k,\eta}(\phi) - \epsilon - \frac{L}{n_k}.$$

Send $k \to \infty$ to get

$$\eta \geq \iint p(\theta, \phi) dv_\eta(\theta) dv_\eta(\phi) - \epsilon \geq \iint \min\{f(\theta - \phi), L\} dv_\eta(\theta) dv_\eta(\phi) - 2\epsilon.$$

Now send $\epsilon \to 0$ and let $L \to \infty$. By monotonicity the limit can pass to the integrand. We obtain $v_\eta \in M_0(\mathbb{T})$ and

$$\eta \geq \iint f(\theta - \phi) dv_\eta(\theta) dv_\eta(\phi) = \iint_{\phi\neq\theta} f(\theta - \phi) dv_\eta(\theta) dv_\eta(\phi) = I[v_\eta].$$

Now let $v_{\eta_k} \to v$ be a convergent sequence with $\eta_k \to 0$. Again fix $L \in \mathbb{R}$ and estimate, using $v_{\eta_k} \in M_0(\mathbb{T})$,

$$\eta_k \geq I[v_{\eta_k}] = \iint f(\theta - \phi) dv_\eta(\theta) dv_\eta(\phi) \geq \iint \min\{L, f(\theta - \phi)\} dv_\eta(\theta) dv_\eta(\phi).$$

Let $\epsilon > 0$. Using Weierstrass we get in the limit $k \to \infty$

$$0 \geq \iint \min\{L,\, f(\theta - \phi)\}d\nu(\theta)d\nu(\phi) - 2\epsilon.$$

Again, we let $\epsilon \to 0$ and $L \to \infty$ to get $\nu \in M_0(\mathbb{T})$ and $0 \geq I[\nu]$. The claim follows. $\square$

Now we can prove Theorem 3. Let h be a continuous function on the torus. Fix $\eta > 0$. By Lemma 7, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}] = \limsup_{n \to \infty} \frac{1}{n} \log \int_{A_{n,\eta}} d\theta\, P_n(\theta)e^{\sum_j h(\theta_j)},$$

and analogously for $\liminf$. Let the continuous, real-valued function $e^{\sum_j h(\theta_j)}$ on the compact set $A_{n,\eta}$ attain its maximum at $\tau^{n,\eta} \in A_{n,\eta}$ and its minimum at $\sigma^{n,\eta} \in A_{n,\eta}$. Denote Dirac measures by $\nu_{n,\eta}$ and $\lambda_{n,\eta}$. We have

$$\frac{1}{n} \log \int_{A_{n,\eta}} d\theta\, P_n(\theta)e^{\sum_j h(\theta_j)} \leq \frac{1}{n} \log P_n[A_{n,\eta}]e^{\sum_j h(\tau_j^{n,\eta})} \leq \int_{\mathbb{T}} d\nu_{n,\eta}(\theta)h(\theta).$$

Similarly,

$$\frac{1}{n} \log \int_{A_{n,\eta}} d\theta\, P_n(\theta)e^{\sum_j h(\theta_j)} \geq \frac{1}{n} \log P_n[A_{n,\eta}]e^{\sum_j h(\sigma_j^{n,\eta})} \geq \int_{\mathbb{T}} d\lambda_{n,\eta}(\theta)h(\theta) + O(\frac{1}{n})$$

since by Lemma 7, $P_n[A_{n,\eta}] \geq 1 - e^{-\eta n^2} \to 1$. Let n_k define a subsequence with

$$\limsup_{n \to \infty} \frac{1}{n} \log \int_{A_{n,\eta}} d\theta\, P_n(\theta)e^{\sum_j h(\theta_j)} = \lim_{k \to \infty} \frac{1}{n_k} \log \int_{A_{n_k,\eta}} d\theta\, P_{n_k}(\theta)e^{\sum_j h(\theta_j)}.$$

Let m_k define a subsequence that realizes the corresponding $\liminf$. By passing to respective subsequences, we can suppose that $\nu_{n_k,\eta} \to \nu_\eta$ and $\lambda_{m_k,\eta} \to \lambda_\eta$ for some Borel probability measures λ_η, ν_η. We obtain

$$\int_{\mathbb{T}} h(\theta)d\lambda_\eta(\theta) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}]$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}] \leq \int_{\mathbb{T}} h(\theta)d\nu_\eta(\theta).$$

The parameter $\eta > 0$ is arbitrary. Now let $\eta_k \to 0$ define a subsequence $\nu_\eta \to \nu$ and $\eta_k' \to 0$ define a subsequence $\lambda_\eta \to \lambda$. By Lemma 8, $\nu = \lambda$ is the normalized Lebesgue measure. We obtain

$$\frac{1}{2\pi} \int_{\mathbb{T}} h(\theta)d\theta \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}]$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n[e^{\sum_j h(\theta_j)}] \leq \frac{1}{2\pi} \int_0^{2\pi} h(\theta)d\theta.$$

The proof of Theorem 3 is complete.

Remark 2. A physical interpretation of Theorem 3 goes as follows: The points of $\mathbb{T}^n$ are the coordinates of n particles on the torus $\mathbb{T}$ that interact via the two body potential $f(\theta_j - \theta_k)$. The integral over $\mathbb{T}^n$ on the left hand side of Eq. (17) is dominated by particle configurations which minimize the potential energy of the system. The even function $f(\theta)$ has a pole at $\theta = 0$ and one minimum θ_0 in $(0, \pi)$. Hence, the particles experience a strong repulsive force once they get close to each other and have a preferred distance θ_0 from each other. For large n they cannot all stay in their preferred distance since the torus is compact. Hence, the repulsive part dominates and in the limit of large n the particles equidistribute. The evaluation of the left hand side of Eq. (17) on those equidistributed points on $\mathbb{T}$ defines a Riemann sum approximating the integral on the right hand side of Eq. (17).

2.8. Identification of coefficients and conclusion. We evaluate the formal parameters of Sect. 2.5 at the complex parameters of Sect. 2.6.1. We set

$$t_i^{-1} = e^{-\lambda \epsilon_i} = q_i, \quad e_\alpha^{-1} = e^{-\lambda a_\alpha} = u_\alpha, \quad b_m = e^{\lambda w_m} = p_m.$$

Under the assumption $\lambda > 0$, the conditions stated at the beginning of Sect. 2.6.1 are satisfied provided we assume $\operatorname{Re} \epsilon_i > 0$ and $\left(\epsilon_1 = \overline{\epsilon_2} \text{ or } \epsilon_1, \epsilon_2 \in \mathbb{R} \right)$ and

$$\max_\alpha \operatorname{Re}(a_\alpha) - \min_\alpha \operatorname{Re}(a_\alpha) < \operatorname{Re} \epsilon_i, \quad i = 1, 2.$$

Hence, $\mathcal{Z}_n(\vec{u}; \vec{p})$ is well-defined. To ensure that each simple residue $\mathcal{Z}_{\vec{Y}}(\vec{u}; \vec{p})$ is well-defined, we have to require conditions (11), which translate to

$$\begin{aligned} a_\alpha - a_\beta &\not\equiv x\,\epsilon_1 + y\,\epsilon_2, & \forall \alpha \neq \beta \in \{1, \ldots, r\} \quad \forall x, y \in \{-n, \ldots, n\}, \\ x\,\epsilon_1 &\not\equiv (y+1)\,\epsilon_2, \ (x+1)\,\epsilon_1 \not\equiv y\,\epsilon_2, & \forall x, y \in \{0, \ldots, n-1\}, \end{aligned} \tag{21}$$

where the inequalities are modulo $\frac{2\pi i}{\lambda} \mathbb{Z}$. Under these conditions, Theorem 1 implies $Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}; \lambda) = \mathcal{Z}_n(\vec{u}; \vec{p})$. By Remark 1, the sum $Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w})$ is well-defined even if we drop condition (21). The representation of the coefficients as integrals allowed us to apply the potential theory in Sect. 2.6.3 to estimate their growth in Theorem 2. We obtain

Theorem 5. *Let $\lambda > 0$ and ϵ_1 and ϵ_2 be a pair of complex numbers with positive real part. Assume either they are complex conjugate or both real. The K-theoretic Nekrasov partition function $Z(\epsilon_1, \epsilon_2, \vec{a}; \vec{w}, \mathsf{q}, \lambda)$ given by Eq. (3) is an analytic function of $\mathsf{q}, \vec{a}, \vec{w}$ in the domain*

$$\vec{w} \in \mathbb{C}^s, \quad \max_\alpha \operatorname{Re}(a_\alpha) - \min_\alpha \operatorname{Re}(a_\alpha) < \operatorname{Re} \epsilon_i, \quad i = 1, 2,$$

$$|\mathsf{q}| < \lambda^{s-2r} e^{\lambda \left(r \frac{\epsilon_1 + \epsilon_2}{2} + \sum_{m=1}^s \min\{-\operatorname{Re} w_m, \operatorname{Re} a_1, \ldots, \operatorname{Re} a_r\} \right)}.$$

There remains to prove the analyticity in $\vec{a}, \vec{w}$. By Lemma 2, we know that in a neighbourhood U of each $(\vec{a}, \vec{w})$ the series defining the partition functions converges to an analytic function for $|\mathsf{q}| < r$ for some small positive radius $r = r(U)$. On the other hand, Theorem 2 tells us that we have convergence of the power series for the indicated range of q and fixed $\vec{a}, \vec{w}$ in the domain. By Hartogs' lemma ([6], Theorem 2, p. 139) the series converges to an analytic function on the whole domain.

3. Generalization

Our technique can be generalized to different parameter ranges and similar types of gauge theory.

3.1. Different parameter ranges. Assume $\vec{p} = \emptyset$. Under the inversion

$$(q_1, q_2, u_1, \ldots, u_r) \mapsto (q_1^{-1}, q_2^{-1}, u_1^{-1}, \ldots, u_r^{-1})$$

the coefficient $\mathcal{Z}_n(\vec{u}; \vec{p} = \emptyset)$ gets multiplied by $(q_1 q_2)^{-nr}$. This can directly be seen from Eq. (13). Using this observation, bounds for $|q_1|, |q_2| > 1$ can be obtained as well.

3.2. Similar types of gauge theory. Our technique readily generalizes to partition functions of gauge theories where the weight factor in line (9) in coordinates $z_j = \rho e^{i\theta_j}$ is of the form

$$\exp\left(-\sum_{j \neq k} f(\theta_j - \theta_k) \right),$$

where $f : \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ is continuous, bounded from below, has a single pole at $\theta = 0$ and its Fourier coefficients $c_k(f)$ satisfy $c_k(f) > 0$ for $k \neq 0$ and $c_0(f) = 0$.

An example would be $\mathcal{N} = 2^*$ supersymmetric gauge theory, i.e. with massive matter in the adjoint representation [18, equation (3.25)]. One has to consider a different polynomial in the numerator of the integrand in (8) and multiply the weight in line (9) by

$$\prod_{1 \leq j < k \leq n} \frac{(z_j - q_1 m^{-1} z_k)(z_j - q_2 m^{-1} z_k)(z_j - q_1^{-1} m z_k)(z_j - q_2^{-1} m z_k)}{(z_j - m z_k)(z_j - m^{-1} z_k)(z_j - q_1 q_2 m^{-1} z_k)(z_j - q_1^{-1} q_2^{-1} m z_k)}.$$

4. Specialization: Norm of Deformed Gaiotto States

In this section, we briefly discuss an application of our results to conformal field theory, namely the finiteness of the norm of deformed Gaiotto states.

4.1. Definition. In this introduction, let q, t denote two generic complex parameters. We follow the exposition of [3]. The deformed Virasoro algebra $Vir_{q,t}$ is defined as the associative algebra topologically generated by $T_n, n \in \mathbb{Z}$ with the defining relations

$$[T_n, T_m] = -\sum_{l=1}^{\infty} r_l(T_{n-l}T_{m+l} - T_{m-l}T_{n+l})$$

$$-\frac{(1-q)(1-t^{-1})}{1-qt^{-1}}(q^n t^{-n} - q^{-n}t^n)\delta_{m+n,0}.$$

The coefficients r_l are determined by the expansion

$$r(x) = \sum_{l \geq 0} r_l x^l = \exp\left(\sum_{n \geq 1} \frac{(1-q^n)(1-t^{-n})}{1+q^n t^{-n}} \frac{x^n}{n} \right).$$

For $h \in \mathbb{C}$, the Verma module M_h is generated by a vector $|h\rangle$ with $T_0|h\rangle = h|h\rangle$ and $T_n|h\rangle = 0$ if $n \geq 1$. The operator T_0 defines a grading $M_h = \bigoplus_{n \geq 0} M_{h,n}$ where $M_{h,n}$ is the eigenspace of T_0 corresponding to the eigenvalue $h + n$. As usual, each $M_{h,n}$ has a basis

$$T_\lambda |h\rangle := T_{-\lambda(1)} \cdots T_{-\lambda(l)} |h\rangle$$

indexed by partitions $\lambda = (\lambda(1), \ldots, \lambda(l))$ of size $|\lambda| = n$. The Shapovalov form $S : M_h \otimes M_h \to \mathbb{C}$ is characterized by $S(|h\rangle, |h\rangle) = 1$ and $S(T_n x, y) = S(x, T_{-n} y)$ for $x, y \in M_h$ and $n \in \mathbb{Z}$. The decomposition $M_h = \bigoplus_{n \geq 0} M_{h,n}$ is orthogonal with respect to the Shapovalov form. In particular, the so-called Kac matrix with entries $S_{\lambda,\mu} := S(T_\lambda |h\rangle, T_\mu |h\rangle)$ is block diagonal with finite blocks $(S_{\lambda,\mu}^{(n)})_{|\lambda|,|\mu|=1,\ldots,n}$.

A deformed Gaiotto state is a formal power series $|G\rangle = \sum_{n \geq 0} \xi^n |G_n\rangle$ whose coefficients $|G_n\rangle \in M_{h,n}$ satisfy $T_1|G_n\rangle = |G_{n-1}\rangle$ with $|G_{-1}\rangle := 0$ and $T_n|G_m\rangle = 0$ for $n \geq 2$ and all m. In terms of the expansion $|G_n\rangle = \sum_{|\lambda|=n} g_\lambda^{(n)} T_\lambda |h\rangle$ these conditions read

$$\sum_{|\lambda|=n} g_\lambda^{(n)} S_{\lambda,\mu}^{(n)} = \delta_{\mu(1^n)}, \quad n = 0, 1, \ldots,$$

where (1^n) is the partition $(1, \ldots, 1)$ of size n.

From now on, we assume $q, t \neq 1$. The zeros of the determinant of the n-th block $S^{(n)}$ of the Kac matrix are located at [23, equation (2.4)]

$$h = \pm(t^{r/2} q^{-s/2} + t^{-r/2} q^{s/2}), \quad r, s \geq 1, rs = n. \tag{22}$$

Outside of these sets, the Kac matrix is invertible, hence the Gaiotto state exists uniquely and its norm with respect to the Shapovalov form is given by the formal power series

$$S(|G\rangle, |G\rangle) = \sum_{n \geq 0} \xi^{2n} (S^{(n)})^{-1}_{(1^n),(1^n)}.$$

Let $Q \in \mathbb{C}$ with $h = Q^{\frac{1}{2}} + Q^{-\frac{1}{2}}$. From the AGT relation [3,23,24] we obtain the formal expansion as a Nekrasov partition function

$$S(|G\rangle, |G\rangle) = \sum_{n \geq 0} \xi^{2n} q^{-n} t^n Z_n(q, t, Q), \tag{23}$$

where the coefficients are sums over pairs (ν, μ) of partitions:

$$Z_n(q, t, Q) = \sum_{|\nu|+|\mu|=n} \frac{1}{N_{\nu,\mu}(Q) N_{\mu,\nu}(Q^{-1}) N_{\nu,\nu}(1) N_{\mu,\mu}(1)},$$

$$N_{\nu,\mu}(Q) = \prod_{t \in \mu} (1 - Q q^{a_\nu(t)} t^{l_\mu(t)+1}) \prod_{s \in \nu} (1 - Q q^{-a_\mu(s)-1} t^{-l_\nu(s)}).$$

4.2. Finiteness of norm. We want to recover the coefficients defined in Eq. (8). We set $q_1 = t, q_2 = q^{-1}, r = 2$ and $u_1 = u_2^{-1} = Q^{\frac{1}{2}}$. The conditions stated at the beginning of Sect. 2.6.1 say $|q| > 1$, $|t| < 1$ and $(t\overline{q} = 1 \text{ or } t, q > 0)$ and

$$|t| \max\{|Q|, |Q|^{-1}\} < \min\{|Q|, |Q|^{-1}\}, \quad \max\{|Q|, |Q|^{-1}\} < |q| \min\{|Q|, |Q|^{-1}\}.$$

The condition (11) on the well-definedness of all simple residues translates to

$$Q \notin \{q^x t^y : x, y \in \mathbb{Z}\}, \quad \text{and} \quad q^x \neq t^y \, \forall (x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

We extend the condition on Q on the left to $-Q$, thus excluding all poles of the Kac determinant in Eq. (22). The condition on the right we drop by remark (1). From Eq. (12) in Theorem 1, we then recover

$$S(|G\rangle, |G\rangle) = \sum_{n \geq 0} z^n \, \mathcal{Z}_n(\vec{u}; p = \emptyset) \tag{24}$$

for $z = \xi^2 t q^{-1}$. Hence, Theorem 2 implies that series (24) converges for $|z| < 1$. We obtain

Theorem 6. *Let q and t be a pair of complex numbers of with $|t| < 1$ and $|q| > 1$ Suppose either $t\overline{q} = 1$ or $t, q > 0$. Let $Q^{\frac{1}{2}}$ be a complex number such that*

$$Q \notin \{q^x t^y : x, y \in \mathbb{Z}\}, \qquad |q| \max\{|Q|, |Q|^{-1}\} < \min\{|Q|, |Q|^{-1}\},$$
$$-Q \notin \{q^x t^y : x, y \in \mathbb{Z}\}, \qquad \max\{|Q|, |Q|^{-1}\} < |t| \min\{|Q|, |Q|^{-1}\}.$$

Set $h = Q^{\frac{1}{2}} + Q^{-\frac{1}{2}}$. The deformed Gaiotto state $|G\rangle$ with formal parameter $\xi \in \mathbb{C}$ exists for the Verma module M_h for the deformed Virasoro algebra $Vir_{q,t}$, and its norm $S(|G\rangle, |G\rangle)$ is an analytic function in the variable ξ for

$$|\xi| < |t|^{-\frac{1}{2}} |q|^{\frac{1}{2}}.$$

5. Open Problems

Here, we briefly describe two formal power series, to which we intended to apply our results. Their coefficients are certain limits of the coefficients we studied earlier. Due to the lack of uniformity in our estimates, we were not able to prove convergence of those power series.

5.1. Homological version of Nekrasov partition function. The K-theoretic Nekrasov partition function defined in Sect. 2 has a counterpart in which K-theory groups are replaced by $\tilde{T}$-equivariant Borel–Moore homology groups $H_*^{\tilde{T}}(-)$. All $H_*^{\tilde{T}}(-)$ are modules for the ring

$$S(\tilde{T}) := H_*^{\tilde{T}}(pt) \cong \mathbb{Z}[\epsilon_1, \epsilon_2, a_1, \dots, a_r].$$

Let $\mathcal{S} \cong \mathbb{Q}(\epsilon_1, \epsilon_2, a_1, \dots, a_r)$ denote its quotient field. The maps ι and $\iota_{\vec{Y}}$, defined in Sect. 2, all define pushforwards in equivariant homology. The localization theorem in equivariant homology says that ι becomes an isomorphism after tensoring with $\mathcal{S}$.

The fixed point theorem says that after tensoring with $\mathcal{S}$ we have

$$(\iota_*)^{-1} = \sum_{|\vec{Y}|=n} \frac{\iota_{\vec{Y}}^*}{e(T_{\vec{Y}}M(r,n))}, \tag{25}$$

where $e(-)$ denotes the Euler class. The homological Nekrasov partition function for pure Yang Mills theory is defined as a formal power series

$$Z(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n \geq 0} \mathfrak{q}^n Z_n(\epsilon_1, \epsilon_2, \vec{a}),$$

with coefficients $Z_n(\epsilon_1, \epsilon_2, \vec{a}) = \sum_{|\vec{Y}|=n}(\iota_*)^{-1}[M(r,n)]$ where $[M(r,n)]$ denotes the fundamental class of $M(r,n)$. Using the fixed point formula (25) and Eq. (2) to compute the Euler classes gives

$$Z_n(\epsilon_1, \epsilon_2, \vec{a}) = \sum_{|\vec{Y}|=n} \frac{1}{\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a})}, \tag{26}$$

where

$$n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}) = \prod_{s \in Y_\alpha} \left(-l_{Y_\beta}(s)\,\epsilon_1 + (a_{Y_\alpha}(s)+1)\,\epsilon_2 + a_\alpha - a_\beta \right)$$

$$\prod_{t \in Y_\beta} \left((l_{Y_\alpha}(t)+1)\,\epsilon_1 - a_{Y_\beta}(t)\,\epsilon_2 + a_\alpha - a_\beta \right).$$

In particular, the coefficients of the homological Nekrasov partition function are limits of the coefficients of the K-theoretic partition function, as defined in Eq. (5):

$$\lambda^{2rn} Z_n(\epsilon_1, \epsilon_2, \vec{a}; \vec{w} = \emptyset; \lambda) \to Z_n(\epsilon_1, \epsilon_2, \vec{a}) \quad (\lambda \to 0). \tag{27}$$

Equation (27) suggests that, under the same conditions on $\epsilon_1, \epsilon_2, a_1, \ldots, a_r$ as stated in Theorem 5, the homological Nekrasov partition function $Z(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q})$ converges for all $\mathfrak{q} \in \mathbb{C}$. This question remains unanswered: Neither is the limit (27) uniform in n, nor is the estimate for the radius of convergence uniform in λ.

Remark 3. In the special case $\epsilon_1 + \epsilon_2 = 0$, one can directly estimate the coefficients Z_n to prove convergence of the expansion $Z(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n \geq 0} \mathfrak{q}^n Z_n(\epsilon_1, \epsilon_2, \vec{a})$: The diagonal term $\alpha = \beta$ in $\prod_{\alpha,\beta=1}^n n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a})$ is a product over all hook lengths occurring in the Young diagram Y_α. By the hook length formula we obtain $\frac{1}{|Y_\alpha|!}$ times the Plancherel measure of the Young diagram Y_α, which allows one to estimate the expansion for all $\mathfrak{q} \in \mathbb{C}$, see [12, proposition 1]. This technique, which requires $\epsilon_1 + \epsilon_2 = 0$, also generalizes [5] to the K-theoretic Nekrasov partition function without additional matter fields.

5.2. Conformal blocks.

5.2.1. The Virasoro algebra and Verma modules.
We consider the Virasoro algebra Vir. It is a complex Lie algebra with generators L_n, $n \in \mathbb{Z}$ and central element C satisfying

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C.$$

Fix $c \in \mathbb{C}$, the so-called *central charge*. It will remain unchanged for the remainder of this introduction. For $h \in \mathbb{C}$ the *Verma module* V_h for the Virasoro algebra of conformal dimension h and central charge c is a module generated by a vector $|h\rangle$ satisfying $L_0|h\rangle = h|h\rangle$, $C|h\rangle = c|h\rangle$ and $L_n|h\rangle = 0$ for $n \geq 1$. A basis is given by $L_\lambda|h\rangle :=$ $L_{-\lambda(1)} \cdots L_{-\lambda(l)}|h\rangle$, where $\lambda = (\lambda(1), \ldots, \lambda(l))$ is a partition. The Verma module comes with a bilinear form, the *Shapovalov form*, $S : V_h \otimes V_h \to \mathbb{C}$ characterized by $S(L_n x, y) = S(x, L_{-n} y)$, for $x, y \in V_h$ and $S(|h\rangle, |h\rangle) = 1$.

5.2.2. Intertwiners.
Fix complex numbers h, h_1, h_2 and set $a = h_2 - h_1 - h$. Given two Verma modules V_{h_1} and V_{h_2} for the same central charge c, an *intertwiner* $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{n+a}$ between V_{h_1} and V_{h_2} of conformal dimension h is a formal power series whose coefficients ϕ_n are linear maps $V_{h_1} \to V_{h_2}$ such that

$$[L_n, \phi(z)] = (z^{n+1}\partial_z + (n + 1)hz^n)\phi(z). \tag{28}$$

In conformal field theory, this property models transformations of fields under infinitesimal conformal transformations. Conformal blocks model vacuum expectation values of such fields. Mathematically they are defined as follows:

5.2.3. Conformal blocks.
Let h_l, H_l, h_m, H_r, h_r be complex numbers. Let $\phi_l(w)$ be an intertwiner of conformal dimension H_l from V_{h_m} to V_{h_l}. Let $\phi_r(z)$ be an intertwiner of conformal dimension H_r from V_{h_r} to V_{h_m}. The object

$$\langle h_l|\phi_l(1)\phi_r(z)|h_r\rangle := \lim_{w \to 1} S\Big(|h_l\rangle, \phi_l(w)\big(\phi_r(z)|h_r\rangle\big)\Big)$$

is called a four-point conformal block. It defined as $z^{h-h_r-H_r}$ times a formal power series in z, whose complex coefficients are in principle determined by Eq. (28) up to normalization. The limit is taken for each coefficient separately.

The AGT relation allows us to express the conformal block as a Nekrasov partition function: One first has to introduce the *Liouville parametrization* of the conformal dimensions and the central charge. We pick $b \in \mathbb{C}$ with

$$c = 1 + 6(b + b^{-1})^2,$$

to parametrize the central charge. The conformal dimensions h_l, h_m, h_r of the Verma modules V_{h_l}, V_{h_m} and V_{h_r} are parametrized as

$$h_l = \frac{(b + b^{-1})^2}{4} - P_l^2, \quad h_m = \frac{(b + b^{-1})^2}{4} - P_m^2, \quad h_l = \frac{(b + b^{-1})^2}{4} - P_l^2,$$

where $P_l, P_m, P_r \in \mathbb{C}$. The conformal dimensions H_l and H_r of the intertwiners $\phi_l(w)$ and $\phi_r(z)$ are parametrized as

$$H_r = \alpha_r(b + b^{-1} - \alpha_r), \quad H_l = \alpha_l(b + b^{-1} - \alpha_l),$$

where $\alpha_r, \alpha_l \in \mathbb{C}$. We have [1]

$$\langle h_l | \phi_l(1) \phi_r(z) | h_r \rangle = (1-z)^{2\alpha_r(b+b^{-1}-\alpha_l)} \sum_{n \geq 0} z^n F_n(b, \alpha_r, \alpha_l, P_r, P_m, P_l),$$

with

$$F_n(b, \alpha_r, \alpha_l, P_r, P_m, P_l)$$
$$= \sum_{|Y_1|+|Y_2|=n} \frac{Z_{bif}\left(\alpha_r \middle| P_r, (\emptyset, \emptyset); P_m, (Y_1, Y_2)\right) Z_{bif}\left(\alpha_l \middle| P_m, (Y_1, Y_2); P_l, (\emptyset, \emptyset)\right)}{Z_{bif}\left(0 \middle| P_m, (Y_1, Y_2); P_m, (Y_1, Y_2)\right)}.$$

Here the value of $Z_{bif}(\alpha | P', \vec{Y}'; P, \vec{Y})$ for pairs of partitions $\vec{Y}, \vec{Y}'$ and complex numbers P, P' and α is given by

$$Z_{bif}(\alpha | P', \vec{Y}'; P, \vec{Y}) = \prod_{i,j=1}^{2} \left(\prod_{s \in Y_i} (P'_j - P_i + b(l_{Y'_j}(s) + 1) - b^{-1} a_{Y_i}(s) - \alpha) \right.$$
$$\left. \prod_{t \in Y'_j} (P'_j - P_i - b l_{Y_i}(t) + b^{-1}(a_{Y'_j}(t) + 1) - \alpha) \right),$$

where, on the right hand side, P_i is the i-th component of the vector $\vec{P} = (P, -P)$ and similarly for P'_i.

5.2.4. Attempt at an estimate. A simple calculation gives

$$Z_{bif}\left(\alpha_r \middle| P_r, (\emptyset, \emptyset); P_m, (Y_1, Y_2)\right) Z_{bif}\left(\alpha_l \middle| P_m, (Y_1, Y_2); P_l, (\emptyset, \emptyset)\right)$$
$$= \prod_{i=1}^{2} \prod_{(x,y) \in Y_i} \prod_{m=1}^{4} ((P_m)_i + b(x-1) + b^{-1}(y-1) + v_m),$$

where $v_1 = \alpha_r + P_r$, $v_2 = \alpha_r - P_r$, $v_3 = -\alpha_l + b + b^{-1} + P_l$ and $v_4 = -\alpha_l + b + b^{-1} - P_l$. Under the identifications

$$a_\alpha = P_\alpha, \quad \alpha = 1, 2, \quad \epsilon_1 = b, \epsilon_2 = b^{-1}, \quad w_m = v_m, \quad m = 1, \ldots, 4$$

and using Eq. (12) in Theorem 1, we recover $F_n(b, \alpha_r, \alpha_l, P_r, P_m, P_l)$ from $Z_n(\epsilon_1, \epsilon_2, \vec{a}, \vec{w}; \lambda)$ in the limit $\lambda \to 0$. The conditions formulated in Theorem 5 on b are $\text{Re } b > 0$ and either $b > 0$ or $|b| = 1$. The ones on P are $|\text{Re } P| < \frac{1}{2} \text{Re } b$ and $|\text{Re } P| < \frac{1}{2} \text{Re } b^{-1}$. Condition (21) translates, for $\lambda = 0$, to

$$P \notin \frac{b}{2} \mathbb{Z} + \frac{b^{-1}}{2} \mathbb{Z} \text{ and } b^2 \notin \mathbb{Q}_{\geq 0}.$$

The condition on P excludes the minimal models. The conditions on b restrict the central charge of the theory to the interval $(1, \infty)$. Since $\lambda^{2r-s} = 1$, Theorem 5 seems to suggest that the conformal block $\langle h_l | \phi_l(1) \phi_r(z) | h_r \rangle$ is analytic in z for $|z| < 1$. However, we neither know that the convergence of the Nekrasov partition function is uniform in λ, nor do we know that the convergence $Z_n(\epsilon_1, \epsilon_2, \vec{a}, \vec{w}; \lambda) \to F_n(b, \alpha_r, \alpha_l, P_r, P_m, P_l)$ as $\lambda \to 0$ is uniform in n.

Appendix A. Evaluation by iterated residues

In this section we prove Theorem 1 by evaluating the integral by residues. In a first step, we find the position of all poles whose residues contribute to the integral and show that they are simple. In a second step we evaluate the residues.

We show that $\mathcal{Z}_n(\vec{u}; \vec{p})$ is a sum

$$\mathcal{Z}_n(\vec{u}; \vec{p}) = \sum_{|\vec{Y}|=n} \mathcal{Z}_{\vec{Y}}(\vec{u}; \vec{p}), \tag{29}$$

of iterated, simple residues

$$\mathcal{Z}_{\vec{Y}}(\vec{u}; \vec{p}) := \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \prod_{j=1}^n \prod_{m=1}^s (\hat{z}_j - p_m)$$

$$\lim_{\substack{z_j \to \hat{z}_j \\ j=1,\ldots,n}} \left(\prod_{j=1}^n (z_j - \hat{z}_j) \; \frac{\mathcal{I}(z_1, \ldots, z_n; \vec{u})}{z_1 \cdots z_n} \right), \tag{30}$$

where

$$\{\hat{z}_1, \ldots, \hat{z}_n\} = \{z_{x,y}^\alpha : (x, y) \in Y_\alpha, \alpha = 1, \ldots, r\}, \tag{31}$$

in any order.

We suppose that $|q_1| = |q_2| + \delta < 1$, where $\delta > 0$ is small enough so that $|q_2| > |q_1|^2 > |q_1|^3 > \cdots$. The general case follows from analytic continuation.

We evaluate the integral $\mathcal{Z}_n(\vec{u}; \vec{p})$ by taking iterated residues. We start with a slightly more general integral. Let U, W be two finite sets of complex numbers and $f(z_1, \ldots, z_n)$ a symmetric function, analytic on the closed ball in $\mathbb{C}^n$ with radius ρ. Our integral is of the form

$$\int_{C_\rho} \frac{dz_n}{2\pi i} \cdots \int_{C_\rho} \frac{dz_1}{2\pi i} I(z_1, \ldots, z_n)$$

$$= \int_{C_\rho} \frac{dz_n}{2\pi i} \frac{\prod_{w \in W}(z_n - w)}{\prod_{u \in U}(z_n - u)} \prod_{n < k} \frac{(z_n - z_k)^2 (z_n - q_1 q_2 z_k)(z_n - q_1^{-1} q_2^{-1} z_k)}{(z_n - q_1 z_k)(z_n - q_2 z_k)(z_n - q_1^{-1} z_k)(z_n - q_2^{-1} z_k)}$$

$$\vdots$$

$$\int_{C_\rho} \frac{dz_2}{2\pi i} \frac{\prod_{w \in W}(z_2 - w)}{\prod_{u \in U}(z_2 - u)} \prod_{2 < k} \frac{(z_2 - z_k)^2 (z_2 - q_1 q_2 z_k)(z_2 - q_1^{-1} q_2^{-1} z_k)}{(z_2 - q_1 z_k)(z_2 - q_2 z_k)(z_2 - q_1^{-1} z_k)(z_2 - q_2^{-1} z_k)}$$

$$\int_{C_\rho} \frac{dz_1}{2\pi i} \frac{\prod_{w \in W}(z_1 - w)}{\prod_{u \in U}(z_1 - u)} \prod_{1 < k} \frac{(z_1 - z_k)^2 (z_1 - q_1 q_2 z_k)(z_1 - q_1^{-1} q_2^{-1} z_k)}{(z_1 - q_1 z_k)(z_1 - q_2 z_k)(z_1 - q_1^{-1} z_k)(z_1 - q_2^{-1} z_k)}$$

$$f(z_1, \ldots, z_n) \tag{32}$$

where $I(z_1, \ldots, z_n)$ is given by

$$f(z_1, \ldots, z_n) \prod_{j=1}^n \left(\frac{\prod_{w \in W}(z_j - w)}{\prod_{u \in U}(z_j - u)} \right)$$

$$\times \prod_{j<k} \frac{(z_j - z_k)^2(z_j - q_1 q_2 z_k)(z_j - q_1^{-1} q_2^{-1} z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)(z_j - q_1^{-1} z_k)(z_j - q_2^{-1} z_k)} \Bigg).$$

When integrating z_1 we can only pick up residues at $q_1 z_j$, $q_2 z_j$ or $\hat{z}_1 = u$, for some $u \in U$ with $|u| \le \rho$.

Assume that we pick up a residue at $\hat{z}_1 = q_i z_j$ for some $i \in \{1, 2\}$ and $j \in \{2, \dots, n\}$. The other case will be treated later on. Let $s \in \{1, 2\}$ be the index complementary to i. The residue is simple. After renaming $z_j \leftrightarrow z_2$ we obtain

$$\int_{C_\rho} \frac{dz_n}{2\pi i} \cdots \int_{C_\rho} \frac{dz_2}{2\pi i} f(q_i z_2, z_2, \dots, z_n)$$

$$\times \prod_{j=3}^{n} \left(\frac{\prod_{w \in W}(z_j - w)}{\prod_{u \in U}(z_j - u)} \prod_{2 < j < k} \frac{(z_j - z_k)^2(z_j - q_1 q_2 z_k)(z_j - q_1^{-1} q_2^{-1} z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)(z_j - q_1^{-1} z_k)(z_j - q_2^{-1} z_k)} \right)$$

$$\times \frac{\prod_{w \in W}(z_2 - w)(q_i z_2 - w)}{\prod_{u \in U}(z_2 - u)(q_i z_2 - u)}$$

$$\times \prod_{2 < k} \frac{(z_2 - z_k)(z_2 - q_i q_s z_k)(z_2 q_i - z_k)(z_2 q_i - q_i^{-1} q_s^{-1} z_k)}{(z_2 - q_i z_k)(z_2 - q_s^{-1} z_k)(z_2 q_i - q_s z_k)(z_2 q_i - q_i^{-1} z_k)}$$

$$\times q_i z_2 \frac{(q_i - 1)^2(1 - q_s)(q_i - q_i^{-1} q_s^{-1})}{(q_i - q_s)(q_i - q_i^{-1})(q_i - q_s^{-1})}.$$

We also have used Fubini's theorem to permute the order of integration, swapping the integrals over dz_2 and dz_j. This is permitted since the integrand does not have poles on the integration contours. Here we use the small perturbation of $|q_1|$ and $|q_2|$. We note that this result does not depend on j anymore, which also follows from the symmetry of $I(z_1, \dots, z_n)$. So we can suppose without loss of generality $j = 2$.

We see that now we can pick up residues at $\hat{z}_2 = u_0, q_i^{-1} u_0$, for some u_0 such that $\hat{z}_2$ lies inside the integration contour or at $\hat{z}_2 = q_i z_j$, where the i is the same as in the previous integral. It appears like there is a residue coming from the pole $(z_2 q_i - q_s z_j)$ if $i = 1$. However, all residues coming from these poles cancel each other in the sum: One directly calculates

$$\lim_{z_k \to q_1^{-1} q_2 z_l} (z_k - q_1^{-1} q_2 z_l) \sum_{i=1,2} \sum_{j=2}^{n} \mathrm{Res}_{\hat{z}_1 = q_i z_j} I(z_1, \dots, z_n) = 0.$$

Assume that we pick $\hat{z}_2 = q_i z_j$, again assume without loss of generality $j = 3$. The residue is simple. The integral over z_3 becomes

$$\int_{C_\rho} \frac{dz_3}{2\pi i} \prod_{l=0}^{2} \frac{\prod_{w \in W}(z_3 q_i^l - w)}{\prod_{u \in U}(z_3 q_i^l - u)}$$

$$\prod_{3 < k} \frac{(z_3 - z_k)(z_3 - q_i q_s z_k)(z_3 q_i^2 - z_k)(z_3 q_i^2 - q_i^{-1} q_s^{-1} z_k)}{(z_3 - q_i z_k)(z_3 - q_s^{-1} z_k)(z_3 q_i^2 - q_s z_k)(z_3 q_i^2 - q_i^{-1} z_k)}$$

$$q_i^3 z_3^2 \frac{(q_i - 1)^3(1 - q_s)^2(q_i^2 - q_s^{-1})(q_i^3 - q_s^{-1})}{(q_i - q_s)(q_i - q_s^{-1})^2(q_i^2 - q_s)(q_i^3 - 1)} f(q_i^2 z_3, q_i z_3, z_3, \dots, z_n).$$

In the next step, we can again only pick up simple residues at $\hat{z}_3 = q_i z_4$ or $\hat{z}_3 = q_i^{-l} u_0$ with $l \in \{0, 1, 2\}$ and $u_0 \in U$ such that $\hat{z}_3$ lies inside the integration contour.

Assume that we have picked and evaluated simple residues at $(\hat{z}_1, \dots, \hat{z}_{J-1})$ during the first $J-1$ integrations with $\hat{z}_j = q_i^{J-j} z_J$, $j = 1, \dots, J-1$, where $i \in \{1, 2\}$ is fixed. Let $s \in \{1, 2\}$ be the other index. The integral over z_J has the integrand

$$\prod_{l=0}^{J-1} \frac{\prod_{w \in W}(z_J q_i^l - w)}{\prod_{u \in U}(z_J q_i^l - u)}$$

$$\times \prod_{J<k} \frac{(z_J - z_k)(z_J - q_i q_s z_k)(z_J q_i^{J-1} - z_k)(z_J q_i^{J-1} - q_i^{-1} q_s^{-1} z_k)}{(z_J - q_i z_k)(z_J - q_s^{-1} z_k)(z_J q_i^{J-1} - q_s z_k)(z_J q_i^{J-1} - q_i^{-1} z_k)}$$

$$\times z_J^{J-1} q_i^{\frac{1}{2}(J+2)(J-1)} \frac{(q_i - 1)^J (1 - q_s)^{J-1}}{(q_i - q_s^{-1})^{J-1}(q_i^J - 1)}$$

$$\times \prod_{j=1}^{J-1} \frac{(q_i^j - q_i^{-1} q_s^{-1})}{(q_i^j - q_s)} f(q_i^{J-1} z_J, \dots, z_J, \dots, z_n), \tag{33}$$

whereas the integrands for $z_{J+1}, \dots, z_n$ in Eq. (32) remain the same. Now pick a residue at $\hat{z}_J = q_i^{-l_0} u_0$ inside the integration contour with $u_0 \in U$ and $l_0 \in \{0, \dots, J-1\}$ and suppose the pole is simple. So we have followed the evaluation steps

$$\hat{z}_1 = q_i z_2, \quad \hat{z}_2 = q_i z_3, \quad \dots, \quad \hat{z}_{J-1} = q_i z_J, \quad \hat{z}_J = q_i^{-l_0} u_0 \tag{34}$$

leading to the residue strip

$$(\hat{z}_1, \dots, \hat{z}_J) = q_i^{-l_0}(q_i^{J-1} u_0, \dots, u_0). \tag{35}$$

When we evaluate the final residue we get the same integral expression we started with as in Eq. (32), except that we only have the variables $z_{J+1}, \dots, z_n$, the sets U, W are changed to

$$U' = U \cup \{q_i^{-1}\hat{z}_J, q_s \hat{z}_J, q_s^{-1}\hat{z}_1, q_i \hat{z}_1\} \qquad W' = W \cup \{\hat{z}_J, q_i^{-1} q_s^{-1} \hat{z}_J, \hat{z}_1, q_i q_s \hat{z}_1\},$$

and $f(z_1, \dots, z_n)$ gets replaced by

$$f'(z_{J+1}, \dots, z_n) = f(q_i^{J-1} q_i^{-l_0} u_0, \dots, q_i^{-l_0} u_0, z_{J+1}, \dots, z_n).$$

This function is again symmetric. Moreover, we have accumulated a prefactor with the value

$$\prod_{l=0}^{J-1} \frac{\prod_{w \in W}(q_i^{l-l_0} u_0 - w)}{\prod_{u \in U \setminus \{u_0\}}(q_i^{l-l_0} u_0 - u)} \prod_{\substack{l=0 \\ l \neq l_0}}^{J-1} \frac{1}{(q_i^{l-l_0} - 1)} q_i^{\frac{1}{2}(J+2)(J-1) - l_0 J}$$

$$\frac{(q_i - 1)^J (1 - q_s)^{J-1}}{(q_i - q_s^{-1})^{J-1}(q_i^J - 1)} \prod_{j=1}^{J-1} \frac{(q_i^j - q_i^{-1} q_s^{-1})}{(q_i^j - q_s)}, \tag{36}$$

coming from the residue evaluation. We see that the evaluation of our integral happens in stages, where one evaluates a strip of residues. We draw the strip (35) for $i = 2$ as follows:

$$\tag{37}$$
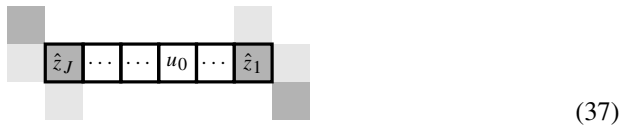
The positive q_i direction goes from west to east. The residues go from east to west over the residue strip (35). Here we have also indicated the poles (light gray) and zeros (dark grey) such a strip adds to the sets U and W. For $i = 1$ the strip is drawn vertically, with positive q_i direction from north to south.

All formulae remain valid for $J = 1$, i.e. when we directly pick up a residue at $\hat{z}_1 = u_0$ for some $u_0 \in U$. We call such a strip of length one a *box*. It is drawn as

$$\tag{38}$$

Here we have used black to mark a zero of order two. We want to compare the result of the evaluation process (34) of the above residues for general $l_0 \in \{0, \dots, J-1\}$ to the result from the procedure where we evaluate

the same final residues (35) by repeated use of the case $J = 1$. Now, the order of the residues will be different. It is clear that the sets U' and W' we end up with agree for both procedures. Moreover, the new symmetric function f' also agrees, since the original function f is symmetric. However the prefactor (36) only agrees up to a sign, as we will now see.

We first treat the case, where we go in positive q_i direction, starting at a pole at some u_1: Suppose we pick up residues using the evaluation process

$$\hat{z}_1 = u_1, \quad \hat{z}_2 = q_i u_1, \quad \ldots, \quad \hat{z}_K = q_i^{K-1} u_1. \tag{39}$$

For $k = 0, \ldots, K$, let U_k^+ and W_k^+ denote the sets U and W after evaluating the residues at $\hat{z}_1, \ldots, \hat{z}_k$. In the first step we pick up the residue at $\hat{z}_1 = u_1$, the sets $U = U_0^+$ and $W = W_0^+$ get changed to

$$U_1^+ = \left(U \setminus \{u_1\}\right) \cup \{q_i^{-1} u_1, q_s u_1, q_s^{-1} u_1, q_i u_1\}$$
$$W_1^+ = W \cup \{u_1, q_i^{-1} q_s^{-1} u_1, q_i q_s u_1\},$$

the symmetric function is $f(u_1, z_2, \ldots, z_n)$ and we get the prefactor

$$\frac{\prod_{w \in W_0^+} (u_1 - w)}{\prod_{u \in U_0^+ \setminus \{u_1\}} (u_1 - u)}.$$

By induction, we see that after evaluating all residues (39), we have changed the original sets U, W to

$$U_K^+ = \left(U \setminus \{u_1\}\right) \cup \{q_i^{-1} u_1, q_s u_1, q_s^{-1} q_i^{K-1} u_1, q_i q_i^{K-1} u_1\}$$
$$W_K^+ = W \cup \{q_i^{-1} q_s^{-1} u_1, q_i^{K-1} u_1, q_i q_s q_i^{K-1} u_1\},$$

the function $f(z_1, \ldots, z_n)$ gets replaced by

$$f_K^+(z_{K+1}, \ldots, z_n) = f(u_1, \ldots, q_i^{K-1} u_1, z_{K+1}, \ldots, z_n)$$

and we have accumulated the prefactor

$$\prod_{k=0}^{K-1} \frac{\prod_{w \in W_k^+} (q_i^k u_1 - w)}{\prod_{u \in U_k^+ \setminus \{q_i^k u_1\}} (q_i^k u_1 - u)} = \prod_{k=0}^{K-1} \frac{\prod_{w \in W} (q_i^k u_1 - w)}{\prod_{u \in U \setminus \{u_1\}} (q_i^k u_1 - u)} \prod_{k=1}^{K-1} \frac{1}{(q_i^k - 1)}$$

$$q_i^{\frac{1}{2}(K+2)(K-1)} \frac{(q_i - 1)^K (1 - q_s)^{K-1}}{(q_i^K - 1)(q_i - q_s^{-1})^{K-1}} \prod_{k=1}^{K-1} \frac{(q_i^k - q_i^{-1} q_s^{-1})}{(q_i^k - q_s)}. \tag{40}$$

Next, we treat the case where we go in negative q_i direction starting at some pole u_2. Suppose we pick up residues using the evaluation steps

$$\hat{z}_1 = u_2, \quad \hat{z}_2 = q_i^{-1} u_2, \quad \ldots, \quad \hat{z}_M = q_i^{-M+1} u_2. \tag{41}$$

For $m = 0, \ldots, M$, let U_m^- and W_m^- denote the sets U and W after evaluating the residues at $\hat{z}_1, \ldots, \hat{z}_m$. In the first step we pick up the residue at $\hat{z}_1 = u_2$, the sets $U = U_0^-$ and $W = W_0^-$ get changed to

$$U_1^- = \left(U \setminus \{u_2\}\right) \cup \{q_i^{-1} u_2, q_s u_2, q_s^{-1} u_2, q_i u_2\}$$
$$W_1^- = W \cup \{u_2, q_i^{-1} q_s^{-1} u_2, q_i q_s u_2\},$$

the symmetric function is $f(u_2, z_2, \ldots, z_n)$ and we get the prefactor

$$\frac{\prod_{w \in W_0^-} (u_2 - w)}{\prod_{u \in U_0^- \setminus \{u_2\}} (u_2 - u)}.$$

Again, after evaluating all residues (41), we have changed the original sets U, W to

$$U_M^- = \left(U \setminus \{u_2\}\right) \cup \{q_i u_2, q_s^{-1} u_2, q_s q_i^{-M+1} u_2, q_i^{-1} q_i^{-M+1} u_2\}$$

$$W_M^- = W \cup \{q_i q_s u_2, q_i^{-M+1} u_2, q_i^{-1} q_s^{-1} q_i^{-M+1} u_2\},$$

the function $f(z_1, \ldots, z_n)$ is replaced by

$$f_M^-(z_{M+1}, \ldots, z_n) = f(u_2, \ldots, q_i^{-M+1} u_2, z_{M+1}, \ldots, z_n)$$

and get the prefactor

$$\prod_{m=0}^{M-1} \frac{\prod_{w \in W_m^-}(q_i^{-m} u_2 - w)}{\prod_{u \in U_m^- \setminus \{q_i^{-m} u_2\}}(q_i^{-m} u_2 - u)} = \prod_{m=0}^{M-1} \frac{\prod_{w \in W}(q_i^{-m} u_2 - w)}{\prod_{u \in U \setminus \{u_2\}}(q_i^{-m} u_2 - u)} \prod_{m=1}^{M-1} \frac{1}{(q_i^{-m} - 1)}$$

$$q_i^{-\frac{1}{2}(M-1)M + M - 1} \frac{(1 - q_i)^M (1 - q_s)^{M-1}}{(q_i - q_s^{-1})^{M-1}(1 - q_i^M)} \prod_{m=1}^{M-1} \frac{(q_i^m - q_i^{-1} q_s^{-1})}{(q_i^m - q_s)}. \tag{42}$$

Recall that were looking at the case where we pick the residues (35) as a strip depicted in figure (37) using the evaluation steps (34) and we want to compare it to the procedure where we apply the case $J = 1$ repeatedly. Set $M = l_0 + 1$ and $K = J - l_0 - 1$. Picking residues according to the stepwise procedures with $u_2 = u_0$ and $u_1 = q_i u_0$ we get the prefactor

$$\prod_{k=0}^{K-1} \frac{\prod_{w \in W_M^-}(q_i^k u_1 - w)}{\prod_{u \in U_M^- \setminus \{u_1\}}(q_i^k u_1 - u)} \prod_{m=0}^{M-1} \frac{\prod_{w \in W}(q_i^{-m} u_2 - w)}{\prod_{u \in U \setminus \{u_2\}}(q_i^{-m} u_2 - u)}$$

$$q_i^{\frac{1}{2}(K+2)(K-1)} q_i^{-\frac{1}{2}(M-1)M + M - 1} \frac{(q_i - 1)^K (1 - q_s)^{K-1}}{(q_i^K - 1)(q_i - q_s^{-1})^{K-1}} \frac{(1 - q_i)^M (1 - q_s)^{M-1}}{(q_i - q_s^{-1})^{M-1}(1 - q_i^M)}$$

$$\prod_{k=1}^{K-1} \frac{(q_i^k - q_i^{-1} q_s^{-1})}{(q_i^k - q_s)} \prod_{m=1}^{M-1} \frac{(q_i^m - q_i^{-1} q_s^{-1})}{(q_i^m - q_s)} \prod_{k=1}^{K-1} \frac{1}{(q_i^k - 1)} \prod_{m=1}^{M-1} \frac{1}{(q_i^{-m} - 1)}. \tag{43}$$

We call this factor the *base value* for our strip of residues. This factor equals the prefactor in Eq. (36) up to a sign of

$$(-1)^{M+1}.$$

(When we first evaluate the residues in positive q_i direction and then in negative q_i direction we get the same result.)

We point out two special cases of our observation:

(1) In the case $l_0 = 0$, where we place the whole strip of residues at u_0 and eastwards of u_0, it does not matter whether we pick the residues one by one or as a strip.
(2) In the case $l_0 = J - 1$, where we place the whole strip of residues at u_0 and westwards of u_0, we get the base value up to a sign equal to $(-1)^{L+1}$ where L is the length of the strip.

Claim 1. *Only strips with $l_0 = 0$ contribute to the integral.*

Suppose, we add a strip of residues of length J with $l_0 > 0$, i.e. part of the residue diagram grows in negative q_i direction. We claim that the total contribution of all possible processes to choose residues leading to the same residue strip is zero. We depict the strip as

$$\boxed{\star}\,\boxed{\star}\,\boxed{\star}\,\boxed{\star}\,\boxed{\star}\,\boxed{\dagger}\,\boxed{\dagger}\,\boxed{\dagger}$$
$$\uparrow$$
$$u_0 \tag{44}$$

for $i = 2$ and vertically for $i = 1$. The number of stars equals $M = l_0 + 1$ and the number of daggers equals $K = J - M$. The residue at u_0 is located at the easternmost star. We draw the corresponding base value, corresponding to a stepwise procedure when picking up residues, as

$$\square\,\square\,\square\,\square\,\square\,\square\,\square\square$$
$$\uparrow$$

$$u_0$$

Of course, this picture does not fully specify the procedure: We can alternate between picking up residues to the east and to the west. We will deal with this multiplicity later.

Another way to evaluate the residues is for example by first evaluating some of the residues as a strip with $l_0 = J - 2$ and then two as boxes and one final strip of length two to the east. We depict this as

$$\uparrow$$
$$u_0 \tag{45}$$

All possible ways to end up with the residues (35) define a partition of the strip (44) into substrips. All those partitions yield the same final sets U', W' and the same final function f'. The contributions of the residue evaluations differ by signs depending on the partition into substrips. To compute these signs we have to cut the partition eastwards of u_0, for example we substitute the partition (45) by

$$\uparrow$$
$$u_0$$

According to our computation above, we then have to count the number of strips of even length westwards of u_0. For each of those, we get a factor of (-1). Then we have to sum over all partitions of the original string, weighted by the number of ways in which one can pick residues to obtain the partition. As we have already noticed, there are multiple ways since one can for example alternate between placing substrips at the west and east end. The total contribution of all procedures leading to the final residues (35) is given by the base value times an expression

$$\sum_{\xi \in \Gamma} w(\xi) s(\xi),$$

which we want to formalize. Here the set Γ represents all the ways to cut the strip (44) into substrips. The integer $w(\xi)$ is the weight corresponding to all possible procedures to pick residues leading to the choice of substrips ξ. Finally the sign $s(\xi)$ is given by the number of substrips of even length to the west of u_0 after cutting directly to the east of u_0. We set

$$\Gamma_J = \{(J_1, \ldots, J_N) : N \in \mathbb{N}, J_1, \ldots, J_N \in \mathbb{N} : J_0 + \cdots + J_N = J\}$$

to be the set of ordered partitions of the integer J. Cutting the strip (44) into substrips according to $\vec{J} \in \Gamma_J$ means cutting it into substrips of length $J_0, \ldots, J_n$ from west to east. For example the cut strip (45) corresponds to the element $\vec{J} = (1, 1, 4, 2) \in \Gamma_8$. Let

$$\mathrm{cut} : \Gamma_J \to \Gamma_M$$

be the map realized by cutting the collection of substrips $\vec{J}$ directly to the east of u_0. Now we can formalize the sign $s(\vec{J})$ as

$$s(\vec{J}) = \prod_a (-1)^{1 + \left(\mathrm{cut}(\vec{J})\right)_a}.$$

In order to describe the weight $w(\vec{J})$, we have to take into account two effects: Firstly, for each substrip of length $J_a > 1$, we have to count the number of choices when selecting a residue $\hat{z}_a = q_i z_b$ that correspond to the freedom to choose any of the remaining variables z_b for $b \in \{1, \ldots, J\}$. Secondly, we can alternate between placing substrips to the west of the already chosen residues or to the east of them. We first have to account for the second effect. To model the second effect, we reorder the vectors $\vec{J}$. We define the map

$$\mathrm{reo} : \Gamma_J \to \Gamma_J$$

by demanding that, for

$$\vec{A} = (A_1, A_2, \ldots, A_N) = \mathrm{reo}(\vec{J}) = (B_0, B_1, \ldots, B_b, C_1, \ldots, C_c),$$

the number B_0 corresponds to the substrip containing u_0, the numbers B_1 up to B_b correspond to the substrips to the west, with increasing index in westward direction, and the numbers C_1 up to C_c correspond to substrips to the east, with increasing index in eastward direction. The indices $b = b(\vec{J})$ and $c = c(\vec{J})$ depend on the collection of substrips $\vec{J} \in \Gamma_J$. For example, the cut strip (45) yields reo$(\vec{J}) = (4, 1, 1, 2)$, $b = 2$ and $c = 1$. We interpret the order of the components of reo$(\vec{J})$ as the order in which we place the residues. All other orderings are obtained by permuting the elements $(\vec{B}, \vec{C}) = (B_1, \ldots, B_b, C_1, \ldots, C_c)$ of this tuple, such that the order of the elements in $\vec{B}$ and in $\vec{C}$ remain the same. This means we can alternate between placing residues to the west and to the east. Let

$$S_{b,c} = \{\sigma \in S_{b+c} : \sigma(1) < \cdots < \sigma(b), \sigma(b+1) < \cdots < \sigma(b+c)\}$$

denote the set of (b, c) shuffles. We define

$$\sigma \vec{A} = \sigma(B_0, B_1, \ldots, B_b, C_1, \ldots, C_c) = (B_0, \sigma(B_1, \ldots, B_b, C_1, \ldots, C_c)),$$

where σ acts on $(B_1, \ldots, B_b, C_1, \ldots, C_c)$ in the usual sense. We see that all possible ways to place the substrips reo$(\vec{J})$ corresponds to the orbit of this tuple under $S_{r,s}$. Hence we have formalized the second effect. Now we can also formalize the first effect described above: Suppose we are placing the substrip $\sigma(\vec{A})_k$. Then the remaining substrips correspond to $\sigma(\vec{A})_{k+1} + \cdots + \sigma(\vec{A})_N$ variables. Each time we choose a pole at $\hat{z}_a = q_i z_b$ in $\sigma(\vec{A})_k$, we can choose z_b to be one of those variables, or one of the variables in $\sigma(\vec{A})_k$, we have not chosen so far. Hence, we get

$$\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right)$$

possibilities in total when placing $\sigma(\vec{A})_k$. In the case $\sigma(\vec{A})_k = 1$, we get an empty product which we interpret as one. This agrees with the observation that $\sigma(\vec{A})_k = 1$ implies we pick a residue from the set U, which leaves no further choices in terms of the remaining variables. Now we can combine the two effects to obtain the expression

$$w(\vec{J}) = \sum_{\sigma \in S_{b,c}} \prod_{j=1}^{N} \left(\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right)$$

where $b = b(\vec{J})$, $c = c(\vec{J})$ and $\vec{A} = \text{reo}(\vec{J})$. In order to prove claim 1, we show

$$\sum_{\vec{J} \in \Gamma_J} s(\vec{J}) w(\vec{J}) = \sum_{\vec{J} \in \Gamma_J} \left(\prod_a (-1)^{1+\left(\text{cut}(\vec{J})\right)_a} \right)$$

$$\times \left(\sum_{\sigma \in S_{b(\vec{J}),c(\vec{J})}} \prod_{j=1}^{N} \left(\prod_{h=1}^{\sigma(\text{reo}(\vec{J}))_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\text{reo}(\vec{J}))_k + h \right) \right) \right) = 0 \quad (46)$$

by exhibiting parts of this sum that cancel each other. Fix one $\vec{J} \in \Gamma_J$ with $B_b > 1$ for

$$\vec{A} = \text{reo}(\vec{J}) = (B_0, B_1, \ldots, B_{b-1}, B_b, C_1, \ldots, C_c).$$

We compare this choice of substrips to

$$\vec{A}' := (B_0, B_1, \ldots, B_{b-1}, B_b - 1, 1, C_1, \ldots, C_c)$$
$$=: (B'_0, B'_1, \ldots, B'_{b+1}, C'_1, \ldots, C'_c).$$

Graphically, $\vec{A}'$ is obtained from $\vec{A}$ by cutting the strip at the westernmost box. Let $\vec{J}' \in \Gamma_J$ with $\vec{A}' = \text{reo}(\vec{J}')$. We claim that

$$w(\vec{J}) s(\vec{J}) = -w(\vec{J}') s(\vec{J}').$$

This suffices: There is a one to one correspondence between collections of substrips whose westernmost substrip has length one and those whose westernmost substrip has length greater than one. A bijection is given by $\vec{A} \mapsto \vec{A}'$. It is clear from the definition of $s(\vec{J})$ that $\vec{J}$ and $\vec{J}'$ have opposite sign, provided $l_0 > 0$. Hence it suffices to show that their weight is equal. We thus want to prove the equality in

$$\sum_{\sigma \in S_{b,c}} \prod_{j=1}^{N} \left(\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right)$$

$$= \sum_{\sigma \in S_{b+1,c}} \prod_{j=1}^{N+1} \left(\prod_{h=1}^{\sigma(\vec{A}')_j - 1} \left(\sum_{k=j+1}^{N+1} \sigma(\vec{A}')_k + h \right) \right).$$

We decompose

$$S_{b+1,c} = \bigcup_{\sigma \in S_{b,c}} \{ \hat{\sigma}_\lambda : \lambda = 1, \ldots, N+1 - \sigma(r) \},$$

where, for $j = 1, \ldots, N+1$,

$$\hat{\sigma}_\lambda(j) = \begin{cases} \sigma(j), & j = 1, \ldots, b \\ \sigma(b) + \lambda, & j = b+1 \\ \sigma(j-1), & j = b+2, \ldots, \sigma(b) + \lambda \\ \sigma(j-1) + 1, & j > \sigma(b) + \lambda + 1. \end{cases}$$

The parameter λ tells us how much we delay the placement of the residue box we cut away from the westernmost residue strip. Fix $\sigma \in S_{b,c}$. We will show

$$\prod_{j=1}^{N} \left(\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right)$$

$$= \sum_{\lambda=1}^{N+1-\sigma(b)} \prod_{j=1}^{N+1} \left(\prod_{h=1}^{\hat{\sigma}_\lambda(\vec{A}')_j - 1} \left(\sum_{k=j+1}^{N+1} \hat{\sigma}_\lambda(\vec{A}')_k + h \right) \right). \tag{47}$$

We set $B := \sigma(b)$. This is step in which we place B'_b. The tuples $\hat{\sigma}_\lambda(\vec{A}')$ and $\sigma(\vec{A})$ are related as follows:

$$\left(\hat{\sigma}_\lambda(\vec{A}')_1, \ldots, \hat{\sigma}_\lambda(\vec{A}')_{B-1}, \hat{\sigma}_\lambda(\vec{A}')_B, \hat{\sigma}_\lambda(\vec{A}')_{B+1}, \ldots, \hat{\sigma}_\lambda(\vec{A}')_{N+1} \right)$$

$$= \left(\sigma(\vec{A})_1, \ldots, \sigma(\vec{A})_{B-1}, \sigma(\vec{A})_B - 1, \left(\sigma(\vec{A})_{B+1}, \ldots, 1, \ldots, \sigma(\vec{A})_N \right) \right),$$

where 1 sits at index λ inside the inner tuple. Using this description, we rewrite the right hand side of Eq. (47) as

$$\sum_{\lambda=1}^{N+1-B} \prod_{j=1}^{B-1} \left(\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right) \left(\prod_{h=2}^{\sigma(\vec{A})_B - 1} \left(\sum_{k=B+1}^{N} \sigma(\vec{A})_k + h \right) \right)$$

$$\prod_{j=B+1}^{B+\lambda-1} \left(\prod_{h=2}^{\sigma(\vec{A})_j} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right) \prod_{j=B+\lambda}^{N} \left(\prod_{h=1}^{\sigma(\vec{A})_j - 1} \left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + h \right) \right).$$

This equals the left hand side of Eq. (47) up to a factor of

$$\left(\sum_{k=B+1}^{N} \sigma(\vec{A})_k + 1 \right)^{-1} \sum_{\lambda=1}^{N+1-B} \prod_{j=B+1}^{B+\lambda-1} \left(\left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + 1 \right)^{-1} \right.$$

$$\left(\sum_{k=j+1}^{N} \sigma(\vec{A})_k + \sigma(\vec{A})_j\right).$$

This factor equals one, since for arbitrary tuples $(x_1, \ldots, x_L)$, we have

$$\sum_{J=1}^{L} \prod_{j=1}^{J} \frac{\sum_{k=j}^{L} x_k}{\sum_{k=j+1}^{L} x_k + 1}$$

$$= \frac{\sum_{k=1}^{L} x_k}{\sum_{k=2}^{L} x_k + 1}\left(1 + \frac{\sum_{k=2}^{L} x_k}{\sum_{k=3}^{L} x_k + 1}\right.$$

$$\times \left(\cdots\left(1 + \frac{x_{k-2} + x_{k-1} + x_k}{x_{k-1} + x_k + 1}\left(1 + \frac{x_{k-1} + x_k}{x_k + 1}\left(1 + \frac{x_k}{1}\right)\right)\right)\cdots\right)\right)$$

$$= \sum_{k=1}^{L} x_k.$$

Hence we have established Eq. (46) and thus proved claim 1. We note two consequences of our argument up to now:

Claim 2. *Since we can only consider strips with $l_0 = 0$, we can only consider the cases were we take single boxes as residues. In particular, the order of their evaluation does not matter.*

Claim 3. *We can therefore discard poles at $q_i^{-1} u_0$ with $u_0 \in U$ since they are either out of the integration contours or realizable by a string with $J = 2$ and $l_0 = 1$ and hence part of a zero sum described above.*

Now we see inductively that the residues $\hat{z}_j$ contributing to the integral are of the form $u_\alpha q_1^{x-1} q_2^{y-1}$ where $(x, y) \in Y_\alpha$ for some Young diagrams Y_α. Because of the condition

$$u_\alpha u_\beta^{-1} \notin \{q_1^x q_2^y : x, y \in \mathbb{Z}\}, \quad \alpha \neq \beta$$

the poles do not interact and we may suppose $r = 1$. The sets U and W we start with are then $U = \{u_1\}$ and $W = \emptyset$. We evaluate a residue at the pole u_1 changing the zeros and poles of the remaining integrand. We depict this process as

the residue   evaluated at pole   yields
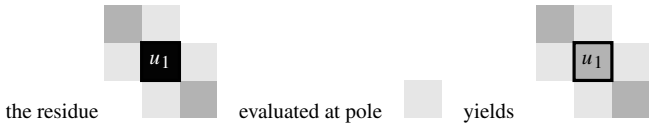
For the induction step, assume we have evaluated residues such that the evaluated residues fill a Young diagram. Also assume that the poles and zeros encoded in the sets U, W are located at the following places in the diagram: at each corner outside the diagram which is open to the south east there is a pole. To the northwest of each of those poles is another pole. At all south-easternmost boxes in the diagram there is a zero and another zero directly south east of it. Finally there is a zero at the coordinate $(-1, -1)$. An example would be for instance

$$(48)$$
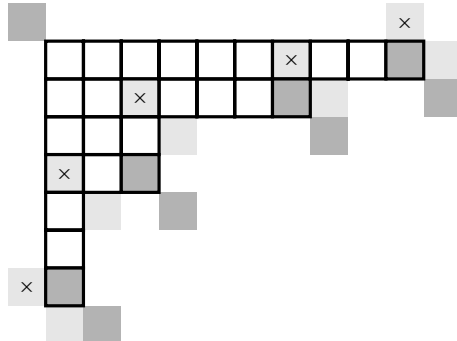
As we have seen before in claim 3, evaluating a residue at the poles marked with a cross does not contribute to the integral. It is clear that placing a residue box as in (38) at one of the remaining poles in diagram (48) yields again a Young diagram with the same structure of the poles and zeros as in diagram (48).

Hence, we have established that all residues are simple residues located at (31). It is clear that different procedures to pick residues at $\{z_s^\alpha : s \in Y_\alpha, \alpha = 1, \ldots, r\}$ correspond to permutation of the variables $\hat{z}_j, j = 1, \ldots, n$. Since the integral is symmetric in $z_1, \ldots, z_n$, all possible permutations occur. This cancels the factor $n!$ in front of the integral.

We have proved Eq. (29). Next, we evaluate the right hand side of Eq. (30). The integrand (8) is invariant under swapping q_1 and q_2. The residues $\{z_s^\alpha : s \in Y_\alpha, \alpha = 1, \ldots, r\}$ defined in Eq. (10) remain invariant if we swap q_1 and q_2 and transpose the diagrams $Y_1, \ldots, Y_r$. Hence

$$\sum_{|\vec{Y}|=n} \mathcal{Z}_{Y_1,\ldots,Y_r}(\vec{u}; \vec{p}) = \sum_{|\vec{Y}|=n} \mathcal{Z}_{Y_1^T,\ldots Y_r^T}(\vec{u}; \vec{p})|_{q_1 \leftrightarrow q_2}.$$

We have $a_Y(x, y) = l_{Y^T}(y, x)$ and hence Eq. (13) follows from Eq. (12).

The proof of Eq. (12) is adapted from [24], where it was performed for the special case $r = 2$. It suffices to consider $\vec{p} = \emptyset$ since we can cancel the factors in the equation we want to prove. Let

$$\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset) = \prod_{\alpha,\beta=1}^r \left(\prod_{s \in Y_\alpha} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{l_{Y_\alpha}(s)+1} q_2^{-a_{Y_\beta}(s)}} \right.$$

$$\left. \prod_{t \in Y_\beta} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{-l_{Y_\beta}(t)} q_2^{a_{Y_\alpha}(t)+1}} \right)$$

be the right hand side of the first equation in Theorem 1.

For any r-tuple $\vec{Y}$ of partitions with $Y_r \neq \emptyset$, define the r-tuple $\vec{Y}'$ of partitions by removing the last box from the last partition in $\vec{Y}$, i.e., we set $Y_\alpha' = Y_\alpha$ for $\alpha = 1, \ldots, r-1$ and $Y_r' = (Y_1, \ldots, Y_{l-1}, Y_l - 1)$, where l is the length of Y_r. In terms of Young diagrams, we go from $\vec{Y}$ to $\vec{Y}'$ by removing the box

$$(l, w) := (l(Y_r), Y_r(l))$$

from the last Young diagram Y_r in $\vec{Y}$. We will prove

$$\frac{\mathcal{Z}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{Z}_{\vec{Y}'}(\vec{u}; \emptyset)} = \frac{\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{R}_{\vec{Y}'}(\vec{u}; \emptyset)}. \tag{49}$$

This already suffices: Using Eq. (49), we can reduce the statement of the theorem to the case $\vec{Y} = (Y_1, \ldots, Y_{r-1}, \emptyset)$. Both $\mathcal{Z}_{\vec{Y}}(\vec{u}; \emptyset)$ and $\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset)$ are invariant under simultaneous permutation of the components of $\vec{Y} = (Y_1, \ldots, Y_r)$ and $u = (u_1, \ldots, u_r)$. This follows directly from the respective definitions. Hence, we can reduce the statement of the theorem to the case $\vec{Y} = (Y_1, \ldots, Y_{r-2}, \emptyset, Y_r)$ and, again using Eq. (49), to the case $\vec{Y} = (Y_1, \ldots, Y_{r-2}, \emptyset, \emptyset)$. Continuing in this fashion, we can reduce the statement to the case $\vec{Y} = (\emptyset, \ldots, \emptyset)$, in which it holds trivially.

In the calculation of both sides of Eq. (49) we have to evaluate telescopic products. In order to group the factors for such evaluations, we will have to keep track when $Y_\alpha(x)$ and $Y_\alpha^T(y)$ remain constant as we vary the row and column indices. We write

$$Y_\alpha = (Y_\alpha(1), \ldots, Y_\alpha(l(Y_\alpha))) = (\underbrace{F_\alpha(1), \ldots, F_\alpha(1)}_{G_\alpha(1)\text{times}}, \ldots, \underbrace{F_\alpha(m_\alpha), \ldots, F_\alpha(m_\alpha)}_{G_\alpha(m_\alpha)\text{times}}).$$

We set $F_\alpha(m_\alpha + 1) = 0$. Note that for any $x, y \in \mathbb{N}$,

$$Y_\alpha(y) \in \{F_\alpha(j) : j = 1, \ldots, m_\alpha + 1\}$$

$$Y_\alpha^T(x) \in \{G_\alpha(1) + \cdots + G_\alpha(j) : j = 0, \ldots, m_\alpha\}.$$

Define the index j_α by

$$G_\alpha(1) + \cdots + G_\alpha(j_\alpha - 1) < l \leq G_\alpha(1) + \cdots G_\alpha(j_\alpha)$$

when this condition can be satisfied and $j_\alpha = m_\alpha + 1$ otherwise. We also introduce the notation $H_\alpha(j) = G_\alpha(1) + \cdots + G_\alpha(j)$. We split products over rows of Young diagrams as

$$\prod_{x=1}^{l} = \prod_{j=1}^{j_\alpha-1} \prod_{x=H_\alpha(j-1)+1}^{H_\alpha(j)} \times \prod_{x=H_\alpha(j_\alpha-1)+1}^{l}. \tag{50}$$

When x comes from the product with index $j \in \{1, \ldots, j_\alpha - 1\}$, we have $Y_\alpha(y) = F_\alpha(j)$. In the remaining product, we have $Y_\alpha(y) = F_\alpha(j_\alpha)$. Products over columns of Young diagrams are grouped as

$$\prod_{y=1}^{Y_\alpha(l)} = \prod_{y=1}^{F_\alpha(j_\alpha)} = \prod_{j=j_\alpha}^{m_\alpha} \prod_{y=F_\alpha(j+1)+1}^{F_\alpha(j)} \tag{51}$$

where we have $Y_\alpha^T(y) = H_\alpha(j)$ if y comes from the factor with value j.

The right hand side of Eq. (49) equals

$$\frac{\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{R}_{\vec{Y}'}(\vec{u}; \emptyset)} = \prod_{\alpha,\beta=1}^{r} \frac{\prod_{s\in Y_\alpha'} 1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha'}(s)+1} q_2^{-a_{Y_\beta'}(s)} \prod_{t\in Y_\beta'} 1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta'}(t)} q_2^{a_{Y_\alpha'}(t)+1}}{\prod_{s\in Y_\alpha} 1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(s)+1} q_2^{-a_{Y_\beta}(s)} \prod_{t\in Y_\beta} 1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(t)} q_2^{a_{Y_\alpha}(t)+1}}.$$

We introduce a variable ξ to be able to ignore poles during the calculation. Regrouping we get

$$\frac{\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{R}_{\vec{Y}'}(\vec{u}; \emptyset)} = \lim_{\xi\to 1} S(\xi) \prod_{\alpha=1}^{r-1} T_\alpha(\xi) U_\alpha(\xi)$$

where

$$S(\xi) = \frac{1}{(\xi - q_1)(\xi - q_2)} \prod_{t\in Y_r'} \frac{(\xi - q_1^{l_{Y_r'}(t)+1} q_2^{-a_{Y_r'}(t)})(\xi - q_1^{-l_{Y_r'}(t)} q_2^{a_{Y_r'}(t)+1})}{(\xi - q_1^{l_{Y_r}(t)+1} q_2^{-a_{Y_r}(t)})(\xi - q_1^{-l_{Y_r}(t)} q_2^{a_{Y_r}(t)+1})}$$

$$T_\alpha(\xi) = \frac{1}{\xi - u_\alpha u_r^{-1} q_1 q_2^{a_{Y_\alpha}(l,w)+1}} \prod_{t\in Y_r} \frac{\xi - u_\alpha u_r^{-1} q_1^{-l_{Y_r'}(t)} q_2^{a_{Y_\alpha}(t)+1}}{\xi - u_\alpha u_r^{-1} q_1^{-l_{Y_r}(t)} q_2^{a_{Y_\alpha}(t)+1}}$$

$$\times \prod_{s\in Y_\alpha} \frac{\xi - u_\alpha u_r^{-1} q_1^{l_{Y_\alpha}(s)+1} q_2^{-a_{Y_r'}(s)}}{\xi - u_\alpha u_r^{-1} q_1^{l_{Y_\alpha}(s)+1} q_2^{-a_{Y_r}(s)}}$$

$$U_\alpha(\xi) = \frac{1}{\xi - u_r u_\alpha^{-1} q_2^{-a_{Y_\alpha}(l,w)}} \prod_{t\in Y_r} \frac{\xi - u_r u_\alpha^{-1} q_1^{l_{Y_r'}(t)+1} q_2^{-a_{Y_\alpha}(t)}}{\xi - u_r u_\alpha^{-1} q_1^{l_{Y_r}(t)+1} q_2^{-a_{Y_\alpha}(t)}}$$

$$\times \prod_{s\in Y_\alpha} \frac{\xi - u_r u_\alpha^{-1} q_1^{-l_{Y_\alpha}(s)} q_2^{a_{Y_r'}(s)+1}}{\xi - u_r u_\alpha^{-1} q_1^{-l_{Y_\alpha}(s)} q_2^{a_{Y_r}(s)+1}}.$$

Using

$$a_{Y_\alpha'}(x, y) = \begin{cases} Y_\alpha(x) - y - 1, & x = l, \alpha = r \\ Y_\alpha(x) - y, & \text{otherwise} \end{cases},$$

$$l_{Y_\alpha'}(x, y) = \begin{cases} Y_\alpha^T(y) - x - 1, & y = w, \alpha = r \\ Y_\alpha^T(y) - x, & \text{otherwise} \end{cases}$$

and the splitting described in Eqs. (50) and (51) we get

$$S(\xi) = \frac{1}{(\xi - q_1 q_2^{-w+1})(\xi - q_2^w)} \frac{(\xi - q_1)(\xi - q_2)}{(\xi - 1)(\xi - q_1 q_2)}$$

$$\prod_{j=1}^{m_r} \frac{(\xi - q_1^{l-H_r(j)} q_2^{-F_r(j)+w})(\xi - q_1^{-l+H_r(j)+1} q_2^{F_r(j)-w+1})}{(\xi - q_1^{l-H_r(j-1)} q_2^{-F_r(j)+w})(\xi - q_1^{-l+H_r(j-1)+1} q_2^{F_r(j)-w+1})},$$

$$T_\alpha(\xi) = \frac{1}{\xi - u_\alpha u_r^{-1} q_1^{l(Y_\alpha)-l+1} q_2^{-w+1}} \prod_{j=1}^{m_\alpha} \frac{\xi - u_\alpha u_r^{-1} q_1^{-l+H_\alpha(j)+1} q_2^{F_\alpha(j)-w+1}}{\xi - u_\alpha u_r^{-1} q_1^{-l+H_\alpha(j-1)+1} q_2^{F_\alpha(j)-w+1}},$$

$$U_\alpha(\xi) = \frac{1}{\xi - u_r u_\alpha^{-1} q_1^{-l(Y_\alpha)+l} q_2^{w}} \prod_{j=1}^{m_\alpha} \frac{\xi - u_r u_\alpha^{-1} q_1^{l-H_\alpha(j)} q_2^{-F_\alpha(j)+w}}{\xi - u_r u_\alpha^{-1} q_1^{l-H_\alpha(j-1)} q_2^{-F_\alpha(j)+w}}.$$

Together

$$\frac{\mathcal{R}_{\vec{Y}}(\vec{u};\emptyset)}{\mathcal{R}_{\vec{Y}'}(\vec{u};\emptyset)} = \lim_{\xi \to 1} \frac{(\xi - q_1)(\xi - q_2)}{(\xi - 1)(\xi - q_1 q_2)}$$

$$\prod_{\alpha=1}^{r} \left(\frac{1}{(\xi - u_\alpha u_r^{-1} q_1^{l(Y_\alpha)-l+1} q_2^{-w+1})(\xi - u_r u_\alpha^{-1} q_1^{-l(Y_\alpha)+l} q_2^{w})} \right.$$

$$\left. \times \prod_{j=1}^{m_\alpha} \frac{(\xi - u_r u_\alpha^{-1} q_1^{l-H_\alpha(j)} q_2^{-F_\alpha(j)+w})(\xi - u_\alpha u_r^{-1} q_1^{-l+H_\alpha(j)+1} q_2^{F_\alpha(j)-w+1})}{(\xi - u_r u_\alpha^{-1} q_1^{l-H_\alpha(j-1)} q_2^{-F_\alpha(j)+w})(\xi - u_\alpha u_r^{-1} q_1^{-l+H_\alpha(j-1)+1} q_2^{F_\alpha(j)-w+1})} \right).$$

For the residue calculation, fix the order of the variables such that the integration over z_n picks up the residue $z_{l,w}^{r}$ coming from the box $(l, w) \in Y_r$ we remove from the last partition in $\vec{Y}$ to get $\vec{Y}'$. The left hand side of Eq. (49) equals

$$\frac{\mathcal{Z}_{\vec{Y}}(\vec{u};\emptyset)}{\mathcal{Z}_{\vec{Y}'}(\vec{u};\emptyset)} = \frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \lim_{z_j \to \hat{z}_j, j=1,\dots,n} (1 - \frac{z_{l,w}^{r}}{z_n}) \frac{\mathcal{I}(z_1,\dots,z_n;\vec{u})}{\mathcal{I}(z_1,\dots,z_{n-1};\vec{u})}.$$

We take the first $(n-1)$ limits separately: The quotient $\frac{\mathcal{I}(z_1,\dots,z_n;\vec{u})}{\mathcal{I}(z_1,\dots,z_{n-1};\vec{u})}$ converges to

$$\prod_{\alpha=1}^{r} \left(\frac{-u_\alpha z_n}{(z_n - u_\alpha)(q_1 q_2 z_n - u_\alpha)} \right.$$

$$\left. \times \prod_{s \in Y_\alpha'} \frac{(z_n - z_s^\alpha)^2 (z_n - q_1 q_2 z_s^\alpha)(z_n - q_1^{-1} q_2^{-1} z_s^\alpha)}{(z_n - q_1 z_s^\alpha)(z_n - q_2 z_s^\alpha)(z_n - q_1^{-1} z_s^\alpha)(z_n - q_2^{-1} z_s^\alpha)} \right)$$

for $z_j \to \hat{z}_j, j = 1, \dots, n-1$. The factors with $\alpha \neq r$ do not have poles for $z_n \to z_{l,w}^{r}$ since $u_\alpha/u_r \notin \{q_1^x q_2^y : x, y \in \mathbb{Z}\}$. We define $\xi = \frac{z_n}{z_{l,w}^{r}}$ and set

$$A_\alpha(\xi) := \prod_{s \in Y_\alpha} \frac{(\xi - \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - q_1 q_2 \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - q_1^{-1} q_2^{-1} \frac{z_s^\alpha}{z_{l,w}^{r}})}{(\xi - q_1 \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - q_2 \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - q_1^{-1} \frac{z_s^\alpha}{z_{l,w}^{r}})(\xi - q_2^{-1} \frac{z_s^\alpha}{z_{l,w}^{r}})}.$$

Hence, the remaining limit $z_n \to z_{l,w}^{r}$ is given by

$$\frac{\mathcal{Z}_{\vec{Y}}(\vec{u};\emptyset)}{\mathcal{Z}_{\vec{Y}'}(\vec{u};\emptyset)} = \lim_{\xi \to 1} \frac{(\xi - q_1^{-1})(\xi - q_2^{-1})}{(\xi - 1)(\xi - q_1^{-1} q_2^{-1})} \prod_{\alpha=1}^{r} \left(\frac{-\xi u_\alpha z_{l,w}^{r}}{(\xi z_{l,w}^{r} - u_\alpha)(q_1 q_2 \xi z_{l,w}^{r} - u_\alpha)} A_\alpha(\xi) \right).$$

Using the splitting described in (50) and (51) we get

$$A_\alpha(\xi) = \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{1-l} q_2^{1-w})(\xi - \frac{u_\alpha}{u_r} q_1^{-l} q_2^{-w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l+1} q_2^{1-w})(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l} q_2^{-w})}$$

$$\times \prod_{j=1}^{m_\alpha} \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l} q_2^{F_\alpha(j)-w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l} q_2^{F_\alpha(j)-w})}.$$

Finally, we use $(\rho - \sigma) = -\rho\sigma(\rho^{-1} - \sigma^{-1})$ repeatedly to conclude

$$
\frac{\mathcal{Z}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{Z}_{\vec{Y}'}(\vec{u}; \emptyset)} = \lim_{\xi \to 1} \frac{(\xi - q_1^{-1})(\xi - q_2^{-1})}{(\xi - 1)(\xi - q_1^{-1}q_2^{-1})}
$$

$$
\prod_{\alpha=1}^{r} \left(\frac{-\xi u_\alpha u_r^{-1} q_1^{-l} q_2^{-w}}{(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l+1} q_2^{1-w})(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l} q_2^{-w})} \right.
$$

$$
\left. \times \prod_{j=1}^{m_\alpha} \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l} q_2^{F_\alpha(j)-w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l} q_2^{F_\alpha(j)-w})} \right)
$$

$$
= \lim_{\xi \to 1} \frac{(\xi^{-1} - q_1)(\xi^{-1} - q_2)}{(\xi^{-1} - 1)(\xi^{-1} - q_1 q_2)}
$$

$$
\prod_{\alpha=1}^{r} \left(\frac{1}{(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l+1} q_2^{1-w})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-l(Y_\alpha)+l} q_2^{w})} \right.
$$

$$
\left. \times \prod_{j=1}^{m_\alpha} \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l+1} q_2^{F_\alpha(j)-w+1})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-H_\alpha(j)+l} q_2^{-F_\alpha(j)+w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l+1} q_2^{F_\alpha(j)-w+1})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-H_\alpha(j-1)+l} q_2^{-F_\alpha(j)+w})} \right)
$$

$$
= \frac{\mathcal{R}_{\vec{Y}}(\vec{u}; \emptyset)}{\mathcal{R}_{\vec{Y}'}(\vec{u}; \emptyset)}
$$

since no factor $(\xi^{+1} - \cdots)$ vanishes in the limit.

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Communicated by N. Nekrasov