Continuous Spectrum or Measurable Reducibility for Quasiperiodic Cocycles in $\mathbb{T}^d \times SU(2)$

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Abstract: We continue our study of the local theory for quasiperiodic cocycles in $\mathbb{T}^d \times G$, where $G = SU(2)$, over a rotation satisfying a Diophantine condition and satisfying a closeness-to-constants condition, by proving a dichotomy between measurable reducibility (and therefore pure point spectrum), and purely continuous spectrum in the space orthogonal to $L^2(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d \times G)$. Subsequently, we describe the equivalence classes of cocycles under smooth conjugacy, as a function of the parameters defining their K.A.M. normal form. Finally, we derive a complete classification of the dynamics of one-frequency ($d = 1$) cocycles over a recurrent Diophantine rotation.

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1. Introduction and Statement of the Results

1.1. Introduction. This article continues the work taken up by the author in [Kar16, Kar17, Kar14]. The present work used and developed the techniques of [Kri99a, Kri01, Eli02, Fra04] among others, and was motivated by [dA13], where conditions for the existence of absolutely continuous spectrum are given.

We study smooth quasi-periodic cocycles in $\mathbb{T}^d \times SU(2)$, i.e., dynamical systems of the form

$$\begin{aligned} (\alpha, A(\cdot)) : \mathbb{T}^d \times SU(2) &\rightarrow \mathbb{T}^d \times SU(2) \\ (x, S) &\mapsto (x + \alpha, A(x).S) \end{aligned}$$

where α is a minimal translation of $\mathbb{T}^d \equiv \mathbb{R}^d / \mathbb{Z}^d$ and $A(\cdot) : \mathbb{T}^d \rightarrow SU(2)$ is of class C^∞ with values in the special unitary group of 2×2 matrices. The dynamics preserves the product of the Lebesgue measure in $\mathbb{T}^d$ with the Haar measure in $SU(2)$, and the product space $\mathbb{T}^d \times SU(2)$ is furnished with this measure. The space of such dynamical systems, with α fixed, will be denoted by $SW_\alpha(\mathbb{T}^d, SU(2))$ (cf. Sect. 3.4.1). We are interested in the spectral properties of such systems and in the topology of equivalence classes up to fibered conjugation.

A particular class can be readily distinguished: when $A(\cdot) \equiv A \in SU(2)$ is a constant mapping, then the phase space is foliated into invariant tori $\mathbb{T}^d \times \mathbb{S}_A^1$ ($\mathbb{S}_A^1 \hookrightarrow SU(2)$ stands for the circle of matrices that are simultaneously diagonalizable with A), and the fibration is precisely the Hopf fibration of S^3 . The dynamics in the invariant tori are quasiperiodic, and such a mapping has discrete spectrum.

The relevant notion of conjugation is fibered conjugation, given by the following action

$$Conj_{B(\cdot)}(\alpha, A(\cdot)) = (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot))$$

where we act by translation within each fiber, and the translation in general depends on the fiber.

A cocycle conjugate to a constant one via a measurable transfer function $B(\cdot)$ is called (*measurably*) *reducible*, and obviously has discrete spectrum. In the present article we investigate the dynamical properties of the complement of this class in the particular case of the cocycles whose frequency α satisfies a Diophantine condition (see Definition 3.1; Diophantine numbers are badly approximated by rationals) and where the cocycle is C^∞ -close to a constant one. These two properties define the *K.A.M. regime* in the space of cocycles.

A cocycle in the K.A.M. regime can be written in the form $(\alpha, A.e^{F(\cdot)})$, where $F(\cdot)$, of class C^∞ , takes its values in the Lie algebra $su(2)$ and it is sufficiently close to the 0 mapping, and $A \in SU(2)$ is a constant. The works of Eliasson [Eli01] and Krikorian [Kri99a] established the convergence of the K.A.M. scheme. The scheme consists in constructing a conjugation acting on

$$(\alpha, A.e^{F(\cdot)}) = (\alpha, A_1.e^{F_1(\cdot)})$$

in order to conjugate it to $(\alpha, A_2.e^{F_2(\cdot)})$ with $F_2 \ll F_1$ and then iterating the construction. Iteration becomes possible under the conditions defining the K.A.M. regime and this proves that every cocycle therein is *almost reducible*, i.e., can be conjugated arbitrarily close to constant ones. The product of conjugations is however expected to diverge “generically” since in each iteration the construction is actually broken down into two steps, the second one of which causes divergence.

Suppose that the scheme has been iterated $n - 1$ times and has produced the cocycle $(\alpha, A_n.e^{F_n(\cdot)})$. The first step is sufficient for the n th iteration to be complete when it so happens that the constant cocycle (α, A_n) is sufficiently Diophantine (cf. Definition 3.3). The step produces a conjugation whose distance to the identity is comparable with F_n . The constant cocycle $(\alpha, A_n) = (\alpha, \{e^{2i\pi a_n}, 0\})$ is Diophantine if a_n is sufficiently away from $\{k\alpha\}_{k \in \mathbb{Z}}$. Under this assumption, we actually obtain that

$$A_{n+1} = A_n \exp \left(\int_{\mathbb{T}^d} F_n(\cdot) \right) = \begin{pmatrix} e^{2i\pi a_n} & 0 \\ 0 & e^{-2i\pi a_n} \end{pmatrix} \cdot \exp \left(\int_{\mathbb{T}^d} F_n(\cdot) \right)$$

is very close to A_n . Since, however, the couples (α, A_n) that are not sufficiently Diophantine are “generic” (they are called *resonant*), the first step may not work as described. The cocycle is resonant if there exists a $k_n \in \mathbb{Z}$, not very big, such that $a_n = k_n\alpha + \epsilon_n$, with ϵ_n very small. Then, the first step actually produces a cocycle $(\alpha, A'_n.e^{F_n^{res}(\cdot)}.e^{F'_{n+1}(\cdot)})$ with a particular structure, since for the first-order perturbation term $F_n^{res}(\cdot)$ only the *resonant mode* k_n is activated. The cocycle $(\alpha, A'_n.e^{F_n^{res}(\cdot)})$ is in fact reducible via a far-from-the-identity conjugation, and the procedure of reducing such a cocycle is called *reduction of a resonant mode*. When this reduction is applied as a second step, the n th iteration is completed and produces a cocycle

$$(\alpha, A_{n+1}.e^{F_{n+1}(\cdot)})$$

that satisfies the desired smallness condition, and the additional property that A_{n+1} is ϵ_n -close to the Id, a lot closer than A_n is. Therefore, when reduction of a resonant mode takes place, the constant around which we will linearize in the next iteration changes substantially, and the circle $\mathbb{S}_{A_{n+1}}^1 = \mathbb{S}_{n+1}^1$ may point to a very different direction than $\mathbb{S}_n^1$. Since the reduction of resonances is quite rare, we label the iterations when it occurs by n_i and conserve the index i for this notation.

Under the standing assumptions, the above procedure can be iterated into a K.A.M. scheme. The scheme may (or may not) produce a converging product of conjugations, but in the case of quasi-periodic cocycles in compact groups it is always possible to extract, even though in a non-canonical way, a converging product in order to obtain the K.A.M. normal form (cf. Sect. 3.8). A cocycle in normal form has the useful property that when the K.A.M. scheme is applied to it, the first step of each iteration is always void: constants are always resonant. In the absence of resonances the circles $\mathbb{S}_n^1$ and $\mathbb{S}_{n+1}^1$ form a very small angle, which is insignificant for the dynamics. Thus, all the important dynamical information is contained in the second step of each K.A.M. iteration, and the normal form allows us to assume that only reduction of resonant modes occurs. When the cocycle is in this normal form, only the indexes n_i of resonant steps are relevant.

A system in normal form is characterized by the fact that it is a perturbation of a constant system (α, A_1) whose eigenvalues are very close to a multiple of α , and, at least to the first order, the perturbation $F_1(\cdot)$ is transversal to $\mathbb{S}_1^1$, and has only one Fourier coefficient (cf. Fig. 1). Such a perturbation can be reduced to a second-order one by a far-from-the-identity conjugation with values in $\mathbb{S}_1^1$. The cocycle is in normal form if, after such a change of coordinates, it is a perturbation of a constant system (α, A_2) with the same properties (for A_2 as well as for the perturbation), and this goes on either until the perturbation is 0 or ad infinitum. The constants A_i thus constructed are indexed by the “resonant” steps of the K.A.M. scheme n_i , i.e. those where close-to-the-identity conjugations are not sufficient for the iteration to continue.

Fig. 1. The n th step of the K.A.M. scheme

The normal forms are actually parametrized by the space formed by the data $\{(k_i, \epsilon_i, \phi_i, \theta_i)\}_{i \in \mathbb{N}^*}$. We denote by k_i the resonant frequency, by ϵ_i the distance of the argument of A_i from the exact resonance, by ϕ_i the argument of the resonant mode of the perturbation $\hat{F}_i(k_i) \in \mathbb{C}$, and by θ_i the angle between $\mathbb{S}_i^1$ and $\mathbb{S}_{i+1}^1$ (cf. Fig. 1).

The fact that the asymptotic relative positions of the circles $\mathbb{S}_i^1$ code in some sense the dynamics of the given cocycle was understood by Eliasson [Eli01]: in an appropriate system of coordinates and for long iterations of the cocycle, the dynamics look like those of (α, A_n) , and are therefore sufficiently isomorphic to quasi-periodic dynamics in $\mathbb{T}^d \times \mathbb{S}_n^1$. When a resonance appears at the iteration $n = n_i$ and we wish to iterate for longer times or to increase the precision with which we calculate the iterates, we need to perform a far-from-the-identity conjugation (which, nonetheless, preserves $\mathbb{T}^d \times \mathbb{S}_{n_i}^1$), and then follow the dynamics by iterating (α, A_{n_i+1}) . Thus, if the angle θ_i is significant, the dynamics are no longer trapped in a neighborhood of $\mathbb{S}_{n_i}^1$.

This observation and the resulting study that proved genericity of Unique Ergodicity were pushed further in our recent work [Kar14].

1.2. Statement of the results. We think that the K.A.M.-theoretical part of the theory of quasiperiodic cocycles in $\mathbb{T}^d \times SU(2)$ from the topological point of view is concluded with the following theorem and with the classification and path connectedness theorems that we state later on. On the other hand, the metric abundance (or *prevalence*) of reducible cocycles in the C^∞ category is an open question (see [Kri99b] for the proof of the full-measure reducibility theorem in the analytic category).

Theorem 1.1. *Let $\alpha \in \mathbb{T}^d$ satisfy a Diophantine condition $DC(\gamma, \tau)$. Then, there exist $\varepsilon > 0$ and $s_0 \in \mathbb{N}^*$, depending on d, γ, τ , such that every cocycle $(\alpha, A.e^{F(\cdot)})$ with $A \in G = SU(2)$, and $F : \mathbb{T}^d \rightarrow su(2)$, C^∞ smooth, such that $\|F\|_0 < \varepsilon$ (i.e., F is small in the C^0 topology) and $\|F\|_{s_0} < 1$ (i.e., the first s_0 derivatives of F are smaller than 1) satisfies the following dichotomy:*

1. *either the cocycle is measurably reducible and therefore the associated Koopman operator has pure-point spectrum,*
2. *or the spectrum of the Koopman operator is purely continuous in the space orthogonal to $L^2(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d \times G, \mathbb{C})$. Such cocycles are weak mixing in the fibers, and the mixing is not strong.*

The norms appearing in the statement are defined formally in Sect. 3.2. Using purely local notation, if T is a measure-preserving mapping of the space (X, μ) , then the Koopman operator associated to T is the unitary operator of $L^2(X)$ given by

$$U_T \cdot \phi \equiv U \cdot \phi = \phi \circ T^{-1}, \phi \in L^2$$

(see [KT06]). Let us also establish some notation that will be of use in the rest of the paper. For the precise definition of the norms, cf. Sect. 3.2.

Notation 1.1. *The neighborhood of constant cocycles described in the statement of the theorem will be referred to as the K.A.M. or the local regime and denoted by*

$$\mathcal{N} = \{(\alpha, A.e^{F(\cdot)}), A \in G, \|F\|_0 < \varepsilon, \|F\|_{s_0} < 1\}.$$

The name “K.A.M. regime” is justified by the fact that such a condition makes the K.A.M. scheme applicable. We now give the following (ad hoc and slightly vague) definition of close-to-the-Id conjugations.

Notation 1.2. *Conjugations of the size of those produced by the K.A.M. scheme without any resonances or of those who reduce a given cocycle to its K.A.M. normal form will be referred to as close-to-the-identity conjugations and will be denoted by $\mathcal{V}$.*

Such conjugations satisfy a condition of the type $\|Y\|_0 < C$ and $\|Y\|_{s_0-\xi\gamma} < 1$, for some constants $\varepsilon \leq C < 1$ and $1 \leq \xi < s_0/\gamma$ depending on γ, τ and d . Close-to-the-identity conjugations form a contractible set in $C^\infty(\mathbb{T}^d, G)$.

We make the following remarks on the statement of Theorem 1.1, assuming that the cocycle is in normal form. Firstly, measurable reducibility, as in item 1, is equivalent to the angles θ_i between the circles $\mathbb{S}_i^1 = \mathbb{S}_{n_i}^1$ and $\mathbb{S}_{i+1}^1$ being square-summable. This is an F_σ -dense condition of empty interior in $\mathcal{N}$. The question of differentiable rigidity of measurable reducibility was investigated in [Kar17], where we proved that measurable reducibility to a full measure set of constants $DC_\alpha \subset G$ implies in fact smooth reducibility (see item 1a in Sect. 2 and Definition 3.3). On the other hand, in [Kar14] it was shown that for constants in the generic set $\mathcal{L}_\alpha = G \setminus DC_\alpha$, reducibility is not rigid.

Secondly, the non-existence of a measurable conjugation reducing the cocycle to a constant, as in item 2, is of course the negation of the condition implying item 1, and thus equally explicit. It is a G_δ -dense condition in $\mathcal{N}$. In the space orthogonal to the one bearing the purely continuous spectrum (i.e. in $L^2(\mathbb{T}^d)$, the Kronecker factor of the cocycle), the spectrum is pure-point, because of the quasiperiodic dynamics within the torus. The Proof of Theorem 1.1 is based on the use of the K.A.M. normal form, in order to prove that the existence of an eigenfunction of the Koopman operator associated to a given cocycle implies the condition for measurable reducibility, provided that the

eigenfunction depends non-trivially on the variable in the fibers. The K.A.M. normal form, though not essential to the proof, greatly simplifies the calculations and elucidates the geometry of the problem. The existence of the K.A.M. normal form is a corollary of the almost reducibility theorem, first obtained in [Eli02], and its function in our study, put informally, is to separate the close-to-the-identity part of the conjugation (the one in $\mathcal{V}$) from the far-from-the-identity part, both constructed almost exactly as in [Eli02], and to keep the latter part, which contains all the interesting information. It allows us to reduce our study to systems of a very particular form (and of “infinite codimension” in the total space).

The purpose of the subsequent analysis is to establish that the necessary and sufficient condition for measurable reducibility is also a necessary and sufficient one for the existence of an eigenfunction of the Koopman operator. Put informally, θ_i estimates the commutator appearing in the following sequence of operations: solve a linearized equation in the coordinates of the i th step of the scheme, make one step of scheme, solve the same equation with greater precision, undo the change of coordinates and check the compatibility of the expressions. If this commutator is small, the expressions are compatible and the algorithm converges. If not, then it diverges and no solution to the equation exists. We remark that cohomology in $C^\infty(\mathbb{T}^d, G)$ over the rotation $x \mapsto x + \alpha$ seems to demand only one step of the scheme (see the proof of thms 1.1 and 1.2 in [Kar14]) for the estimate to be effective, while cohomology in C^∞ or in $L^2(\mathbb{T}^d \times G, \mathbb{C})$ over any given cocycle seems to demand two steps of the scheme (see e.g. the proof of item 2 of Theorem 1.1).

The proof that cocycles with continuous spectrum are not strong mixing uses the fact that a dynamical system is strong mixing iff the iterates of the associated Koopman operator converge to 0 in the weak operator topology (cf. prop. 2.8 in [KT06]). This property is incompatible with *rigidity*, the following strong recurrence property. A diffeomorphism f of a smooth compact manifold M is said to be C^∞ -*rigid* iff there exists a sequence of iterates $m_j \rightarrow \infty$ such that

$$f^{m_j} \xrightarrow{C^\infty} \text{Id}.$$

In Proposition 4.3 we prove this property for each cocycle $(\alpha, A(\cdot)) \in \mathcal{N}$, i.e., that for some subsequence of iterates

$$(\alpha, A(\cdot))^{m_j} \xrightarrow{C^\infty} (0, \text{Id}).$$

The following theorems concern the topology of the different conjugacy classes, and the way they lie in $\mathcal{N}$.

Theorem 1.2. *Given any cocycle $(\alpha, A'(\cdot)) \in \mathcal{N}$, one can construct a continuous path $[0, 1] \rightarrow \mathcal{N}$ such that $(\alpha, A_0(\cdot)) = (\alpha, \text{Id})$, for all $t \in [0, 1)$ the cocycle $(\alpha, A_t(\cdot))$ is C^∞ reducible, and $(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$.*

The path is in fact piecewise C^∞ . If we allow the path to exit the K.A.M. regime into the total space of cocycles, $SW_\alpha^\infty(\mathbb{T}^d, G)$, we can obtain more.

Theorem 1.3. *Given any cocycle $(\alpha, A'(\cdot)) \in \mathcal{N}$, one can construct a continuous path $[0, 1] \rightarrow SW_\alpha^\infty(\mathbb{T}^d, G)$ such that $(\alpha, A_0(\cdot)) = (\alpha, \text{Id})$, for all $t \in [0, 1)$ the cocycle $(\alpha, A_t(\cdot))$ is C^∞ conjugate to (α, Id) , and $(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$.*

The path $\gamma : [0, 1) \rightarrow C^\infty(\mathbb{T}^d, G)$ acting on (α, Id) and producing

$$(\alpha, A_t(\cdot)) = \text{Conj}_{\gamma(t)}(\alpha, \text{Id})$$

is continuous (in fact piecewise C^∞), and degenerates in a prescribed way when the target cocycle is not C^∞ reducible. At time $t = 1^-$ it may exit C^∞ into a function space of lower regularity, or a space of distributions.¹ The path in the conjugacy space may exit $\mathcal{V}$, and in general will do so. The path in SW_α^∞ will consequently exit $\mathcal{N}$ and reenter, in general an infinite number of times. Since the space of conjugations taking cocycles to their respective normal forms is contractible (it is the set $\mathcal{V}$), we immediately obtain the following corollary.

Corollary 1.4. *The target of the path can be the cocycle $(\alpha, A'(\cdot))$ itself, while the same as in Theorem 1.2 (resp. Theorem 1.3) holds for all times $t \in [0, 1)$.*

We can in fact obtain the following, stronger theorem, which establishes a way in which the topology of any two classes share some properties.

Theorem 1.5. *Given any two cocycles $(\alpha, A_0(\cdot))$ and $(\alpha, A'(\cdot))$, in $\mathcal{N}$, one can construct a continuous path $[0, 1] \rightarrow \mathcal{N}$, and such that:*

1. *for every $t \in [0, 1)$ the normal form of the cocycle $(\alpha, A_t(\cdot))$ has the same tail as that of $(\alpha, A_0(\cdot))$, and thus the same dynamical properties.*
2. *$(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$, and thus conjugate to it via a conjugation in $\mathcal{V}$.*

As before, the target cocycle can be the cocycle $(\alpha, A'(\cdot))$.

The first item of the previous theorem states that the two K.A.M. normal forms coincide, save for a finite number of terms. Since the dynamical properties of a cocycle depend on the asymptotic properties of the normal form, the dynamics of the two cocycles are essentially the same.

Informally, we can connect any two types of dynamical behavior without exiting $\mathcal{N}$. The following theorem says that if we allow the path to exit $\mathcal{N}$, we can do the same thing with conjugacy classes.

Theorem 1.6. *Given any two cocycles $(\alpha, A_0(\cdot))$ and $(\alpha, A'(\cdot))$, in $\mathcal{N}$, one can construct a continuous path $[0, 1] \rightarrow SW_\alpha^\infty$ such that:*

1. *for every $t \in [0, 1)$, the cocycle $(\alpha, A_t(\cdot))$ is conjugate to $(\alpha, A_0(\cdot))$.*
2. *$(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$, and thus conjugate to it via a conjugation in $\mathcal{V}$.*

As before, the target cocycle can be the cocycle $(\alpha, A'(\cdot))$.

Again, a path in the space of conjugations acting on $(\alpha, A_0(\cdot))$ is constructed with the same properties as in Theorem 1.3.

These two last theorems illustrate the necessity of a transversality condition for obtaining a full-measure reducibility theorem for one-parameter families of cocycles as in [Kri99b]. On the other hand, the K.A.M. normal form should be expected to depend badly on parameters along a generic family, so our approach is not expected to be well adapted to the metric point of view.

Finally, we give a satisfactory classification of conjugation classes with the following, more technical theorem. The sequence N_n is a parameter of the K.A.M. scheme and is a rapidly increasing sequence in $\mathbb{N}$, and N_i stands for N_{n_i} .

¹ In fact the limit is always well defined in $H^{-d/2}(\mathbb{T}^d, G)$, but we ignore whether this has any dynamical content.

Theorem 1.7. *Two cocycles $(\alpha, A_j(\cdot))$, $j = 1, 2$, in K.A.M. normal form are C^∞ conjugate iff the parameters defining the normal forms satisfy the following properties.*

1. *The resonant steps n_i^j , $j = 1, 2$, are the same, except for a finite number.*
2. *The angles between the successive circles $\mathbb{S}_{i,j}^1$ and $\mathbb{S}_{i+1,j}^1$*

$$\theta_i^j = \arctan \frac{|\hat{F}_i^j(k_i^j)|}{|\epsilon_i^j|}, \quad j = 1, 2$$

are equal up to $O(N_{i-1}^{-\infty})$.

3. *The arguments of $\hat{F}_i^j(k_i^j) \in \mathbb{C} \setminus \{0\}$, φ_i , are equal up to $O(N_{i-1}^{-\infty})$.*

Thus, the cocycles in the K.A.M. regime are parametrized by the action of conjugations in $\mathcal{V}$ composed with constant ones on the right, and the parameters θ_i , k_i and φ_i , that have nonetheless to respect the limitations of a K.A.M. scheme with given parameters.

Classification up to H^σ -smooth conjugation is obtained by replacing the $O(N_i^{-\infty})$ by the respective $O(N_i^{-\sigma})$ ones.

We observe that, by [Kar17], every representative of a constant cocycle (α, A_d) with $A_d \in DC_\alpha$ has a finite normal form, modulo the action of conjugations in $\mathcal{V}$, and are therefore defined by a finite number of parameters in the parameter space. Therefore, in a certain sense, the orbits of Liouvillean cocycles in the local regime, even under C^∞ conjugations, are bigger than the orbits of Diophantine ones, since they have representatives in the parameter space that are not finitely determined for any choice of parameters for the K.A.M. scheme.

In order to make this fact precise, we introduce the space $\mathcal{P}$, which is obtained by reducing the space of normal forms. It can be seen (cf. Sect. 5.1) that the arguments φ_i of the resonant mode at each step n_i , and jointly $\{\epsilon_i, \theta_i\}$ are moduli. However, the argument of the resonant constant at the step n_{i+1} , A_{i+1} , given by

$$a_{i+1} = (\epsilon_i^2 + |\hat{F}_i(k_i)|^2)^{1/2}$$

is not a modulus of the normal form.

Notation 1.3. *We call $\mathcal{P}$ the reduced parameter space of the normal forms, formed by the data $\{(k_i, a_{i+1})\}_{i \in \mathbb{N}^*}$.*

Loosely speaking, the space $\mathcal{P}$ is the space of normal forms quotiented by the action of $\mathcal{V}$ on the space of normal forms.

Corollary 1.8. *The orbit of each constant cocycle (α, A_d) with $A_d \in DC_\alpha$ has countably many representatives in $\mathcal{P}$. The orbit of each constant cocycle (α, A_l) with $A_l \in \mathcal{L}_\alpha$ has uncountably many representatives in the same space.*

This corollary shows in fact that the orbit of a Liouville constant exits the K.A.M. regime and re-enters many more times than the orbit of a Diophantine one, which is constrained by differentiable rigidity, see item 1a in the classification (Sect. 2), to leave to infinity as the norms of conjugations grow. We also obtain the following corollary.

Corollary 1.9. *Every conjugacy class is dense and totally disconnected in $\mathcal{P}$.*

The density could be viewed as an infinite-dimensional analogue of the density of $x + \beta\mathbb{Z} \bmod 1$ in $[0, 1]$, for each $x \in [0, 1]$ and for $\beta \in \mathbb{R} \setminus \mathbb{Q}$ fixed, though the analogy is quite loose.

We topologize the parameter space in a way that is compatible with the mapping of the parameters into a space of C^∞ functions, i.e., closeness means $O(N_{n_i}^{-\infty})$ -closeness, and the h^s norms use $N_{n_i}^s$ as weights. The formal definition would be tedious and we omit it.

The infinite dimensionality comes from the infinite number of significant “rotation numbers” a_i at each step of the K.A.M. scheme. This theorem and its corollary show that there should be no reasonable way of defining a fibered rotation number for non-reducible cocycles, as we can for $SL(2, \mathbb{R})$ cocycles, see [Her83, JM82].²

Theorem 1.10. *Every conjugacy class is dense in $\mathcal{N}$. Total disconnectedness is lost because of the action of conjugations in $\mathcal{V}$. Conjugacy classes are not locally connected around any point.*

The proof of these last results implies the next theorem, which illustrates at what point all classes and all dynamical behaviours are indistinguishable, at least before having iterated the dynamical system an infinite number of times.

Theorem 1.11. *Given any cocycle $(\alpha, A(\cdot)) \in \mathcal{N}$ and every conjugacy class $\mathcal{C}$ represented in $\mathcal{N}$, the cocycle $(\alpha, A(\cdot))$ is almost conjugate to $\mathcal{C}$.*

The precise definition of *almost conjugation*, a generalization of almost reducibility, is given in Definition 3.7, and roughly says that every cocycle can be conjugated arbitrarily close to any other one. The theorem is in fact slightly stronger than a corollary of the almost reducibility theorem, and we stress it since the analysis of the K.A.M. normal form shows that, in fact, any class of cocycles can serve as the linear model, admittedly using as a basis the class of constant cocycles.

2. Classification

In order to avoid stating a large number of theorems whose proofs are outside the scope of the present article, we present the following classification of the K.A.M regime in the form of a list, hoping that the reader will find it useful. The classification combines the results obtained in the literature [Kri99a, Eli02, Kar17, Kar14, Kar16] with Theorem 1.1.

Firstly, we remind that every cocycle in the K.A.M. regime is almost reducible, i.e. can be conjugated arbitrarily close to constant ones [Kri99a, Eli02, Kar14]. Moreover, every cocycle can be conjugated to its normal form, which is an infinite (in general) product of resonant perturbations.

1. If the sequence of angles θ_i between $\mathbb{S}_{A_i}^1$ and $\mathbb{S}_{A_{i+1}}^1$ is summable in ℓ^2 , then the cocycle is measurably reducible, and such a conjugation can be constructed as a sequence of C^∞ conjugations converging in L^2 . Such cocycles have discrete spectrum, and this condition is an F_σ -dense condition in the K.A.M. regime.

² Actually, the series $\sum k_i \alpha \bmod \mathbb{Z}$ converges to, say, a_∞ and it is possible to conjugate the cocycle arbitrarily close to (α, A_∞) by applying conjugations as in item 3 of Sect. 5.1. However, a_∞ is well-defined only in $\mathbb{T}^1 \bmod \mathbb{Z}^d \cdot \alpha$, which is a pathological space.

- (a) There exists a full measure set $DC_\alpha \subset SU(2)$ (the set of matrices whose eigenvalues are Diophantine with respect to α), such that if a cocycle is measurable reducible to a system (α, A_d) with $A_d \in DC_\alpha$, then it is actually smoothly reducible to it. This property is known as differentiable rigidity, (see [Kar17]). This condition is equally an F_σ -dense condition with empty interior.
 - (b) If the system is measurably conjugate to $(\alpha, A_\mathcal{L})$, with $A_\mathcal{L} \in \mathcal{L}_\alpha$, the complement of DC_α (the set of matrices Liouvillean with respect to α), then the transfer function may be of any regularity H^s , $0 \leq s \leq \infty$. For each such s , this condition is equally an F_σ -dense condition with empty interior.
2. All cocycles that are not measurably reducible are weak mixing in the fibers, and therefore uniquely ergodic in the whole space for the product of the Haar measures. Non-reducibility is a G_δ -dense condition.
- (a) There exists a G_δ -dense set of cocycles such that for a cocycle Φ in this set, we have

$$\{\psi \circ \Phi - \psi, \psi \in C^\infty(\mathbb{T}^d \times SU(2))\}^{cl_{C^\infty}} = C_0^\infty(\mathbb{T}^d \times SU(2))$$

This property is known as Distributional Unique Ergodicity, see [Kar14]. It is equivalent to an explicit condition on the asymptotic repartition of the angles between $\mathbb{S}_{A_i}^1$ and $\mathbb{S}_{A_{i+1}}^1$.

- (b) When the condition of the previous item degenerates in a controlled way,

$$\text{codim}\{\psi \circ \Phi - \psi, \psi \in C^\infty(\mathbb{T}^d \times SU(2))\}^{cl_{C^\infty}} = \infty$$

This condition is F_σ -dense with empty interior.

- (a) When the condition degenerates totally, we only have

$$\{\psi \circ \Phi - \psi, \psi \in C^\infty(\mathbb{T}^d \times SU(2))\}^{cl_{L^2}} = L_0^2(\mathbb{T}^d \times SU(2))$$

3. The existence of a conjugation between any two given cocycles can be determined by the parameters defining their respective K.A.M. normal forms
4. One can construct a continuous path $[0, 1] \rightarrow SW_\alpha^\infty$ such that $(\alpha, A_0(\cdot)) = (\alpha, \text{Id})$, for all $t \in [0, 1)$ the cocycle $(\alpha, A_t(\cdot))$ is C^∞ reducible, and for $t = 1$ it belongs to any given conjugacy class with a representative in the local regime.
5. Given any two cocycles $(\alpha, A_0(\cdot))$ and $(\alpha, A_1(\cdot))$, in the K.A.M. regime, one can construct a continuous path $[0, 1] \rightarrow SW_\alpha^\infty$ connecting the two cocycles such that:
 - (a) The path stays within the K.A.M. regime and for $t \in [0, 1)$ the cocycle $(\alpha, A_t(\cdot))$ has the same dynamical properties as $(\alpha, A_0(\cdot))$
 - (b) The path may exit the K.A.M. scheme and for $t \in [0, 1)$ the cocycle $(\alpha, A_t(\cdot))$ is conjugate to $(\alpha, A_0(\cdot))$
6. In the parameter space of K.A.M. normal forms, every conjugacy class is dense and totally disconnected.
7. Every conjugacy class is dense in the K.A.M. regime.

All of the above theorems hold for any fixed number of frequencies $d \in \mathbb{N}^*$ and a bigger d only results in a smaller neighborhood of constants where they hold true, due to Sobolev injection theorems. If, now, we restrict ourselves to the one-frequency case ($d = 1$), we can use the powerful tool of renormalization ([Kri01, AK06, FK09], see also [Kar16]) which we can combine with the work of [Fra00, Fra04, dA13] and obtain the following picture, which fills the total space $SW_\alpha^\infty(\mathbb{T}, G)$, provided that $\alpha \in RDC$ (this is a full measure arithmetic condition (cf. Definition 3.2)).

1. The picture described above holds true in an open dense subset of the total space $SW_\alpha^\infty(\mathbb{T}, SU(2))$, provided that $\alpha \in RDC$ (we stress that $d = 1$ in the global theorems).
2. The rest of the total space $SW_\alpha^\infty(\mathbb{T}, SU(2))$, $\alpha \in RDC$, is filled up by the conjugacy classes of periodic geodesics of $SU(2)$ of degree $r \in \mathbb{N}^*$. Such cocycles
 - (a) form a countable union of immersed Fréchet manifolds of codimension $2r$.
 - (b) have purely absolutely continuous spectrum when restricted in appropriate subspaces of $L^2(\mathbb{T} \times SU(2))$.

Finally, we think that a picture similar to the one above should hold when $SU(2)$ is replaced by any semisimple compact Lie group (cf. [Kri99a, Kar16]), at least in the K.A.M. regime. The phenomena observed in the more general case should consist of combinations and interactions between different behaviors observed in $SU(2)$, respecting the conditions of linear dependence and (non-)commutativity between the different root spaces. However, in the neighborhood of singular geodesics interesting phenomena may appear, caused by the interaction of the strong mixing with the weak mixing part of the dynamics. The analysis of such systems seems to be difficult.

Further, and certainly non-exhaustive, literature in the subject includes the works of Chavaudret [Cha11, Cha12, Cha13], also in collaboration with St. Marmi [CM12] and with Stolovich [CS], Eliasson ([Eli88, Eli92a, Eli01], see also [Eli09]), Hou and You [HY09, HY12] and of Popov [HP13], Zhou [YZ13], and the paper of Avila-Fayad-Kocsard [AFK12], which triggered this finer study that we took up in our recent papers.

A digression on entropy In the overall classification obtained for $\alpha \in RDC$, we have two types of models. The first one consists of (continuously) reducible cocycles with quasi-periodic dynamics in $\mathbb{T} \times G$, and the entropy of such a system is 0. The other model, given by the periodic geodesics of the group, gives dynamics that fiber over the parabolic system

$$(x, z) \mapsto (x + \alpha, e^{2i\pi r x} z) \quad (1)$$

defined on $\mathbb{T} \times \mathbb{S}^1$ for $r \in \mathbb{N}^*$. The entropy of such a system is still 0, but with a linear growth of derivatives.

The question whether the entropy of cocycles not conjugate to any of the models is positive is natural. Such cocycles are actually almost reducible, and they are exactly the measurably but not continuously reducible cocycles plus the weak mixing cocycles. The answer to the question is given by the following proposition.

Proposition 2.1. *Cocycles in $SW(\mathbb{T}^d, G)$ have 0 entropy.*

This is due to a sublinear growth of derivatives. In fact, we show moreover that if the cocycle $(\alpha, A(\cdot))$ is almost reducible, the derivative of $A_n(\cdot)$ grows sublinearly. For single-frequency cocycles, the rate of linear growth is the degree of a cocycle as it was defined in [Kar16], and it is essentially the natural number r in Eq. 1.

Proof. The cocycle acts by isometry on each fiber $\{x\} \times G$, and consequently eventual dynamical complexity lives in the transversal direction. It suffices to estimate the evolution of distances in the slice $\mathbb{T}^d \times \{\text{Id}\}$. We thus consider two points (x, Id) and $(x + \delta, \text{Id})$ for some small $\delta \in \mathbb{R}^d$. After k iterations, the distance between the two points is

$$(\delta^2 + d^2(A_k(x + \delta)A_k^*(x), \text{Id}))^{1/2}$$

When $\delta \in \mathbb{R}^d$ is small, the above quantity is bounded above by

$$|\delta|(1 + \|D_x A_k(x)\|)^{1/2}$$

The differential of a mapping $\mathbb{T}^d \rightarrow G$ can be represented (see, e.g. [Kar16]) by a mapping $C^\infty(\mathbb{T}^d, \mathbb{R}^d) \rightarrow C^\infty(\mathbb{T}^d, g)$ mapping sections of the tangent bundle $\mathbb{T}\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$ to mappings $\mathbb{T}^d \rightarrow g$ in the following canonical way. Given a vector field on $\mathbb{T}^d$, we calculate its push-forward to a vector field on G (i.e. a section of $TG \simeq G \times g$) and use right multiplication in order to move the base point of each vector to the Id: for $\delta(\cdot) : \mathbb{T}^d \rightarrow \mathbb{R}^d$ and $A(\cdot) : \mathbb{T}^d \rightarrow G$, we define

$$L(A(\cdot), \delta(\cdot)) = (D_x A(x) \cdot \delta(x)) \cdot A^{-1}(x)$$

It is a direct consequence of the definition that

$$L(A(\cdot) \cdot B(\cdot), \delta(\cdot))(x) = L(A(\cdot), \delta(\cdot))(x) + Ad_*(A(x)) \cdot L(B(\cdot), \delta(\cdot))(x)$$

which in turn, since the adjoint action is an isometry, implies that the norm of the differential grows at most linearly. This fact readily proves that the entropy of any cocycle³ is 0.

The calculation can actually be made more precise. If a cocycle is C^1 -reducible, the norm of the differentials of its iterates remain uniformly bounded above. If it is almost reducible with the estimates provided by the K.A.M. scheme, then it can be written in the form

$$(\alpha, A(\cdot)) = (\alpha, H_n^{-1}(\cdot + \alpha) \cdot A_n \cdot H_n(\cdot) \cdot e^{F_n(\cdot)})$$

with $F_n(\cdot)$ very small compared to $H_n(\cdot)$ (cf. paragraph 3.4.3). This directly implies sublinear growth of the derivatives by the same calculation as above. $\square$

3. Notation and Definitions

3.1. The group $SU(2)$. The matrix group $G = SU(2) \approx S^3 \subset \mathbb{C}^2$ is the multiplicative group of unitary 2×2 matrices of determinant 1. We will denote the matrix $S \in G$, $S = \begin{pmatrix} z & w \\ -\bar{z} & \bar{w} \end{pmatrix}$, where $(z, w) \in \mathbb{C}^2$ and $|z|^2 + |w|^2 = 1$, by $\{z, w\}_G$. The subscript will be suppressed from the notation, unless necessary. For any given $A \in G$, coordinates in $\mathbb{C}^2$ can be chosen so that the action of A preserves each copy of $\mathbb{C}$ in $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Then, the circle S^1 is naturally embedded in G as S^1_A , the group of matrices also preserving each copy of $\mathbb{C}$. They are the matrices simultaneously diagonalizable with A .

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is naturally isomorphic to $\mathbb{R}^3 \approx \mathbb{R} \times \mathbb{C}$ equipped with its vector and scalar product. The element $s = \begin{bmatrix} it & u \\ -\bar{u} & -it \end{bmatrix}$ will be denoted by $\{t, u\}_g \in \mathbb{R} \times \mathbb{C}$, where the real line represents the direction of diagonal matrices in the chosen coordinates. The scalar product is defined by

$$\langle \{t_1, u_1\}, \{t_2, u_2\} \rangle = t_1 t_2 + \mathcal{R}(u_1 \bar{u}_2) = t_1 t_2 + \mathcal{R} u_1 \cdot \mathcal{R} u_2 + \mathcal{I} u_1 \cdot \mathcal{I} u_2$$

In these coordinates, mappings with values in $\mathfrak{g}$ will be denoted by

$$U(\cdot) = \{U_t(\cdot), U_z(\cdot)\}_g$$

where $U_t(\cdot)$ is a real-valued and $U_z(\cdot)$ is a complex-valued function.

The adjoint action of the group on its algebra is pushed-forward to the action of $SO(3)$ on $\mathbb{R} \times \mathbb{C}$. In particular, the diagonal matrices, of the form $S = \exp(\{2i\pi s, 0\}_g)$, we have $Ad(S) \cdot \{t, u\} = \{t, e^{4i\pi s} u\}$.

Finally, we will consistently use the following notation.

³ Actually $SU(2)$ can be replaced by any compact Lie group.

Notation 3.1. Whenever a capital letter denotes a matrix in G , the corresponding lowercase letter will denote the argument of (one of) its eigenvalues: for any $A \in G$ there exists $D \in G$ such that

$$A = D \cdot \exp(\{2i\pi a, 0\}_g) \cdot D^* = D \cdot \{e^{2i\pi a}, 0\}_G \cdot D^*$$

3.2. Functional Spaces. We will consider the space $C^\infty(\mathbb{T}, g)$ equipped with the standard maximum norms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} \max_{\mathbb{T}} |\partial^\sigma U(\cdot)|$$

for $s \geq 0$, and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{U}(k)|^2$$

where $\hat{U}(k) = \int U(\cdot) e^{-2i\pi kx}$ are the Fourier coefficients of $U(\cdot)$. The fact that the injections $H^{s+d/2}(\mathbb{T}^d, g) \hookrightarrow C^s(\mathbb{T}^d, g)$ and $C^s(\mathbb{T}^d, g) \hookrightarrow H^s(\mathbb{T}^d, g)$ for all $s \geq 0$ are continuous is classical. By abusing the notation, we will denote $H^0 = L^2$. We will denote the corresponding spaces of complex sequences by lowercase letters,

$$h^s = \{f \in \ell^2, \sum (1+n)^{2s} |f_n|^2 < \infty\}$$

For this part, see [Fol95, SW71]. In view of the identification $G \approx \mathbb{S}^3$, with normalized measure, the space $L^2((G))$ of smooth $\mathbb{C}$ -valued functions defined on G , can be isometrically identified with $L^2(\mathbb{S}^3)$.

Let us give a convenient basis for $L^2(\mathbb{S}^3)$. Given a system of coordinates (ζ, ω) in $\mathbb{C}^2$, we can define an orthonormal basis for $\mathcal{P}_m$, the space of homogeneous polynomials of degree m , by $\{\psi_{l,m}\}_{0 \leq l \leq m}$ where $\psi_{l,m}(\zeta, \omega) = \sqrt{\frac{(m+1)!}{l!(m-l)!}} \zeta^l \omega^{m-l}$. The group G acts on $\mathcal{P}_m$ by

$$\{z, w\} \cdot \phi(\zeta, \omega) = \phi(z\zeta + w\omega, -\bar{w}\zeta + \bar{z}\omega)$$

and the resulting representation is noted by π_m . For m fixed, we can define the matrix coefficients relative to the basis by $\pi_m^{j,p} \{z, \bar{z}, w, \bar{w}\} \mapsto \langle \{z, w\} \cdot \psi_{j,m}, \psi_{p,m} \rangle$. The matrix coefficients are harmonic functions of $z, \bar{z}, w, \bar{w}$, and are of bidegree $(m-p, p)$, i.e. they are homogeneous of degree $m-p$ in (z, w) , and homogeneous of degree p in $(\bar{z}, \bar{w})$. They generate the space $\mathcal{E}_m$, thus yielding the decomposition $L^2 = \oplus_{m \in \mathbb{N}} \mathcal{E}_m$.

Therefore, given a system of coordinates in $\mathbb{C}^2$, a function $f \in L^2(\mathbb{S}^3)$ can be written in the form

$$f(z, \bar{z}, w, \bar{w}) = \sum_{m \in \mathbb{N}} \sum_{0 \leq p \leq m} \sum_{0 \leq j \leq m} f_{j,p}^m \pi_m^{j,p}(z, \bar{z}, w, \bar{w})$$

where $f_{j,p}^m \in \mathbb{C}$ are the Fourier coefficients, which are square summable. The functions $\pi_m^{j,p}(z, \bar{z}, w, \bar{w})$ are the eigenfunctions of the Laplacian on $\mathbb{S}^3$ and consequently smooth (in fact real analytic), and they form an orthonormal basis for $L^2(\mathbb{S}^3)$.

The group G acts on $L^2(G) \equiv L^2(\mathbb{S}^3)$ by pullback: if $A \in G$ and $\begin{pmatrix} \zeta \\ \omega \end{pmatrix} \in \mathbb{S}^3$, then, for $\phi : \mathbb{S}^3 \rightarrow \mathbb{C}$,

$$(A \cdot \phi) \left(\begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right) = \phi \left(A^* \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right)$$

If coordinates are chosen so that $A = \{e^{2i\pi a}, 0\}$ is diagonal, then

$$A.\phi(\zeta, \omega) = \phi(e^{-2i\pi a}\zeta, e^{2i\pi a}\omega)$$

and A then acts on harmonic functions by

$$A.\pi_m^{j,p}(z, \bar{z}, w, \bar{w}) = e^{-2i\pi(m-2p)a}\pi_m^{j,p}(z, \bar{z}, w, \bar{w})$$

where $m - 2p = m - p - p$ is the difference of the degree of homogeneity in $(\bar{z}, \bar{w})$ and (z, w) . Therefore, the harmonics in these coordinates are eigenvectors for the associated operator. In particular, if a is irrational, the eigenvectors for the eigenvalue 1 are exactly the elements $\pi_m^{j,m/2}$, $0 \leq j \leq m$.

The group of symmetries of $\mathbb{C}.\psi_{m/2,m}$ is exactly the normalizer of $\mathbb{S}_A^1$. We revisit the following lemma from [Kar14]. It examines the effect of changes of coordinates on the eigenvectors for the eigenvalue 1, $\psi_{m/2,m}$.

Lemma 3.1. *For a given $m > 0$ and even, $\pi_{m/2,m}(D.\psi_{m/2,m}) = 1$ iff $D \in \mathcal{N}_A$, the normalizer of $\mathbb{S}_A^1$. The derivative of the norm of the projection at $D \equiv Id \in G \pmod{\mathcal{N}_A}$ is negative and $\pi_{m/2,m}(D.\psi_{m/2,m}) < 1$ when $D \notin \mathcal{N}_A$.*

Proof. Call $l = m/2$ and calculate the projection:

$$\pi_{l,m}(\{z, w\}_G.\psi_{l,m}) = \sum_0^l (-1)^i \binom{l}{i}^2 |z|^{2(l-i)} |w|^{2i} \psi_{l,m} = p_l(|z|, |w|) \psi_{l,m}$$

The factor of the projection, p_l , is a Legendre polynomial in the variable $|z|^2$ and $|w|^2 = 1 - |z|^2$. The conclusion follows from the properties of Legendre polynomials. $\square$

Finally, we will use the truncation operators for mappings $\mathbb{T}^d \rightarrow g$:

$$T_N f(\cdot) = \sum_{|k| \leq N} \hat{f}(k) e^{2i\pi k \cdot}$$

$$\dot{T}_N f(\cdot) = T_N f(\cdot) - \hat{f}(0)$$

$$R_N f(\cdot) = \sum_{|k| > N} \hat{f}(k) e^{2i\pi k \cdot}$$

These operators satisfy the estimates

$$\|T_N f(\cdot)\|_{C^s} \leq C_s N^{d/2} \|f(\cdot)\|_{C^s} \quad (2)$$

$$\|R_N f(\cdot)\|_{C^s} \leq C_{s,s'} N^{s-s'+d} \|f(\cdot)\|_{C^{s'}} \quad (3)$$

3.3. Arithmetics, continued fraction expansion. For this section we refer the reader to [Khi63]. Let us introduce some notation. For $\alpha \in \mathbb{R}^*$, define $||\alpha||_{\mathbb{Z}} = \text{dist}(\alpha, \mathbb{Z}) = \min_{\mathbb{Z}} |\alpha - l|$, $[\alpha]$ the integer part of α , $\{\alpha\}$ its fractional part and $G(\alpha) = \{\alpha^{-1}\}$, the Gauss map.

The following definition is classical and central in K.A.M. theory.

Definition 3.1. We will denote by $DC(\gamma, \tau)$ the set of vectors α in $\mathbb{T}^d$ such that for any $k \neq 0$, $|\alpha \cdot k|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^{\tau}}$. Such numbers are called *Diophantine*.

The set $DC(\gamma, \tau)$, for $\tau > d$ fixed and $\gamma \in \mathbb{R}_+^*$ small is of positive Haar measure in $\mathbb{T}^d$, and $\cup_{\gamma>0} DC(\gamma, \tau)$ is of full Haar measure. The numbers that do not satisfy any Diophantine condition are called *Liouvillean*. They form a residual set of 0 Lebesgue measure.

The following definition concerns the preservation of Diophantine properties when the algorithm of continued fractions is applied to the number. We need the following notation: for $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, let $\alpha_n = G^n(\alpha) = G(\alpha_{n-1})$.

Definition 3.2. We will denote by $RDC(\gamma, \tau)$ the full measure set of *recurrent Diophantine* numbers, i.e. the α in $\mathbb{T} \setminus \mathbb{Q}$ such that $G^n(\alpha) \in DC(\gamma, \tau)$ for infinitely many n .

In contexts where the parameters γ and τ are not significant, they will be omitted in the notation of both sets.

Finally, we will need to approximate the eigenvalues of matrices in G with iterates of α , and thus need the following notion, which is looser than $(\alpha, a) \in \mathbb{T}^{d+1}$ being Diophantine.

Definition 3.3. We will denote by $DC_\alpha(\gamma, \tau)$ the set of elements A of G satisfying the following property. If $A = D\{e^{2i\pi a}, 0\}D^*$ for some $D \in G$, then

$$|||a - k\alpha||| \geq \frac{\gamma^{-1}}{|k|^\tau}, \forall k \neq 0$$

Such numbers (or matrices) are called *Diophantine with respect to α* .

3.4. Cocycles in $\mathbb{T}^d \times SU(2)$.

3.4.1. Definition of the dynamics. Let $\alpha \in \mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}^*$, be an irrational rotation. If we also let $A(\cdot) \in C^\infty(\mathbb{T}^d, G)$, the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T}^d \times G \rightarrow \mathbb{T}^d$ by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S), (x, S) \in \mathbb{T}^d \times G$$

We will call such an action a *quasiperiodic cocycle over α* (or simply a cocycle). The space of such actions is denoted by $SW_\alpha^\infty(\mathbb{T}^d, G)$, most times abbreviated to SW_α^∞ . The number $d \in \mathbb{N}^*$ is the number of frequencies of the cocycle.

Remark 3.1. All results are stated sharply in terms of d , but in the proofs we will most times assume that $d = 1$ for simplicity in notation and expression.

The space $\bigcup_{\alpha \in \mathbb{T}^d} SW_\alpha^\infty(\mathbb{T}^d, G)$ will be denoted by $SW^\infty(\mathbb{T}^d, G)$. The space SW_α^∞ inherits the topology of $C^\infty(\mathbb{T}^d, G)$, and SW^∞ has the standard product topology of $\mathbb{T}^d \times C^\infty(\mathbb{T}^d, G)$. We note that cocycles are defined over more general maps and in more general contexts of regularity and structure of the basis and fibers.

The cocycle acts on any product space $E \times F$ (E, F are vector spaces), provided that $\mathbb{T}^d$ acts on E and G act on F , in an obvious way. The particular case which will be important in this article is the representation of G on $L^2(G)$, and the resulting action of the cocycle on $L^2(\mathbb{T}^d \times G)$.

The n th iterate of the action is given by

$$\begin{aligned}(\alpha, A(\cdot))^n.(x, S) &= (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S) \\ &= (x + n\alpha, A(\cdot + (n-1)\alpha) \dots A(\cdot).S)\end{aligned}$$

if $n > 0$. Negative iterates are the inverses of positive ones:

$$(\alpha, A(\cdot))^{-n} = ((\alpha, A(\cdot))^n)^{-1} = (-n\alpha, A^*(\cdot - n\alpha) \dots A^*(\cdot - \alpha))$$

3.4.2. Conjugation and reducibility. The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices, $(\alpha, A)^n = (n\alpha, A^n)$.

The group $C^\infty(\mathbb{T}^d, G) \equiv SW_0^\infty(\mathbb{T}^d, G) \hookrightarrow SW^\infty(\mathbb{T}^d, G)$ acts by *fibered conjugation*: Let $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$. Then we define

$$\begin{aligned}Conj_{B(\cdot)}(\alpha, A(\cdot)) &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} \\ &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot))\end{aligned}$$

The dynamics of $Conj_{B(\cdot)}(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

$$(Conj_{B(\cdot)}(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B^{-1}(\cdot))$$

Definition 3.4. Two cocycles $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ in SW_α^∞ are C^s -conjugate iff there exists $B(\cdot) \in C^s(\mathbb{T}^d, G)$ such that $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$. We will use the notation

$$(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$$

to state that the two cocycles are conjugate to each other.

Since constant cocycles are a class whose dynamics can be analysed, we give the following definition.

Definition 3.5. A cycle will be called *reducible* iff it is conjugate to a constant.

In contrast with the greater part of the literature, in this article reducible means that the transfer function is at least measurable, whenever its regularity is not mentioned. Cocycles are always C^∞ smooth, but the smoothness of conjugations may vary from $H^0 \equiv L^2$ to C^∞ .

Due to the fact that not all cocycles are reducible (e.g. generic cocycles in $\mathbb{T} \times \mathbb{S}^1$ over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this result is hard to obtain, see [Eli02, Kri01]), we also need the following concept, which has proved to be central in the study of such dynamical systems.

Definition 3.6. A cocycle $(\alpha, A(\cdot))$ is said to be *almost reducible* if there exists a sequence of conjugations $B_n(\cdot) \in C^\infty$, such that $Conj_{B_n(\cdot)}(\alpha, A(\cdot))$ becomes arbitrarily close to constants in the C^∞ topology, i.e. iff there exists (A_n) , a sequence in G , such that

$$A_n^*(B_n(\cdot + \alpha)A(\cdot)B_n^*(\cdot)) \xrightarrow{C^\infty} \text{Id}$$

When this property is established in a K.A.M. constructive way, we can compare the size of $F_n(\cdot) \in C^\infty(\mathbb{T}^d, g)$, the error term which makes this last limit into an equality, with the rate of growth of the conjugation B_n , and obtain that

$$Ad(B_n(\cdot)).F_n(\cdot) = B_n(\cdot).F_n(\cdot).B_n^*(\cdot) \xrightarrow{C^\infty} 0$$

In this case, almost reducibility in the sense of the definition above and almost reducibility in the sense that “the cocycle can be conjugated arbitrarily close to reducible cocycles” are equivalent.

Herein, we will prove a more general statement, concerning conjugation close to any conjugacy class, where the same considerations on the error term apply.

Definition 3.7. Let $(\alpha, A(\cdot))$ be a given cocycle, and $\mathcal{C}$ a given class of cocycles. The cocycle $(\alpha, A(\cdot))$ is said to be *almost conjugate* to $\mathcal{C}$ if there exists a sequence of conjugations $B_n(\cdot) \in C^\infty$ and a sequence of cocycles $(\alpha, C_n(\cdot)) \in \mathcal{C}$, such that $Conj_{B_n(\cdot)}(\alpha, A(\cdot))$ becomes arbitrarily close to $(\alpha, C_n(\cdot))$ in the C^∞ topology.

3.4.3. Review of the K.A.M. scheme and of the normal form. Local conjugation. Let $(\alpha, Ae^{F(\cdot)}) = (\alpha, A_1e^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ be a cocycle over a Diophantine rotation satisfying some smallness conditions to be made more precise later on, and suppose, moreover, that $A = \{e^{2i\pi a}, 0\}$ is diagonal. The goal is to conjugate the cocycle ever closer to constant cocycles by means of an iterative scheme. This is obtained by iterating the following lemma, for the detailed proof of which we refer to [Kri99a, Eli02] or [Kar16]. For the sake of completeness, we sketch the proof, following the notation of [Kar14].

Lemma 3.2. Let $\alpha \in DC(\gamma, \tau)$ and $K \geq C\gamma N^\tau$. Let, also, $(\alpha, Ae^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}^d, G)$ with

$$c_0 K N^{s_0} \varepsilon_{1,0} < 1$$

where c_0, s_0 depend on γ, τ and d , and $\varepsilon_{1,s} = \|F_1\|_s$. Then, there exists a conjugation $G(\cdot) = G_1(\cdot) \in C^\infty(\mathbb{T}^d, G)$ such that

$$G_1(\cdot + \alpha).A_1.e^{F_1(\cdot)}.G_1^*(\cdot) = A_2.e^{F_2(\cdot)}$$

and such that the mappings $G_1(\cdot)$ and $F_2(\cdot)$ satisfy the following estimates

$$\begin{aligned} \|G_1(\cdot)\|_s &\leq c_{1,s}(N^s + K N^{s+d/2} \varepsilon_{1,0}) \\ \varepsilon_{2,s} &\leq c_{2,s} K^2 N^{2\tau+d} (N^s \varepsilon_{1,0} + \varepsilon_{1,s}) \varepsilon_{1,0} + C_{s,s'} K^2 N^{s-s'+2\tau+d} \varepsilon_{1,s'} \end{aligned}$$

where $s' \geq s$, and $\varepsilon_{2,s} = \|F_2(\cdot)\|_s$

If we suppose that $Y(\cdot) : \mathbb{T} \rightarrow g$ can conjugate $(\alpha, A_1e^{F_1(\cdot)})$ to $(\alpha, A_2e^{F_2(\cdot)})$, with $\|F_2(\cdot)\| \ll \|F_1(\cdot)\|$, then it must satisfy the functional equation

$$A_1^* e^{Y(\cdot+\alpha)} A_1 e^{F_1(\cdot)} e^{-Y(\cdot)} = A_1^* A_2 e^{F_2(\cdot)}$$

Linearization of this equation under the assumption that all C^0 norms are smaller than 1 gives

$$Ad(A_1^*)Y(\cdot + \alpha) + F_1(\cdot) - Y(\cdot) = \exp^{-1}(A_1^* A_2)$$

The equation in the frequency domain, after truncation to the order N , reads

$$\begin{cases} (e^{2i\pi k\alpha} - 1)\hat{Y}_t(k) = -\hat{F}_{1,t}(k), & 0 < |k| \leq N \\ (e^{2i\pi(k\alpha - 2a_1)} - 1)\hat{Y}_z(k) = -\hat{F}_{1,z}(k), & 0 \leq |k| \leq N \end{cases} \quad (4)$$

The rest satisfies the estimate of Eq. 2, and therefore is of order 2 provided that N is big enough. The mean value $\hat{F}_{1,t}(0)$ is an obstruction and is integrated in $\exp^{-1}(A_1^* A_2)$.

If for some k_1 we have

$$|k_1\alpha - 2a_1|_{\mathbb{Z}} < K^{-1} = N^{-\nu}$$

with $\nu > \tau$ to be fixed, we declare the corresponding Fourier coefficient $\hat{F}_{1,z}(k_1)$ a resonance, and integrate it to the obstructions. We know by [Eli02] that such a k_1 (called a *resonant mode*), if it exists and satisfies $0 < |k_1| \leq N$, is unique in $\{k \in \mathbb{Z}, |k - k_1| \leq 2N\}$. We can thus write

$$a_1 = k_1\alpha + \epsilon_1 \pmod{\mathbb{Z}}$$

and call ϵ_1 the *distance to the exact resonance*. Therefore, we can solve, with the estimates of the statement of the lemma, the equation

$$\begin{cases} (e^{2i\pi k\alpha} - 1)\hat{Y}_t(k) = -\hat{F}_{1,t}(k), & 0 < |k| \leq N \\ (e^{2i\pi(k\alpha - 2a_1)} - 1)\hat{Y}_z(k) = -\hat{F}_{1,z}(k), & 0 < |k - k_1| \leq 2N \end{cases} \quad (5)$$

Consequently, there exists $F'_2(\cdot)$, a “quadratic” term, such that

$$e^{Y(\cdot+\alpha)} A_1 e^{F_1(\cdot)} e^{-Y(\cdot)} = \{e^{2i\pi(a+\hat{F}_{1,t}(0))}, 0\}_G \cdot e^{\{0, \hat{F}_{1,z}(k_1)e^{2i\pi k_1 \cdot}\}_g} e^{F'_2(\cdot)}$$

If k_1 exists and is non-zero, iteration of local conjugation is impossible. On the other hand, the mapping $B_1(\cdot) = \{e^{-2i\pi k_1/2}, 0\}$ is such that

$$\begin{aligned} \text{Conj}_{B(\cdot)}(\alpha, A_1 \cdot \exp(\{\hat{F}_t(0), \hat{F}_z(k_1)e^{2i\pi k_1 \cdot}\})) &= (\alpha, \exp(\{\epsilon_1 + \hat{F}_t(0), \hat{F}_z(k_1)\})) \\ &= (\alpha, A_2) \end{aligned}$$

that is, $B(\cdot)$ reduces the initial constant perturbed by the obstructions to a cocycle close to (α, Id) .⁴ Calling $G_1(\cdot) = B_1(\cdot)e^{Y_1(\cdot)}$, we obtain the conclusion of the lemma with $F_2(\cdot) = B_1(\cdot) \cdot F'_2(\cdot) B_1^*(\cdot)$.

The K.A.M. scheme and normal form An appropriate choice of parameters, as in [Kar14], allows us to iterate Lemma 3.2. Let $N_{n+1} = N_n^{1+\sigma} = N^{(1+\sigma)^{n-1}}$, where $N = N_1$ is big enough and $0 < \sigma < 1$, and $K_n = N_n^\nu$, for some $\nu > \tau$. If we suppose that $(\alpha, A_n e^{F_n(\cdot)})$ satisfies the hypotheses of Lemma 3.2 for the corresponding parameters, then we obtain a mapping $G_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)}$ that conjugates it to $(\alpha, A_{n+1} e^{F_{n+1}(\cdot)})$, and we use the notation $\varepsilon_{n,s} = \|F_n\|_s$.

If we suppose that the initial perturbation small in small norm: $\varepsilon_{1,0} < \epsilon < 1$, and not big in some bigger norm: $\varepsilon_{1,s_0} < 1$, where ϵ and s_0 depend on the choice of parameters, then we can prove (see [Kar16] and, through it, [FK09]), that the lemma can be iterated into a scheme, and moreover

⁴ If $B(\cdot)$ happens to be 2-periodic, periodicity can be regained with no cost in the estimates. We refer the reader to [Kar16] or [Kar14] for this fact.

$$\begin{aligned}\varepsilon_{n,s} &= O(N_n^{-\infty}) \text{ for every fixed } s \text{ and} \\ \|G_n\|_s &= O(N_n^{s+\lambda}) \text{ for every } s \text{ and some fixed } \lambda > 0\end{aligned}$$

We sum these inequalities up by saying that the norms of perturbations decay exponentially, while conjugations grow polynomially.

This fact allows us to obtain the normal form as follows. The product of conjugations produced by the scheme at the n th step is written in the form $H_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)} \dots B_1(\cdot)e^{Y_1(\cdot)}$, where the $B_j(\cdot)$ reduce the resonant modes. We can rearrange the terms of the product into

$$B_n(\cdot) \dots B_1(\cdot) \cdot e^{\tilde{Y}_n(\cdot)} \dots e^{\tilde{Y}_2(\cdot)} e^{Y_1(\cdot)}$$

where $\tilde{Y}_j(\cdot) = \prod_{l=j-1}^1 \text{Ad}(B_l^*(\cdot)) \cdot Y_j(\cdot)$. Since the $Y_j(\cdot) = O(N_j^{-\infty})$ (they are conjugations comparable with $F_j(\cdot)$ with a fixed loss of derivatives), and since $\prod_{j-1}^1 \text{Ad}(B_i^*(\cdot))$ deteriorates the C^s norms by a factor of the order of N_{j-1}^{s+d} ,

$$\prod_{\infty}^1 \exp(\tilde{Y}_j(\cdot)) \rightarrow D(\cdot) \in C^\infty(\mathbb{T}^d, G)$$

even if the $H_n(\cdot)$ do not converge.

Definition 3.8. The cocycle $\text{Conj}_{D(\cdot)}(\alpha, A^{F(\cdot)})$ is the *K.A.M. normal form* of the cocycle $(\alpha, A^{F(\cdot)})$

The K.A.M. normal form is defined in an ad-hoc way, and its usefulness stems from the following remark: the K.A.M. scheme applied to a cocycle in normal form consists only in the reduction of resonant modes.

Notation 3.2. For a cocycle in normal form, we relabel the indices as $(\alpha, A_{n_i} e^{F_{n_i}}) = (\alpha, A_i e^{F_i})$, where n_i is a step where a reduction of a resonant mode takes place.

In the language of Fig. 1, a cocycle in normal form, after the successive conjugations up to the step i and in the first order of magnitude looks like a circle around the origin in the plane tangent to a resonant sphere $\{S \cdot \{2\pi(k_i\alpha + \epsilon_i, 0)\}_g \cdot S^*\}_{S \in G}$. The radius of the circle is $|\hat{F}_i(k_i)|$. The reduction of the resonant mode drives $k_i\alpha$ to 0, and reduces the perturbation to the point of coordinates $\{2\pi\epsilon_i, \hat{F}_i(k_i)\}_g$. The picture repeats itself if we zoom in order to see the finer scales of the dynamics, and the first part of the picture (the reduction by the close-to-the-identity transformation $Y_n(\cdot)$) never occurs.

At the step i , we will assume that the constant $A_i = \{e^{2i\pi k_i \alpha}, 0\}$ is the exact resonance, and the first order perturbation

$$e^{F_i(\cdot)} = e^{\{2i\pi\epsilon_i, 0\}} \cdot e^{\{0, \hat{F}_i(k_i)\} e^{2i\pi k_i \cdot}}$$

contains the distance from the exact resonance, $\{2i\pi\epsilon_i, 0\}$.⁵

⁵ This choice interferes with the estimates only when the cocycle is C^∞ reducible.

4. Proof of Spectral Dichotomy

Let $f \in L^2(\mathbb{T} \times G, \mathbb{C})$ be an eigenfunction of the Koopman operator $U = U_{(\alpha, A(\cdot))}$, and suppose that $f(x, S)$ depends explicitly on $S = \{z, w\}_G \in G$. We remark that any eigenfunction depending non-trivially on S also depends non-trivially on x , unless $A(\cdot) \equiv A \in G$ is constant. Since each subspace $L^2(\mathbb{T}) \times \mathcal{E}_m$ is invariant under U , for each m fixed, we can suppose that there exists $m \in \mathbb{N}^*$ such that $f \in L^2(\mathbb{T}) \times \mathcal{E}_m$. The function f then admits a development

$$f(x, S) = \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq j, p \leq m}} f_{j,p}^{m,k} e^{2i\pi kx} \pi_m^{j,p}(z, \bar{z}, w, \bar{w}) = \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq j, p \leq m}} f_{j,p}^k e^{2i\pi kx} \pi^{j,p}(z, \bar{z}, w, \bar{w})$$

where we have dropped m from the notation since it is considered to be fixed. It will be replaced by i , the index of the step of the K.A.M. scheme.

The equation satisfied by an eigenfunction of the Koopman operator U is

$$f(x - \alpha, A^{-1}(x).S) = \lambda f(x, S)$$

for some fixed $\lambda \in \mathbb{S}^1$.

For a diagonal constant cocycle $(\alpha, A(\cdot)) \equiv (\alpha, A) = (\alpha, \{e^{2i\pi a}, 0\})$, the following lemma is immediate.

Lemma 4.1. *Let (α, A) be a constant cocycle, and consider the canonical basis of $\mathcal{E}_m$, formed by the functions $\pi^{j,p} = \pi_m^{j,p}$. Then, for every $k \in \mathbb{Z}$ and $0 \leq j, p \leq m$, the function*

$$e^{2i\pi kx} \pi^{j,p}(z, \bar{z}, w, \bar{w}) \in L^2(\mathbb{T}) \times \mathcal{E}_m$$

is an eigenfunction of the Koopman operator $U_{(\alpha, A)}$, with eigenvalue

$$e^{-2i\pi(k\alpha + (m-2p)a)}$$

The proof of the lemma is by immediate calculation (or see [Kar14]) and points to the proof of the main theorem of the paper, where the assumption that the cocycle $(\alpha, A(\cdot))$ be in K.A.M. normal form becomes relevant. If f is an eigenfunction of the operator associated to the cocycle $(\alpha, A(\cdot))$, and if $(\alpha, A_i e^{F_i(\cdot)}) \stackrel{H_i^*}{\sim} (\alpha, A(\cdot))$ at the n_i th step of the K.A.M. scheme, then, $f_i = f \circ (Id, H_i^*)$ is an eigenfunction of $U_i = U_{(\alpha, A_i e^{F_i(\cdot)})}$. Since $F_i(\cdot)$ is very small, f_i should be close to an eigenfunction of the operator $U'_i = U_{(\alpha, A_i)}$. The corresponding eigenvalues of the exact eigenfunctions will be distinct, since α is supposed Diophantine. Since this approximation converges exponentially fast for $i \rightarrow \infty$, and since the support in the frequencies in $L^2(\mathbb{T})$ is related with the summability of the angles between the successive constants A_i , we obtain the announced theorem.

We now make the argument precise. Clearly, $f_0 = f$ is an eigenfunction of the operator $U_0 = U_{(\alpha, A(\cdot))}$ and for the eigenvalue λ iff $f_i = f \circ (Id, H_i^*)$ is an eigenfunction of the operator

$$U_i = U_{(\alpha, A_i e^{F_i(\cdot)})}, \quad i \in \mathbb{N}$$

for the same eigenvalue. Then, linearization with respect to the dynamics gives

$$\tilde{U}_i f_i = \lambda f_i + O_{L^2}(N_i^{-\infty}) \quad (6)$$

where $\tilde{U}_i = U_{(\alpha, A_i)}$ and the constants on the O_{L^2} depend on the norm of the function f . Linearization is possible because f depends in a C^∞ (in fact real analytic) way on the variable in G , and the L^2 character of the function may only be due to the slow decay of the Fourier coefficients in the variable in $\mathbb{T}$.

We will also use the fact that, if the cocycle is not measurably reducible, resonances appear rarely.

Lemma 4.2. *Let $(\alpha, A(\cdot))$ be in normal form and not measurably reducible. Then,*

$$n_{i+1} - n_i \rightarrow \infty$$

Proof. The condition that the cocycle is not measurably reducible (cf. item 2 in Sect. 2) is equivalent to

$$\{\theta_i\} = \{\arctan \frac{|\hat{F}(k_i)|}{|\epsilon_i|}\} \notin \ell^2$$

Since $|\hat{F}(k_i)| = O(|k_i|^{-\infty})$ and $|\hat{F}(k_i)| > 0$ for all i , we obtain that, also, $|\epsilon_i| = O(|k_i|^{-\infty})$. This implies that $||k_{i+1}\alpha|| = O(|k_i|^{-\infty})$. Since, however

$$\frac{\log |k_{i+1}|}{\log |k_i|} \approx (1 + \sigma)^{n_{i+1} - n_i}$$

and $\alpha \in DC$, we can conclude. $\square$

In fact, this argument can be applied as soon as the cocycle is not C^∞ reducible.

Let us now apply the operator T_{N_i} on Eq. 6 and use the fact that it commutes with $\tilde{U}_i$ to obtain

$$\tilde{U}_i T_{N_i} f_i = \lambda T_{N_i} f_i + O_{L^2}(N_i^{-\infty})$$

Consequently, since the inverse of $\tilde{U}_i \circ T_{N_i}$ does not magnify the error term outside of $O_{L^2}(N_i^{-\infty})$, $T_{N_i} f_i$ is $O_{L^2}(N_i^{-\infty})$ -close to an eigenfunction of $\tilde{U}_i \circ T_{N_i}$. Since the eigenvalues of $\tilde{U}_i \circ T_{N_i}$ are separated by $\gamma^{-1} N_i^{-m\tau}$, we find that the eigenvalues of $\tilde{U}_i \circ T_{N_i}$ and their corresponding eigenfunctions are $O(N_i^{-\infty})$ and $O_{L^2}(N_i^{-\infty})$ good approximations of the corresponding objects for U_i . Therefore, since L^2 -norms in the original coordinates and those of the i th step of the K.A.M. scheme are the same, the same approximation holds also for the operator U and the eigenfunctions transformed accordingly.

We now compare the Eq. 6 at the steps n_i and $n_{i+1} \gg n_i$, and examine how the divergence of the product of conjugations sends L^2 mass to infinity, thus contradicting the initial assumption that $f \in L^2$.

Let us express the eigenfunctions of $\tilde{U}_i$ in a coordinate system where (α, A_i) is diagonal. There exists $l_i \in \mathbb{N}$, such that, up to $O(N_i^{-\infty})$,⁶

$$f_i(x, S_i) = \sum_{\substack{k+(m-2p)k_i=l_i \\ 0 \leq j, p \leq m \\ |k| < N_i}} f_{j,p}^{i,k} e^{2i\pi kx} \pi_i^{j,p}(S_i)$$

⁶ The upper index k in the Fourier coefficients $f_{j,p}^{i,k}$ is in fact redundant.

Let us, now, apply the transformation $B_i(\cdot)$ which conjugates $(\alpha A_i e^{F_i(\cdot)})$ to $(\alpha A_{i+1} e^{F_{i+1}(\cdot)})$. In the new coordinates,

$$\begin{aligned} f_i(x, \tilde{S}_i) &= \sum_{\substack{k+(m-2p)k_i=l_i \\ 0 \leq j, p \leq m \\ |k| < N_i}} f_{j,p}^{i,k} e^{2i\pi(k+(m-2p)k_i)x} \pi_i^{j,p}(\tilde{S}_i) \\ &= \sum_{0 \leq j, p \leq m} \tilde{f}_{j,p}^i e^{2i\pi l_i x} \pi_i^{j,p}(\tilde{S}_i) \end{aligned}$$

If we let D_i be such that $D_i A_i D_i^*$ is diagonal in the coordinates where A_{i+1} is diagonal, then D_i is θ_i -away from a diagonal matrix in the same coordinates. If we now apply D_i , we obtain the new coordinates (x, S_{i+1}) where the cocycle $(\alpha, A(\cdot))$ is represented by $(\alpha, A_{i+1} e^{F_{i+1}(\cdot)})$ and A_{i+1} is diagonal, as well as the expression

$$f_i(x, S_{i+1}) = \sum_{\substack{0 \leq j, p \leq m \\ |k| < N_i}} \tilde{f}_{j,p}^{i+1,k} e^{2i\pi l_i x} \pi_{i+1}^{j,p}(S_{i+1})$$

By our observation, the formula above should coincide up to $O_{L^2}(N_{i+1}^{-\infty})$ with $T_{2mN_i} f_{i+1}(x, S_{i+1})$, where

$$\tilde{U}_{i+1} T_{N_{i+1}} f_{i+1} = \lambda T_{N_{i+1}} f_{i+1} + O_{L^2}(N_{i+1}^{-\infty})$$

for the same eigenvalue λ , eventually up to $O(N_{i+1}^{-\infty})$.

The incompatibility between the two representations arises from the transformation rule of the $\pi^{j,p}$ under a change of basis. More precisely, the only functions that are eigenfunctions for the operator $\tilde{U}_{i+1}$ for the eigenvalue $e^{2i\pi l_i \alpha}$ are the functions $e^{2i\pi l_i} \pi_{i+1}^{j,m/2}$, $0 \leq j \leq m$, and m even number. Therefore, the compatibility of the two expressions for the eigenfunction would impose that

$$f_i(x, S_{i+1}) = \sum_{\substack{0 \leq j \leq m \\ |k| < N_i}} \tilde{f}_{j,m/2}^{i+1,k} e^{2i\pi l_i x} \pi_{i+1}^{j,m/2}(S_{i+1})$$

When we insert D_i^* in this expression in order to undo the change of coordinates $\tilde{S}_i \mapsto S_{i+1}$, we constrain the coefficients $f_{j,p}^{i,k}$ in the image of $\oplus_{0 \leq j \leq m} \mathbb{C} \pi_{i+1}^{j,m/2}$ under the change of coordinates, always up to $O(N_{i+1}^{-\infty})$.

Now, the same must hold when we compare the expressions obtained at the steps $i+1$ and $i+2$. Comparison between the constraint on the coefficients at the step $i+2$, i.e.

$$f_i(x, S_{i+2}) = \sum_{\substack{0 \leq j \leq m \\ |k| < N_i}} \tilde{f}_{j,m/2}^{i+2,k} e^{2i\pi l_i x} \pi_{i+2}^{j,m/2}(S_{i+2})$$

shows that the space admissible at the step $i+1$ is restricted. When we project the preimage of the vector $\zeta_{i+2}^{m/2} \omega_{i+2}^{m/2}$ under the change of coordinates $\tilde{S}_{i+1} \mapsto S_{i+2}$, the norm of the vector shrinks by a factor $\geq O(\theta_{i+1}^2)$, i.e.

$$|\langle (D_{i+1})_*(\zeta_{i+2}^{m/2} \omega_{i+2}^{m/2}), \zeta_{i+1}^{m/2} \omega_{i+1}^{m/2} \rangle| \leq \frac{(m/2)!}{(m+1)!} (1 - O(\theta_{i+1}^2))$$

(see lem. 2.2 of [Kar14] and Lemma 3.1).

Since the different constraints on the coefficients are imposed in different scales of the dynamics for every different i , or equivalently since they correspond to frequencies in $\mathbb{Z}^d$ belonging to distant shells, these constraints are independent from one another. Therefore, if the angles are not summable in ℓ^2 , the intersection of the constraints is empty and there exists no eigenfunction in L^2 . On the other hand, if the angles are summable in ℓ^2 , the procedure converges and produces an eigenfunction as should be expected.

Finally, for any given cocycle in $\mathcal{N}$, we prove the existence of a subsequence of iterates accumulating to $(0, \text{Id})$ in the C^∞ topology.

Proposition 4.3. *All cocycles in the K.A.M. regime are rigid.*

Proof. Every cocycle is almost reducible to a resonant one

$$(\alpha, A(\cdot)) \stackrel{H_i^*}{\sim} (\alpha, A_i) + O(N_{i+1}^{-\infty})$$

where $A_i = \{e^{2i\pi k_{i+1}\alpha}, 0\}$ up to $O(N_i^{-\infty})$. Since, in the case where $(\alpha, A(\cdot))$ is not C^∞ reducible, $n_{i+1} \gg n_i$, there exists an iterate $n_i < m_i \ll n_{i+1}$ such that $(\alpha, A(\cdot))^{m_i} = (m_i\alpha, O(N_i^{-\infty}))$, and $m_i\alpha \rightarrow 0$ when $i \rightarrow \infty$.⁷ $\square$

5. The Topology of Conjugacy Classes

In this section we sketch a Proof of Theorems 1.2, 1.3, 1.5, 1.6 and 1.7, Corollary 1.9 and Theorem 1.10.

5.1. The parameter space of the normal form. The conjugations that act on a K.A.M. normal form at step i of the scheme are:

1. Far-from-the-identity conjugations commuting with the constant A_i

$$B'(\cdot) = \begin{pmatrix} e^{2i\pi k'_i \cdot / 2} & 0 \\ 0 & e^{-2i\pi k'_i \cdot / 2} \end{pmatrix}$$

where $k'_i \in \mathbb{Z}$ is such that $N_{i-1} < |k_i + k'_i| \leq N_i$.

2. Constant conjugations commuting with A_i .
3. Consider a one-parameter subgroup $\{D_i^t\}_{t \in [0,1]}$, of minimal length such that $D_i A_{i+1} D_i^*$ is diagonal in the coordinates where A_i is diagonal and $D_i = D_i^1$. Then, the path

$$t \mapsto B_i^*(\cdot) e^{t D_{i+1}} B_i(\cdot)$$

when it acts by conjugation on $(\alpha, A_i e^{F_i(\cdot)})$

4. Conjugations regaining periodicity, if necessary.

These conjugations act as follows.

⁷ In fact, $\nu > \tau$ is sufficient for obtaining rigidity for any cocycle, but it seems cumbersome, unless the cocycle is not C^∞ reducible.

1. Translation of the resonance by $k'_i\alpha/2$ and of the corresponding resonant frequency by k'_i . They satisfy the estimate $N_{i-1}^{s+1/2} \lesssim \|B'\|_s \lesssim N_i^{s+1/2}$.
2. Multiplication of $\hat{F}_i(k_i)$ by a complex number in $\mathbb{S}^1$. This only changes the argument of $\hat{F}_i(k_i)$, and such a conjugation can be introduced as a phase in $B_{i-1}(\cdot)$. They satisfy the estimate $\|B''\|_s \lesssim N_{i-1}^{s+1/2} |\theta_{i-1}|$.
1. Conjugations of the third kind transform collapse the cone on which the perturbation lives to a line:

$$\begin{aligned}\theta_i^t &= (1-t)\theta_i \\ \frac{|\hat{F}_i^t(k_i)|}{|\epsilon_i^t|} &= \arctan \theta_i^t \\ \sqrt{|\hat{F}_i^t(k_i)|^2 + (\epsilon_i^t)^2} &\equiv \sqrt{|\hat{F}_i^0(k_i)|^2 + (\epsilon_i^0)^2} \simeq a_{i+1} = d(\exp^{-1}(A_{i+1}), \text{Id})\end{aligned}$$

i.e. the angle between A_i and A_{i+1} is driven to zero, the rotation number at the step $i+1$ is added to the one at the step i and the following resonances are translated by k_i . These conjugations affect the norms by a factor

$$O_{H^s}(|\theta_i^t - \theta_i^0| N_{i-1}^s) = O_{H^s}(|t\theta_i^0| N_{i-1}^s)$$

when θ_i is close to 0, and therefore will be close to the identity whenever t is small enough.

The fact that these three conjugations are the only ones who act on the parameter space of normal forms follows from the following. In view of item 1 of paragraph 5.1, we can assume that the resonant mode is the same for both forms. Then, we can apply item 3 to each one, in order to obtain a resonant constant, but with the resonant mode deactivated. If these two cocycles are conjugate to each other, then their arguments $a_i^j + a_{i+1}^j$, for $j = 1, 2$ must be equal up to $k'_i\alpha$, with k'_i not too big (see again item 1). Since resonances are unique for this size of k'_i , no other conjugation can act on the space of normal forms.

These facts prove Theorem 1.7, by applying the same procedure as for the construction of the K.A.M. normal form. They also prove its Corollary 1.9, and Theorem 1.10. Corollary 1.8 is proved by combining the estimates above with the proof of differentiable rigidity (item 1a of the classification), where it is proved that the K.A.M. scheme produces only finitely many resonances for cocycles reducible to a constant in CD_α , and, “generically”, infinitely many if the constant is in $\mathcal{L}_\alpha$.

5.2. Path connectedness. The construction of the paths is carried out by partitioning the interval $[0, 1]$ into dyadic intervals, and then continuously deforming the parameters of the i th step of the K.A.M. scheme for $t \in [2^{i-1}, 2^i]$ in a continuous way from those of the original cocycle to those corresponding to the normal form of the target.

Let us sketch the Proof of Theorem 1.2. First, connect the Id with A_1 with a continuous path, say the shortest one-parameter group connecting the two elements. This part cannot be obtained by the action of a conjugation, since the argument of the constant changes. Then, activate corresponding mode of the normal form by a reparametrization of, say

$$t \rightarrow A_1 \cdot \{e^{2i\pi t \epsilon_1}, 0\} \cdot \{0, e^{\{0, t \hat{F}_1 e^{2i\pi k_1 \cdot}\}}\}$$

The path of cocycles thus obtained is reducible for all $t \in [0, 1]$, cf. item 3 of the previous paragraph. We can proceed by induction in the coordinates where the target cocycle at $t = 1$ is constant (and equal to (α, A_2)).

The Proof of Theorem 1.3 replaces the first step by the following one. Let $B_{12}^t(\cdot): [0, 1/4] \rightarrow C^\infty$ be a path such that $B_{12}^0(\cdot) \equiv Id$, and $B_{12}^{1/4}(\cdot) \equiv B_1.B_2(\cdot)$. This is possible, since $B_1.B_2(\cdot)$ is homotopic to constant mappings. This conjugation transforms the constant cocycle (α, Id) into $(\alpha, \{e^{2i\pi(k_1\alpha+k_2\alpha)}, 0\})$. However, the path exits $\mathcal{N}$, since the homotopy $B_{12}^t(\cdot)$ cannot be made to take values in a unique circle $\mathbb{S}^1 \hookrightarrow G$.

Then, we can activate the mode in the normal form with a path $[1/4, 1/2] \rightarrow C^\infty$ acting by a conjugation as in item 3 of the previous paragraph. This will drive the constant part to $\{e^{2i\pi k_1\alpha}, 0\} = A_1$, and add the perturbation

$$\{e^{2i\pi\epsilon_1}, 0\}.\{0, e^{i0, \hat{F}_1 e^{2i\pi k_1 \cdot}}\}$$

Proceed by induction.

The proofs of the two other similar Theorems 1.5 and 1.6, are only slightly more complicated.

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