



Irreducibility and Exponential Mixing of Some Stochastic Hydrodynamical Systems Driven by Pure Jump Noise

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Abstract: In this paper, we derive several results related to the long-time behavior of a class of stochastic semilinear evolution equations in a separable Hilbert space H :

$$du(t) + [Au(t) + \mathbf{B}(u(t), u(t))]dt = dL(t), \quad u(0) = x \in H.$$

Here A is a positive self-adjoint operator and \mathbf{B} is a bilinear map, and the driving noise L is basically a $D(A^{-1/2})$ -valued Lévy process satisfying several technical assumptions. By using a density transformation theorem type for Lévy measure, we first prove a support theorem and an irreducibility property of the Ornstein–Uhlenbeck processes associated to the nonlinear stochastic problem. Second, by exploiting the previous results we establish the irreducibility of the nonlinear problem provided that for a certain $\gamma \in [0, 1/4]$ \mathbf{B} is continuous on $D(A^\gamma) \times D(A^\gamma)$ with values in $D(A^{-1/2})$. Using a coupling argument, the exponential ergodicity is also proved under the stronger assumption that \mathbf{B} is continuous on $H \times H$. While the latter condition is only satisfied by the nonlinearities of GOY and Sabra shell models, the assumption under which the irreducibility property holds is verified by several hydrodynamical systems such as the 2D Navier–Stokes, Magnetohydrodynamics equations, the 3D Leray- α model, the GOY and Sabra shell models.

1. Introduction

Motivated by the need of rigorous mathematical results to understand the turbulence phenomenon in fluid dynamics, several prominent mathematicians have intensively studied the ergodicity of stochastic hydrodynamical systems driven by Wiener noise. Their studies have generated many important results. We refer, for instance, to [5, 20, 22–24, 27, 28, 38–40] and references therein for the results obtained and the advances that have been made so far.

In contrast to the case of SPDEs with Wiener noise, there are not so many results related to the long-time behavior of the stochastic version of hydrodynamical systems

with Lévy noise. The main reason is that, in general, all the available results for the SPDEs with Wiener noise do not apply to the treatment of SPDEs driven by Lévy noise. In fact, as shown in [31] and [32], the dynamics of Lévy noise driven SPDEs differ essentially from dynamics of systems driven by Brownian noise, and thus require different techniques. However, several results about the qualitative or long-time behavior of solution of SPDEs driven by Lévy noise have been obtained during the last decade. We refer, among others, to [18, 19, 33, 38, 43, 45, 46, 48, 49] for results related to the ergodicity, irreducibility and mixing property of several classes of stochastic evolution equations driven by Lévy processes.

In this paper we study the long-time behavior of some hydrodynamical systems driven by Lévy noise, which are written in the form of an abstract stochastic evolution equation on a separable Hilbert space H :

$$\begin{aligned} du(t) + [\kappa Au(t) + \mathbf{B}(u(t), u(t))]dt \\ = \sum_{k=1}^{\infty} \left(\int_{|z|<1} \beta_k z e_k d\tilde{\eta}_k(z, t) + \int_{|z|\geq 1} \beta_k z e_k d\eta_k(z, t) \right) \\ u(0) = x \in H, \end{aligned} \tag{1}$$

where $\kappa > 0$ is a constant, A is a positive and self-adjoint operator with dense domain in H , and \mathbf{B} is a bilinear map defined on dense subset of H . The sequence $\{e_j; j \in \mathbb{N}\}$ represents an orthonormal basis of H consisting of the eigenfunctions of A , $\{\beta_j; j \in \mathbb{N}\}$ a sequence of positive numbers. Additional notations and assumptions on the linear map A , the bilinear map \mathbf{B} and $\{\beta_j; j \in \mathbb{N}\}$ will be given later on. Throughout, the noise entering the system is represented by

$$L(t) = \sum_{k=1}^{\infty} \left(\int_0^t \int_{|z|<1} \beta_k z e_k d\tilde{\eta}_k(z, s) + \int_0^t \int_{|z|\geq 1} \beta_k z e_k d\eta_k(z, s) \right), \tag{2}$$

where the η_j -s are mutually independent and identically distributed (i.i.d.) Poisson random measures on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ which basically represent the random counting measures associated to a sequence of mutually i.i.d. tempered stable processes with intensity measure ν . This class of Lévy noises is very important in Mathematics of Finance, see, for instance, [35] and [50]. They were also introduced in statistical physics, see for *e.g.* [37], to study the phenomenon of self-similar intermittency of turbulent flows. For further applications to other fields in physics we refer, for instance, to [34] and [36]. We should note, however, that the family of truncated Lévy flights studied in the latter paper is not a suitable example for our purpose, because the *tempering* function therein does not satisfy Assumption 2.5(ii) of our article.

The main results in the present paper can be summarized in the following three items.

- (i) By using a change of measure method we first prove in Theorem 4.1 that the real-valued Ornstein–Uhlenbeck (O-U) process $y := \{\int_0^t e^{-\kappa(t-s)} d\ell(s); t \in [0, T]\}$, where $\{\ell(t); t \in [0, T]\}$ is a tempered stable process, has full support on $L^p(0, T; \mathbb{R})$ for any $p > 0$ and that it is irreducible on \mathbb{R} . We exploit these results to establish a support theorem and irreducibility property (see Theorem 4.3) for the H -valued O-U process $\mathfrak{G} := \{\int_0^t e^{-\kappa(t-s)A} dL(s); t \in [0, T]\}$.
- (ii) Under fairly general assumptions on the nonlinear term \mathbf{B} , irreducibility property of the solution of (1) is proved using the above results and the exact controllability

of the deterministic version of (1). This result, which can be applied to 2D Navier–Stokes, Magnetohydrodynamics equations, the 3D Leray- α (all with the periodic boundary condition), and the GOY and Sabra shell models, is stated in Theorem 3.5.

- (iii) With much stronger conditions on \mathbf{B} , which are satisfied by GOY and Sabra shell models, we show in Theorem 3.7 by using the coupling method as in [38] the exponential ergodicity of (1).

To our knowledge these results are new and extend existing results related to the long-time behavior of stochastic hydrodynamical systems driven by Lévy noise. In fact, the polynomial mixing of the 2D Navier–Stokes equations driven by compound Poisson processes was treated in [38], but their approach does not apply to our situation as we consider stochastic evolution equation driven by Lévy noise of infinite activity. By adapting the tools for the ergodicity of SPDEs driven by Wiener noise developed in [15,22] and [24], the authors of [19] proved the ergodicity of the 2D Navier–Stokes equations driven by Lévy noise with a non-degenerate Wiener noise. Recently, H. Bessaih and the last two authors proved in [7] the ergodicity of stochastic shell models with tempered stable process. Due to the lack of irreducibility they could not prove any convergence rate to the invariant measure.

To prove our results we were inspired by [46,48] and [49], which treated stable processes driven SPDEs with bounded nonlinearity which does not include the examples we treat in this paper. It is clear from the sketch of our results and the assumptions on our driving noise that irreducibility and ergodicity for 2D Navier–Stokes, Magnetohydrodynamics equations and the 3-dimensional Leray- α driven by stable processes do not follow from our results and are still an open problem. Finally, while the support theorem and irreducibility for Ornstein–Uhlenbeck processes is true for any tempered stable measure $\nu(dz) = |z|^{-1-\theta} e^{-|z|}$ with $\theta \in (0, 2)$, the exponential ergodicity and the irreducibility property of the nonlinear problem is only true for $\theta \in (0, 1)$ (see Remark 2.8, Theorem 4.1 and Corollary 4.4). The main reason is that, in order to ensure the existence of solution (see Proposition 3.3) and the strong Feller property (see [7, Proposition 3.6]), we require the finiteness of the moment of order $p \geq 1$ of the measure ν (see Assumption 2.5(iii)).

Let us now close this introduction with the layout of the paper. In Sect. 2 we introduce several notations and all the assumptions that we need in this paper. We also give several motivating examples in the same section. In Sect. 3 we state two of our main results, which are the irreducibility and exponential mixing of (1). From these results and the preparatory steps in Sect. 2.2 we derive the irreducibility of all our motivating examples. We also derive from the first two main results the exponential mixing of the GOY and Sabra shell models driven by tempered stable noise. The proofs of the main theorems of our work are given in Sects. 4 and 5. The statement and proofs of our third and fourth results, which are the support theorems and irreducibility of the O-U processes, are given in Sect. 4. The proof of the irreducibility of the general model (1) is also given in Sect. 4. By using the coupling approach and following closely [38] the exponential mixing of (1) is proved in Sect. 5.

2. Notation, Assumptions and Motivating Examples

2.1. Notations and assumptions. Let H be a separable Hilbert space with norm and scalar product denoted by $|\cdot|$ and (u, v) , respectively. Let A be a (possibly unbounded)

linear map with domain $D(A)$, which is endowed with the graph norm, with values on H . We impose the following set of conditions on A .

Assumption 2.1. We assume that A is a positive self-adjoint operator and its domain $D(A)$ is densely and compactly embedded in H .

Observe that in view of the above assumption, we can and will assume that the eigenfunctions $\{e_1, e_2, \dots\}$ of A form an orthonormal basis of H . Throughout this paper the eigenvalues associated with the eigenfunctions of A are denoted by $\lambda_1 < \lambda_2 < \dots$. The fractional power operators $A^\gamma, \gamma \geq 0$ are well-defined; they are also self-adjoint, positive and invertible with inverse $A^{-\gamma}$. We denote by $V_\gamma := D(A^{\frac{\gamma}{2}}), \gamma \geq 0$ the domain of A^γ . It is a Hilbert space endowed with the graph norm. The dual space V_γ^* of $V_\gamma, \gamma \geq 0$, wrt to the inner product of H can be identified with $D(A^{-\frac{\gamma}{2}})$. For $\gamma = 1$ we set $V := V_1$ and we denote its norm by $\|\cdot\| := |\cdot| + |A^{\frac{1}{2}} \cdot|$. Since Assumption 2.1 also implies the following Poincaré type inequality

$$|\cdot| \leq \lambda_1^{-1/2} |A^{1/2} \cdot| \leq \lambda_1^{-1/2} \|\cdot\|, \tag{3}$$

the norm $\|\cdot\|$ is equivalent to $|A^{\frac{1}{2}} \cdot|$ on V .

When identifying H with its dual H^* we have the Gelfand triple $V \subset H \subset V^*$. We denote by $\langle u, v \rangle$ the duality between V^* and V such that $\langle u, v \rangle = (u, v)$ for $u \in H$ and $v \in V$.

Now, let $\mathbf{B} : V \times V \rightarrow V^*$ be a bilinear map satisfying the following set of conditions.

Assumption 2.2. (a) We assume that $\mathbf{B} : V \times V \rightarrow V^*$ is a continuous bilinear map satisfying

$$\langle \mathbf{B}(u, v), v \rangle = 0, \quad \text{for any } u \in V, v \in V.$$

(b) The map $\mathbf{B}(\cdot, \cdot)$ admits an extension, still denoted by the same symbol, on $\mathcal{H} \times \mathcal{H}$, where $\mathcal{H} = D(A^\gamma)$ for some $\gamma \in [0, \frac{1}{4}]$. Furthermore, there exists a constant $C_1 > 0$ such that

$$\|\mathbf{B}(u, v)\|_{V^*} \leq C_1 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \quad \text{for any } u, v \in \mathcal{H}. \tag{4}$$

Remark 2.3. Observe that the embedding $\mathcal{H} \subseteq H$ is dense, and there exists $C_0 \in (0, \infty)$ such that

$$\|u\|_{\mathcal{H}} \leq C_0 |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}}, \quad \text{for any } u \in V. \tag{5}$$

Furthermore, there exists a sequence of positive numbers $(\gamma_k)_{k \in \mathbb{N}}$ such that $\{\varphi_k := \gamma_k e_k; k \in \mathbb{N}\}$ is an orthonormal basis of \mathcal{H} .

Let $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Before we proceed to the assumptions on the noise L , let us recall the following definition.

Definition 2.4. Let Z be a metric space and \mathcal{Z} be its Borel σ -algebra, ν be a positive σ -finite measure on (Z, \mathcal{Z}) . Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}, \mathbb{R}_+ = [0, \infty)$ and $M_I(Z)$ be the family of all $\bar{\mathbb{N}}$ -valued measures on (Z, \mathcal{Z}) .

A Poisson random measure, with intensity measure ν, η defined on (Z, \mathcal{Z}) over \mathfrak{P} is a measurable map $\eta : (\Omega, \mathcal{F}) \rightarrow (M_I(Z \times \mathbb{R}_+), \mathcal{M}_I(Z \times \mathbb{R}_+))$ satisfying the following conditions:

- (i) for all $B \in \mathcal{B}(Z \times \mathbb{R}_+)$, $\eta(B) : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;
- (ii) η is independently scattered, i.e. if the sets $B_j \in \mathcal{B}(Z \otimes \mathbb{R}_+)$, $j = 1, \dots, n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, \dots, n$, are independent;
- (iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\mathbb{R}_+)$

$$\mathbb{E}[\eta(U \times I)] = \text{Leb}(I)v(U),$$

where Leb is the Lebesgue measure;

- (iv) for all $U \in \mathcal{Z}$ the $\bar{\mathbb{N}}$ -valued process $(N(U, t))_{t \geq 0}$ defined by $N(U, t) := \eta(U \times (0, t])$, $t \geq 0$, is \mathbb{F} -adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then the random variable $N(U, t) - N(U, s) = \eta(U \times (s, t])$ is independent of \mathcal{F}_s .

We will denote by $\tilde{\eta}$ the *compensated Poisson random measure* defined by $\tilde{\eta} := \eta - \gamma$, where *the compensator* $\gamma : \mathcal{B}(Z \times \mathbb{R}_+) \rightarrow \mathbb{R}_+$ defined by

$$\gamma(A \times I) = \text{Leb}(I)v(A), \quad I \in \mathcal{B}(\mathbb{R}_+), \quad A \in \mathcal{Z}.$$

While items (i) and (ii) are the classical definition, see for e.g. [45, Definition 6.1], of a Poisson Random measure η , the remaining items implicitly indicate that our η is associated to a certain Lévy process \tilde{L} , see, for instance [45, Proposition 4.16].

Throughout this paper an intensity measure is always positive and σ -finite.

Now, let $\eta := \{\eta_1, \eta_2, \dots\}$ be a family of mutually independent Poisson random measures defined on $(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))$ over \mathfrak{P} with intensity measures $\{\nu_1, \nu_2, \dots\}$. Throughout this paper, we denote by $\{\nu_1(dz)dt, \nu_2(dz)dt, \dots\}$ the family of compensators of the elements of η and $\{\tilde{\eta}_1, \tilde{\eta}_2, \dots\}$ the family of compensated Poisson random measures associated to the elements of η . To shorten notation, we will use the following notations $d\eta_j(z, t) := \eta_j(dz, dt)$ and $d\tilde{\eta}_j(z, t) := \tilde{\eta}_j(dz, dt)$ for any $j \in \{1, 2, \dots\}$. We will also use the notation

$$d\tilde{\eta}_j(z, t) = \mathbb{1}_{|z| \leq 1}d\tilde{\eta}_j(z, t) + \mathbb{1}_{|z| > 1}d\eta_j(z, t), \quad j \in \{1, 2, \dots\}.$$

Now, we introduce all the assumptions on noise L . The first of these are given in the following set of conditions which basically implies the strong Feller property of (1) (see Proposition 5.1).

Assumption 2.5. (i) The Poisson random measures η_j , $j \in \mathbb{N}$ are independent and identically distributed. This means in particular that there exists a positive σ -finite measure ν such that

$$\nu_j(dz) = \nu(dz) \quad \text{for any } j = 1, 2, \dots$$

- (ii) There exists a strictly monotone and C^1 function $q : (0, \infty) \rightarrow (0, \infty)$ such that

$$\lim_{r \nearrow \infty} q(r) = 0, \quad \lim_{r \searrow 0} q(r) = 1, \quad \text{and} \quad \nu(dz) = q(|z|)|z|^{-1-\theta} dz, \quad \theta \in [0, 2).$$

Moreover, for any $p \geq 1$ there exist two constants $K_0 > 0$, $K_1 \geq 0$ such that

$$\left| \frac{q'(z)}{q(z)} \right|^p \leq K_0 + K_1 z^{-p}, \quad z \in (0, \infty).$$

(iii) We have $|z|^{1-\theta}q(|z|) \rightarrow 0$ as $|z| \rightarrow \infty$. We also impose that

$$\int_{\mathbb{R}_0} |z|^p \nu(dz) < \infty, \quad \text{for any } p \geq 1. \tag{6}$$

Remark 2.6. Setting $g(x) = q(|x|)|x|^{-1-\theta}$, $\theta \in [0, 2)$, it is easy to see that Assumption 2.5 implies the following items:

(I) There exists a C^1 function $g : \mathbb{R}_0 \rightarrow (0, \infty)$ such that $\nu(dz) = g(z)dz$ and for any $p \geq 1$ there exists a constant $C > 0$ so that

$$\left| \frac{g'(z)}{g(z)} \right|^p \leq C(1 + |z|^{-p}), \quad z \in \mathbb{R}_0.$$

(II) As $|z| \rightarrow \infty$ we have $z^2g(z) \rightarrow 0$ and the Lévy measure ν satisfies (6).

(III) Furthermore, there exists a constant $\theta_1 \in (0, 2]$ such that for any $y \in \mathbb{R}_0$

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{\theta_1} \int_{\mathbb{R}_0} (|z \cdot y/\varepsilon|^2 \wedge 1) \nu(dz) > 0.$$

The items (II)–(III) are very similar to [7, Assumption 2.3(ii)–2.3(iv)] (see also [54, Assumption 1]) and ensure the validity of a Bismut–Elworthy–Li (BEL) formula (see [7, Lemma A.3]) which is used in [7, Proposition 3.6] to prove the strong Feller property of GOY and Sabra shell models. The item (III) is called in some literature the order and non-degeneracy condition (see [54, Remark 2.2]).

The following assumptions, independently of several items of Assumptions 2.5, imply the irreducibility property in H of the mild solution (also known as stochastic convolution) \mathfrak{S} of the problem

$$dZ(t) + \kappa AZ(t) = dL(t), \quad Z(0) = 0. \tag{7}$$

Assumption 2.7. Let ν be the intensity measure in the first part of Assumption 2.5(ii). We suppose that $\theta \in (0, 2)$ and

$$\int_{\mathbb{R}} (1 - q^{\frac{1}{2}}(|z|))^2 \nu(dz) < \infty. \tag{8}$$

Before we state the final assumption for the paper we give some basic examples that satisfy Assumptions 2.5 and 2.7.

Remark 2.8. (a) The function $q(z) = e^{-\beta z}$, for any $z > 0$ and $\beta > 0$ is an example of function satisfying items (ii) and (iii) of Assumption 2.5. Moreover, any measure $\nu(dz) = |z|^{-1-\theta} e^{-\beta|z|}$ satisfies Assumptions 2.5 and 2.7 with $\theta \in (0, 1)$.

(b) The components of the noise in (1) can be replaced with the following ones

$$\ell_k(t) = \sigma W_k(G_k(t)), \quad \sigma > 0, t \in [0, \infty), k \in \mathbb{N}, \tag{9}$$

where $\{W_k; k \in \mathbb{N}\}$ is a family of i.i.d standard Brownian motions and $\{G_k; k \in \mathbb{N}\}$ is a family of i.i.d Gamma processes with Lévy measure $\nu_G(dz) = z^{-1} e^{-z} \mathbb{1}_{z>0} dz$. In fact, it was shown in [30, Chapter 10] that each ℓ_k is a pure jump Lévy noise which is identical in law to a variance gamma process $\tilde{\ell}_k$ having a Lévy measure

$$\nu(dz) = |z|^{-1} e^{-\beta|z|} dz,$$

with $\beta = \sqrt{2}/\sigma$. That is, we are in the situation of symmetric tempered stable process with $\theta = 0$ which satisfies only Assumption 2.5.

The final assumption on our model is the following.

Assumption 2.9. On the family of positive numbers $\{\beta_j; j = 1, 2, \dots\}$ we assume that

$$\sum_{j=1}^{\infty} \left(\beta_j + \beta_j^2 \lambda_j^{-2\vartheta} \right) + \left(\beta_j^2 \lambda_j^{\varepsilon-1} + \beta_j^4 \lambda_j^{\varepsilon} \right) < \infty,$$

for a certain $\varepsilon \in (0, 2)$ and $\vartheta \in [0, \frac{1}{2})$.

To close this subsection we also introduce the following additional notations. For a Banach space B we denote respectively by $B_b(B)$, $C_b(B)$, and $C_b^2(B)$ the space of bounded and measurable functions, the space of continuous and bounded functions, and the space of bounded and twice Fréchet differentiable functions on B and taking values in \mathbb{R} . The supremum norm of a map $\varphi \in B_b(B)$ is denoted by $\|\varphi\|_{\infty}$. For two Banach spaces B_1 and B_2 we denote by $C_b^2(B_1, B_2)$ the space of bounded and twice Fréchet differentiable functions on B_1 and taking values in B_2 . For any measurable space (M, \mathcal{M}) we denote by $L^q(M, B)$, $q \in [1, \infty)$, the space of Bochner integrable functions defined on M and taking values in B . The space $D([0, T]; B)$ denotes the space of all right continuous functions $\ell : [0, T] \rightarrow B$ with left limits. The space $D([0, T]; B)$ equipped with the Skorokhod topology J_1 , which is the finest of Skorokhod topologies, is both separable and complete. For more information about the Skorokhod space and the J_1 -topology we refer to Ethier and Kurtz [21, Chapter 3, Section 5].

Let $\mathcal{P}(B)$ be the set of Borel probability measures on $(B, \mathcal{B}(B))$, where $\mathcal{B}(B)$ is the Borel σ -algebra on B . The total variation distance of two probability measures $\mu_1, \mu_2 \in \mathcal{P}(B)$ is defined by

$$\begin{aligned} \|\mu_1 - \mu_2\|_{TV} &= \frac{1}{2} \sup_{\substack{\varphi \in B_b(B) \\ \|\varphi\|_{\infty}=1}} \left| \int_B \varphi(x) \mu_1(dx) - \int_B \varphi(x) \mu_2(dx) \right| \\ &= \sup_{\Gamma \in \mathcal{B}(B)} |\mu_1(\Gamma) - \mu_2(\Gamma)|. \end{aligned}$$

2.2. Motivating examples. In this subsection we give few examples of evolution equations which can be treated with our results. We will mainly treat the Sabra shell models, the GOY shell models, the 2D Navier–Stokes, Magnetohydrodynamics equations and the 3D Leray- α model of turbulence. To keep the presentation short we will impose the periodic boundary condition on the last three examples.

2.2.1. The 2D Navier–Stokes equations with periodic boundary condition. We consider the Navier–Stokes equations (NSEs) subjected to the periodic boundary condition on the torus $\mathcal{O} = [0, 2\pi]^2$:

$$\begin{cases} du - \kappa \Delta u + u \cdot \nabla u + \nabla p = F, \\ \nabla \cdot u = 0, \\ \int_{\mathcal{O}} u(t, z) dz = 0, \\ u(0) = x, \end{cases} \tag{10}$$

where u and p are unknown vector field and scalar periodic functions in the space variable, representing, respectively, the fluid velocity and the pressure. The term F represents an

external forcing. Finally, x is a given initial velocity. We will briefly outline in this subsection how we put the NSEs in an abstract evolution equation of the form (1); for the detail we refer, among others, to [55] and [25].

Let \mathcal{V} be the set of periodic, divergence free and infinitely differentiable function with zero mean. In what follows, we denote by H and V the closures of \mathcal{V} in $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$, respectively. We also set

$$D(A) = [H^2(\mathcal{O})]^2 \cap V, \quad Av = -\Delta v, \quad v \in D(A).$$

It is well-known that the Stokes operator A is positive self-adjoint with compact resolvent and its eigenfunctions $\{e_1, e_2, \dots\}$, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$, form an orthonormal basis of H . It is also well-known that $V = D(A^{\frac{1}{2}})$, see [25, Appendix A.1 of Chapter II]. Furthermore, we see from [55, Chapter II, Section 1.2] and [25, Appendix A.3 of Chapter II] that one can define a continuous bilinear map \mathbf{B} from $V \times V$ with values in V^* such that

$$\langle \mathbf{B}(u, v), w \rangle = \int_{\mathcal{O}} [u(z) \cdot \nabla v(z)] \cdot w(z) dz \quad \text{for any } u, v, w \in V, \tag{11}$$

$$\langle \mathbf{B}(u, v), v \rangle = 0, \quad \text{for any } u, v \in V, \tag{12}$$

$$|\langle \mathbf{B}(u, v), w \rangle| \leq C_0 \|u\|_{L^4} \|v\|_{L^4} \|w\|, \quad \text{for } u, v \in L^4(\mathcal{O}), w \in V. \tag{13}$$

From the last line along with the embedding $D(A^{1/4}) \subset L^4(\mathcal{O})$ we infer that Assumption 2.2(b) is satisfied with $\mathcal{H} = D(A^{1/4})$.

With all these notations the Navier–Stokes equations (10) can be written in the abstract form

$$\frac{du}{dt} + \kappa Au(t) + \mathbf{B}(u, u) = \Pi F, \tag{14a}$$

$$u(0) = x \in H. \tag{14b}$$

Thanks to the above preliminary results we see that $A, \mathbf{B}(\cdot, \cdot)$ and \mathcal{H} satisfy Assumptions 2.1 and 2.2.

2.2.2. The 2D magnetohydrodynamics equations. In the torus $\mathcal{O} = [0, 2\pi]^2$, the dynamic of an incompressible conducting fluid in presence of a magnetic field is described by the 2D Magnetohydrodynamics (MHD) system

$$\begin{cases} \frac{\partial u}{\partial t} - \kappa_1 \Delta u + (u \cdot \nabla)u + \nabla p + \frac{1}{2} \nabla |m|^2 - (m \cdot \nabla)m = F_1 \\ \frac{\partial m}{\partial t} - \kappa_2 \Delta m + (u \cdot \nabla)m - (m \cdot \nabla)u = F_2, \\ \nabla \cdot u = \nabla \cdot m = 0, \\ \int_{\mathcal{O}} u(x, t) dx = \int_{\mathcal{O}} m(x, t) dx = 0, \\ u(x, 0) = u_0, \quad m(x, 0) = m_0, \end{cases} \tag{15}$$

where $u = (u_1, u_2)$, $m = (m_1, m_2)$ and p are unknown functions defined on $[0, T] \times \mathcal{O}$, representing, respectively, the fluid velocity, the magnetic field and the pressure, at each point of $[0, T] \times \mathcal{O}$. Throughout we assume that u, m and p are periodic functions in the space variable. The terms F_1 and F_2 represent external perturbations acting on the system. Finally, u_0 and m_0 are given initial velocity and magnetic field, respectively.

Since we are assuming that all functions are periodic in the space variable, we can use the same function spaces and notations defined in the subsection for the NSEs. Moreover, following the argument in [52] we can rewrite (15) into the abstract form

$$\frac{d\mathbf{u}}{dt} + \mathbf{A}\mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = F, \tag{16}$$

where $\mathbf{u} = (u, m)$ is the unknown and $F = (\Pi F_1, \Pi F_2)$. Setting $\mathbf{H} = \mathbb{H} \times \mathbb{H}$, $\mathbf{V} = \mathbb{V} \times \mathbb{V}$, the operator \mathbf{A} with domain $D(\mathbf{A}) = D(A) \times D(A)$ and the bilinear map $\mathcal{B} : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}^*$ are respectively defined by

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} \kappa_1 A & 0 \\ 0 & \kappa_2 A \end{pmatrix} \begin{pmatrix} u \\ m \end{pmatrix},$$

for any $\mathbf{u} = (u, m) \in D(\mathbf{A})$, and

$$\begin{aligned} \langle \mathcal{B}(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_3 \rangle &= \langle \mathbf{B}(u_1, u_2), u_3 \rangle - \langle \mathbf{B}(m_1, m_2), u_3 \rangle + \langle \mathbf{B}(u_1, m_2), m_3 \rangle \\ &\quad - \langle \mathbf{B}(m_2, u_2), m_3 \rangle, \end{aligned}$$

for any $\mathbf{u}_i = (u_i, m_i) \in \mathbf{V}$, $i = 1, 2, 3$.

Since A is positive self-adjoint with compact inverse, so is \mathbf{A} . Using the properties we mentioned in the case for the NSEs, it is not difficult to check that $\mathcal{B}(\cdot, \cdot)$ satisfies Assumption 2.2 where $\mathcal{H} = D(A^{1/4}) \times D(A^{1/4})$. Therefore, the MHD model (15) is also one example we can study in this paper. For more information on the mathematical theory of MHD equations we refer, for instance to, [4,52] and references therein.

2.2.3. The 3D Leray- α model with periodic boundary condition. We can also analyse a 3D model, in particular we can treat the 3D Leray- α model with periodic boundary condition. On the 3D torus $\mathcal{O} = [0, 2\pi]^3$ this model is given by

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \kappa \Delta \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u} + \nabla p = F, \\ (\mathbf{I} - \alpha \Delta) \mathbf{v} = \mathbf{u}, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0, \\ \int_{\mathcal{O}} u(t, x) dx = \int_{\mathcal{O}} v(t, x) dx = 0, \\ \mathbf{u}(0) = x. \end{cases} \tag{17}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are unknown vector fields, p is the unknown pressure.

Here we can also use the spaces of functions and notations used in the mathematical theory of NSEs. Moreover, we set $L_\alpha = (\mathbf{I} + \alpha A)^{-1}$ and define a bilinear map $\mathcal{B}(\cdot, \cdot)$ on $\mathbf{V} \times \mathbf{V}$ by setting

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathbf{B}(L_\alpha \mathbf{u}, \mathbf{v}),$$

for any $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v} \in \mathbf{V}$. With these notations we can rewrite the system (17) in the following form

$$\frac{d\mathbf{u}}{dt} + \kappa \mathbf{A}\mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = F, \quad \mathbf{u}(0) = x \in \mathbf{H}. \tag{18}$$

It is proved in [13] that $\mathcal{B}(\cdot, \cdot)$ satisfies the following property

$$\begin{aligned} \langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle &= \langle \mathbf{B}(L_\alpha \mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0 \quad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ |\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle| &\leq C |L_\alpha \mathbf{u}|_{L^6} |\mathbf{v}|_{L^3} \|\mathbf{w}\| \quad \text{for any } \mathbf{u} \in \mathbf{H}, \mathbf{w} \in \mathbf{V}, \mathbf{v} \in L^6. \end{aligned}$$

Using the Sobolev embedding $H^1 \subset L^6$ and the continuity of the map $A^{\frac{1}{2}}L_\alpha : H \rightarrow H^1$ we obtain that

$$|\langle \mathcal{B}(u, v), \mathbf{w} \rangle| \leq C \|u\| \|v\|_{L^3} \|\mathbf{w}\|, \tag{19}$$

which shows the continuity of the bilinear map \mathcal{B} on $V \times V$. We also derive from (19) that $\mathcal{B}(\cdot, \cdot)$ admits an extension, still denoted by the same symbol, on $H \times L^3$. Since $D(A^{\frac{1}{4}}) \subset L^3$, $\mathcal{B}(\cdot, \cdot)$ admits also a continuous extension, denoted again by $\mathcal{B}(\cdot, \cdot)$, on $\mathcal{H} \times \mathcal{H}$, where $\mathcal{H} = D(A^{\frac{1}{4}})$. Thus, A and \mathcal{B} respectively satisfy Assumptions 2.1 and 2.2.

For more information on the 3D Leray- α model we refer to [10, 13, 16] and references therein.

2.2.4. GOY and Sabra shell models of turbulence. In this section we denote by \mathbb{C} the field of complex numbers and $\mathbb{C}^{\mathbb{N}}$ be the set of all \mathbb{C} -valued sequences $(u_n)_{n \in \mathbb{N}}$. Furthermore, we denote by H the set of all $u := (u_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=1}^\infty |u_n|^2 < \infty$. Let k_0 be a positive number and $\lambda_n = k_0 2^n$ be a sequence of positive numbers. We set

$$D(A) = \{u \in H; \sum_{n=1}^\infty \lambda_n^4 |u_n|^2 < \infty\}, \quad Au = (\lambda_n^2 u_n)_{n \in \mathbb{N}}, \quad \text{for } u \in D(A).$$

It is not hard to check that A is a positive self-adjoint operator. It is well-known, see, for instance, [14], that $D(A^r)$, $r \in \mathbb{R}$, can be identified with the set of all sequences $u = (u_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=1}^\infty \lambda_n^{4r} |u_n|^2 < \infty$; the embedding $D(A^r) \subset D(A^{r+\varepsilon})$ is compact for any $r \in \mathbb{R}$ and $\varepsilon > 0$. In what follows we set $V = D(A^{\frac{1}{2}})$.

The evolution equation describing the GOY and Sabra shell models is given by

$$\begin{aligned} \frac{du}{dt} + \kappa Au + \mathbf{B}(u, u) &= F, \\ u(0) &= u_0, \end{aligned} \tag{20}$$

where $F = (F_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is an external perturbation. The map $\mathbf{B}(\cdot, \cdot)$ is a bilinear map defined on $V \times V$ taking values in the dual space V^* and is defined by

$$\begin{aligned} b_n(u, v) &:= (\mathbf{B}(u, v))_n \\ &:= i\lambda_n \left(\frac{1}{4} \bar{v}_{n-1} \bar{u}_{n+1} - \frac{1}{2} (\bar{u}_{n+1} \bar{v}_{n+2} + \bar{v}_{n+1} \bar{u}_{n+2}) + \frac{1}{8} \bar{u}_{n-1} \bar{v}_{n-2} \right), \end{aligned}$$

for the GOY shell model, see for *e.g.* [26], and by

$$\begin{aligned} b_n(u, v) &:= (\mathbf{B}(u, v))_n := \frac{i}{3} \lambda_{n+1} [\bar{v}_{n+1} u_{n+2} + 2\bar{u}_{n+1} v_{n+2}] \\ &\quad + \frac{i}{3} \lambda_n [\bar{u}_{n-1} v_{n+1} - \bar{v}_{n-1} u_{n+1}] \\ &\quad + \frac{i}{3} \lambda_{n-1} [2u_{n-1} v_{n-2} + u_{n-2} v_{n-1}], \end{aligned}$$

for the Sabra shell model, see for *e.g.* [42] and [17].

It was shown in [14, Proposition 1] that the nonlinear term $\mathbf{B}(\cdot, \cdot)$ for the GOY and Sabra shell models satisfies Assumption 2.2 with $\mathcal{H} = H$. For more mathematical results related to shell models we refer to [3, 5, 6, 14] and references therein.

3. The Main Results and Their Applications

In this section we give the main results of the paper. Their proofs are postponed to the two forthcoming sections. We start with the introduction of the notion of solution.

Definition 3.1. An \mathbb{F} -adapted process u is called a weak solution of Eq. (1) if the following conditions are satisfied

- (i) $u \in L^2(0, T; V)$ \mathbb{P} -almost surely,
- (ii) the following equality holds for every $t \in [0, T]$ and \mathbb{P} -a.s.

$$(u(t), \phi) = (x, \phi) - \int_0^t (\kappa Au(s) + \mathbf{B}(u(s), u(s)), \phi) ds + \langle \phi, L(t) \rangle, \tag{21}$$

for any $\phi \in V$.

Remark 3.2. (a) Owing to Assumption 2.2 the nonlinear term $\int_0^t \langle \mathbf{B}(u(s), u(s)), \phi \rangle ds$ makes sense whenever $\phi \in V$ and $u \in L^2(0, T; V)$.

(b) For any $t \geq 0$ let

$$\ell_k(t) := \int_0^t \int_{\mathbb{R}_0} z d\tilde{\eta}_k(z, s), k \in \mathbb{N}.$$

We can rewrite the definition of L as follows

$$L(t) = \sum_{j=1}^{\infty} \tilde{\ell}_j(t) \lambda_j^\vartheta e_j,$$

where each $\tilde{\ell}_j = \sigma_j \ell_j$ is a Lévy process with Lévy measure μ_j defined by $\mu_j(dz_j) = \nu(\sigma_j^{-1} dz)$ with $\sigma_j = \beta_j \lambda_j^{-\vartheta}$. Now, thanks to Assumption 2.9 and the fact that $\{\lambda_j^\vartheta e_j; j \in \mathbb{N}\}$ is an orthonormal basis of $D(A^{-\vartheta})$, we derive from [45, Theorem 4.40] that the Lévy process L lives in $D(A^{-\vartheta})$ with $\vartheta \in (0, \frac{1}{2})$. Thus $\langle \phi, L(t) \rangle$ makes sense for any $\phi \in V$.

We recall the following result which can be proved using similar ideas as in [7, Proofs of Proposition 3.3 and Lemma 3.9].

Proposition 3.3. *If, in addition to Assumptions 2.1, 2.2, 2.9 and Assumption 2.5(i), the estimate (6) is verified, then the problem (1) has a unique solution $u \in D([0, T]; H)$ and there exists two constants $c_0, c_1 > 0$ such that*

$$\mathbb{E} \sup_{t \in [0, T]} |u(t, x)|^2 + \mathbb{E} \int_0^t |A^{\frac{1}{2}} u(s, x)|^2 ds \leq c_0 e^{c_1 t} (1 + |x|^2). \tag{22}$$

In particular, there exists a constant $\tilde{C} > 0$ such that for any $t > 0$ and $x \in H$ we have

$$\mathbb{E} |u(t, x)|^2 \leq (|x|^2 + \tilde{C}t) e^{-\frac{\kappa}{\lambda_1} t}, \tag{23}$$

$$\mathbb{E} \int_0^t |A^{\frac{1}{2}} u(s, x)|^2 ds \leq (|x|^2 + \tilde{C}t) + \tilde{C} \frac{\lambda_1}{\kappa}. \tag{24}$$

Moreover, u is a Markov process having the Feller property in H .

The result in Proposition 3.3 enables us to define a Markov semigroup as in the following definition.

Definition 3.4. Let $\mathcal{P}_t, t \geq 0$, be the Markov semigroup defined by

$$[\mathcal{P}_t \Phi](x) = \mathbb{E}[\Phi(u(t, x))], \quad \Phi \in B_b(\mathbb{H}), x \in \mathbb{H}, t \geq 0,$$

where $u(\cdot, x)$ is the unique solution to (1) with initial condition $x \in \mathbb{H}$. Throughout this paper, for simplicity we will write

$$\mathcal{P}_t \Phi(x) := [\mathcal{P}_t \Phi](x), \quad \Phi \in B_b(\mathbb{H}), x \in \mathbb{H}, t \geq 0.$$

We also denote by $\mathcal{P}_t^*, t \geq 0$, the dual semigroup acting on $\mathcal{P}(\mathbb{H})$ of the Markov semigroup $\mathcal{P}_t, t \geq 0$.

The first of our main results is stated in the following theorem. The proof of this result, which needs some new tools that are interesting in themselves, will be carried out in Sect. 4.

Theorem 3.5. *For any positive number R and $a \in \mathbb{H}$ we set*

$$\Gamma_{a,R} = \{v \in \mathbb{H}; |v - a| < R\}$$

Suppose that Assumptions 2.1, 2.2, 2.7 and 2.9 are verified. Then, for any $x \in \mathcal{H}, a \in \mathbb{H}$ and any positive number $\varepsilon > 0$ we have

$$\mathcal{P}_t \mathbb{1}_{\Gamma_{a,\varepsilon}}(x) = \mathbb{P}(|u(t, x) - a| < \varepsilon) > 0,$$

for any $t > 0$.

We apply the above theorem to infer the irreducibility of the 2D Navier–Stokes (NSEs), Magnetohydrodynamics (MHD) equations and the 3D Leray- α model driven by a pure jump Lévy process L . These models were introduced in Sects. 2.2.1, 2.2.2, and 2.2.3, respectively.

Corollary 3.6. *1. Let us consider the NSEs and MHD equations driven by a Lévy process L defined by (2). Assume that Assumption 2.7 is satisfied and suppose that $\beta_k = \lambda_k^{-(1+\gamma)}$ where $\gamma > 0$ is a small number and $(\lambda_k)_{k \in \mathbb{N}}$ is the family of eigenvalues of the 2D Stokes operator with periodic boundary condition. If $u(\cdot, x)$ and $(u(\cdot, x), m(\cdot, y))$ are the solution of the stochastic NSEs and MHD equations starting at x and (x, y) , respectively, then for any $x \in D(A^{\frac{1}{4}})$ (resp. $(x, y) \in D(A^{\frac{1}{4}}) \times D(A^{\frac{1}{4}})$) and $a \in \mathbb{H}$ (resp. $(a, c) \in D(A^{\frac{1}{4}}) \times D(A^{\frac{1}{4}})$) and $\varepsilon > 0$ we have*

$$\begin{aligned} &\mathbb{P}(|u(t, x) - a| < \varepsilon) > 0, \\ &\mathbb{P}(|u(t, x) - a| + |m(t, y) - c| < \varepsilon) > 0, \end{aligned}$$

for any $t > 0$.

2. Now, we consider the case of the 3D Leray- α model (17). We suppose that the noise entering the system is as above but with $\beta_k = \lambda_k^{-(\frac{3}{2}+\gamma)}$, where $\gamma > 0$ is a small number and $(\lambda_k)_{k \in \mathbb{N}}$ is the family of eigenvalues of the 3D Stokes operator with periodic boundary condition. If Assumption 2.7 is satisfied and if $u(\cdot, x)$ is the solution of the 3D stochastic Leray- α model (17), then for any $\varepsilon > 0$, $x \in D(A^{\frac{1}{4}})$, and $a \in \mathbf{H}$ we have

$$\mathbb{P}(|u(t, x) - a| < \varepsilon) > 0,$$

for any $t > 0$.

Proof. We have seen in Sect. 2.2 that both the 2D NSEs, MHD equations and the 3D Leray- α model can be written as an abstract evolution equation of the form (1). Moreover, we have also seen in Sect. 2.2 that the linear and nonlinear terms involved in these systems satisfy Assumptions 2.1 and 2.2 with $\mathcal{H} = D(A^{\frac{1}{4}})$ for the NSEs and 3D Leray- α model, and $\mathcal{H} = D(A^{\frac{1}{4}}) \times D(A^{\frac{1}{4}})$ for the MHD equations. Now, notice that $\lambda_k \sim k^{\frac{2}{d}}$, $d = 2, 3$, thus, owing to our assumption we have

$$\sum_{k=1}^{\infty} (\beta_k + \beta_k^2 \lambda_k^{\varepsilon-1} + \beta_k^4 \lambda_k^{\varepsilon}) < \infty,$$

for any $\varepsilon \in (0, 2)$ if $d = 2$ and $\varepsilon \in (0, 1)$ if $d = 3$. It is clear that if the sequence $\{\beta_k; k \in \mathbb{N}\}$ is defined as above then $\sum_{k=1}^{\infty} \beta_k^2 \lambda_k^{-2\vartheta} < \infty$ for any $\vartheta \in [0, \frac{1}{2})$. Therefore, the 2D NSEs, MHD equations and the 3D Leray- α model satisfy Assumption 2.9. Now, the corollary follows from the application of Theorem 3.5. \square

We will show in the next theorem, which is our second main result and whose proof will be carried out in Sect. 5, that one can say more about the Markov semigroup associated when much stronger conditions than in Theorem 3.5 are imposed on the nonlinear term $\mathbf{B}(\cdot, \cdot)$.

Theorem 3.7. *In addition to Assumptions 2.1, 2.5, 2.7 and 2.9, we also suppose that Assumption 2.2 is satisfied with $\mathcal{H} = \mathbf{H}$ and the sequence $\{\beta_j; j \in \mathbb{N}\}$ satisfies*

$$\sum_{j=1}^{\infty} \beta_j^{-2} \lambda_j^{-1} < \infty. \tag{25}$$

Then, the system (1) is ergodic and exponential mixing. That means, there exists a unique $\mu \in \mathcal{P}(\mathbf{H})$ and two constants $C, \tilde{C} > 0$ such that for any measure $\mathfrak{m} \in \mathcal{P}(\mathbf{H})$, we have

$$\|\mathcal{P}_T^* \mathfrak{m} - \mu\|_{\text{TV}} \leq \tilde{C} e^{-CT} \left(1 + \int_{\mathbf{H}} |x|^2 \mathfrak{m}(dx) \right),$$

for any $T > 0$.

The nonlinear terms of the 2D Navier–Stokes, MHD and 3D Leray- α models do not satisfy the assumption of the above theorem. However, we can apply Theorem 3.7 to the GOY and Sabra shell models, because their nonlinear terms verify Assumption 2.2 with $\mathcal{H} = \mathbf{H}$.

Corollary 3.8. *Let us consider the GOY and Sabra shell models driven by a Lévy process L defined by (2). Assume that Assumptions 2.5 and 2.7 are satisfied and suppose that $\beta_k = \lambda_k^{-\gamma}$ where $\gamma \in (0, \frac{1}{2})$ is a real number and $(\lambda_k)_{k \in \mathbb{N}} = (k_0 2^k)_{k \in \mathbb{N}}$ is the family of eigenvalues of the operator A defined in Sect. 2.2.4. Then, the semigroup $\mathcal{P}_t, t \geq 0$, associated to the unique solution of the shell models admits a unique invariant measure μ whose support is included in V . Moreover, there exist two constants $C, \tilde{C} > 0$ such that for any measure $m \in \mathcal{P}(\mathbb{H})$, we have*

$$\|\mathcal{P}_T^* m - \mu\|_{TV} \leq \tilde{C} e^{-CT} \left(1 + \int_{\mathbb{H}} |x|^2 m(dx) \right),$$

for any $T > 0$.

Proof. Owing to the compact embedding $V \subset H$ and the estimates (23) and (24) the existence of an invariant measure μ follows from the Krylov–Bogolyubov theorem, see, for instance, the proofs in [12, Theorem 2.2] or [29, Theorem 5.3]. One can also argue as in [29, Theorem 5.3] to show that the support of μ is included in V .

We have seen in Sect. 2.2.4 that the GOY and Sabra shell models satisfy Assumptions 2.1 and 2.2 with $\mathcal{H} = H$. It is not difficult to check that Assumption 2.9 is satisfied when $(\lambda_k)_{k \in \mathbb{N}} = (k_0 2^k)_{k \in \mathbb{N}}$ and $\beta_k = \lambda_k^{-\gamma}$ for any real number $\gamma > 0$, in particular for $\gamma \in (0, \frac{1}{2})$ it is easy to see that (25) is verified. Hence, the uniqueness and the exponential convergence follows from Theorem 3.7. \square

Remark 3.9. It was proved in [7, Theorem 3.10] that (1), with nonlinearity \mathbf{B} satisfying Assumption 2.2(b) with $\mathcal{H} = H$, admits a unique invariant measure provided that for each $k \in \mathbb{N}$ the Lévy process $\ell_k(t)$ satisfies the small deviation property, that is, for any $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(\sup_{t \in [0, T]} |\ell_k(t)| < \varepsilon) > 0, \quad k \in \mathbb{N}.$$

Thanks to [53, Théorème 1, pp. 157], [2, Proposition 1.1], any Lévy process having intensity measure ρ satisfying $\int_{|z| \leq 1} z \rho(dz) = 0$, which is verified by the intensity measure ν in Remark 2.8(b), has the small deviation property. Thus, the assumptions of [7, Theorem 3.10] are much weaker than those of Corollary 3.8 and allow us to take a large class of tempered stable processes. However, we saw in the previous theorem that we get stronger result (exponential mixing) under the much restrictive assumptions of Corollary 3.8.

4. Support Theorem and Irreducibility for Some Stochastic Evolution Equations

This section is devoted to the proof of Theorem 3.5. We will start with the statements and proofs of some support and irreducibility theorems for finite and infinite dimensional Ornstein–Uhlenbeck processes with tempered stable Lévy noise. These results, which are part of our main results, are new and interesting in themselves.

4.1. Support theorem and irreducibility for O - U process with tempered stable Lévy noise. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. It is well-known (see, for instance, [51, Corollary 8.3]) that the characteristic function of any \mathbb{R}^d -valued Lévy process $\{\ell(t); t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ has the Lévy -Khintchine representation

$$\mathbb{E}[e^{i\langle x, \ell(t) \rangle}] = e^{t\psi(x)}, \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{26}$$

$$\psi(x) = -\frac{1}{2} \langle Mx, x \rangle + i \langle \varkappa, x \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i \langle x, z \mathbf{1}_{|z| \leq 1} \rangle) \nu(dz), \tag{27}$$

where $\varkappa \in \mathbb{R}^d$, M is a symmetric nonnegative-definite $d \times d$ -matrix, ν is a positive σ -finite measure on \mathbb{R}^d satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} [|x|^2 \wedge 1] \nu(dx) < \infty.$$

The triplet (M, ν, \varkappa) is called the generating triplet of the Lévy process $\{\ell(t); t \geq 0\}$. In this subsection we will mainly work with real-valued symmetric tempered stable process (TSP) whose generating triplet is $(0, \nu, 0)$ and the intensity measure ν satisfies Assumption 2.7. We state and prove our third main result in the following theorem.

Theorem 4.1. *Let $\kappa > 0$ be a real number and $T \in (0, \infty)$ be fixed. Let $\{\ell(t); t \geq 0\}$ be a TSP process with generating triplet $(0, \nu, 0)$ and $\{y(t); t \in [0, T]\}$ be the real-valued stochastic convolution solving*

$$dy(t) = -\kappa y(t)dt + d\ell(t), \quad y(0) = 0.$$

If the measure ν satisfies Assumption 2.7, then for any $p > 0$ the pair $(y, y(T))$ has full support in $L^p(0, T; \mathbb{R}) \times \mathbb{R}$. The stochastic convolution is irreducible on \mathbb{R} , that is, for any open set $\mathcal{O} \subset \mathbb{R}$ and $t > 0$ we have $\mathbb{P}(y(t) \in \mathcal{O}) > 0$.

In particular, the above results hold for any real-valued stochastic convolution y driven by a Lévy process with intensity measure ν as in Remark 2.8(a) but with $\theta \in (0, 2)$.

The proof of this theorem relies on the following transformation theorem for Lévy density which is a corollary of [51, Theorem 33.2] or [51, Theorem 33.1].

Lemma 4.2. *Let $\{\ell(t); t \geq 0\}$ be a TSP with generating triple $(0, \nu, 0)$. If the measure ν satisfies all the assumptions of Theorem 4.1, then there exists a new probability measure \mathbf{P} under which the symmetric TSP $\{\ell(t); t \geq 0\}$ is a stable process with generating triplet $(0, \varrho, 0)$ with $\varrho(dx) = |x|^{-1-\theta} dx$. Moreover, the measures \mathbb{P} and \mathbf{P} are equivalent.*

Now, we proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\varepsilon > 0$, $\phi \in L^p(0, T; \mathbb{R})$ and $x \in \mathbb{R}$. Let C_ε be the set

$$C_\varepsilon = \{\omega; \|y(\cdot, \omega) - \phi\|_{L^p(0, T; \mathbb{R})} + |y(\cdot, \omega) - x| \leq \varepsilon\}.$$

Owing to Assumption 2.7 we can apply Lemma 4.2 to infer that there exists a probability measure \mathbf{P} under which ℓ is a stable process. Since ℓ is under \mathbf{P} a stable process, the stochastic convolution y is an Ornstein–Uhlenbeck process driven by a stable process under \mathbf{P} . Thus, we derive from [46, Proposition 4.8] that under \mathbf{P} the random variable $(y, y(T))$ has full support on $L^p(0, T; \mathbb{R}) \times \mathbb{R}$, in particular, $\mathbf{P}(C_\varepsilon) > 0$. Now, let us assume that $\mathbb{P}(C_\varepsilon) = 0$. From this assumption and the equivalence of the measures \mathbb{P} and \mathbf{P} (see Lemma 4.2), we infer that $\mathbf{P}(C_\varepsilon) = 0$, which contradicts the fact that $\mathbf{P}(C_\varepsilon) > 0$. Therefore, $\mathbb{P}(C_\varepsilon) > 0$, which completes the proof of the lemma. \square

Now we will study the irreducibility property of the mild solution \mathfrak{S} to the problem 7. From Assumption 2.9 and an application of [44, Corollary 3.3] we infer that (7) has a unique solution \mathfrak{S} (called stochastic convolution) defined by

$$\mathfrak{S}(t) = \sum_{k=1}^{\infty} \mathfrak{S}_k(t) e_k, \tag{28}$$

where each \mathfrak{S}_k is the solution to

$$d\mathfrak{S}_k(t) = -\kappa \lambda_k \mathfrak{S}_k(t) dt + \beta_k \int_{\mathbb{R}_0} z d\bar{\eta}_k(dz, dt). \tag{29}$$

Moreover, $\mathfrak{S} \in D([0, T]; H)$ with probability 1 and

$$\mathbb{E} \sup_{t \in [0, T]} |\mathfrak{S}(t)|^2 < \infty.$$

Now we state and prove our fourth main theorem.

Theorem 4.3. *Let $\mathfrak{S}(\cdot)$ be the stochastic convolution defined by (28)-(29).*

- (i) *If Assumptions 2.1, 2.7 and 2.9 are verified, then the random variable $(\mathfrak{S}, \mathfrak{S}(T))$ has full support in $L^p(0, T; H) \times H$ for any $p > 0$.*
- (ii) *If, in addition to all hypotheses of item (i), we suppose that condition (6) of Assumption 2.5(iii) is satisfied, then the stochastic convolution \mathfrak{S} has full support in $L^4(0, T; \mathcal{H})$.*

As a direct consequence of Theorem 4.3(i), we have the following corollary.

Corollary 4.4. (i) *If the assumptions of Theorem 4.3(i) are satisfied, then the stochastic convolution \mathfrak{S} is irreducible on H , i.e., for any open set $\mathcal{O} \subset H$ and $t > 0$ we have $\mathbb{P}(\mathfrak{S}(t) \in \mathcal{O}) > 0$. In particular, the current result and those in Theorem 4.3(i) hold for the stochastic convolution \mathfrak{S} whenever each $\ell_k, k \in \mathbb{N}$, has an intensity measure ν as in Remark 2.8(a) but with $\theta \in (0, 2)$.*

- (ii) *The result of Theorem 4.3(ii) holds whenever the intensity measure ν of each $\ell_k, k \in \mathbb{N}$ is as in Remark 2.8(a).*

Remark 4.5. One should note that, since the support of our symmetric Lévy measures contains 0 and the Assumptions 2.1 and (i) are satisfied, the item (ii) of the above corollary can be also proved by using a simple yet powerful result which is established in [47, Theorem 3.3].

Before we proceed to the proof of Theorem 4.3 let us state the following lemma whose proof is postponed to the end of the current subsection.

Lemma 4.6. *If, in addition to Assumptions 2.1 and 2.9, the estimate (6) in Assumption 2.5(iii) is verified, then the stochastic convolution \mathfrak{S} belongs to $L^4(0, T; \mathcal{H})$ almost surely.*

Now, we are ready to give the promised proof of Theorem 4.3.

Proof of Theorem 4.3. Proof of item (i). We have already seen that if Assumptions 2.1 and 2.9 hold, then $\mathfrak{S}(\cdot) = \sum_{k=1}^{\infty} \mathfrak{S}_k(\cdot)e_k$ belongs to $D([0, T]; \mathbb{H})$, hence to $L^p(0, T; \mathbb{H})$ for any $p > 0$. We will only prove the result for $p > 2$ which implies the case $p \in (0, 2]$. For this purpose let $N > 0$ be an integer, $(\phi_k)_{k=1}^N$ (resp. $(a_k)_{k=1}^N$) be a family of continuous functions (resp. real numbers). We set

$$\phi(\cdot) = \sum_{k=1}^N \phi_k(\cdot)e_k \quad \text{and} \quad a = \sum_{k=1}^N a_k e_k.$$

For any $\varepsilon > 0$ we also put

$$\begin{aligned} C_\varepsilon &= \left\{ \omega; \int_0^T \left(\sum_{k=1}^N |\phi_k(t) - \mathfrak{S}_k(t, \omega)|^2 \right)^{\frac{p}{2}} dt < \varepsilon, \sum_{k=1}^N |a_k - \mathfrak{S}_k(T, \omega)|^2 < \varepsilon \right\}, \\ \tilde{D}_\varepsilon &= \left\{ \omega; \sum_{k=1}^N \int_0^T |\phi_k(t) - \mathfrak{S}_k(t, \omega)|^p dt < \frac{\varepsilon}{N^{\frac{p}{2}-1}}, \sum_{k=1}^N |a_k - \mathfrak{S}_k(T, \omega)|^2 < \varepsilon \right\}, \\ D_{\varepsilon,k} &= \left\{ \omega; \int_0^T |\phi_k(t) - \mathfrak{S}_k(t, \omega)|^p dt < \frac{\varepsilon}{N^{\frac{p}{2}}}, |a_k - \mathfrak{S}_k(T, \omega)|^2 < \frac{\varepsilon}{N} \right\}. \end{aligned}$$

Thanks to the inequality

$$\left(\sum_{k=1}^N b_k^2 \right)^{\frac{p}{2}} \leq N^{\frac{p}{2}-1} \sum_{k=1}^N b_k^p, \quad b_k > 0 \forall k \in \mathbb{N},$$

we obtain

$$\mathbb{P}(C_\varepsilon) \geq \mathbb{P}(\tilde{D}_\varepsilon) \geq \mathbb{P}(\cap_{k=1}^N D_{\varepsilon,k}).$$

Since $(\ell_k)_{k=1}^N$ is a family of i.i.d Lévy processes, the members of the sequence $(\mathfrak{S}_k)_{k=1}^N$ (resp. $(\mathfrak{S}(T))_{k=1}^N$) are also mutually independent. Therefore,

$$\mathbb{P}(C_\varepsilon) \geq \prod_{k=1}^N \mathbb{P}(D_{\varepsilon,k}),$$

from which along with Theorem 4.1 we infer that $\mathbb{P}(C_\varepsilon) > 0$. Now, by using a standard density argument we obtain

$$\mathbb{P} \left(\|\phi - \mathfrak{S}\|_{L^p(0,T;\mathbb{H})}^p < \varepsilon, |a - \mathfrak{S}(T)|^2 < \varepsilon \right) > 0,$$

for any $\phi \in L^p(0, T; \mathbb{H})$, $a \in \mathbb{H}$ and $\varepsilon > 0$. From the last estimate we easily conclude the proof of part (i).

Proof of item (ii). Since, in addition to Assumptions 2.1 and 2.9, the estimate (6) in Assumption 2.5(iii) is verified, we infer from Lemma 4.6 that the stochastic convolution \mathfrak{S} belongs to $L^4(0, T; \mathcal{H})$ almost surely. Moreover, from Remark 2.3 we can rewrite \mathfrak{S} in the following way

$$\mathfrak{S}(\cdot) = \sum_{k=1}^{\infty} \gamma_k^{-1} \mathfrak{S}_k(\cdot) \varphi_k = \sum_{k=1}^{\infty} \tilde{\mathfrak{S}}_k(\cdot) \varphi_k,$$

where each real-valued process $\tilde{\mathfrak{S}}_k$ is a stochastic convolution solving

$$d\tilde{\mathfrak{S}}_k(t) + \kappa \lambda_k \tilde{\mathfrak{S}}_k = \gamma_k^{-1} \beta_k \int_{\mathbb{R}_0} z d\tilde{\eta}_k(dz, dt).$$

Now we can argue as above to show that for any $\varepsilon > 0$, $\Phi \in L^4(0, T; \mathcal{H})$ we have

$$\mathbb{P}(\|\mathfrak{S} - \Phi\|_{L^4(0, T; \mathcal{H})} < \varepsilon) > 0,$$

from which we conclude the proof of (ii) and Theorem 4.3. □

The proof of Lemma 4.6 is given below.

Proof of Lemma 4.6. A weak solution to (7) is a stochastic process $Z \in L^2(0, T; V)$ almost surely such that for all t and almost surely

$$\langle Z(t), \Phi \rangle + \kappa \int_0^t \langle AZ(s), \Phi \rangle ds = \langle L(t), \Phi \rangle,$$

for any $\Phi \in V$. If Assumptions 2.1 and the estimate (6) in Assumption 2.5(iii) are verified, then, by using the Galerkin approximation, one can prove as in Proposition 3.3, see also [9], that (7) has a unique weak solution Z satisfying

$$\mathbb{E} \sup_{t \in [0, T]} |Z(t)|^2 + \kappa \mathbb{E} \int_0^T |A^{\frac{1}{2}} Z(t)|^2 dt \leq C,$$

for some positive constant $C > 0$. From the above estimate and the inequality (5) in Remark 2.3 we infer that

$$\mathbb{E} \left(\int_0^T \|Z(t)\|_{\mathcal{H}}^4 dt \right)^{\frac{1}{2}} \leq \left(\mathbb{E} \sup_{t \in [0, T]} |Z(t)|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T |A^{\frac{1}{2}} Z(t)|^2 dt \right)^{\frac{1}{2}} \leq C. \tag{30}$$

Thanks to [45, Theorem 9.15], any weak solution to (7) is also a mild solution (7). Hence Z is also a mild solution of (7), and by uniqueness of the mild solution we have $Z = \mathfrak{S}$ almost surely. From Assumption (2.9) and (30) we infer that $\mathfrak{S} \in D([0, T]; H) \cap L^4(0, T; \mathcal{H})$ and

$$\mathbb{E} \left(\int_0^T \|\mathfrak{S}(t)\|_{\mathcal{H}}^4 dt \right)^{\frac{1}{2}} \leq C,$$

from which we conclude the proof of the lemma. □

4.2. Irreducibility of the problem (1): Proof of Theorem 3.5. In this subsection we will prove that the stochastic model (1) is irreducible provided that Assumptions 2.1 to 2.9 are satisfied. For this purpose, let us fix $x \in H$ and $Z \in L^4(0, T; \mathcal{H})$, and consider the problem

$$\frac{d\mathbf{v}(t)}{dt} + \kappa A\mathbf{v}(t) + \mathbf{B}(\mathbf{v}(t) + Z(t), \mathbf{v}(t) + Z(t)) = 0, \quad \mathbf{v}(0) = x \in H. \tag{31}$$

We have the following existence and uniqueness result.

Lemma 4.7. *For any $x \in \mathbf{H}$ and $Z \in L^4(0, T; \mathcal{H})$, there exists a unique solution $\mathbf{v}(\cdot, x) \in C([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ to (31). Moreover, there exists a constant $C > 0$, independent of x and Z , such that*

$$\sup_{s \in [0, T]} |\mathbf{v}(s, x)|^2 + \kappa \int_0^T |A^{\frac{1}{2}} \mathbf{v}(s, x)|^2 ds \leq C \left(|x|^2 + \int_0^T \|Z(s)\|_{\mathcal{H}}^4 ds \right) e^{C \int_0^T \|Z(s)\|_{\mathcal{H}}^4 ds}. \tag{32}$$

Proof. The proof of existence and uniqueness follows the same lines as in [24, Appendix] or [8, Proof of Theorem 4.5]. The estimate (32) can be proved using the same idea as in [8, Proof of (5.5)] or as in [24, Proof of (21)]. \square

Remark 4.8. Note that if Z is the stochastic convolution \mathfrak{S} defined by (28)-(29) then the process $\mathbf{v} + \mathfrak{S}$ is the unique solution to (1).

In the next lemma we establish the continuous dependence of \mathbf{v} on $Z \in L^4(0, T; \mathcal{H})$.

Lemma 4.9. *Let $\{Z_n; n \in \mathbb{N}\} \subset L^4(0, T; \mathcal{H})$ and $\{\mathbf{v}_n; n \in \mathbb{N}\} \subset C([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ be two sequences such that for each $n \in \mathbb{N}$ the function $\mathbf{v}_n(\cdot, x)$ is the unique solution to (31) with Z replaced by Z_n . If the sequence $\{Z_n; n \in \mathbb{N}\}$ converges in $L^4(0, T; \mathcal{H})$ to an element $Z \in L^4(0, T; \mathcal{H})$, then the sequence $\{\mathbf{v}_n(\cdot, x); n \in \mathbb{N}\}$ converges in $C([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ to the unique solution $\mathbf{v}(\cdot, x)$ of (31).*

Proof. We omit the proof because it follows the same lines as the proof of [8, Theorem 4.6]. \square

For $c \in L^4(0, T; \mathcal{H})$ and $x \in \mathbf{H}$, let $y_c(\cdot, x)$ be the mild solution of (31) with Z replaced by c . As in [24], we also set

$$u_c(t, x) = y_c(t, x) + c(t), \quad \text{for } t \geq 0. \tag{33}$$

We also need the following lemma whose proof is given after the proof of Theorem 3.5.

Lemma 4.10. *For any $x, x_f \in \mathcal{H}$ there exists $c \in L^4(0, T; \mathcal{H}) \cap C([0, T]; \mathbf{H})$ such that $u_c(T, x) = x_f$.*

After these preparatory lemmata we are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. Let $u(\cdot, x)$ be the solution of (1) and $\mathbf{v}(\cdot, x)$ be that of (31) with Z replaced by the stochastic convolution \mathfrak{S} . Since, by Remark 2.3, the embedding $\mathcal{H} \subset \mathbf{H}$ is dense, one can find $\bar{a} \in \mathcal{H}$ such that $|\bar{a} - a| < \frac{\varepsilon}{3}$. Next, from Lemma 4.10 we infer the existence of a control $c \in L^4(0, T; \mathcal{H}) \cap C([0, T]; \mathbf{H})$ such that $u_c(T, x) = \bar{a}$ where $u_c(\cdot, x)$ is defined by (33). Thus, using the decomposition $u(\cdot, x) = \mathbf{v}(\cdot, x) + \mathfrak{S}$ and the definition of $u_c(\cdot, x)$ we infer that

$$|u(T, x) - a| \leq |\mathbf{v}(T, x) - y_c(T, x)| + |\mathfrak{S}(T) - c(T)| + \frac{\varepsilon}{3}.$$

Owing to Lemma 4.9, for any $\varepsilon > 0$ one can find $\delta > 0$ such that if $\|\mathfrak{S} - c\|_{L^4(0, T; \mathcal{H})} < \delta$, then $\|\mathbf{v} - y_c\|_{C(0, T; \mathbf{H})} < \frac{\varepsilon}{3}$. Hence, for $\delta_1 < \min\{\frac{\varepsilon}{3}, \delta\}$ we have

$$\begin{aligned} \mathbb{P}(|u(T, x) - a| < \varepsilon) &\geq \mathbb{P}(\|\mathbf{v} - y_c\|_{C(0, T; \mathbf{H})} + |\mathfrak{S}(T) - c(T)| < \frac{2\varepsilon}{3}) \\ &\geq \mathbb{P}\left(\|\mathfrak{S} - c\|_{L^4(0, T; \mathcal{H})} < \delta, |\mathfrak{S}(T) - c(T)| < \frac{\varepsilon}{3}\right) \\ &\geq \mathbb{P}(\|\mathfrak{S} - c\|_{L^4(0, T; \mathcal{H})} + |\mathfrak{S}(T) - c(T)| < \delta_1). \end{aligned}$$

Since $\mathfrak{c} \in C([0, T]; \mathbf{H}) \cap L^4(0, T; \mathcal{H})$, we have $\mathfrak{S}_\mathfrak{c}(T) \in \mathbf{H}$ which altogether with Theorem 4.3 implies

$$\mathbb{P}(|\mathfrak{u}(T, x) - a| < \varepsilon) > \mathbb{P}(\|\mathfrak{S} - \mathfrak{c}\|_{L^4(0, T; \mathcal{H})} + |\mathfrak{S}(T) - \mathfrak{c}(T)| < \delta_1) > 0,$$

from which we easily conclude the proof of Theorem 3.5. □

The promised proof of Lemma 4.10 is given below.

Proof of Lemma 4.10. We will closely follow [24, Proof of Lemma 5.3-(c)]. Let $x, x_f \in \mathcal{H}$ and $T > 0$. We take arbitrary $t_0, t_1 \in (0, T)$ such that $t_0 < t_1$. We define a function X by

$$\begin{aligned} X(t) &= e^{-\kappa t A} x, \quad t \in [0, t_0], \\ X(t) &= e^{-\kappa(T-t)A} x_f, \quad t \in [t_1, T], \\ X(t) &= X(t_0) + \frac{t - t_0}{t_1 - t_0} (X(t_1) - X(t_0)), \quad t \in (t_0, t_1). \end{aligned}$$

It is clear that $X \in C([0, T]; \mathcal{H})$ which along with Assumption 2.2(b) implies that $-\mathbf{B}(X, X) \in L^2(0, T; \mathbf{V}^*)$. With this in mind, by using standard Galerkin and compactness methods, we easily prove that the linear problem

$$\frac{dY(t)}{dt} + \kappa AY(t) = -\mathbf{B}(X(t), X(t)), \quad Y(0) = x. \tag{34}$$

has a unique solution $Y \in C([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$. The inequality (5) implies that $Y \in L^4(0, T; \mathcal{H})$. Now, the function $\mathfrak{c} := X - Y \in L^4(0, T; \mathcal{H}) \cap C([0, T]; \mathbf{H})$ satisfies all the requirements of the lemma. □

5. Proof of Theorem 3.7: Exponential Mixing by Coupling Method

In this section we will prove the exponential ergodicity stated in Theorem 3.7. For this purpose we will use the coupling method and closely follow [38]. The section is divided into two steps. The first one consist of the proof of a crucial preparatory proposition and the second the actual proof of Theorem 3.7.

Before proceeding further, we shall introduce few notations and concepts that we need in this section. For each $n \in \mathbb{N}$ let $\mathbf{H}_n := \text{Linspan}\{e_1, \dots, e_n\}$ and $\Pi_n : \mathbf{V}^* \rightarrow \mathbf{H}_n$ be the orthogonal projection defined by

$$\Pi_n \mathfrak{v} := \sum_{k=1}^n \langle \mathfrak{v}, e_k \rangle e_k, \quad \text{for any } \mathfrak{v} \in \mathbf{V}^*.$$

Throughout this paper, we will identify \mathbf{H}_n with \mathbb{R}^n .

We will need the following system of SDEs which is nothing but the Galerkin approximation of (1):

$$d\mathfrak{u}_n(t) + [\kappa A \mathfrak{u}_n(t) + \Pi_n \mathbf{B}(\mathfrak{u}_n(t), \mathfrak{u}_n(t))]dt = \sum_{k=1}^n \beta_k d\ell_k(t) e_k, \quad \mathfrak{u}_n(0) = \Pi_n x, \tag{35}$$

where the sequence $\{\ell_k; k \in \mathbb{N}\}$ is defined by (2). Next, let $\rho(\cdot) : [0, \infty) \rightarrow [0, 1]$ be a C^∞ function satisfying

$$\rho(r) = \begin{cases} 1 & \text{if } r \in [0, 1], \\ 0 & \text{if } r \in [2, \infty), \\ \in [0, 1] & \text{if } r \in [1, 2], \end{cases}$$

and $|\rho'(r)| \leq 2$. For any real number $R > 0$ and $u \in H$, we set

$$\mathbf{B}^R(u, u) := \rho\left(\frac{|u|^2}{R}\right) \mathbf{B}(u, u).$$

Let us also consider the following modified problem

$$\begin{aligned} du^R(t) + [\kappa Au^R(t) + \mathbf{B}^R(u^R(t), u^R(t))]dt &= \sum_{k=1}^{\infty} \beta_k d\ell_k(t) e_k, \\ u^R(0) &= x \in H. \end{aligned} \tag{36}$$

Finally, let u_n^R be the solution of the following

$$\begin{aligned} du_n^R + [\kappa Au_n^R + \mathbf{B}_n^R(u_n^R, u_n^R)]dt &= \sum_{k=1}^n \int_{\mathbb{R}_0} z d\bar{\eta}_k(dz, dt) \beta_k e_k, \\ u_n^R(0) &= \Pi_n x \in H_n. \end{aligned} \tag{37}$$

The system (35) (resp. (36) and (37)) is a stochastic evolution equation with locally (resp. globally) Lipschitz coefficients. In particular, (37) and (35) have respectively unique solutions u_n^R and u_n which are càdlàg Markov processes taking values in H_n , see, for instance, [1]. We have also seen in [7, Proposition 4.3] that (36) has a unique solution which is a càdlàg Markov process taking values in H . We denote by $\mathcal{P}_{t,n}$, \mathcal{P}_t^R and $\mathcal{P}_{t,n}^R$, $t \geq 0$, the Markov semigroups associated with u_n , u^R and u_n^R , respectively.

Since the coefficients of (37) belong to $C^2(H_n; H_n)$ the map $x_n \ni H_n \mapsto u_n^R$ is C^1 differentiable and the derivative $U_n^R(s, x) := \nabla_x u_n^R(s, x)$ in the direction of $x \in \mathbb{R}^n$ at point $x_n \in H_n$ is the solution of the linearized equation

$$\begin{aligned} dU_n^R(t, x) + [\kappa AU_n^R(t, x) + \nabla \mathbf{B}_n^R(u_n^R(t, x), u_n^R(t, x))][U_n^R(t, x)]dt &= 0, \\ U_n^R(0) &= x. \end{aligned} \tag{38}$$

Throughout this section $B_H(\xi, \delta)$ denotes the ball centered at $\xi \in H$ with radius δ .

5.1. Preparatory result. In this subsection we will state and prove the following proposition which is one of the crucial results needed for the construction of the coupling and the proof of the exponential mixing.

Proposition 5.1. *If all, but Assumptions 2.7, hypotheses of Theorem 3.7 are satisfied, then $\mathcal{P}_t, t \geq 0$, has the strong Feller property. Furthermore, there exists a constant $\delta > 0$ such that*

$$|\mathcal{P}_1 \Phi(x) - \mathcal{P}_1 \Phi(y)| \leq \frac{1}{2}, \tag{39}$$

for any $x, y \in B_H(0, \delta)$, and $\Phi \in B_b(H)$ with $\|\Phi\|_\infty \leq 1$.

The proof of this proposition is based on the truncation of the nonlinearity of (1) and the Galerkin approximation of the truncated problem (36). The advantage of this approach is that since the Galerkin approximation (37) is a system of SDEs with globally Lipschitz coefficients, we can apply a BEL formula for pure jump noise driven SDEs recently proved in [7, Lemma A.3] to estimate the gradient of the Markovian semigroup $\mathcal{P}_{t,n}^R$ and show that $\mathcal{P}_{t,n}^R$ has the strong Feller property. By carefully passing to the limit we also prove that the Markovian semigroup \mathcal{P}_t^R associated with (36) has the strong Feller property too. Since we have good moment estimates of the solution to the original equation (1), we show by using stopping time argument that (1) has also the strong Feller property from which we easily conclude the proof of the Proposition 5.1.

To rigorously implement this approach we need two sets of results one of which is stated in the following lemma.

Lemma 5.2. *Let all assumptions of Proposition 5.1 be satisfied. Then, system (38) has a unique solution U_n^R such that $U_n^R \in C(0, t; \mathbb{H}) \cap L^2(0, t; \mathbb{V})$ for any $t > 0$ and $x \in \mathbb{H}_n$. Moreover, there exists a constant $C > 0$ such that*

$$\sup_{\substack{x \in \mathbb{H}_n, \\ |x| \leq 1}} \left[\mathbb{E} |U_n^R(s, x)|^2 + \kappa \mathbb{E} \int_0^t |A^{\frac{1}{2}} U_n^R(s, x)|^2 ds \right] \leq e^{\frac{4cR^2}{\kappa} t}, \quad t > 0. \tag{40}$$

Proof. The proof is very similar to the proof of [7, Lemma 4.3] and we omit it. □

The results in Lemma 5.2 will be used below to estimate the Markov semigroups $\{\mathcal{P}_{t,n}^R; t \geq 0\}$ and $\{\mathcal{P}_t^R; t \geq 0\}$. The proof of the following results will be given after the proof of Proposition 5.1.

Lemma 5.3. *Let all the assumptions of Theorem 3.7 be satisfied. Let*

$$C_p(t) = t^{-2p} + t^{-\frac{4p}{\alpha}}, \quad t > 0.$$

Then, there exists a constant $K > 0$ such that

$$|\mathcal{P}_{t,n}^R \Phi(x) - \mathcal{P}_{t,n}^R \Phi(y)| < K e^{\frac{4cR^2}{\kappa} t} (C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t)) \|\Phi\|_{\infty} |x - y|, \tag{41}$$

for any $n \in \mathbb{N}$, $R > 0$, $t > 0$, $x, y \in \mathbb{H}_n$ and $\Phi \in B_b(\mathbb{H}_n)$. Moreover,

$$|\mathcal{P}_t^R \Phi(x) - \mathcal{P}_t^R \Phi(y)| < K e^{\frac{4cR^2}{\kappa} t} (C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t)) \|\Phi\|_{\infty} |x - y|, \tag{42}$$

After these few preparatory lemmata we are now ready to give the promised proof of the main result of this subsection.

Proof of Proposition 5.1. Let $\varepsilon > 0$ be an arbitrary positive number, $x, y \in \mathbb{H}$ and $\Phi \in B_b(\mathbb{H})$. We set

$$C(t, R) := K e^{\frac{4cR^2}{\kappa} t} (C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t)), \quad C(t, x) := c_0 e^{c_1 t} (|x|^2 + 1).$$

Let $u(x) := u(\cdot, x)$ and $u(y) := u(\cdot, y)$ be solutions of (1) with the initial conditions $x \in \mathbb{H}$ and $y \in \mathbb{H}$, respectively. For any $x \in \mathbb{H}$, let $\{\vartheta_R(x); R \in \mathbb{N}\}$ be the sequence of stopping times defined by

$$\vartheta_R(x) := \inf\{t \geq 0; |u(t, x)| \geq R\}.$$

We have

$$\begin{aligned}
 & |\mathcal{P}_t \Phi(x) - \mathcal{P}_t \Phi(y)| \\
 & \leq |\mathcal{P}_t \Phi(x) - \mathcal{P}_t^R \Phi(x)| + |\mathcal{P}_t^R \Phi(x) - \mathcal{P}_t^R \Phi(y)| + |\mathcal{P}_t^R \Phi(y) - \mathcal{P}_t \Phi(y)| \\
 & \leq 2\|\Phi\|_\infty \mathbb{P}(\vartheta_R(x) < t) + 2\|\Phi\|_\infty \mathbb{P}(\vartheta_R(y) < t) + |\mathcal{P}_t^R \Phi(x) - \mathcal{P}_t^R \Phi(y)|.
 \end{aligned}$$

From the inequality (22) of Proposition 3.3 we infer that for any $x \in \mathbb{H}$ and $t > 0$

$$\mathbb{P}(\vartheta_R(x) < t) \leq \frac{1}{R^2} C(t, x), \tag{43}$$

from which altogether with (42) we infer that for any $\Phi \in B_b(\mathbb{H})$, $x, y \in \mathbb{H}$

$$|\mathcal{P}_t \Phi(x) - \mathcal{P}_t \Phi(y)| \leq \|\Phi\|_\infty \left[\frac{4C(t, x)}{R^2} + C(R, t) |x - y| \right]. \tag{44}$$

Choosing $R \geq (8C(t, x)/\varepsilon)^{1/2}$ and $\delta \leq \varepsilon/2C(R, t)$, we infer that for any $\varepsilon > 0$ and $x \in \mathbb{H}$

$$|\mathcal{P}_t \Phi(x) - \mathcal{P}_t \Phi(y)| \leq \varepsilon,$$

for any $y \in B_{\mathbb{H}}(x, \delta)$ and $\Phi \in B_b(\mathbb{H})$ with $\|\Phi\|_\infty \leq 1$. This completes the strong Feller property.

We easily conclude the proof of (39) by choosing $R > 0$ sufficiently large and δ sufficiently small in (44). □

Now, we give the proof of the Lemma 5.3.

Proof of Lemma 5.3. The idea is to use the estimate for the gradient of the Markovian semigroup $\mathcal{P}_{t,n}^R$. We are allowed to use this gradient estimate because of Assumption 2.5 and Remark 2.6 the assumptions of the Bismut–Elworthy–Li lemma and the gradient estimate in [7, Lemma A.3 and Lemma B.1] are met. Let $\Phi \in C_b^2(\mathbb{H}_n)$ and $\nabla_x \mathcal{P}_{t,n}^R \Phi(x)$ be the derivative in the direction of $x \in \mathbb{H}_n$ at a point $x \in \mathbb{H}_n$ of $\mathcal{P}_{t,n}^R \Phi(\cdot)$. Notice that when identifying \mathbb{H}_n with \mathbb{R}^n the linear operator A^δ , $\delta \in [0, \infty)$, can be identified with the diagonal matrix $[A_{jk}^\delta; j, k = 1, \dots, n]$ defined by

$$A_{jk}^\delta = \begin{cases} \lambda_j^\delta & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to [7, Lemma A.3 and Lemma B.1] we have

$$\sup_{\substack{n \in \mathbb{N}, x \in \mathbb{H}_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(x)| \leq C_0(t) \left(\sum_{j=1}^\infty \beta_j^{-2} \lambda_j^{-1} \right)^{\frac{1}{2}} \|\Phi\|_\infty \left[\mathbb{E} \int_0^t |A^{\frac{1}{2}} U_n^R(s)|^2 ds \right]^{\frac{1}{2}},$$

where we have used the shorthand notation $U_n^R(\cdot) := U_n^R(\cdot, x)$ and

$$\tilde{C}_0(t) = C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t).$$

Owing to (40) we obtain the following estimate

$$\sup_{\substack{n \in \mathbb{N}, x \in H_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(x)| \leq \tilde{C}_0(t) \left(\sum_{j=1}^{\infty} \beta_j^{-2} \lambda_j^{-1} \right)^{\frac{1}{2}} \|\Phi\|_{\infty} e^{\frac{4cR^2}{\kappa} t}.$$

Now we derive that for any real number $R > 0, t > 0$ there exists a constant K such that

$$\sup_{\substack{n \in \mathbb{N}, x \in H_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(x)| \leq K e^{\frac{4cR^2}{\kappa} t} (C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t)) \|\Phi\|_{\infty},$$

for any $\Phi \in C_b^2(H_n)$. Now we easily see that the estimate (41) holds for $\Phi \in C_b^2(H_n)$. Owing to the equivalence lemma [45, Lemma 2.2] it follows that (41) also holds for $\Phi \in B_b(H_n)$. Now arguing as in [7, Section 4.3] we can let n tend to ∞ and recover (42). \square

5.2. *Construction of coupling and exponential estimates of hitting times.* Now we shall construct a coupling of Markov processes in the extended phase space $\mathbf{H} = H \times H$. For this aim we closely follow [38, Chapter 3] and [48]. Let δ the constant given in Proposition 5.1.

For any $\mathbf{x} := (x, y) \in \mathbf{H}$ we denote by $\mathbf{u} := (u, \bar{u})$ a pair of solutions starting at \mathbf{x} . Let $\mathbf{u}_{nT} := (\mathbf{u}_{nT}, \bar{\mathbf{u}}_{nT})_{n \in \mathbb{N}}$ be the underlying Markov chain in \mathbf{H} . Denote by $\mathbf{U}_x := (U_x, U_y)$ be the maximal coupling of $(\mathcal{P}_T)^* \delta_x$ and $(\mathcal{P}_T)^* \delta_y$ for any $\mathbf{x} := (x, y) \in \mathbf{H}$. Now we can define a transition function $\widehat{\mathcal{P}}_T(\mathbf{x}, \cdot)$ on the phase space $\mathbf{H} = H \times H$ as follows:

$$\widehat{\mathcal{P}}_T(\mathbf{x}, \mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \mathcal{P}_T(x, \mathcal{O}_1 \cap \mathcal{O}_2) & \text{if } x = y, \\ \mathcal{D}(U_x, U_y)(\mathcal{O}_1 \times \mathcal{O}_2) & \text{if } x, y \in B_H(0, \delta) \text{ and } x \neq y, \\ \mathcal{P}_T(x, \mathcal{O}_1) \mathcal{P}_T(y, \mathcal{O}_2) & \text{otherwise,} \end{cases}$$

for any $\mathbf{O} = \mathcal{O}_1 \times \mathcal{O}_2 \in \mathcal{B}(\mathbf{H})$. Here $\mathcal{P}_T(x, \cdot)$ is the transition probability of $u_x(T)$, and $\mathcal{D}(\varphi)$ is the distribution of a random variable φ . Since our results in this section is valid for any fixed T , we set $T = 1$ for the rest of the calculations.

Let $\delta > 0$ be as above and

$$\rho_{\delta} = \inf\{t \geq 0; |\mathbf{u}(t)| \leq \delta\},$$

where $|\mathbf{v}|^2 = |\mathbf{v}|^2 + |\bar{\mathbf{v}}|^2$ for any $\mathbf{v} = (\mathbf{v}, \bar{\mathbf{v}}) \in H \times H$. For any $M > 0$, let

$$\rho_M = \inf\{t \geq 0; \|\mathbf{u}(t)\| \leq M\},$$

where $\|\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\bar{\mathbf{v}}\|^2$ for any $\mathbf{v} = (\mathbf{v}, \bar{\mathbf{v}}) \in V \times V$. For any $p \geq 1$, we also set

$$\mathfrak{m}_p := \int_{\mathbb{R}_0} |z|^p \nu(dz), \quad \mathfrak{n}_p := \sum_{k=1}^{\infty} \beta_j^p.$$

Before proceeding further, we shall recall the following definition which is borrowed from [38, Section 1.3].

Definition 5.4. A family of probability measures $\{\mathbb{P}_{\mathbf{v}} : \mathbf{v} \in \mathbf{H}\}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ associated with the Markov process $\{\mathbf{u}(t) \in \mathbf{H} : t \geq 0\}$ is defined as follows.

- (i) The mapping $\mathbf{v} \mapsto \mathbb{P}_{\mathbf{v}}(A)$ is universally measurable [38, see Section A.3] for any $A \in \mathcal{F}$.
- (ii) The process $\{\mathbf{u}(t) \in \mathbf{H} : t \geq 0\}$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and satisfying the condition below for any $\mathbf{v} \in \mathbf{H}$, $\Gamma \in \mathcal{B}(\mathbf{H})$ and $t, s \geq 0$,
 - (a) $\mathbb{P}_{\mathbf{v}}\{\mathbf{u}(0) = \mathbf{v}\} = 1$,
 - (b) $\mathbb{P}_{\mathbf{v}}\{\mathbf{u}(t+s) \in \Gamma | \mathcal{F}_s\} = \widehat{\mathcal{P}}_t(\mathbf{u}(s), \Gamma)$,
for $\mathbb{P}_{\mathbf{v}}$ -a.s.. Where $\widehat{\mathcal{P}}_t(\mathbf{u}(s), \Gamma) := \mathbb{P}_{\mathbf{v}}\{\mathbf{u}(t) \in \Gamma\}$ for any $\mathbf{v} \in \mathbf{H}$.

$\mathbb{E}_{\mathbf{v}}$ is the corresponding expectation associated with the probability measure $\mathbb{P}_{\mathbf{v}}$ for any $\mathbf{v} \in \mathbf{H}$.

The following proposition is very important for our purpose.

Proposition 5.5. *For any $\kappa > 0$, there exists $M > 0$ such that if Assumption 2.9 is satisfied, then there exist two constants $\gamma > 0$ and $K > 0$ such that*

$$\mathbb{E}_{\mathbf{x}} e^{\gamma \rho M} \leq 1 + K |\mathbf{x}|^2, \tag{45}$$

for any $\mathbf{x} \in \mathbf{H}$.

Proof. To prove this proposition we will first derive some estimates for the Galerkin solution $\mathbf{u}_n, n \in \mathbb{N}$, defined by (35). Since the argument is very similar to [38, Proof of Lemma 3.3.10] we only sketch our calculation. Applying Itô’s formula to $|\mathbf{u}_n(t)|^2$, then to $e^{\gamma_1 \kappa t} |\mathbf{u}_n(t)|^2$, choosing $t = t_M := \rho_M \wedge N, N \geq 1$ and taking the mathematical expectation in the final equations yield

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}} \left[e^{\gamma_1 \kappa t_M} |\mathbf{u}_n(t_M)|^2 + 2\kappa \int_0^{t_M} e^{\gamma_1 \kappa s} \|\mathbf{u}_n(s)\|^2 ds \right] - \gamma_1 \kappa \mathbb{E}_{\mathbf{x}} \int_0^{t_M} e^{\gamma_1 \kappa s} |\mathbf{u}_n(s)|^2 ds - |\mathbf{x}_n|^2 \\ &= \mathbb{E}_{\mathbf{x}} \sum_{k=1}^n \left[\int_0^{t_M} \int_{\mathbb{R}_0} e^{\gamma_1 \kappa s} \beta_k \left(\beta_k |z|^2 + 2\langle \mathbf{u}_n(s), e_k \rangle z \right) \nu(dz) ds \right. \\ & \quad \left. - 2 \int_0^{t_M} \int_{|z| < 1} e^{\gamma_1 \kappa s} \beta_k z \langle \mathbf{u}_n(s), e_k \rangle \nu(dz) ds \right]. \end{aligned} \tag{46}$$

Observe that for $\varepsilon > 0$ we have

$$\begin{aligned} & \left| \int_{\mathbb{R}_0} \beta_k (\beta_k |z|^2 + 2\langle \mathbf{u}_n(s), e_k \rangle z) \nu(dz) dt - 2\beta_k \int_{|z| < 1} \langle \mathbf{u}_n(s), e_k \rangle z \nu(dz) \right| \\ & \leq \beta_k^2 m_2 + 4\beta_k |\mathbf{u}_n(s)| m_1 \\ & \leq \beta_k m_2 + \frac{16\beta_k}{\varepsilon} m_1^2 + \varepsilon \beta_k |\mathbf{u}_n(s)|^2. \end{aligned}$$

By choosing $\varepsilon = \frac{\kappa}{2\lambda_1^2 n_1}$ in the last line of the above chain of inequalities, plugging the resulting estimate in (46) and using (3) we obtain

$$\mathbb{E}_{\mathbf{x}} [e^{\gamma_1 \kappa t_M} |\mathbf{u}_n(t_M)|^2] + \mathbb{E}_{\mathbf{x}} \int_0^{t_M} e^{\gamma_1 \kappa s} \left(\frac{\kappa}{2} \|\mathbf{u}_n(s)\|^2 - m_2 n_2 - \frac{32}{\kappa} n_1^2 m_1^2 \lambda_1^2 \right) ds \leq |\mathbf{x}_n|^2.$$

Now, letting $n \rightarrow \infty$ and using Fatou’s lemma we have

$$\mathbb{E}_{\mathbf{x}}[e^{\gamma_1 \kappa t_M} |\mathbf{u}(t_M)|^2] + \mathbb{E}_{\mathbf{x}} \int_0^{t_M} e^{\gamma_1 \kappa s} \left(\frac{\kappa}{2} \|\mathbf{u}(s)\|^2 - m_2 n_2 - \frac{32}{\kappa} n_1^2 m_1^2 \lambda_1^2 \right) ds \leq |\mathbf{x}|^2,$$

which altogether with the choice of M such that $M^2 > \frac{4}{\kappa} \left(m_2 n_2 + \frac{32}{\kappa} n_1^2 m_1^2 \lambda_1^2 \right)$ and the fact that $\|\mathbf{u}(s)\|^2 \geq M^2$ on $\{s \leq \rho_M\}$ implies the following estimate

$$\frac{\lambda_1^2}{\kappa} \mathbb{E}_{\mathbf{x}}[e^{\lambda_1^{-2} \kappa t_M} - 1] \leq \frac{4}{\kappa M^2} |\mathbf{x}|^2.$$

Now, letting $N \rightarrow \infty$ and using Fatou’s lemma we have

$$\mathbb{E}_{\mathbf{x}} e^{\lambda_1^{-2} \kappa \rho_M} \leq \frac{4}{\lambda_1^{-2} M^2} |\mathbf{x}|^2 + 1,$$

from which and the choices $\gamma = \lambda_1^{-2} \kappa$ and $K = \frac{4}{\lambda_1^{-2} M^2}$ we complete the proof of the proposition. □

The following lemma is useful for proving the forthcoming proposition.

Lemma 5.6. *For any compact set $\mathcal{A} \subset \mathbb{H} \times \mathbb{H}$ and any $\delta > 0$ there exists a number $p \in (0, 1)$ such that*

$$\inf_{\mathbf{x} \in \mathcal{A}} \mathbb{P}_{\mathbf{x}}\{\mathbf{u}_1 = (u_1, \bar{u}_1) \in B_{\mathbb{H}}(0, \delta) \times B_{\mathbb{H}}(0, \delta)\} > p.$$

Proof. Thanks to the Propositions 3.5, 5.1 one can use the same argument as in [48, Proof of Lemma 6.7] to prove the lemma. □

Proposition 5.7. *Suppose that all the assumptions of Proposition 5.5 are verified. Then, for any $\kappa > 0$ there exist $\widehat{\gamma} > 0$ and $\widehat{K} > 0$ such that*

$$\mathbb{E}_{\mathbf{x}} e^{\widehat{\gamma} \rho_{\delta}} \leq 1 + \widehat{K} |\mathbf{x}|^2. \tag{47}$$

Proof. Firstly, let us introduce an increasing sequence of stopping times as follows:

$$\tilde{\rho}_M(0) = \rho_M, \quad \tilde{\rho}_M(j + 1) = \inf\{t \geq \tilde{\rho}_M(j) + 1; \|\mathbf{u}(t)\| \leq M\} \text{ with } j \geq 0.$$

We also set $\rho_M(j) = \tilde{\rho}_M(j) + 1$ and define the integer valued stopping time n_{δ} by

$$n_{\delta} = \min\{n \geq 1; |\mathbf{u}(\rho_M(n))| \leq \delta\}.$$

Secondly, we compute the probability of the event $\{n_{\delta} > k\}$, $k \geq 1$ as follows:

$$\mathbb{P}_{\mathbf{x}}(n_{\delta} > k) = \mathbb{P}_{\mathbf{x}} \left(\bigcap_{i=1}^k \{|\mathbf{u}(\rho_M(i))| > \delta\} \right) \leq \prod_{i=1}^k \left(1 - \mathbb{P}_{\mathbf{x}}(|\mathbf{u}(\rho_M(i))| \leq \delta) \right). \tag{48}$$

Thirdly, we derive a lower bound for $\mathbb{P}_{\mathbf{x}}(|\mathbf{u}(\rho_M(i))| \leq \delta)$:

$$\begin{aligned} \mathbb{P}_{\mathbf{x}}(|\mathbf{u}(\rho_M(i))| \leq \delta) &= \mathbb{P}_{\mathbf{x}}\left(|\mathbf{u}(\tilde{\rho}_M(i) + 1)| \leq \delta | \mathcal{F}_{\tilde{\rho}_M(i)}\right) \\ &= \mathbb{P}_{\mathbf{x}}\left(\mathbf{u}(\tilde{\rho}_M(i) + 1) \in B_{\mathbf{H}}(0, \delta) \times B_{\mathbf{H}}(0, \delta) | \mathbf{u}(\tilde{\rho}_M(i)) \in B_{\mathbf{V}}(0, M) \times B_{\mathbf{V}}(0, M)\right) \geq p. \end{aligned} \tag{49}$$

The above estimate holds due to the compactness of the ball $B_{\mathbf{V}}(0, M)$ of \mathbf{V} in \mathbf{H} , Lemma 5.6 and the strong Markov property of $\{\tilde{\rho}_M(i); i \in \mathbb{N}\}$. Combining (48) and (49) we obtain

$$\mathbb{P}_{\mathbf{x}}(n_{\delta} > k) \leq (1 - p)^k \quad \text{for any } \mathbf{x} \in \mathbf{H}, k \geq 1. \tag{50}$$

Now, we claim that

$$\mathbb{E}_{\mathbf{x}} e^{\gamma \rho_M(n)} \leq e^{\gamma T} C_1^n (1 + K|\mathbf{x}|^2), \tag{51}$$

where C_1 depends only on M . To this end, we set $\hat{\rho}_M = \inf\{t \geq 1; \|\mathbf{u}(t)\| \leq M\}$ and estimate its exponential moment by using the strong Markov property of $\hat{\rho}_M$ and Proposition 5.5. More precisely,

$$\mathbb{E}_{\mathbf{x}} e^{\gamma \hat{\rho}_M} = \mathbb{E}_{\mathbf{x}} \mathbb{E}(e^{\gamma \hat{\rho}_M} | \mathcal{F}_1) = e^{\gamma} \mathbb{E}_{\mathbf{x}} \mathbb{E}_{\mathbf{u}(T)} e^{\gamma \hat{\rho}_M} \leq e^{\gamma} \mathbb{E}_{\mathbf{x}} (1 + K|\mathbf{x}|^2) \leq C_1 < \infty. \tag{52}$$

By means of the argument in [38, Proof of Proposition 3.3.6, Step 2] and the fact that $\{\tilde{\rho}_M(i); i \in \mathbb{N}\}$ satisfies the strong Markov property together with (52), we obtain

$$\mathbb{E}_{\mathbf{x}} e^{\gamma \tilde{\rho}_M(n)} = \mathbb{E}_{\mathbf{x}} e^{\gamma(\tilde{\rho}_M(n-1) + \hat{\rho}_M)} = \mathbb{E}_{\mathbf{x}} \mathbb{E}\left(e^{\gamma(\tilde{\rho}_M(n-1) + \hat{\rho}_M)} | \mathcal{F}_{\tilde{\rho}_M(n-1)}\right) \leq C_1^n \mathbb{E}_{\mathbf{x}} e^{\gamma \hat{\rho}_M}. \tag{53}$$

The above result (53) and the equality $\hat{\rho}_M = \rho_M + 1$, immediately imply the estimate (51). Finally, we estimate $\mathbb{P}_{\mathbf{x}}(\rho_{\delta} > l)$ as follows: By following the argument in [38, Proof of Proposition 3.3.6, Step 3] and using the estimates (50), (51) together with Chebyshev’s inequality,

$$\mathbb{P}_{\mathbf{x}}(\rho_{\delta} > \rho_M(n)) \leq \mathbb{P}_{\mathbf{x}}(n_{\delta} > n) \leq (1 - p)^n,$$

and

$$\mathbb{P}_{\mathbf{x}}(\rho_M(n) \geq l) \leq e^{-\gamma l} \mathbb{E}_{\mathbf{x}} e^{\gamma \rho_M(n)} \leq C_1^n e^{-\gamma(l-1)} (1 + K|\mathbf{x}|^2).$$

For n sufficiently large, we can bound $(1 - p)^n$ by $e^{-\hat{\gamma}(l-1)} (1 + K|\mathbf{x}|^2)$ with $\hat{\gamma} < \gamma$. Then we have

$$\mathbb{P}_{\mathbf{x}}(\rho_{\delta} > l) \leq C_2 e^{-\hat{\gamma}(l-1)} (1 + K|\mathbf{x}|^2). \tag{54}$$

By applying exponential type estimates in [41, Section 7.1] together with the result (54) we easily conclude the proof of the proposition. \square

5.3. *Proof of Theorem 3.7.* After these few preparatory results, we are now ready to prove Theorem 3.7. For this aim, it is enough to prove that for any $\mathbf{x} = (x, y) \in H \times H$,

$$\|\mathcal{P}_l(x, \cdot) - \mathcal{P}_l(y, \cdot)\|_{TV} \leq \tilde{C}e^{-cl}(1 + K|\mathbf{x}|^2) \quad \text{for any } l \in \mathbb{N}. \tag{55}$$

The constant \tilde{C}, c are independent of $x, y \in H$ and $l \in \mathbb{N}$. Introduce the stopping time

$$\vartheta = \min\{l > 0; \text{ such that } u_l = \bar{u}_l \text{ with } l \in \mathbb{N}\}.$$

Now, as an intermediate step of the proof we will establish the estimate

$$\mathbb{P}_{\mathbf{x}}(\vartheta > l) \leq \tilde{C}e^{-cl}(1 + K|\mathbf{x}|^2). \tag{56}$$

Thanks to the Proposition 5.1, there exists a constant $\delta > 0$ such that

$$|\mathcal{P}_1\Phi(x) - \mathcal{P}_1\Phi(y)| \leq \frac{1}{2},$$

for any $x, y \in B_H(0, \delta)$, $\Phi \in B_b(H)$ with $\|\Phi\|_{\infty} \leq 1$. This leads to

$$\|\mathcal{P}_1^*\delta_x - \mathcal{P}_1^*\delta_y\|_{TV} \leq \frac{1}{2},$$

for any $x, y \in B_H(0, \delta)$. Since (u_1, \bar{u}_1) is a maximal coupling for the pair $(\mathcal{P}_1(x, \cdot), \mathcal{P}_1(y, \cdot))$, we get

$$\mathbb{P}_{\mathbf{x}}\{u_1 \neq \bar{u}_1\} \leq \frac{1}{2}, \tag{57}$$

for any $x, y \in B_H(0, \delta)$. Now we define a sequence of stopping times $\{\rho_{\delta}(j); j \in \mathbb{N}\}$ associated with stopping time ρ_{δ} as follows

$$\rho_{\delta}(0) = \rho_{\delta}, \quad \rho_{\delta}(j + 1) = \inf\{t \geq \rho_{\delta}(j) + 1; |\mathbf{u}(t)| \leq \delta\} \text{ with } j \geq 0.$$

By using similar steps as in the proof of (51), we can obtain

$$\mathbb{E}_{\mathbf{x}}e^{\gamma\rho_{\delta}(n)} \leq e^{\gamma}C_3^n(1 + K|\mathbf{x}|^2),$$

where $\gamma > 0$ and $C_3 > 0$ are independent of $x, y \in H$. Apply Chebyshev's inequality to get

$$\mathbb{P}_{\mathbf{x}}\{\rho_{\delta}(n) > l\} \leq e^{-\gamma(l-1)}C_3^n(1 + K|\mathbf{x}|^2), \tag{58}$$

for $n, l \in \mathbb{N}$. Now, define the event

$$\vartheta_n = \{u_{\rho_{\delta}(m)+1} \neq \bar{u}_{\rho_{\delta}(m)+1}; 1 \leq m \leq n\}.$$

By means of (57) and the strong Markov property,

$$\mathbb{P}_{\mathbf{x}}\{u_{\rho_{\delta}(n)+1} \neq \bar{u}_{\rho_{\delta}(n)+1} | \mathcal{F}_{\rho_{\delta}(n)}\} \leq \mathbb{P}_{\mathbf{u}_{\rho_{\delta}(n)}}\{u_1 \neq \bar{u}_1\} \leq \frac{1}{2}.$$

Then,

$$\begin{aligned} \mathbb{P}_{\mathbf{x}}(\vartheta_n) &= \mathbb{P}_{\mathbf{x}}\left(\vartheta_{(n-1)} \cap \{u_{\rho_{\delta}(n)+1} \neq \bar{u}_{\rho_{\delta}(n)+1}\}\right) \\ &= \mathbb{E}_{\mathbf{x}}\left(\mathbb{1}_{\vartheta_{(n-1)}}\mathbb{P}_{\mathbf{x}}\{u_{\rho_{\delta}(n)+1} \neq \bar{u}_{\rho_{\delta}(n)+1} | \mathcal{F}_{\rho_{\delta}(n)}\}\right) \\ &\leq \frac{1}{2}\mathbb{P}_{\mathbf{x}}(\vartheta_{n-1}) \leq \frac{1}{4}\mathbb{P}_{\mathbf{x}}(\vartheta_{n-2}) \leq \dots \leq 2^{-n}, \text{ for any } n \in \mathbb{N}. \end{aligned} \tag{59}$$

Combining (58) and (59),

$$\begin{aligned} \mathbb{P}_x(\vartheta > l) &= \mathbb{P}_x(\vartheta > l, \rho_\delta(n) < l) + \mathbb{P}_x(\vartheta > l, \rho_\delta(n) > l) \\ &\leq \mathbb{P}_x(\vartheta_n) + \mathbb{P}_x(\rho_\delta(n) > l) \\ &\leq 2^{-n} + e^{-\gamma(l-1)} C_3^n (1 + K|x|^2), \text{ for any } n, l \in \mathbb{N}. \end{aligned}$$

For large n we can bound 2^{-n} by $e^{-\hat{\gamma}(l-1)}(1 + K|x|^2)$ with $\hat{\gamma} < \gamma$ which leads to (56). Finally, we have

$$\begin{aligned} |\mathcal{P}_l(x, \Gamma) - \mathcal{P}_l(y, \Gamma)| &\leq \mathbb{E}_x \left(\mathbb{1}_{\{\vartheta > l\}} |\mathbb{1}_\Gamma(u_l) - \mathbb{1}_\Gamma(\tilde{u}_l)| \right) \\ &\leq \mathbb{P}_x\{\vartheta > l\} \leq \tilde{C} e^{-cl} (1 + K|x|^2). \end{aligned}$$

By taking the supremum over all $\Gamma \in \mathcal{B}(H)$, we obtain (55) from which we readily complete the proof of Theorem 3.7.

Remark 5.8. The noise we consider in this paper is non-degenerate, i.e., $\beta_k > 0$ for all $k \in \mathbb{N}$ and it was pointed out by one of the anonymous referees that it might also be possible to establish Theorem 3.7 by using the Harris approach (see, for instance, [48]) which would considerably simplify and shorten the proof.

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