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Erratum

Erratum to: Spectral Simplicity and Asymptotic Separation of Variables

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The statement of Lemma A.3 in [HllJdg11] is false.[1] This lemma is used in the proof of Lemma 8.2 which is also incorrect and is used in the proof of Proposition 9.1. Despite this gap, Proposition 9.1 is correct and, in order to rectify the situation, we provide here the correct estimates needed to derive Proposition 9.1. As a consequence, the statements of the main results of the article [HllJdg11] are correct.

We would like to emphasize that the overall strategy of the proof of Proposition 9.1 remains unchanged. In particular, Proposition 9.1 is a statement about concentration properties of quasimodes for the quadratic form a_t^μ coming from separation of variables. More precisely, it relies on the fact that a quasimode of order t for a_t^μ at a non-critical energy cannot concentrate on the turning point (and thus must have some mass in the classically allowed region). In the exposition given in [HllJdg11], this non-concentration was hidden behind Lemmas 9.4 and 9.5. We will make it more transparent here by directly using the Langer-Cherry transform and the following estimate for solutions to the semiclassical Airy equation:[2]

$$t^2 \cdot W''(y) - y \cdot W(y) = R(y). \tag{1}$$

Lemma 0.1. *Let $a < 0 < b$. For each $\varepsilon > 0$, there exists $C > 0$, $\delta_0 > 0$ and a positive function T such that if $\delta < \delta_0$, $t < T(\delta)$ and W satisfies (1), then*

$$\int_{-\delta}^{\delta} |W|^2 \leq \varepsilon \cdot t^{-2} \int_a^b |R|^2 + C \cdot \left(\int_{-2\delta}^{-\delta} |W|^2 + \int_{\delta}^{2\delta} |W|^2 \right). \tag{2}$$

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[1] To correct the statement, one could replace $\sqrt{\alpha}$ with α.

[2] Lemma 0.1 may be viewed as a correction of Lemma 8.2 in [HllJdg11].

Proof. Let $\widetilde{W}$ be the solution to (1) that is defined in Lemma 8.1 of [HllJdg11] :

$$\widetilde{W}(y) := \frac{t^{-\frac{4}{3}}}{w}\left[A_+(t^{-\frac{2}{3}}y)\int_y^b A_-(t^{-\frac{2}{3}}z)R(z)dz \,+\, A_-(t^{-\frac{2}{3}}y)\int_0^y A_+(t^{-\frac{2}{3}}z)R(z)dz\right],$$

where $A_\pm$ are the linearly independent Airy functions that are defined in the appendix of [HllJdg11], and w is their Wronskian determinant. We use the Cauchy-Schwarz inequality and rescale the integrals with $A_\pm$ by $t^{-\frac{2}{3}}$. Using the asymptotic behavior of $A_\pm$ near $\pm\infty$, we observe that $X \mapsto \|A_-\|_{[X,\infty)}|A_+(X)| + \|A_+\|_{[0,X]}|A_-(X)|$ is bounded on $\mathbf{R}$. We thus obtain a constant C' such that

$$\sup_{[a,b]}\left|\widetilde{W}\right| \le C' \cdot t^{-1}\|R\|_{[a,b]}. \tag{3}$$

The rescaled difference $A(x) = \left(W - \widetilde{W}\right)\left(t^{\frac{2}{3}} \cdot x\right)$ is a solution to the Airy equation $A''(x) - x \cdot A(x) = 0$. Using estimates for Airy functions in the same manner as in the proof of Lemma A.4 of [HllJdg11]—but with greater care on how the constant depends on a and b—we find a constant $C > 0$ that is independent of δ and a positive function t_0 such that, for any $\delta > 0$ and any $t < t_0(\delta)$ and for any A which is a solution to Airy's equation,

$$\int_{-\delta}^{\delta}\left|A\left(t^{-\frac{2}{3}}y\right)\right|^2 dy \le C \cdot \left(\int_{-2\delta}^{-\delta}\left|A\left(t^{-\frac{2}{3}}y\right)\right|^2 dy + \int_{\delta}^{2\delta}\left|A\left(t^{-\frac{2}{3}}y\right)\right|^2 dy\right). \tag{4}$$

The desired estimate (2) then follows from straightforward estimations: Use the inequality $|W|^2 \le 2(|W - \widetilde{W}|^2 + |\widetilde{W}|^2)$. The second term is estimated by integrating (3) over $[-\delta, \delta]$. To bound the first term, first apply estimate (4) to $|W - \widetilde{W}|^2$, then use the inequality $|W - \widetilde{W}|^2 \le 2(|W|^2 + |\widetilde{W}|^2)$, and finally apply (3) integrated over the intervals $[-2\delta, -\delta]$ and $[\delta, 2\delta]$. Since estimate (3) is integrated over intervals of width δ, the prefactor ε results from choosing δ_0 small enough. □

Proposition 0.2. *Given a compact set* $K \subset \left(\frac{\mu}{\sigma(0)}, \infty\right)$ *and* $C > 0$, *there exist positive constants* C', s_0 *and* t_0 *such that if* $0 < t < t_0$, $E \in K$ *and for each* v *we have*

$$\left|a_t^{\mu}(w, v) - E \cdot \langle w, v\rangle_\sigma\right| \le C \cdot t \cdot \|w\|_\sigma \cdot \|v\|_\sigma, \tag{5}$$

then

$$\|w\|_\sigma^2 \le C' \int_0^{x_E^{-s_0}} |w(x)|^2\, dx.$$

Proof. Let W_E denote the Langer-Cherry transform of w at energy E, and let ϕ_E : $[0, \infty) \to \mathbf{R}$ denote the associated change of variables (see §7 in [HllJdg11]). Let $a = \frac{1}{2}\sup\{\phi_E(0) \mid E \in K\}$. For each $E \in K$, there exists $s_E > 0$ so that $a = \phi_E\left(x_E^{-s_E}\right)$. Let $b = \sup\{\phi_E\left(x_E^{s_E}\right) \mid E \in K\}$.

By Proposition 7.3 in [HllJdg11], W_E satisfies (1) on $[a, b]$ with a right-hand side R_E that can be estimated using Lemma 7.5. Using the latter lemma, the assumptions on w, and Lemma 6.2[3], we get the following bound:

$$\int_a^b |R_E(y)|^2 dy \le C \cdot t^2 \cdot \|w\|_\sigma^2. \tag{6}$$

By Lemma 7.4 in [HllJdg11] and using the compactness of K, there exists $M > 0$ so that for each interval $I \subset [a, b]$,

$$M^{-1} \int_{\phi_E^{-1}(I)} |w|^2 \, dx \le \int_I |W_E|^2 \, dy \le M \int_{\phi_E^{-1}(I)} |w|^2 \, dx. \tag{7}$$

Using Lemma 0.1 and (6) and choosing ε small enough, we find δ_0, C and a positive function T such that, for any $\delta < \delta_0$ and any $t \le T(\delta)$ we have

$$\int_{\phi_E^{-1}([-\delta,\delta])} |w(x)|^2 \sigma(x) dx \le \frac{1}{4} \|w\|_\sigma^2 + C \cdot \int_{\phi_E^{-1}([-\delta,\delta]^c)} |w(x)|^2 \sigma(x) dx,$$

where we have set $[-\delta, \delta]^c := [a, b] \backslash [-\delta, \delta]$. We choose $\delta = \frac{\delta_0}{2}$ and fix some s_0 such that $\phi_E([x_E^{-s_0}, x_E^{s_0}]) \subset]-\delta, \delta[$. Since $[x_E^{-s_0}, x_E^{s_0}] \subset \phi_E^{-1}([-\delta, \delta])$ and $\phi_E^{-1}([-\delta, \delta]^c) \subset \mathbf{R} \backslash (x_E^{-s_0}, x_E^{s_0})$, we obtain the following estimate:

$$\int_{x_E^{-s_0}}^{x_E^{s_0}} |w(x)|^2 \sigma(x) dx \le \frac{1}{4} \|w\|_\sigma^2 + C \cdot \int_{\mathbf{R} \backslash (x_E^{-s_0}, x_E^{s_0})} |w(x)|^2 \sigma(x) dx.$$

The claim now follows in a quite standard way: We split the integral defining $\|w\|_\sigma^2$ into three parts : $[0, x_E^{-s_0}]$, $[x_E^{-s_0}, x_E^{s_0}]$ and $[x_E^{s_0}, \infty)$. We use the preceding bound for the second integral and we use Lemma 6.2 to bound each integral over $[x_E^{s_0}, \infty)$. We obtain the following estimate:

$$\|w\|_\sigma^2 \le \left(\frac{1}{4} + C \cdot t \right) \|w\|_\sigma^2 + (C + 1) \cdot \int_0^{x_E^{-s_0}} |w(x)|^2 \sigma(x) dx.$$

For t small enough, the term $\|w\|_\sigma^2$ on the right can be absorbed on the left. The claim follows since σ is bounded. □

Proof of Proposition 9.1 in [HllJdg11]. It suffices to prove that there exists $\kappa > 0$ so that under the assumptions of Proposition 0.2 we have

$$\int_0^\infty (E \cdot \sigma(x) - \mu) \cdot |w(x)|^2 \, dx \ge \kappa \cdot \|w\|_\sigma^2. \tag{8}$$

Let s_0, t_0 be as in Proposition 0.2. For any $s < s_0$, split the integral on the left of (8) into the integrals corresponding to the intervals $[0, x_E^{-s_0}]$, $[x_E^{-s_0}, x_E^{-s}]$, $[x_E^{-s}, x_E^{s}]$, and

[3] In Lemma 6.2, the integral on the right can be replaced by the integral from 0 to x_E^s. This allows us to put the weight σ on both sides.

$[x_E^s, \infty)$. Observe that the integral over the second interval is positive. This yields the following lower bound:

$$\int_0^\infty (E \cdot \sigma(x) - \mu) \cdot |w(x)|^2 \, dx$$
$$\geq s_0 \int_0^{x_E^{-s_0}} |w(x)|^2 \, dx - 2Cs\|w\|_\sigma^2 - C \int_{x_E^s}^\infty |w(x)|^2 dx.$$

By Proposition 0.2, the first term is bounded below by $\frac{s_0}{C'}\|w\|_\sigma^2$, where C' is the constant in Proposition 0.2. We choose s small enough so that the second term is bounded below by $-\frac{s_0}{4C'}\|w\|_\sigma^2$ and we choose t small enough so that, using Lemma 6.2 in [HllJdg11], the third is term is also bounded by the latter quantity. The claim follows with $\kappa = s_0/2C'$. □

Reference

[HllJdg11] Hillairet, L., Judge, C.: Spectral simplicity and asymptotic separation of variables. Commun. Math. Phys. **302**(2), 291–344 (2011)

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