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On the Dimension of the Singular Set of Solutions to the Navier–Stokes Equations

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Abstract: In this paper we prove that if a suitable weak solution u of the Navier–Stokes equations is an element of $L^w(0, T; L^s(\mathbb{R}^3))$, where $1 \leq 2/w + 3/s \leq 3/2$ and $3 < w, s < \infty$, then the box-counting dimension of the set of space-time singularities is no greater than $\max\{w, s\}(2/w+3/s-1)$. We also show that if $\nabla u \in L^w(0, T; L^s(\Omega))$ with $2 < s \leq w < \infty$, then the Hausdorff dimension of the singular set is bounded by $w(2/w + 3/s - 2)$. In this way we link continuously the bounds on the dimension of the singular set that follow from the partial regularity theory of Caffarelli, Kohn, & Nirenberg (Commun. Pure Appl. Math. 35:771–831, 1982) to the regularity conditions of Serrin (Arch. Ration. Mech. Anal. 9:187–191, 1962) and Beirão da Veiga (Chin. Ann. Math. Ser. B 16(4):407–412, 1995).

1. Introduction

The flow of an incompressible fluid in a domain $\Omega \subseteq \mathbb{R}^3$ is governed by the system of the Navier–Stokes equations:

$$u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0, \qquad \text{div } u = 0, \tag{1}$$

$$u(0) = u_0, \quad u|_{\partial\Omega} = 0, \tag{2}$$

where u is the velocity of the fluid and p is the pressure. Since the works of Leray [10], Hopf [7] and Ladyzhenskaya [9], it is known that each divergence-free initial condition $u_0 \in \mathbb{L}^2(\Omega) := [L^2(\Omega)]^3$ gives rise to a weak solution $u \in L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}^1_0)$ that satisfies the Navier–Stokes equations in the distributional sense. The open question is whether or not each initial condition $u_0 \in \mathbb{H}^1_0$ gives rise to a weak solution that is actually strong: $u \in L^\infty(0, T; \mathbb{H}^1_0) \cap L^2(0, T; \mathbb{H}^2)$.

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There are many results that give sufficient conditions for regularity of a weak solution u. The simplest is due to Serrin [17] and says that u is regular if $u \in L^w(0, T; L^s(\Omega))$ with

$$\frac{2}{w} + \frac{3}{s} \leq 1. \tag{3}$$

(In fact Serrin's proof requires strict inequality in (3); that the condition in (3) is sufficient was shown by Fabes, Jones, & Rivière [5] with the exception of the endpoint cases; Struwe [18] gave an alternative proof including $u \in L^2(0, T; L^\infty)$; and Escauriaza, Seregin, & Šverák showed relatively recently in [4] that $u \in L^\infty(0, T; L^3)$ implies regularity).

Notice that the Sobolev embedding result $H^1(\Omega) \subset L^6(\Omega)$ implies that any weak solution is an element of $L^2(0, T; L^6)$ which gives $2/w + 3/s = 3/2$. Even though this 'regularity gap' of $1/2$ prevents one from proving the regularity of weak solutions, it has been shown by many authors that the putative set $\mathcal{S}$ of points at which u is not regular must be very small. For example, it is known that for a suitable weak solution the set of singular points in space-time has one-dimensional parabolic Hausdorff measure zero (Caffarelli, Kohn, & Nirenberg, [2]) and box-counting dimension no greater than 5/3 (Robinson & Sadowski [15]; see also Kukavica [8], for a finer result).

In this paper we present a link between these partial regularity results bounding the dimension of the set of space-time singularities, and the regularity result of Serrin (et al.). More precisely, in our first result we consider a suitable weak solution u with $u \in L^w(0, T; L^s(\mathbb{R}^3))$ where

$$1 \leq \frac{2}{w} + \frac{3}{s} \leq \frac{3}{2}$$

and $3 < w, s < \infty$; as a consequence of the results of Caffarelli et al. we prove that the box-counting dimension of the set of space-time singularities is no greater than

$$\alpha = \max\{w, s\} \left(\frac{2}{w} + \frac{3}{s} - 1 \right).$$

Specialising this to the case $w = s$ shows that if $u \in L^s((0, T) \times \mathbb{R}^3)$, then $d_B(\mathcal{S}) \leq 5 - s$. Observing that any weak solution belongs to $L^{10/3}((0, T) \times \mathbb{R}^3)$ since

$$\int_0^T \|u(t)\|_{L^{10/3}}^{10/3} \, \mathrm{d}t \leq \int_0^T \|u(t)\|_{L^2}^{4/3} \|u(t)\|_{L^6}^2 \, \mathrm{d}t$$

(by Hölder's inequality), we recover the bound $d_B(\mathcal{S}) \leq 5/3$ for weak solutions, which decreases as s increases until we reach the critical value $s = 5$ known to guarantee regularity from (3).

While this result requires us to consider the equations on $\mathbb{R}^3$ due to problems estimating the pressure, we can circumvent this and consider the equations on a bounded domain if we instead impose conditions on ∇u. In our second result we consider a suitable weak solution u such that

$$\nabla u \in L^w(0, T; L^s(\Omega)) \qquad \text{with} \quad 2 \leq \frac{2}{w} + \frac{3}{s} \leq \frac{5}{2}.$$

We prove that if $2 < s \leq w < \infty$, then the Hausdorff dimension of the singular set of u is no greater than

$$\beta = w \left(\frac{2}{w} + \frac{3}{s} - 2 \right).$$

This result provides a link between the partial regularity result in its standard form (a bound on the Hausdorff dimension of the singular set) and the condition for regularity due to Beirão da Veiga [1], that u is regular if $2/w+3/s=2$. Again, in the case of weak solutions we have $w=s=2$ which yields the partial regularity result in its standard form (the Hausdorff dimension of the singular set is no larger than one), and we recover regularity when $w=s=5/2$.

2. Notation and Auxiliary Results

Throughout the paper we use standard notation for Lebesgue and Sobolev spaces. The Bochner space $L^w(0,T;L^s(\Omega))$ is endowed with the norm:

$$\|u\|_{L^w(0,T;L^s(\Omega))}=\int_0^T \|u(t)\|^w_{L^s(\Omega)}\,\mathrm{d}t.$$

We denote by $Q_r(x,t)$ the space-time cylinder

$$Q_r(x,t)=B_r(x)\times(t-r^2,t),$$

where $B_r(x)$ is a three-dimensional ball of radius $r>0$ centred at x.

In what follows we consider only suitable weak solutions, which are weak solutions that in addition satisfy a local energy inequality and for which the associated pressure belongs to $L^{5/3}(Q_T)$, where Q_T is the space-time domain.

We say that $z\in\mathbb{R}^3\times\mathbb{R}_+$ is a regular point of a suitable weak solution u if u is bounded in some neighbourhood of z. A point is singular if it is not regular, and the set of all singular points of a suitable weak solution u we denote by $\mathcal{S}$.

For our main results we will need the following two lemmas. The first formalises the fact that in some sense $p\sim u^2$. We do not know how to prove a similar result on a bounded domain; this is the reason that we restrict to the whole of $\mathbb{R}^3$ in Theorem 1. However, a recent result due to Wolf [19] shows that one can remove the pressure term from the condition of regularity used in the proof of Lemma 2 allowing its generalisation to bounded domains.

Lemma 1. *If $u\in L^w(0,T;L^s(\mathbb{R}^3))$, then $p\in L^{w/2}(0,T;L^{s/2}(\mathbb{R}^3))$.*

Proof. Equations (1) and (2) imply formally that

$$\Delta p=-\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(u^iu^j).$$

From the Calderon-Zygmund theorem we can now deduce that for all $2<s<\infty$,

$$\int_{\mathbb{R}^3}|p|^{s/2}\,\mathrm{d}x\le C(s)\int_{\mathbb{R}^3}|u|^s\,\mathrm{d}x,$$

and therefore

$$\int_0^T \|p(t)\|^{w/2}_{L^{s/2}}\,\mathrm{d}t\le\int_0^T \|u(t)\|^w_{L^s}\,\mathrm{d}t$$

(see Caffarelli et al., 1982, for details).

The second simple lemma – essentially a version of the regularity criterion of Caffarelli et al. obtained by repeated application of Hölder's inequality – is the key observation that allows us to prove our main results.

Lemma 2. *Assume that u is a suitable weak solution to the Navier–Stokes equations and* $u \in L^w(0,T;L^s(\mathbb{R}^3))$ *with* $3 \le w, s < \infty$. *There exists an absolute constant* $\varepsilon > 0$ *such that if* $z = (x,t)$ *is a singular point of u then*

$$\varepsilon r^{w\left(\frac{3}{s}+\frac{2}{w}-1\right)} \le \int_{t-r^2}^{t} \left(\int_{B_r(x)} |u|^s \,\mathrm{d}x\right)^{w/s} \mathrm{d}t + \int_{t-r^2}^{t} \left(\int_{B_r(x)} |p|^{s/2} \,\mathrm{d}x\right)^{w/s} \mathrm{d}t$$

for all sufficiently small $r > 0$.

Proof. The fundamental regularity result of Caffarelli et al. [2] is that there exists an absolute constant ϵ_0 such that if for any $r > 0$ such that $Q_r(x,t) \subset \mathbb{R}_+ \times \mathbb{R}^3$,

$$\frac{1}{r^2}\int_{Q_r(x,t)} |u|^3 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{r^2}\int_{Q_r(x,t)} |p|^{3/2} \,\mathrm{d}x\,\mathrm{d}t < 2\epsilon_0,$$

then $z = (x,t)$ is regular. It follows that if z is a singular point, then for all sufficiently small $r > 0$ we have

$$\epsilon_0 \le \frac{1}{r^2}\int_{Q_r(x,t)} |u|^3 \,\mathrm{d}x\,\mathrm{d}t \qquad \text{or} \qquad \epsilon_0 \le \frac{1}{r^2}\int_{Q_r(x,t)} |p|^{3/2} \,\mathrm{d}x\,\mathrm{d}t. \tag{4}$$

From Hölder's inequality it follows that for all $s \ge 3$:

$$\begin{aligned} r^{-2}\int_{Q_r(x,t)} |u|^3 \,\mathrm{d}x\,\mathrm{d}t &\le r^{-2}\int_{t-r^2}^{t} \left(\int_{B_r(x)} |u|^s \,\mathrm{d}x\right)^{\frac{3}{s}} \left(\int_{B_r(x)} \mathrm{d}x\right)^{1-\frac{3}{s}} \mathrm{d}t \\ &\le c r^{1-\frac{9}{s}} \left[\int_{t-r^2}^{t} \left(\int_{B_r(x)} |u|^s \mathrm{d}x\right)^{w/s} \mathrm{d}t\right]^{3/w} \left[\int_{t-r^2}^{t} \mathrm{d}t\right]^{1-\frac{3}{w}}. \end{aligned}$$

Hence

$$\frac{1}{r^2}\int_{Q_r(x,t)} |u|^3 \mathrm{d}x\,\mathrm{d}t \le c r^{3-\frac{9}{s}-\frac{6}{w}} \left[\int_{t-r^2}^{t} \left(\int_{B_r(x)} |u|^s \,\mathrm{d}x\right)^{w/s} \mathrm{d}t\right]^{3/w}. \tag{5}$$

Similarly, for $w \ge 3$, we obtain:

$$\frac{1}{r^2}\int_{Q_r(x,t)} |p|^{3/2} \mathrm{d}x\,\mathrm{d}t \le c r^{3-\frac{9}{s}-\frac{6}{w}} \left[\int_{t-r^2}^{t} \left(\int_{B_r(x)} |p|^{s/2} \mathrm{d}x\right)^{w/s} \mathrm{d}t\right]^{3/w}. \tag{6}$$

Therefore taking $\varepsilon = \epsilon_0^{w/3} c^{-w/3}$ we obtain the assertion of the lemma.

Notice that from Lemma 2 it follows immediately that if $u \in L^w(0,T;L^s)$ with $3/w + 2/s = 1$ and $w, s > 3$ then it is regular. Indeed, it follows from (3),(4) and (5) that if $2/w + 3/s = 1$ then there exits a sequence $r_n \to 0$ such that for each n we have

$$\begin{aligned} \varepsilon &\le \int_{t-r_n^2}^{t} \left(\int_{B_{r_n}(x)} |u|^s \mathrm{d}x\right)^{w/s} \mathrm{d}t + \int_{t-r_n^2}^{t} \left(\int_{B_{r_n}(x)} |p|^{s/2} \mathrm{d}x\right)^{w/s} \mathrm{d}t \\ &\le \int_{t-r_n^2}^{t} \|u\|_{L^s(\mathbb{R}^3)}^w + \|p\|_{L^{s/2}(\mathbb{R}^3)}^{w/2} \,\mathrm{d}t, \end{aligned}$$

which contradicts the fact that $u \in L^w(0,T;L^s(\mathbb{R}^3))$ and $p \in L^{w/2}(0,T;L^{s/2}(\mathbb{R}^3))$.

3. Main Results

In this section we give our main results concerning the dimension of the singular set. The classical result of Caffarelli, Kohn, & Nirenberg is given in terms of the parabolic Hausdorff measure. For a given $s \geq 0$, let

$$\mathcal{P}^s_\delta(X) = \inf \left\{ \sum_j r_j^s \, : \, X \subset \cup_{j=1}^\infty Q_{r_j}(x_j, t_j), \; r_j < \delta \right\},$$

and define the s-dimensional parabolic Hausdorff measure as

$$\mathcal{P}^s(X) = \lim_{\delta \to 0} \mathcal{P}^s_\delta(X).$$

One can define a parabolic Hausdorff dimension $d_{PH}(X) = \inf\{s \, : \, \mathcal{P}^s(X) = 0\}$; if $\mathcal{P}^s(X) < \infty$ then $d_{PH}(X) \leq s$. This quantity also bounds the standard Hausdorff dimension d_H (which can be defined in the same way but with cylinders replaced by balls), $d_H(X) \leq d_{PH}(X)$.

We will also make use of the upper box-counting dimension $d_B(X)$. Let $N(X, \epsilon)$ denote the minimum number of balls of radius ϵ necessary to cover X; then

$$d_B(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon}.$$

A useful observation here is that one obtains the same quantity if $N(X, \epsilon)$ instead denotes the maximum number of disjoint balls of radius ϵ with centres in X. One always has $d_H(X) \leq d_B(X)$. See Falconer [6] or Robinson [13] for details.

Theorem 1. *Assume that u is a suitable weak solution with $u \in L^w(0, T; L^s(\mathbb{R}^3))$ for $3 < w, s < \infty$. Then the box-counting dimension of its singular set $\mathcal{S}$ is no greater than*

$$\alpha = \max\{w, s\} \left(\frac{2}{w} + \frac{3}{s} - 1 \right).$$

Proof. For a given sufficiently small $r > 0$ let $N(r)$ be the maximal number of disjoint 4-dimensional balls of radius $2r$ centred at points $z_i = (x_i, t_i) \in \mathcal{S}$, where $i = 1, 2, 3, \ldots, N(r)$. Observe that for all sufficiently small r the cylinders $Q_r(x_i, t_i)$ are disjoint, too.

For $i = 1, 2, \ldots, N(r)$ we define the function a_i by

$$a_i(t) = \int_{B_r(x_i)} |u(t, x)|^s \, \mathrm{d}x \qquad \text{if} \quad t_i - r^2 \leq t \leq t_i,$$

and we let $a_i(t) = 0$ for all other values of t. Similarly, for $i = 1, 2, \ldots, N(r)$ we define functions b_i by

$$b_i(t) = \int_{B_r(x_i)} |p(t, x)|^{s/2} \mathrm{d}x \qquad \text{if} \quad t_i - r^2 \leq t \leq t_i,$$

and $b_i(t) = 0$ otherwise.

Notice that for each $i = 1, 2, \ldots, N(r)$ we have

$$\int_{t_i - r^2}^{t_i} \left(\int_{B_r(x_i)} |u|^s \, \mathrm{d}x \right)^{w/s} \mathrm{d}t = \int_0^T [a_i(t)]^{w/s} \, \mathrm{d}t$$

and

$$\int_{t_i-r^2}^{t_i}\left(\int_{B_r(x_i)}|p|^{s/2}\,\mathrm{d}x\right)^{w/s}\mathrm{d}t=\int_0^T[b_i(t)]^{w/s}\,\mathrm{d}t.$$

Since the cylinders are disjoint we have

$$\int_0^T\left(\sum_{i=1}^{N(r)}a_i(t)\right)^{w/s}+\left(\sum_{i=1}^{N(r)}b_i(t)\right)^{w/s}\mathrm{d}t\le M,\tag{7}$$

where

$$M=\int_0^T\left(\int_{\mathbb{R}^3}|u|^s\,\mathrm{d}x\right)^{w/s}\mathrm{d}t+\int_0^T\left(\int_{\mathbb{R}^3}|p|^{s/2}\,\mathrm{d}x\right)^{w/s}\mathrm{d}t.$$

Now we are going to consider the cases $w\ge s$ and $s>w$ separately.

Case 1. If $w\ge s$ then (3) implies that

$$\sum_{i=1}^{N(r)}\int_0^T[a_i(t)]^{w/s}+[b_i(t)]^{w/s}\,\mathrm{d}t\le M.$$

From Lemma 2 it now follows that

$$N(r)\varepsilon r^{w(\frac{2}{w}+\frac{3}{s}-1)}\le M.\tag{8}$$

If the box-counting dimension of $\mathcal{S}$ was greater than $\alpha=w(\frac{2}{w}+\frac{3}{s}-1)$, then for some constant $\delta>0$ there would exist a sequence $r_n\to 0$ such that $N(\mathcal{S},r_n)>r_n^{-\alpha-\delta}$. For $n\to\infty$ the left-hand side of (8) would tend to infinity, giving a contradiction.

Case 2. If $w<s$ then from Hölder's inequality it follows that

$$N^{w/s-1}\left(\sum_{i=1}^{N}[a_i]^{w/s}+[b_i]^{w/s}\right)\le\left(\sum_{i=1}^{N}a_i\right)^{w/s}+\left(\sum_{i=1}^{N}b_i\right)^{w/s}.$$

Thus we have

$$N(r)N(r)^{r/s-1}\varepsilon r^{w(\frac{2}{w}+\frac{3}{s}-1)}\le M,$$

and the proof is concluded as before.

As remarked in the Introduction, every suitable weak solution belongs to $L^{10/3}((0,T)\times\mathbb{R}^3)$, and so has a singular set whose box-counting dimension is no greater than 5/3 (cf. Robinson & Sadowski [15]). At the other extreme, if u satisfies Serrin's condition (e.g. $u\in L^5((0,T)\times\mathbb{R}^3)$ then the singular set has box-counting dimension zero. (In fact we showed after the proof of Lemma 2 that in this case the singular set is empty.)

We now provide a related result which, inspired by Beirão da Veiga [1], makes an assumption on ∇u rather than on u itself. The advantage of this is that the following theorem is also valid on bounded domains, since the regularity condition involved does not require any properties of the pressure.

Theorem 2. *Assume that u is a suitable weak solution with*

$$\nabla u \in L^w(0,T;L^s(\Omega)),$$

where $2 < s \leq w < \infty$. *Then*

$$\mathcal{P}^\beta(\mathcal{S}) = 0,$$

where

$$\beta = w\left(\frac{2}{w} + \frac{3}{s} - 2\right).$$

Proof. For this result we use Caffarelli et al.'s second regularity theorem (which is what allows them to deduce that $\mathcal{P}^1(\mathcal{S}) = 0$ for any weak solution), namely that there exists an absolute constant $\varepsilon > 0$ such that if

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_r(x,t)} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t \leq \varepsilon$$

then (x,t) is regular.

Fix $\delta > 0$, and for each singular point $z = (x,t) \in \mathcal{S}$ choose a cylinder $Q_r(x,t)$ with $r < \delta/5$ such that

$$\frac{1}{r}\int_{Q_r(x,t)} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t \geq \varepsilon;$$

let $\mathcal{C}$ be the family of all these cylinders. Using the covering lemma from Caffarelli et al. we can choose a countable subfamily $\mathcal{C}'$ of disjoint cylinders $Q_{r_i}(z_i)$ such that $Q_{5r_i}(z_i)$ still covers the singular set $\mathcal{S}$.

To show that $d_{PH}(\mathcal{S}) \leq \beta$, it is sufficient to show that $\mathcal{P}^\beta(\mathcal{S}) < \infty$. To this end we notice that

$$\begin{aligned}\int_{Q_{r_i}(x_i,t_i)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq \int_{t_i-r_i^2}^{t_i}\left(\int_{B_{r_i}(x_i)} |\nabla u|^s \,\mathrm{d}x\right)^{2/s}\left(\int_{B_{r_i}(x_i)} \mathrm{d}x\right)^{1-\frac{2}{s}} \mathrm{d}t\\ &\leq c(r_i)^{3-\frac{6}{s}}\left(\int_{t_i-r_i^2}^{t_i}\left(\int_{B_{r_i}(x_i)} |\nabla u|^s \mathrm{d}x\right)^{w/s} \mathrm{d}t\right)^{2/w}\left(\int_{t_i-r_i^2}^{t_i} \mathrm{d}t\right)^{1-\frac{2}{w}}.\end{aligned}$$

Hence

$$\varepsilon r_i \leq c(r_i)^{3(1-\frac{2}{s})+2(1-\frac{2}{w})}\left(\int_{t_i-r_i^2}^{t_i}\left(\int_{B_{r_i}(x_i)} |\nabla u|^s \,\mathrm{d}x\right)^{w/s} \mathrm{d}t\right)^{2/w}$$

and finally

$$c^{-w/2}\varepsilon^{w/2}(r_i)^\beta \leq \int_{t_i-r_i^2}^{t_i}\left(\int_{B_{r_i}(x_i)} |\nabla u|^s \,\mathrm{d}x\right)^{w/s} \mathrm{d}t,$$

where $\beta = w\left(\frac{3}{s} + \frac{2}{w} - 2\right)$.

Now for $i = 1, 2, 3, \ldots$, let

$$\varphi_i(t) = \int_{B_{r_i}(x_i)} |\nabla u(x,t)|^s \mathrm{d}x \quad \text{ if } \; t_i - r_i^2 \le t \le t_i,$$

and $\varphi_i(t) = 0$ otherwise. Since

$$\int_{t_i - r_i^2}^{t_i} \left(\int_{B_{r_i}(x_i)} |\nabla u|^s \mathrm{d}x \right)^{w/s} \mathrm{d}t = \int_0^T [\varphi_i(t)]^{w/s} \, \mathrm{d}t,$$

it follows that for $C = c^{w/2}\varepsilon^{-w/2}$ we have

$$\sum_{i=1}^{\infty} r_i^{\beta} \le C \sum_{i=1}^{\infty} \int_0^T [\varphi_i(t)]^{w/s} \, \mathrm{d}t \le C \int_0^T \left[\sum_{i=1}^{\infty} \varphi_i(t) \right]^{w/s} \mathrm{d}t.$$

Since the cylinders are disjoint, for each $0 \le t \le T$ we have

$$\sum_{i=1}^{\infty} \varphi_i(t) \le \int_{\Omega} |\nabla u(x,t)|^s \mathrm{d}x.$$

Now let $\Pi\mathcal{S}$ be the projection of $\mathcal{S}$ onto $(0, T)$ and let $\mathcal{T}$ be the set of singular times:

$$\mathcal{T} = \mathbb{R} \setminus \bigcup_{q=1}^{\infty} J_q,$$

where J_q are intervals of regularity of a weak solution u (the existence of such "epochs of regularity" dates back to [10]). The set $\mathcal{T}$ has box-counting dimension no greater than $1/2$ ([14]). Moreover[1], $\Pi\mathcal{S} \subseteq \mathcal{T}$. Indeed, if $t_0 \in J_q$ for some q, then for sufficiently small $\varepsilon > 0$ the weak solution u is uniformly bounded for all $(x, t) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times \Omega$. It follows that if D is the projection of $\mathcal{C}'$ onto $(0, T)$ then $D \subseteq O_\delta(\mathcal{T})$, where $O_\delta(\mathcal{T})$ is δ-neighbourhood of $\mathcal{T}$. In particular, since $d_B(\mathcal{T}) \le 1/2$, for any $\theta > 1/2$, one can cover $O_\delta(\mathcal{T})$ and hence D by $c\delta^{-\theta}$ intervals of length 4δ, and hence $\mu(D) \le c\delta^{-\theta}(4\delta) \to 0$ as $\delta \to 0$.

We have

$$\sum_{i=1}^{\infty} r_i^{\beta} \le C \int_0^T \left[\sum_{i=1}^{\infty} \varphi_i(t) \right]^{w/s} \mathrm{d}t \le C \int_D \left(\int_{\Omega} |\nabla u|^s \mathrm{d}x \right)^{w/s} \mathrm{d}t,$$

where the right-hand side tends to zero as $\delta \to 0$ (since the integrand is in $L^1(0, T)$).

To summarise, given any $\delta > 0$ we have found a covering of $\mathcal{S}$ by sets $Q_{5r_i}(x_i, t_i)$ such that $5r_i < \delta$ and

$$\sum_{i=1}^{\infty} (5r_i)^{\beta} \le 5^{\beta} C \int_D \left(\int_{\Omega} |\nabla u|^s \mathrm{d}x \right)^{w/s} \mathrm{d}t,$$

where the right hand side of the above inequality tends to zero as δ tends to zero. It follows that $\mathcal{P}^{\beta}(\mathcal{S}) = 0$, and hence that $d_H(\mathcal{S}) \le d_{PH}(\mathcal{S}) \le \beta$.

[1] One can show equality in the case of periodic boundary conditions, see [12]. The proof there - which relies on a compactness argument - does not obviously generalise to $\mathbb{R}^3$ or a bounded domain.

Note that if $\beta = 0$ then the set of singular points is empty. Indeed, if (x, t) is a singular point, then taking into account that

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r(x,t)} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}t > \varepsilon$$

and reasoning as above we can show that there exists a sequence $r_n \to 0$ such that

$$\varepsilon^{w/2} \leq \int_{t-r_n^2}^{t} \left(\int_{B_{r_n}(x)} |\nabla u|^s \, \mathrm{d}x \right)^{w/s} \mathrm{d}t \leq \int_{t-r_n^2}^{t} \left(\int_{\Omega} |\nabla u|^s \, \mathrm{d}x \right)^{w/s} \mathrm{d}t;$$

but the right-hand side tends to zero as n tends to infinity, which is a contradiction.

Conclusion

In this paper we presented the upper bounds on the box-counting and Hausdorff dimension of the singular set of a suitable weak solution to the Navier–Stokes equations that has some additional regularity. As the border cases of these bounds we have obtained the well-known conditions for regularity of weak solutions.

Some natural questions arise from these results. It would be interesting to relax the assumption $w > 2$ in Theorem 1 and obtain the same bound for any $w \geq 2$; similarly in Theorem 2 one would like to relax the condition $w \geq s$. In order to obtain Theorem 1 in a bounded domain we would require the analogue of Lemma 2 (estimates for the pressure when $u \in L^w(0, T; L^s(\Omega))$).

An order of magnitude harder is to determine whether any of these partial regularity results can be proved for general weak solutions, and not only suitable weak solutions.

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