

Erratum

Erratum to: Integral Formulas for the Asymmetric Simple Exclusion Process

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This is a correction to the proof of Theorem 2.1 of [1]. An error occurred in the “proof” of Lemma 2.2, which is false. The following should replace the proof of (c) (Sect. II, p. 821ff.), which will correct the proof of the theorem:

The initial condition is satisfied by the summand in (2.3) coming from the identity permutation id . So what we have to show is

$$\sum_{\sigma \neq id} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} d\xi_1 \cdots d\xi_N = 0,$$

or equivalently

$$\sum_{\sigma \neq id} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma \prod_i \xi_i^{x_{\sigma^{-1}(i)} - y_i - 1} d\xi_1 \cdots d\xi_N = 0,$$

when $x_1 < \cdots < x_N$. We write $I(\sigma)$ for the integral corresponding to σ , so that the above becomes

$$\sum_{\sigma \neq id} I(\sigma) = 0. \tag{1}$$

For $1 \leq n < N$ fix $n - 1$ distinct numbers $i_1, \dots, i_{n-1} \in [1, N - 1]$. Define

$$A = \{i_1, \dots, i_{n-1}\},$$

and then

$$\mathbb{S}_N(A) = \{\sigma \in \mathbb{S}_N : \sigma(1) = i_1, \dots, \sigma(n-1) = i_{n-1}, \sigma(n) = N\}.$$

When $n = 1$ this consists of all permutations with N in position 1, and when $n = N - 1$ it consists of a single permutation. If B is the complement of $A \cup \{N\}$ in $[1, N]$, then $\sigma \in \mathbb{S}_N(A)$ is determined by the restriction

$$\sigma|_{[n+1, N]} : [n+1, N] \rightarrow B.$$

Lemma 2.1. *For each A ,*

$$\sum_{\sigma \in \mathbb{S}_N(A)} I(\sigma) = 0.$$

Start of the Proof. When $\sigma \in \mathbb{S}_N(A)$ the inversions involving N are the (N, i) with $i \in B$. Therefore the integrands in $I(\sigma)$ involving these σ may be written

$$\prod_{i \in B} S(\xi_N, \xi_i) \times \prod_{i \leq N} \xi_i^{x_{\sigma^{-1}(i)} - y_i - 1} \times \prod \{S(\xi_k, \xi_\ell) : N > k > \ell, \sigma^{-1}(k) < \sigma^{-1}(\ell)\}.$$

The integrals are taken over \mathcal{C}_r with r so small that all the denominators in the S -factors are nonzero on and inside the contour. In these integrals we make the substitution

$$\xi_N \rightarrow \frac{\eta}{\prod_{i < N} \xi_i},$$

so that η runs over a circle of radius r^N . The integrand becomes

$$(-1)^{N-n} \prod_{i \in B} \frac{p + q\eta \prod_{\ell \neq i, N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{p + q\eta \prod_{\ell \neq i, N} \xi_\ell^{-1} - \xi_i} \tag{2}$$

$$\times \eta^{x_N - y_N - 1} \prod_{i < N} \xi_i^{x_{\sigma^{-1}(i)} - x_n + y_N - y_i - 1} \tag{3}$$

$$\times \prod \{S(\xi_k, \xi_\ell) : N > k > \ell, \sigma^{-1}(k) < \sigma^{-1}(\ell)\}. \tag{4}$$

(The reason that we still have -1 in the exponents in (3) is that $d\xi_N = \prod_{i < N} \xi_i^{-1} d\eta$.)

Sublemma 2.1. *When $n = N - 1$ we have $I(\sigma) = 0$.*

Proof. There is a single $i \in B$ and (2) is analytic inside the ξ_i -contour except for a simple pole at $\xi_i = 0$. The power of ξ_i in (3) is

$$\xi_i^{x_N - x_{N-1} + y_N - y_i - 1},$$

and since $x_N > x_{N-1}$ and $y_N > y_i$, the exponent is positive. Therefore the integrand is analytic inside the ξ_i -contour, and so the integral is zero.

Sublemma 2.2. *When $n < N - 1$ all $I(\sigma)$ with $\sigma \in \mathbb{S}_N(A)$ are sums of lower-order integrals in each of which (2) is replaced by a factor depending on A . The other factors remain the same. In each integral some ξ_i with $i \in B$ is equal to another ξ_j with $j \in B$.*

Proof. We may assume that $q \neq 0$. This case follows by a limiting argument. We are going to shrink some of the ξ_i -contours with $i \in B$. Due to the defining property of r , the only poles we pass will come from the product (2). In fact, to avoid double poles later we take $\xi_i \in \mathcal{C}_{r_i}$ with the r_i all slightly different.

Take $j = \max B$ and shrink the ξ_j -contour. The product (2) has a simple pole at $\xi_j = 0$ (the j -factor has the pole and the i -factors with $i \neq j$ are analytic there) and the power of ξ_j in (3) is positive as before, so the integrand is analytic at $\xi_j = 0$. For each $k \in B$ with $k \neq j$ we pass the pole at

$$\xi_j = \frac{q\eta \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}{\xi_k - p}, \tag{5}$$

coming from the k -factor in (2). (Our assumption on the r_i assures that there are no double poles.) For the residue we replace the k -factor by

$$- \frac{p + q\eta \prod_{\ell \neq k,N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{q\eta \xi_j^{-2} \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}, \tag{6}$$

where in this and the j -factor we replace ξ_j by the right side of (5). When $i \neq j$, k the i -factor becomes

$$\frac{p + q\eta \prod_{\ell \neq i,N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{p(1 - \xi_i \xi_k^{-1})},$$

and we replace ξ_j in the numerator by the right side of (5).

We now shrink the ξ_k -contour. There is a pole of order 2 at $\xi_k = 0$ coming from (6) and the j -factor in (2). Since $k < j = \max B < N$, we have $y_N - y_k \geq 2$, so the exponent of ξ_k in (3) is at least 2. Therefore the integrand is analytic at $\xi_k = 0$. The factor (6) has no other poles inside \mathcal{C}_{r_k} . An i -factor with $i \neq j, k$ will have a pole at $\xi_k = \xi_i$ if $r_i < r_k$. There is also the pole at

$$\xi_k = \frac{q\eta \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}{\xi_j - p}$$

coming from the j -factor. But this relation and (5) imply $\xi_j = \xi_k$.

Thus when we shrink the ξ_j -contour and the ξ_k -contours with $k \neq j$ we obtain $(N - 2)$ -dimensional integrals in each of which two of the ξ -variables corresponding to indices in B are equal. This proves the sublemma.

Sublemma 2.3. *For each integral of Sublemma 2.2 there is a partition of $\mathbb{S}_N(A)$ into pairs σ, σ' such that $I(\sigma) + I(\sigma') = 0$ for each pair.*

Proof. Consider an integral in which $\xi_i = \xi_j$. We pair σ and σ' if $\sigma^{-1}(i) = \sigma'^{-1}(j)$ and $\sigma^{-1}(j) = \sigma'^{-1}(i)$, and $\sigma^{-1}(k) = \sigma'^{-1}(k)$ when $k \neq i, j$. The factor (3) is clearly the same for both when $\xi_i = \xi_j$, and we shall show that the σ - and σ' -factors in (4) are negatives of each other when $\xi_i = \xi_j$.

Assume for definiteness that

$$i < j \text{ and } \sigma^{-1}(i) < \sigma^{-1}(j). \tag{7}$$

(Otherwise we reverse the roles of σ and σ' .) Then the factor $S(\xi_j, \xi_i)$ does not appear for σ in (4) but it does appear for σ' . This factor equals -1 when $\xi_i = \xi_j$.

To complete the proof it is enough to show that for any $k \neq i, j$ the product of S -factors involving k and either i or j is the same for σ and σ' when $\xi_i = \xi_j$. There are nine cases, depending on the position of k relative to i and j and the position of $\sigma^{-1}(k)$ relative to $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$. If k is outside the interval $[i, j]$ and $\sigma^{-1}(k)$ is outside the interval $[\sigma^{-1}(i), \sigma^{-1}(j)]$ then the products of S -factors for σ and σ' are clearly the same. There are five remaining cases, with the results displayed in the table below. The first column gives the position of k relative to i and j , the second column gives the position of $\sigma^{-1}(k)$ relative to $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$, the third column gives the product of S -factors involving k and either i or j for σ , and the fourth column gives the corresponding product for σ' . Keep (7) in mind.

$i < k < j$	$\sigma^{-1}(k) < \sigma^{-1}(i)$	$S(\xi_k, \xi_i)$	$S(\xi_k, \xi_j)$
$i < k < j$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	1	$S(\xi_k, \xi_j) S(\xi_i, \xi_k)$
$i < k < j$	$\sigma^{-1}(j) < \sigma^{-1}(k)$	$S(\xi_j, \xi_k)$	$S(\xi_i, \xi_k)$
$k < i$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	$S(\xi_i, \xi_k)$	$S(\xi_j, \xi_k)$
$k > j$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	$S(\xi_k, \xi_j)$	$S(\xi_k, \xi_i)$

In all cases but the second the S -factors are exactly the same for σ and σ' when $\xi_i = \xi_j$. For the second we use $S(\xi_k, \xi) S(\xi, \xi_k) = 1$.

Sublemmas 2.1–2.3 give Lemma 2.1.

To prove (1) we use induction on N . When $N = 2$ it follows from Sublemma 2.1. Assume $N > 2$ and that the result holds for $N - 1$. For those permutations for which $\sigma(N) = N$, we use the fact that N appears in no involution, so we may integrate with respect to ξ_1, \dots, ξ_{N-1} and use the induction hypothesis. The set of permutations with $\sigma(N) < N$ is the disjoint union of the various $\mathbb{S}_N(A)$, and for these we apply Lemma 2.1.

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Reference

1. Tracy, C.A., Widom, H.: Integral Formulas for the Asymmetric Simple Exclusion Process. *Commun. Math. Phys.* **279**, 815–844 (2008)

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