Commun. Math. Phys. 295, 877–884 (2010)
Digital Object Identifier (DOI) 10.1007/s00220-010-1004-0

Communications in
Mathematical Physics

Erratum

Erratum to: Global Wellposed Problem for the 3-D Incompressible Anisotropic Navier-Stokes Equations in an Anisotropic Space

Ting Zhang

Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: zhangting79@hotmail.com

Received: 1 October 2009 / Accepted: 16 October 2009
Published online: 5 February 2010 –

Commun. Math Phys. **287**, 211–224 (2009)

As it was pointed out by Ping Zhang, there is an error in the proof of Proposition 3.2 in our paper [1]. Hereby we would like to correct it and obtain Theorem 1.1 in [1] with the initial condition (1.4) being replaced by

$$C_1\nu_h^{-1}\|u_0^h\|_{B^{0,\frac{1}{2}}}\exp\{C_1(\nu_h^{-1}\|u_0^3\|_{B^{0,\frac{1}{2}}}+1)^8\}\leq 1. \tag{0.1}$$

We apologize to the reader for this inconvenience.

At first, we obtain the following three lemmas.

Lemma 0.1. *Let $u(t)$ and $\nabla_h u(t)$ be in $B^{0,\frac{1}{2}}$. We have*

$$\begin{aligned}\|\Delta_j^v(u^3\partial_h u^h)(t)\|_{L_h^{\frac{4}{3}}(L_v^2)} \leq C\|\nabla_h u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}}\Bigg[&\|u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}}\sum_{|j-j'|\leq 5}\|\Delta_{j'}^v\partial_h u^h\|_{L^2(\mathbb{R}^3)}\\ &+\sqrt{d_j(t)}\sqrt{d_{1j}(t)}2^{-\frac{j}{2}}\|\nabla_h u^h\|_{B^{0,\frac{1}{2}}}\left(\|u^3\|_{B^{0,\frac{1}{2}}}+\varepsilon_1\right)^{\frac{1}{2}}\Bigg],\end{aligned}$$

where $\sum d_j(t)\leq 1$, ε_1 is any positive constant, C is independent of ε_1, and

$$d_{1j}(t) := \sum_{j'\geq j-N_0}\frac{2^{\frac{j}{2}}\|\Delta_{j'}^v u^3\|_{L^2}}{\|u^3\|_{B^{0,\frac{1}{2}}}+\varepsilon_1},$$

The online version of the original article can be found under doi:10.1007/s00220-008-0631-1.

satisfying

$$\sum_j d_{1j} \le C,\ \sum_j \sup_{t\in[0,T]} d_{1j} \le C\frac{\|u^3\|_{\widetilde{L}^\infty_T(B^{0,\frac12})}}{\varepsilon_1}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta^v_j(u^3\partial_h u^h) = \sum_{|j-j'|\le 5}\Delta^v_j(S^v_{j'-1}u^3\partial_h\Delta^v_{j'}u^h) + \sum_{j'\ge j-N_0}\Delta^v_j(\Delta^v_{j'}u^3\partial_h S^v_{j'+2}u^h).$$

Using Hölder's inequality and Lemma 2.3 in [1], we get

$$\begin{aligned}\|\Delta^v_j(S^v_{j'-1}u^3\partial_h\Delta^v_{j'}u^h)\|_{L_h^{\frac43}(L^2_v)} &\lesssim \|S^v_{j'-1}u^3\|_{L^4_h(L^\infty_v)}\|\Delta^v_{j'}\partial_h u^h\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \|u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|\Delta^v_{j'}\partial_h u^h\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

and

$$\begin{aligned}&\|\Delta^v_j(\Delta^v_{j'}u^3\partial_h S^v_{j'+2}u^h)(t)\|_{L_h^{\frac43}(L^2_v)}\\ &\quad\lesssim \|S^v_{j'+2}(\partial_h u^h)(t)\|_{L^2_h(L^\infty_v)}\|\Delta^v_{j'}u^3(t)\|_{L^4_h(L^2_v)}\\ &\quad\lesssim \|\nabla_h u^h\|_{B^{0,\frac12}}\left(\frac{2^{\frac{j'}{2}}\|\Delta^v_{j'}u^3\|_{L^2}}{\|u^3\|_{B^{0,\frac12}}+\varepsilon_1}\right)^{\frac12}2^{-\frac{j'}{4}}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)^{\frac12}\sqrt{d_{j'}(t)}2^{-\frac{j'}{4}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}.\end{aligned}$$

Then, we can immediately finish the proof.

Lemma 0.2. *Let u be in* $B^{0,\frac12}(T)$. *We have*

$$\begin{aligned}&\|\Delta^v_j(u^3u^h)\|_{L^2_T(L^2(\mathbb{R}^3))}\\ &\quad\le C2^{-\frac{j}{2}}\sqrt{d_j}\|\nabla_h u^h\|^{\frac12}_{\widetilde{L}^2_T(B^{0,\frac12})}\left\|d_{2j}(t)\|\nabla_h u^3\|_{B^{0,\frac12}}\|u^3\|_{B^{0,\frac12}}\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)\right\|^{\frac12}_{L^2_T}\\ &\qquad + C2^{-\frac{j}{2}}\left\|\sqrt{d_j(t)}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^h\|^{\frac12}_{B^{0,\frac12}}\sqrt{d_{1j}(t)}\|u^h\|^{\frac12}_{B^{0,\frac12}}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)^{\frac12}\right\|_{L^2_T},\end{aligned}$$

where $\sum d_j \le 1$, ε_1 *and* ε_2 *are any positive constants,* C *is independent of* ε_1 *and* ε_2, $d_{1j}(t)$ *is defined in Lemma 0.1 and*

$$d_{2j}(t) := \sum_{|j-j'|\le 5}\frac{2^{\frac{j'}{2}}\|\Delta^v_{j'}u^h\|_{L^2}}{\|u^h\|_{B^{0,\frac12}}+\varepsilon_2},\ \sum_j \sup_{t\in[0,T]} d_{2j} \le C\frac{\|u^h\|_{\widetilde{L}^\infty_T(B^{0,\frac12})}}{\varepsilon_2}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta^v_j(u^3u^h) = \sum_{|j-j'|\le 5}\Delta^v_j(S^v_{j'-1}u^3\Delta^v_{j'}u^h) + \sum_{j'\ge j-N_0}\Delta^v_j(S^v_{j'+2}u^h\Delta^v_{j'}u^3).$$

Using Hölder's inequality and Lemma 2.3 in [1], we get

$$
\begin{aligned}
&\|\Delta_j^v(S_{j'-1}^v u^3 \Delta_{j'}^v u^h)\|_{L_T^2(L^2(\mathbb{R}^3))} \\
&\lesssim \Big\| \|S_{j'-1}^v u^3\|_{L_v^\infty(L_h^4)} \|\Delta_{j'}^v u^h\|_{L_v^2(L_h^4)} \Big\|_{L_T^2} \\
&\lesssim \Bigg\| \|u^3\|_{B^{0,\frac12}}^{\frac12} \|\nabla_h u^3\|_{B^{0,\frac12}}^{\frac12} \left(\frac{2^{\frac{j'}{2}} \|\Delta_{j'}^v u^h\|_{L^2}}{\|u^h\|_{B^{0,\frac12}} + \varepsilon_2} \right)^{\frac12} \\
&\quad \times 2^{-\frac{j'}{4}} \left(\|u^h\|_{B^{0,\frac12}} + \varepsilon_2 \right)^{\frac12} \|\nabla_h \Delta_{j'}^v u^h\|_{L^2}^{\frac12} \Bigg\|_{L_T^2} \\
&\lesssim \Bigg\| \|u^3\|_{B^{0,\frac12}}^{\frac12} \|\nabla_h u^3\|_{B^{0,\frac12}}^{\frac12} \left(\frac{2^{\frac{j'}{2}} \|\Delta_{j'}^v u^h\|_{L^2}}{\|u^h\|_{B^{0,\frac12}} + \varepsilon_2} \right)^{\frac12} 2^{-\frac{j'}{4}} \left(\|u^h\|_{B^{0,\frac12}} + \varepsilon_2 \right)^{\frac12} \Bigg\|_{L_T^4} \\
&\quad \times d_{j'}^{\frac12} 2^{-\frac{j'}{4}} \|\nabla_h u^h\|_{\widetilde{L}_T^2(B^{0,\frac12})}^{\frac12}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Delta_j^v(S_{j'+2}^v u^h \Delta_{j'}^v u^3)(t)\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \|S_{j'+2}^v u^h\|_{L_v^\infty(L_h^4)} \|\Delta_{j'}^v u^3\|_{L_v^2(L_h^4)} \\
&\lesssim \|u^h\|_{B^{0,\frac12}}^{\frac12} \|\nabla_h u^h\|_{B^{0,\frac12}}^{\frac12} \left(\frac{2^{\frac{j'}{2}} \|\Delta_{j'}^v u^3\|_{L^2}}{\|u^3\|_{B^{0,\frac12}} + \varepsilon_1} \right)^{\frac12} \\
&\qquad \times 2^{-\frac{j'}{4}} \left(\|u^3\|_{B^{0,\frac12}} + \varepsilon_1 \right)^{\frac12} \sqrt{d_{j'}(t)} 2^{-\frac{j'}{4}} \|\nabla_h u^3\|_{B^{0,\frac12}}^{\frac12}.
\end{aligned}
$$

Then, we can immediately finish the proof.

Similarly, one can obtain the following lemma.

Lemma 0.3. *Let u and w be in $B^{0,\frac12}(T)$. We have*

$$
\|\Delta_j^v(uw)\|_{L_T^2(L^2(\mathbb{R}^3))} \lesssim d_j \nu_h^{-\frac12} 2^{-\frac{j}{2}} \|u\|_{B^{0,\frac12}(T)} \|w\|_{B^{0,\frac12}(T)}.
$$

Then, we can correct Proposition 3.2 in [1] as follows.

Proposition 0.1. *Let u be a divergence free vector filed in $B^{0,\frac12}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$
\begin{aligned}
G_j(T) &:= \int_0^T \left| \sum_{k,l} \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_l \partial_k (u^l u^k) \Delta_j^v \partial_h w dx \right| dt \\
&\le C d_j^2 \nu_h^{-1} 2^{-j} \|u^h\|^3_{B^{0,\frac{1}{2}}(T)} + C 2^{-\frac{j}{2}} \|\Delta_j^v \partial_h w\|_{L_T^2(L^2)} \\
&\quad \times \Bigg[d_j^{\frac{1}{2}} \|\nabla_h u^h\|^{\frac{1}{2}}_{\widetilde{L}_T^2(B^{0,\frac{1}{2}})} \left\| d_{2j}(t) \|\nabla_h u^3\|_{B^{0,\frac{1}{2}}} \|u^3\|_{B^{0,\frac{1}{2}}} \left(\|u^h\|_{B^{0,\frac{1}{2}}} + \varepsilon_2 \right) \right\|^{\frac{1}{2}}_{L_T^2} \\
&\quad + \left\| \sqrt{d_j(t)} \|\nabla_h u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u^h\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \sqrt{d_{1j}(t)} \|u^h\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \left(\|u^3\|_{B^{0,\frac{1}{2}}} + \varepsilon_1 \right)^{\frac{1}{2}} \right\|_{L_T^2} \Bigg] \\
&\quad + C \int_0^T \|\nabla_h u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \Bigg[\sqrt{d_j(t)} \sqrt{d_{1j}(t)} 2^{-\frac{j}{2}} \|\nabla_h u^h\|_{B^{0,\frac{1}{2}}} \left(\|u^3\|_{B^{0,\frac{1}{2}}} + \varepsilon_1 \right)^{\frac{1}{2}} \\
&\quad + \|u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \sum_{|j-j'|\le 5} \|\Delta_{j'}^v \partial_h u^h\|_{L^2(\mathbb{R}^3)} \Bigg] \sqrt{d_{3j}(t)} 2^{-\frac{j}{4}} \left(\|u^h\|_{B^{0,\frac{1}{2}}} + \varepsilon_2 \right)^{\frac{1}{2}} \|\Delta_j^v \nabla_h w\|^{\frac{1}{2}}_{L^2} dt,
\end{aligned}
$$

where $w = u^1$ or u^2, C is independent of ε_1 and ε_2, and

$$
d_{3j}(t) := \frac{2^{\frac{j}{2}} \|\Delta_j^v u^h\|_{L^2}}{\|u^h\|_{B^{0,\frac{1}{2}}} + \varepsilon_2}, \quad \sum_j \sup_{t\in[0,T]} d_{3j} \le \frac{\|u^h\|_{\widetilde{L}_T^\infty(B^{0,\frac{1}{2}})}}{\varepsilon_2}.
$$

Proof. We distinguish the terms with horizontal derivatives from the terms with vertical ones, writing

$$
G_j(T) \le G_j^h(T) + 2G_j^{v1}(T) + G_j^{v2}(T),
$$

where

$$
\begin{aligned}
G_j^h(T) &:= \sum_{l=1}^2 \sum_{k=1}^2 \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_l \partial_k (u^l u^k) \Delta_j^v \partial_h w dx \right| dt, \\
G_j^{v1}(T) &:= \sum_{k=1}^2 \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_3 \partial_k (u^3 u^k) \Delta_j^v \partial_h w dx \right| dt,
\end{aligned}
$$

and

$$
G_j^{v2}(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_3 (2u^3 \partial_3 u^3) \Delta_j^v \partial_h w dx \right| dt.
$$

Using Hölder's inequality and Lemma 0.3, we get

$$
\begin{aligned}
G_j^h(T) &\lesssim \sum_{l=1}^2 \sum_{k=1}^2 \|\Delta_j^v (u^l u^k)\|_{L_T^2(L^2)} \|\Delta_j^v \partial_h u^h\|_{L_T^2(L^2)} \\
&\lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|u^h\|^2_{B^{0,\frac{1}{2}}(T)} \|\Delta_j^v \partial_h u^h\|_{L_T^2(L^2)} \\
&\lesssim d_j^2 \nu_h^{-1} 2^{-j} \|u^h\|^3_{B^{0,\frac{1}{2}}(T)}.
\end{aligned}
$$

Similarly, using Hölder's inequality and Lemma 0.2, we have

$$
\begin{aligned}
&G_j^{v1}(T)\\
&\lesssim \sum_{k=1}^{2} \|\Delta_j^v(u^3u^k)\|_{L_T^2(L^2)}\|\Delta_j^v\partial_h w\|_{L_T^2(L^2)}\\
&\lesssim 2^{-\frac{j}{2}}\|\Delta_j^v\partial_h w\|_{L_T^2(L^2)}\Bigg[d_j^{\frac12}\|\nabla_h u^h\|^{\frac12}_{\widetilde{L}_T^2(B^{0,\frac12})}\Big\|d_{2j}(t)\|\nabla_h u^3\|_{B^{0,\frac12}}\|u^3\|_{B^{0,\frac12}}\Big(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\Big)\Big\|^{\frac12}_{L_T^2}\\
&\quad+\Big\|\sqrt{d_j(t)}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^h\|^{\frac12}_{B^{0,\frac12}}\sqrt{d_{1j}(t)}\|u^h\|^{\frac12}_{B^{0,\frac12}}\Big(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\Big)^{\frac12}\Big\|_{L_T^2}\Bigg].
\end{aligned}
$$

Since $\mathrm{div} u = 0$, we obtain

$$
\begin{aligned}
G_j^{v2}(T) &= \int_0^T \left|\int_{\mathbb{R}^3} \Delta_j^v(-\Delta)^{-1}\partial_3(2u^3\mathrm{div}_h u^h)\Delta_j^v\partial_h w dx\right| dt\\
&= \int_0^T \left|\int_{\mathbb{R}^3} \Delta_j^v(2u^3\mathrm{div}_h u^h)\Delta_j^v(-\Delta)^{-1}\partial_3\partial_h w dx\right| dt.
\end{aligned}
$$

Then, using Hölder's inequality, Minkowski's inequality, Lemma 2.3 in [1] and Lemma 0.1, we get

$$
\begin{aligned}
G_j^{v2}(T) &\lesssim \int_0^T \|\Delta_j^v(u^3\mathrm{div}_h u^h)\|_{L_h^{\frac43}(L_v^2)}\|\Delta_j^v(-\Delta)^{-1}\partial_h\partial_3 w\|_{L_h^4(L_v^2)}dt\\
&\lesssim \int_0^T \|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\Bigg[\sqrt{d_j(t)}\sqrt{d_{1j}(t)}2^{-\frac{j}{2}}\|\nabla_h u^h\|_{B^{0,\frac12}}\Big(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\Big)^{\frac12}\\
&\quad+\|u^3\|^{\frac12}_{B^{0,\frac12}}\sum_{|j-j'|\le 5}\|\Delta_{j'}^v\partial_h u^h\|_{L^2(\mathbb{R}^3)}\Bigg]\|\Delta_j^v w\|^{\frac12}_{L^2}\|\Delta_j^v\nabla_h w\|^{\frac12}_{L^2}dt.
\end{aligned}
$$

This completes the proof of Proposition 0.1.

Similarly, we can obtain the following proposition and omit the details.

Proposition 0.2. *Let u be a divergence free vector field in $B^{0,\frac12}(T)$. Then, for any $j\in\mathbb{Z}$, we have*

$$
\begin{aligned}
G_j(T) &:= \int_0^T \left|\sum_{k,l}\int_{\mathbb{R}^3} \Delta_j^v(-\Delta)^{-1}\partial_l\partial_k(u^lu^k)\Delta_j^v\partial_h u^h dx\right| dt\\
&\lesssim d_j^2\nu_h^{-1}2^{-j}\left[\|u^h\|^3_{B^{0,\frac12}(T)}+\|u^h\|^2_{B^{0,\frac12}(T)}\|u^3\|_{B^{0,\frac12}(T)}\right].
\end{aligned}
$$

Proof of the existence part of Theorem 1.1 in [1] with the initial condition (0.1). Applying the operator Δ_j^v to (3.1) in [1] and taking the L^2 inner product of the resulting equation with $\Delta_j^v u_n^1$, from Proposition 3.1 in [1], Proposition 0.1 and the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
&2^j\left(\|\Delta_j^v u_n^1(t)\|_{L^2}^2+\nu_h\|\nabla_h\Delta_j^v u_n^1\|_{L_t^2(L^2)}^2+2\nu_3\|\partial_3\Delta_j^v u_n^1\|_{L_t^2(L^2)}^2\right)\\
&\quad\le 2^j\|\Delta_j^v u_n^1(0)\|_{L^2}^2+Cd_j^2\nu_h^{-1}\|u_n^h\|_{B^{0,\frac12}(t)}^3\\
&\qquad+C\nu_h^{-1}d_j\|\nabla_h u^h\|_{\widetilde{L}_t^2(B^{0,\frac12})}\left(\int_0^t d_{2j}^2(s)\|\nabla_h u^3\|_{B^{0,\frac12}}^2\|u^3\|_{B^{0,\frac12}}^2\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)^2ds\right)^{\frac12}\\
&\qquad+C\nu_h^{-1}\int_0^t d_j(s)\|\nabla_h u^3\|_{B^{0,\frac12}}\|\nabla_h u^h\|_{B^{0,\frac12}}d_{1j}(s)\|u^h\|_{B^{0,\frac12}}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)ds\\
&\qquad+C\nu_h^{-\frac13}\int_0^t\|\nabla_h u^3\|_{B^{0,\frac12}}^{\frac23}d_j^{\frac23}(s)d_{1j}^{\frac23}(s)\|\nabla_h u^h\|_{B^{0,\frac12}}^{\frac43}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)^{\frac23}d_{3j}^{\frac23}(s)\\
&\qquad\times\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)^{\frac23}ds\\
&\qquad+C\nu_h^{-\frac13}\left(\int_0^t d_{3j}^2(t)\|\nabla_h u^3\|_{B^{0,\frac12}}^2\|u^3\|_{B^{0,\frac12}}^2\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)^2ds\right)^{\frac13}\\
&\qquad\times\left(\sum_{|j-j'|\le5}2^{j'}\|\Delta_{j'}^v\partial_h u^h\|_{L_t^2(L^2(\mathbb{R}^3))}^2\right)^{\frac23}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_n^1(t)\|_{B^{0,\frac12}}+\sqrt{\nu_h}\|\nabla_h u_n^1\|_{\widetilde{L}_t^2(B^{0,\frac12})}+\sqrt{\nu_3}\|\partial_3\Delta_j^v u_n^1\|_{\widetilde{L}_t^2(B^{0,\frac12})}\\
&\quad\le C\|u_n^1(0)\|_{B^{0,\frac12}}+C\nu_h^{-\frac12}\|u_n^h\|_{B^{0,\frac12}(t)}^{\frac32}\\
&\qquad+C\nu_h^{-\frac12}\|\nabla_h u^h\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12}\left\|\|\nabla_h u^3\|_{B^{0,\frac12}}\|u^3\|_{B^{0,\frac12}}\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)\right\|_{L_t^2}^{\frac12}\|\sup_{s\in[0,t]}d_{2j}(s)\|_{l_j^1}^{\frac14}\\
&\qquad+C\nu_h^{-\frac12}\left(\int_0^t\|\nabla_h u^h\|_{B^{0,\frac12}}\|\nabla_h u^3\|_{B^{0,\frac12}}\|u^h\|_{B^{0,\frac12}}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)ds\right)^{\frac12}\|\sup_{s\in[0,t]}d_{1j}(s)\|_{l_j^1}^{\frac12}\\
&\qquad+C\nu_h^{-\frac16}\left(\int_0^t\|\nabla_h u^3\|_{B^{0,\frac12}}^{\frac23}\|\nabla_h u^h\|_{B^{0,\frac12}}^{\frac43}\left(\|u^3\|_{B^{0,\frac12}}+\varepsilon_1\right)^{\frac23}\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)^{\frac23}ds\right)^{\frac12}\\
&\qquad\times\|\sup_{s\in[0,t]}d_{1j}^{\frac23}(s)d_{3j}^{\frac13}(s)\|_{l_j^1}^{\frac12}\\
&\qquad+C\nu_h^{-\frac32}\left(\int_0^t\|\nabla_h u^3\|_{B^{0,\frac12}}^2\|u^3\|_{B^{0,\frac12}}^2\left(\|u^h\|_{B^{0,\frac12}}+\varepsilon_2\right)^2ds\right)^{\frac12}\|\sup_{s\in[0,t]}d_{3j}(s)\|_{l_j^1}^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|u_n^1(0)\|_{B^{0,\frac12}} + C\nu_h^{-\frac12}\|u_n^h\|^{\frac32}_{B^{0,\frac12}(T)} + \frac{\sqrt{\nu_h}}{4}\|\nabla_h u_n^h\|_{\widetilde{L}^2_t(B^{0,\frac12})} \\
&+C\nu_h^{-\frac32}\|u^3\|_{\widetilde{L}^\infty_T(B^{0,\frac12})}\left(\int_0^t \|\nabla_h u^3\|^2_{B^{0,\frac12}}\|u^h\|^2_{B^{0,\frac12}}\,ds\right)^{\frac12}\left(1+\varepsilon_2^{-\frac12}\|u^h\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\right) \\
&+C\nu_h^{-\frac32}\|\nabla_h u^3\|_{\widetilde{L}^2_t(B^{0,\frac12})}\|u^3\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}\varepsilon_2^{\frac12}\|u^h\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}, \qquad (0.2)
\end{aligned}
$$

where $t \in (0, T]$ and we choose $\varepsilon_1 = \|u^3\|_{\widetilde{L}^\infty_T(B^{0,\frac12})}$. Similarly, we obtain the same estimate on u_n^2. Letting

$$
\varepsilon_2 = \frac{\|u^h\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}}{2C\nu_h^{-3}\|\nabla_h u^3\|^2_{\widetilde{L}^2_T(B^{0,\frac12})}\|u^3\|^2_{\widetilde{L}^\infty_T(B^{0,\frac12})}},
$$

using Gronwall's inequality, we get

$$
\begin{aligned}
&\|u_n^h\|^2_{B^{0,\frac12}(T)} \\
&\quad\leq \left(2C_0\|u_n^h(0)\|^2_{B^{0,\frac12}} + C\nu_h^{-1}\|u_n^h\|^3_{B^{0,\frac12}(T)}\right)\exp\left\{C\nu_h^{-8}\left(\|u_n^3\|_{B^{0,\frac12}(T)}+\nu_h\right)^8\right\}.
\end{aligned}
$$

Similarly, we obtain

$$
\|u_n^3\|_{B^{0,\frac12}(T)} \leq 2C_0\|u_n^3(0)\|_{B^{0,\frac12}} + C\nu_h^{-\frac12}\|u_n^h\|^{\frac12}_{B^{0,\frac12}(T)}\|u_n^3\|_{B^{0,\frac12}(T)} + C\nu_h^{-\frac12}\|u_n^h\|^{\frac32}_{B^{0,\frac12}(T)}.
$$

Then, we obtain

$$
\begin{aligned}
\|u_n^h\|^2_{B^{0,\frac12}(T)} \leq e^{C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac12}}+2\nu_h)^8}\Big(&2C_0\|u_0^h\|^2_{B^{0,\frac12}} \\
&+C\nu_h^{-1}(4C_0\|u_0^h\|^2_{B^{0,\frac12}})^{\frac32}e^{\frac32 C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac12}}+2\nu_h)^8}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_n^3\|_{B^{0,\frac12}(T)} \\
&\leq 2C_0\|u_0^3\|_{B^{0,\frac12}} + C\nu_h^{-\frac12}(4C_0\|u_0^h\|^2_{B^{0,\frac12}})^{\frac34}e^{\frac34 C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac12}}+2\nu_h)^8} \\
&\quad+C\nu_h^{-\frac12}(4C_0\|u_0^h\|^2_{B^{0,\frac12}})^{\frac14}e^{\frac14 C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac12}}+2\nu_h)^8}\left(4C_0\|u_0^3\|_{B^{0,\frac12}}+\nu_h\right),
\end{aligned}
$$

for all $T < T_n$, where

$$
\begin{aligned}
T_n := \sup\{t > 0;\ &\|u_n^h\|^2_{B^{0,\frac12}(t)} \leq 4C_0\|u_0^h\|^2_{B^{0,\frac12}}e^{C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac12}}+2\nu_h)^8}, \\
&\|u_n^3\|_{B^{0,\frac12}(t)} \leq 4C_0\|u_0^3\|_{B^{0,\frac12}}+\nu_h\},
\end{aligned}
$$

Then, if u_0 satisfies

$$C_1 \nu_h^{-1} \|u_0^h\|_{B^{0,\frac{1}{2}}} \exp\{C_1(\nu_h^{-1} \|u_0^3\|_{B^{0,\frac{1}{2}}} + 1)^8\} \leq 1,$$

where

$$C_1 = 4^9 C^2 C_0^8, \tag{0.3}$$

we get that for any n and for any $T < T_n$,

$$\|u_n^h\|_{B^{0,\frac{1}{2}}(T)} \leq \frac{5}{2} C_0 \|u_0^h\|^2_{B^{0,\frac{1}{2}}} e^{C\nu_h^{-8}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}} + 2\nu_h)^8}$$

and

$$\|u_n^3\|_{B^{0,\frac{1}{2}}(t)} \leq \frac{5}{2} C_0 \|u_0^3\|_{B^{0,\frac{1}{2}}} + \frac{1}{2}\nu_h.$$

Thus, $T_n = +\infty$. Then, the existence follows from the classical compactness method. □

Finally, we would like to thank Ping Zhang for pointing out this gap.

Reference

1. Zhang, T.: Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space. Commun. Math. Phys. **287**, 211–224 (2009)

Communicated by P. Constantin