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A Characterization of Vertex Operator Algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$

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Abstract: We study a simple, rational and C_2-cofinite vertex operator algebra whose weight 1 subspace is zero, the dimension of weight 2 subspace is greater than or equal to 2 and with $c = \tilde{c} = 1$. Under some additional conditions it is shown that such a vertex operator algebra is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$.

1. Introduction

The vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ is characterized in [ZD] as a unique simple rational, C_2-cofinite vertex operator algebra with $c = \tilde{c} = 1$, weight one subspace being zero and weight two subspace being 2 dimensional. In this paper we strengthen this result by allowing the dimensions of weight two subspace to be greater than or equal to 2. This proves the conjecture given in [ZD].

The importance of $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ was first noticed in [DMZ] (also see [M2,DGH]) for the study of the moonshine vertex operator algebra $V^\natural$ [FLM]. In fact, it was essentially proved in [DMZ] that the fixed point vertex operator subalgebra V_L^+ under the involution induced from the -1 isometry of L is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ if L is a rank one lattice generated by a vector whose squared length is 4 and $V^\natural$ contains $L(\frac{1}{2}, 0)^{\otimes 48}$. This led to the theory of code vertex operator algebras [M1,M2,M3] and framed vertex operator algebras [DGH]. A new construction of the moonshine vertex operator algebra $V^\natural$ is given in [M4] using the theory of code and framed vertex operator

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algebras. Furthermore, the recent progress in [DGL and LY] on proving the uniqueness of $V^\natural$ depends largely on the theory of framed vertex operator algebras and code vertex operator algebras. Also see [KL] for the study of conformal nets arising from framed vertex operator algebras.

The characterization of $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ given in this paper is a necessary step in the classification of rational vertex operator algebras with $c = 1$. It is a well known conjecture (cf. [K,ZD]) that any simple rational vertex operator algebra with $c = 1$ is either V_L, V_L^+ or $V_{L_{A_1}}^G$ where L is a rank one positive definite even lattice, L_{A_1} is the root lattice of type A_1 and G is a subgroup of $SO(3)$ isomorphic to A_4, S_4 or A_5. As pointed out in [ZD], the correct conjecture should also assume c is equal to the effective central charge $\tilde{c}$. A characterization of V_L for an arbitrary positive definite even lattice is obtained in [DM1]. Although there was some progress at the q-character level on the classification of rational vertex operator algebras with $c = 1$ in the physics literature [K], there is still a long way to prove the conjecture completely by a lack of characterization of V_L^+. It is desirable that the characterization of $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ may help to understand V_L^+ in general.

If the weight one subspace of a vertex operator algebra is 0, then its weight two subspace is a commutative (non-associative) algebra (cf. [FLM,DGL]). Since the weight two subspace V_2 in [ZD] is assumed to be 2-dimensional, it is necessarily a commutative associative algebra. The main result in [ZD] was based on the study of the vertex operator algebra $W(2, 2)$ and the growth of the graded dimensions of vertex operator algebras. But in this paper we assume $\dim V_2 \geq 2$. So V_2 is not an associative algebra and the situation is much more complicated. By a result from [R], V_2 either has two nontrivial idempotent elements or has a nontrivial nilpotent element. The former case basically follows from the argument in [ZD]. The key point in this paper is to use the fusion rules for the Virasoro algebra with $c = 1$ to deal with the later case. This should explain why we need the assumption in the main theorem that the vertex operator algebra is a sum of highest weight modules for the Virasoro algebra. This assumption is expected to be established for all rational vertex operator algebras with $c = 1$. This leads us to the study of fusion rules for the Virasoro algebra with $c = 1$. The fusion rules for the Virasoro algebra with $c = 1$ have been investigated from different points of view [RT,X]. The fusion rules among irreducible modules $L(1, m^2/4)$ with $m \in \mathbb{Z}$ for the Virasoro algebra have been given in [M] based on the $A(V)$-theory developed in [Z,FZ and L2]. We extend these results to include irreducible modules $L(1, n)$ for $n \in \mathbb{Z}$. We certainly believe that the fusion rules computed in this paper will play important roles in the future classification of rational vertex operator algebras with $c = 1$.

The paper is organized as follows: In Sect. 2 we review the various notions of modules and define rational vertex operator algebras. Section 3 is about the Virasoro vertex operator algebras and some results on the structure of highest weight modules for the Virasoro algebra with $c = 1$. We also prove that any simple vertex operator algebra with $c > 1$ is a completely reducible module for the Virasoro algebra. In Sect. 4 we first review the $A(V)$-theory including how to use the bimodules to compute the fusion rules. The new results in this section are the fusion rules for the Virasoro algebra with $c = 1$. The most difficult case is the fusion rules for the irreducible modules $L(1, m^2)$ for integers m as they are not the Verma modules. These fusion rules are fundamental later in the proof of the main theorem. Section 5 is devoted to the proof of the main theorem. In the case that V_2 has a nontrivial nilpotent element we need to construct some highest weight vectors with certain properties. Then we use the fusion rules to prove this is impossible. This forces the dimension of V_2 to be 2 and the result in [ZD] applies.

2. Preliminaries

Let $V=(V,Y,\mathbf{1},\omega)$ be a vertex operator algebra [B,FLM]. We review various notions of V-modules (cf. [FLM,Z,DLM1]) and the definition of rational vertex operator algebras. We also discuss some consequences following [DLM1].

Definition 2.1. *A weak V module is a vector space M equipped with a linear map*

$$Y_M: V\to \mathrm{End}(M)[[z,z^{-1}]],$$
$$v\mapsto Y_M(v,z)=\sum_{n\in\mathbb{Z}} v_n z^{-n-1},\quad v_n\in \mathrm{End}(M),$$

satisfying the following:

1) $v_n w=0$ *for* $n>>0$, *where* $v\in V$ *and* $w\in M$,
2) $Y_M(\mathbf{1},z)=Id_M$,
3) *The Jacobi identity holds:*

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(u,z_1)Y_M(v,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(v,z_2)Y_M(u,z_1)$$
$$=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(u,z_0)v,z_2). \tag{2.1}$$

Definition 2.2. *An admissible V module is a weak V module which carries a $\mathbb{Z}_+$-grading $M=\bigoplus_{n\in\mathbb{Z}_+}M(n)$, such that if $v\in V_r$ then $v_m M(n)\subseteq M(n+r-m-1)$.*

Definition 2.3. *An ordinary V module is a weak V module which carries a $\mathbb{C}$-grading $M=\bigoplus_{\lambda\in\mathbb{C}}M_\lambda$, such that:*

1) $dim(M_\lambda)<\infty$,
2) $M_{\lambda+n}=0$ *for fixed* λ *and* $n<<0$,
3) $L(0)w=\lambda w=\mathrm{wt}(w)w$ *for* $w\in M_\lambda$, *where* $L(0)$ *is the component operator of* $Y_M(\omega,z)=\sum_{n\in\mathbb{Z}}L(n)z^{-n-2}$.

Remark 2.4. It is easy to see that an ordinary V-module is an admissible one. If W is an ordinary V-module, we simply call W a V-module.

We call a vertex operator algebra rational if the admissible module category is semisimple. We have the following result from [DLM2] (also see [Z]).

Theorem 2.5. *If V is a rational vertex operator algebra, then V has finitely many irreducible admissible modules up to isomorphism and every irreducible admissible V-module is ordinary.*

Suppose that V is a rational vertex operator algebra and let $M^1,\dots,M^k$ be the irreducible modules such that

$$M^i=\oplus_{n\geq 0}M^i_{\lambda_i+n},$$

where $\lambda_i\in\mathbb{Q}$ [DLM3], $M^i_{\lambda_i}\neq 0$ and each $M^i_{\lambda_i+n}$ is finite dimensional. Let λ_{min} be the minimum of λ_i's. The effective central charge $\tilde{c}$ is defined as $c-24\lambda_{min}$. For each M^i we define the q-character of M^i by

$$\mathrm{ch}_q M^i=q^{-c/24}\sum_{n\geq 0}(\dim M^i_{\lambda_i+n})q^{n+\lambda_i}.$$

A vertex operator algebra is called C_2-cofinite if $C_2(V)$ has finite codimension where $C_2(V) = \langle u_{-2}v | u, v \in V\rangle$.

Take a formal power series in q or a complex function $f(z) = q^{\lambda}\sum_{n\geq 0} a_n q^n$. We say that the coefficients of $f(q)$ satisfy the *polynomial growth condition* if there exist positive numbers A and α such that $|a_n| \leq An^{\alpha}$ for all n.

If V is rational and C_2-cofinite, then $\mathrm{ch}_q M^i$ converges to a holomorphic function on the upper half plane [Z]. Using the modular invariance result from [Z] and results on vector valued modular forms from [KM] we have (see [DM1])

Lemma 2.6. *Let V be rational and C_2-cofinite. For each i, the coefficients of $\eta(q)^{\tilde{c}}$ $\mathrm{ch}_q M^i$ satisfy the polynomial growth condition where*

$$\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n).$$

3. Virasoro Vertex Operator Algebras

We will review vertex operator algebras associated to the highest weight representations for the Virasoro algebra and study a general vertex operator algebra viewed as a module for the Virasoro vertex operator algebra.

We first recall some basic facts about the highest weight modules for the Virasoro algebra Vir. Let $c, h \in \mathbb{C}$ and $V(c,h)$ be the corresponding highest weight module for the Virasoro algebra Vir with central charge c and highest weight h. We set $\bar{V}(c,0) = V(c,0)/U(Vir)L(-1)v$, where v is a highest weight vector with highest weight 0 and denote the irreducible quotient of $V(c,h)$ by $L(c,h)$. We have (see [KR,FZ]):

Proposition 3.1. *Let c be a complex number.*

(1) *$\bar{V}(c,0)$ is a vertex operator algebra and $L(c,0)$ is a simple vertex operator algebra.*
(2) *For any $h \in \mathbb{C}$, $V(c,h)$ is a module for $\bar{V}(c,0)$.*
(3) *$V(c,h) = L(c,h)$, $\bar{V}(c,0) = L(c,0)$, for $c > 1$ and $h > 0$.*
(4) *$V(1,h) = L(1,h)$ if and only if $h \neq \frac{m^2}{4}$ for $m \in \mathbb{Z}$. In case $h = m^2$ for a nonnegative integer m, the unique maximal submodule of $V(1,m^2)$ is generated by a highest weight vector with highest weight $(m+1)^2$ and is isomorphic to $V(1,(m+1)^2)$.*

We next study a general simple vertex operator algebra as a module for the Virasoro algebra.

Lemma 3.2. *Let V be a simple vertex operator algebra such that $V_0 = \mathbb{C}\mathbf{1}$ and $L(1)V_1 = 0$. Let $h > 0$ be such that the Verma module $V(c,h)$ for the Virasoro algebra is irreducible. Let U be the sum of irreducible submodules of V isomorphic to $V(c,h)$. Then $V = U \oplus U^{\perp}$, where $U^{\perp} = \{v \in V | (v,U) = 0\}$ and $(,)$ is the canonical non-degenerate symmetric invariant bilinear form on V such that $(\mathbf{1},\mathbf{1}) = 1$ [FHL], [L1].*

Proof. It is enough to prove that $U \cap U^{\perp} = 0$. First note that U is a completely reducible module for the Virasoro algebra. Also, $U^{\perp}$ is a module for the Virasoro algebra. Suppose that $U \cap U^{\perp} \neq 0$. Let W be an irreducible submodule of $U \cap U^{\perp}$. Then $X = V/W^{\perp}$ is an irreducible module for the Virasoro algebra isomorphic to $V(c,h)$ and can be identified with the graded dual W' of W. Let $v \in V_h$ be such that $v + W^{\perp}$ is the highest weight vector of $V/W^{\perp}$. Let M be the module for the Virasoro algebra generated by v. Then $M\cap W^{\perp}$ is a submodule of M, $M/(M\cap W^{\perp})$ is isomorphic to X and

$$M\cap V_h=\mathbb{C}v\oplus(M\cap W^{\perp}\cap V_h)\ \text{(direct sum of subspaces)}.$$

Note that there are only finitely many composition factors in $M\cap W^{\perp}$. We have the following exact sequences for modules of the Virasoro algebra:

$$0\to M\cap W^{\perp}\to M\to L(c,h)\to 0$$

and

$$0\to L(c,h)\to M'\to(M\cap W^{\perp})'\to 0.$$

Since $(W,v)\neq 0$, it follows that M can not be a direct sum of submodules $L(c,h)$ and $M\cap W^{\perp}$ for the Virasoro vertex operator algebra. So M' can not be a direct sum of submodules $L(c,h)$ and $(M\cap W^{\perp})'$. Therefore there exists a highest weight submodule Z of M' such that $L(c,h)$ is a submodule of Z. But from the module structure theory in [KR], $L(c,h)$ can never be a submodule of any highest weight module if $V(c,h)=L(c,h)$. This is a contradiction. The proof is complete. □

Proposition 3.3. *If V is a simple vertex operator algebra such that $V_0=\mathbb{C}\mathbf{1}$, $L(1)V_1=0$ and $c>1$. Then V is a completely reducible module for the Virasoro algebra.*

Proof. Recall from [KR] or Proposition 3.1 that $V(c,h)=L(c,h)$ if $h>0$ and $L(c,0)=\bar{V}(c,0)$. It is clear that the vertex operator subalgebra of V generated by $\mathbf{1}$ is isomorphic to $L(c,0)$. So we can regard $L(c,0)$ as a subalgebra of V. Then we have the decomposition $V=L(c,0)\oplus L(c,0)^{\perp}$ as $(\mathbf{1},\mathbf{1})=1$ and $L(c,0)\cap L(c,0)^{\perp}=0$. Let U^n be the $L(c,0)$-submodule of V generated by the highest weight vectors with highest weight n. Then U^n is a completely reducible module for the Virasoro algebra and $V=\oplus_{n\geq 0}U^n$ by Lemma 3.2. □

We remark that in the case $c=1$ we cannot establish the result in Proposition 3.3 although we strongly believe it is true if we also assume that V is rational and C_2-cofinite. We need this assumption for $c=1$ later to characterize the vertex operator algebra $L(1/2,0)\otimes L(1/2,0)$. This is also the original motivation for us to study the complete reducibility of vertex operator algebras as modules for the Virasoro algebra.

It has been studied extensively on how to decompose an arbitrary vertex operator algebra and its modules as a sum of indecomposable modules for $sl(2,\mathbb{C})=\mathbb{C}L(1)+\mathbb{C}L(-1)+\mathbb{C}L(0)$ in [DLiM]. It seems that decomposing an arbitrary vertex operator algebra into a sum of indecomposable modules for the Virasoro algebra is much more difficult. But such a decomposition is definitely important in the study of vertex operator algebras and their representations.

4. $A(V)$-Theory and Fusion Rules

Let V be a vertex operator algebra. An associative algebra $A(V)$ has been introduced and studied in [Z]. It turns out that $A(V)$ is very powerful and useful in representation theory for vertex operator algebras. One can use $A(V)$ not only to classify the irreducible admissible modules [Z], but also to compute the fusion rules using $A(V)$-bimodules [FZ]. We will first review the definition of $A(V)$ and some important results about $A(V)$

from [Z,FZ and L2]. We then apply the $A(V)$-theory to the vertex operator algebra $L(1,0)$ to compute the fusion rules for $L(1,0)$. The central task is to determine the $A(L(1,0))$-bimodule $A(L(1,m^2))$ for any integer m.

As a vector space, $A(V)$ is a quotient space of V by $O(V)$, where $O(V)$ denotes the linear span of elements

$$u \circ v = \mathrm{Res}_z(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u}}{z^2}v) = \sum_{i\geq 0}\binom{\mathrm{wt}\,u}{i}u_{i-2}v \tag{4.1}$$

for $u, v \in V$ with u being homogeneous. Product in $A(V)$ is induced from the multiplication

$$u * v = \mathrm{Res}_z(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u}}{z}v) = \sum_{i\geq 0}\binom{\mathrm{wt}\,u}{i}u_{i-1}v \tag{4.2}$$

for $u, v \in V$ with u being homogeneous. $A(V) = V/O(V)$ is an associative algebra with identity $\mathbf{1} + O(V)$ and with $\omega + O(V)$ being in the center of $A(V)$. The most important result about $A(V)$ is that for any admissible V-module $M = \oplus_{n\geq 0}M(n)$ with $M(0) \neq 0$, $M(0)$ is an $A(V)$-module such that $v + O(V)$ acts as $o(v)$, where $o(v) = v_{\mathrm{wt}v-1}$ for homogeneous v.

For an admissible V-module W, we also define $O(W) \subset W$ to be the linear span of elements of type

$$\mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v}}{z^2}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v}{i}v_{i-2}w \tag{4.3}$$

for homogeneous $v \in V$ and $w \in W$. Let $A(W) = W/O(W)$. Then $A(W)$ has an $A(V)$-bimodule structure [FZ] induced by the following bilinear operations $V \times W \to W$ and $W \times V \to W$: for $w \in W$ and homogeneous $v \in V$,

$$v * w = \mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v}}{z}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v}{i}v_{i-1}w, \tag{4.4}$$

$$w * v = \mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v-1}}{z}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v-1}{i}v_{i-1}w. \tag{4.5}$$

We quote the following proposition from [FZ]:

Proposition 4.1. *If W is an admissible module for a vertex operator algebra V and M is a submodule of W, then the image $\bar{M}$ of M in $A(W)$ is a sub-$A(V)$-bimodule of $A(W)$, and the quotient $A(W)/\bar{M}$ is isomorphic to the $A(V)$-bimodule $A(W/M)$ associated to the quotient V-module W/M.*

Let W^i $(i = 1, 2, 3)$ be ordinary V-modules. We denote by $I_V\binom{W^3}{W^1\ W^2}$ the vector space of all intertwining operators of type $\binom{W^3}{W^1\ W^2}$. For a V-module W, let W' denote the graded dual of W. Then W' is also a V-module [FHL]. It is well known that fusion rules have the following symmetry (see [FHL]).

Proposition 4.2. *Let W^i $(i=1,2,3)$ be V-modules. Then*

$$\dim I_V\begin{pmatrix} W^3 \\ W^1\, W^2\end{pmatrix} = \dim I_V\begin{pmatrix} W^3 \\ W^2\, W^1\end{pmatrix}, \quad \dim I_V\begin{pmatrix} W^3 \\ W^1\, W^2\end{pmatrix} = \dim I_V\begin{pmatrix} (W^2)' \\ W^1\, (W^3)'\end{pmatrix}.$$

Let $W^i = \oplus_{n\geq 0} W^i(n)$ $(i=1,2,3)$ be V-modules such that $L(0)|_{W^i(0)} = \lambda_i$. Let $\mathcal{Y}(\cdot,z)$ be an intertwining operator of type $\begin{pmatrix} W^3 \\ W^1\, W^2\end{pmatrix}$. Define the following bilinear map:

$$f_{\mathcal{Y}}:\quad A(W^1)\otimes_{A(V)} W^2(0)\to W^3(0),$$
$$u^1\otimes u^2 \to o(u^1)u^2,\quad u^1\in A(W^1),\quad u^2\in W^2(0),$$

where $o(u^1)$ is the component operator of $\mathcal{Y}(u^1,z)$ such that $o(u^1)$ maps $W^2(0)$ to $W^3(0)$. Then $f_{\mathcal{Y}}$ is an $A(V)$-module homomorphism [FZ]. To state the next result we need to define the Verma type admissible module $M(U)$ associated to an $A(V)$-module U:

Definition 4.3. *Let V be a vertex operator algebra and U an $A(V)$-module. An admissible V-module $M=\bigoplus_{n=0}^{\infty} M(n)$ is called the Verma type module generated by U if $M(0)=U$ as $A(V)$-module and for any admissible V-module $W=\bigoplus_{n=0}^{\infty} W(n)$ with $W(0)=U$ as $A(V)$-module, the identity map from $M(0)$ to $W(0)$ lifts to a V-module homomorphism from M to W.*

The existence of a Verma type admissible module was given in [Z] (also see [DLM2]). The following result comes from [L2]:

Lemma 4.4. *Let W^i be V-modules for $i=1,2,3$. If W^3 is an irreducible V-module, then the linear map $\mathcal{Y}\mapsto f_{\mathcal{Y}}$ is an injective map from the space of intertwining operators of type $\begin{pmatrix} W^3 \\ W^1\, W^2\end{pmatrix}$ to $Hom_{A(V)}(A(W^1)\otimes_{A(V)} W^2(0), W^3(0))$. Furthermore, $\mathcal{Y}\mapsto f_{\mathcal{Y}}$ is an isomorphism, if both W^2 and $(W^3)'$ are Verma type modules for V.*

We quote a result about the vertex operator algebra $\bar{V}(c,0)$ from [FZ].

Proposition 4.5. (1) *The associative algebra $A(\bar{V}(c,0))$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$, with the isomorphism being given by $x^n\in\mathbb{C}[x]\mapsto[(L(-2)+L(-1))^n\mathbf{1}]$, where $[a]=a+O(\bar{V}(c,0))$ for $a\in\bar{V}(c,0)$.*

(2) *For the Verma module $V(c,h)$, the $A(\bar{V}(c,0))$-bimodule $A(V(c,h))$ is $\mathbb{C}[x,y]$ with x and y acting on the left and right as multiplications by x and y respectively. The isomorphism from $\mathbb{C}[x,y]$ to $A(V(c,h))$ is given by $x^my^n\mapsto[(L(-2)+2L(-1)+L(0))^m(L(-2)+L(-1))^n\mathbf{1}_h]$, where $\mathbf{1}_h$ is a fixed nonzero highest weight vector of $V(c,h)$.*

We now discuss the relation between the Verma module for the Virasoro algebra and the Verma type admissible module for vertex operator algebra $\bar{V}(c,0)$. By Proposition 4.5, $A(\bar{V}(c,0))=\mathbb{C}[x]$. So any irreducible $A(\bar{V}(c,0))$-module is one dimensional such that $[\omega]$ acts as a constant h. Denote this module by U. It is clear that the Verma type admissible $\bar{V}(c,0)$-module generated by U is exactly the Verma module $V(c,h)$.

We next turn our attention to the fusion rules for the vertex operator algebra $L(1,0)$. The following theorem is the foundation in our computation of the fusion rules.

Theorem 4.6. *Let r be a positive integer. Then*

$$A(L(1,r^2)) = \mathbb{C}[x,y]/\bar{I},$$

where

$$\bar{I} = < (x-y)\prod_{i=1}^{r}[(x-y)^2 - 2i^2(x+y) + i^4] >$$

is a two-sided ideal of $\mathbb{C}[x,y]$ *generated by* $(x-y)\prod_{i=1}^{r}[(x-y)^2 - 2i^2(x+y)+i^4]$.

Proof. Since $\bar{V}(1,0) = L(1,0)$, by Proposition 4.5, the associative algebra $A(L(1,0))$ is $\mathbb{C}[x]$ and the $A(L(1,0))$-bimodule $A(V(1,r^2))$ is isomorphic to $\mathbb{C}[x,y]$ with x and y acting on the left and right as multiplications by x and y respectively. By Proposition 4.1, as an $A(L(1,0))$-bimodule,

$$A(L(1,r^2)) \cong \mathbb{C}[x,y]/\bar{I},$$

where $\bar{I}$ is the image in $A(V(1,r^2))$ of the maximal proper submodule I of $V(1,r^2)$. Since I is generated by a non-zero element $v^{(r+1)}$ in $V(1,r^2)$ such that

$$L(0)v^{(r+1)} = (r+1)^2 v^{(r+1)}, \quad L(k)v^{(r+1)} = 0, \quad 0 < k \in \mathbb{Z}_+,$$

it follows that $\bar{I}$ is generated by a polynomial $f(x,y)$ in $\mathbf{C}[x,y]$ with degree $s \leq 2r+1$. Assume that

$$f(x,y) = \sum_{i=0}^{s} a_i(x)y^i,$$

where $a_i(x)$, $i = 0,1,\ldots,s$ are polynomials in x of degrees at most $2r+1-i$.

We need to use the vertex operator algebra V_L associated to the rank one even positive definite lattice $L = \mathbb{Z}\alpha$ with $(\alpha,\alpha) = 2$ [FLM]. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, and $\hat{\mathfrak{h}}_{\mathbb{Z}}$ be the corresponding Heisenberg algebra. Denote by $M(1) = \mathbb{C}[\alpha(-n)|n > 0]$ the associated irreducible induced module for $\hat{\mathfrak{h}}_{\mathbb{Z}}$ such that the canonical central element of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ acts as 1. Let $\mathbb{C}[L]$ be the group algebra of L with a basis e^{γ} for $\gamma \in L$. Let $\beta \in \mathfrak{h}$ be such that $(\beta,\beta) = 1$. It is known that $V_L = M(1) \otimes \mathbb{C}[L]$ is a simple rational vertex operator algebra with $\mathbf{1} = 1 \otimes e^0$ and $\omega = \frac{1}{2}\beta(-1)^2\mathbf{1}$ [B,FLM,D,DLM1]. The subalgebra generated by ω of V_L is isomorphic to $L(1,0)$ and

$$M(1) = \bigoplus_{p\geq 0} L(1,p^2),$$
$$V_L = \bigoplus_{m\geq 0} (2m+1)L(1,m^2), \tag{4.6}$$

as modules for the Virasoro algebra (cf. [DG]).

It is well-known that V_L is isomorphic to the fundamental representation $L(\Lambda_0)$ for the affine Kac-Moody algebra $A_1^{(1)}$ [FK]. Note that the weight one subspace $(V_L)_1$ of V_L forms a Lie algebra $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{C})$, where the Lie bracket in $(V_L)_1$ is defined as $[u,v] = u_0 v$ and u_0 is the component operator of $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$. $\mathfrak{g}$ acts on V_L via v_0 for $v \in (V_L)_1$. The $\mathfrak{g}$-invariant elements $V_L^{\mathfrak{g}} = \{v \in V_L | \mathfrak{g}\cdot v = 0\}$ form a simple vertex operator algebra and is isomorphic to $L(1,0)$ (see [DG]).

Let W_m be the unique $m+1$-dimensional highest weight module for $\mathfrak{g}$ with highest weight $m\in\mathbb{Z}_{\geq 0}$. Let $V_L^{W_m}$ be the sum of irreducible $\mathfrak{g}$-submodules of V_L isomorphic to W_m, and $(V_L)_{W_m}$ the space of highest weight vectors in $V_L^{W_m}$. Then by [DG], as a $(V_L^{\mathfrak{g}},\mathfrak{g})$-module V_L has decomposition

$$V_L=\bigoplus_{m\geq 0}V_L^{W_{2m}}=\bigoplus_{m\geq 0}(V_L)_{W_{2m}}\otimes W_{2m} \tag{4.7}$$

and $(V_L)_{W_{2m}}$ is an irreducible module for $V_L^{\mathfrak{g}}$. Moreover, $(V_L)_{W_{2k}}$ and $(V_L)_{W_{2m}}$ are isomorphic if and only if $k=m$. By [DG], $(V_L)_{W_{2m}}$ is isomorphic to $L(1,m^2)$ as $L(1,0)$-module. For $m,n\in\mathbb{Z}_{+}$, $m\geq n$, let

$$W_{2m,2n}=span\{u_jv|u\in W_{2m},v\in W_{2n},j\in\mathbb{Z}\}.$$

Then $W_{2m,2n}$ is a $\mathfrak{g}$-module. Let $u\in W_{2m}$ and $v\in W_{2n}$ such that

$$\alpha(0)u=(2m-2i)u,\ \ \alpha(0)v=(2n-2j)v,$$

for some $0\leq i\leq 2m$, $0\leq j\leq 2n$, where $\alpha(0)=(\alpha(-1)\mathbf{1})_0$ is the component operator of $\alpha(z)=Y(\alpha(-1)\mathbf{1},z)=\sum_{k\in\mathbb{Z}}\alpha(k)z^{-k-1}$. Then

$$\alpha(0)u_pv=(\alpha(0)u)_pv+u_p\alpha(0)v=(2m+2n-2i-2j)u_pv,$$

for all $p\in\mathbb{Z}$. This means that $W_{2m,2n}$ is a sum of irreducible $\mathfrak{g}$-modules in $\{W_{2k}|0\leq k\leq m+n\}$. On the other hand, we have the following well-known tensor product decomposition:

$$W_{2m}\otimes W_{2n}=W_{2(m-n)}\oplus W_{2(m-n)+2}\oplus\cdots\oplus W_{2(m+n)-2}\oplus W_{2(m+n)}. \tag{4.8}$$

By Lemma 2.2 of [DM2], for small enough integer p, the map $\psi_p:W_{2m}\otimes W_{2n}\to W_{2m,2n}$ defined by $\psi_p: u\otimes v\mapsto\sum_{i=p}^{\infty}u_iv$, $u\in W_{2m}$, $v\in W_{2n}$ is injective. Therefore in the decomposition of $W_{2m,2n}$ into irreducible $\mathfrak{g}$-modules, each W_{2k} appears for $m-n\leq k\leq m+n$. Denote by $U_{m,n}$ the $L(1,0)$-submodule of V_L generated by $W_{2m,2n}$. Then by (4.7), we have

$$U_{m,n}\supseteq\bigoplus_{m-n\leq k\leq m+n}(V_L)_{W_{2k}}\otimes W_{2k}.$$

This proves that

$$I_{L(1,0)}\begin{pmatrix} L(1,k^2)\\ L(1,m^2)\ L(1,n^2)\end{pmatrix}\neq 0,$$

for all $m,n,k\in\mathbb{Z}_{+}$ such that $|m-n|\leq k\leq n+m$.

Let $m=r$, then we have $f(n^2,k^2)=0$, for all $n,k\in\mathbb{Z}_{+}$ satisfying $|r-n|\leq k\leq n+r$. Thus for $n\in\mathbb{Z}_{+}$ with $n-r\geq 0$, we have

$$\begin{bmatrix} 1 & (n-r)^2 & (n-r)^4 & (n-r)^6 & \cdots & (n-r)^{2s}\\ 1 & (n-r+1)^2 & (n-r+1)^4 & (n-r+1)^6 & \cdots & (n-r+1)^{2s}\\ 1 & (n-r+2)^2 & (n-r+2)^4 & (n-r+2)^6 & \cdots & (n-r+2)^{2s}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & (n+r)^2 & (n+r)^4 & (n+r)^6 & \cdots & (n+r)^{2s}\end{bmatrix}\begin{bmatrix} a_0(n^2)\\ a_1(n^2)\\ a_2(n^2)\\ \vdots\\ a_s(n^2)\end{bmatrix}=0. \tag{4.9}$$

If $s \leq 2r$, then for each $n \in \mathbb{Z}_+$ such that $n \geq r$, the coefficient matrix of (4.9) contains a $(s+1) \times (s+1)$-minor which is a non-singular Vandermonde determinant, it follows that (4.9) has only zero solution. This implies that $a_i(x) = 0$ for all i, a contradiction. So we have

$$s = 2r + 1.$$

We may assume that $a_{2r+1}(x) = 1$. Then we have

$$A_{(n)} \begin{bmatrix} a_0(n^2) \\ a_1(n^2) \\ a_2(n^2) \\ \vdots \\ a_{2r}(n^2) \end{bmatrix} = \begin{bmatrix} -(n-r)^{2(2r+1)} \\ (n-r+1)^{2(2r+1)} \\ -(n-r+2)^{2(2r+1)} \\ \vdots \\ (n+r)^{2(2r+1)} \end{bmatrix}, \tag{4.10}$$

where

$$A_{(n)} = \begin{bmatrix} 1 & (n-r)^2 & (n-r)^4 & (n-r)^6 & \cdots & (n-r)^{4r} \\ 1 & (n-r+1)^2 & (n-r+1)^4 & (n-r+1)^6 & \cdots & (n-r+1)^{4r} \\ 1 & (n-r+2)^2 & (n-r+2)^4 & (n-r+2)^6 & \cdots & (n-r+2)^{4r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (n+r)^2 & (n+r)^4 & (n+r)^6 & \cdots & (n+r)^{4r} \end{bmatrix}.$$

This shows that (4.10) has a unique solution for each $n \in \mathbb{Z}_+$ such that $n \geq r$. Since $a_i(x)$, $i = 0, 1, \ldots, 2r+1$ are polynomials in x with degrees at most $2r+1$, it follows that $f(x, y)$ is uniquely determined (up to a non-zero scalar) by the condition that $f(n^2, k^2) = 0$ for all $n, k \in \mathbb{Z}_+$ such that $|n - r| \leq k \leq n + r$. Let

$$f_i(x, y) = (x-y)^2 - 2i^2(x+y) + i^4, \; i = 1, 2, \cdots, r.$$

Then we have

$$f_i(n^2, (n \pm i)^2) = 0.$$

This proves that the polynomial

$$(x-y) \prod_{i=1}^{r} [(x-y)^2 - 2i^2(x+y) + i^4]$$

satisfies the above condition. So we have

$$f(x, y) = (x-y) \prod_{i=1}^{r} [(x-y)^2 - 2i^2(x+y) + i^4],$$

as expected. □

We are now in a position to give the fusion rules for the vertex operator algebra $L(1, 0)$.

Theorem 4.7. *We have*

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\ L(1,n^2)\end{pmatrix} = 1,\quad k\in\mathbb{Z}_+,\ |n-m|\le k\le n+m, \tag{4.11}$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\ L(1,n^2)\end{pmatrix} = 0,\quad k\in\mathbb{Z}_+,\ k<|n-m| \text{ or } k>n+m, \tag{4.12}$$

where $n,m\in\mathbb{Z}_+$. For $n\in\mathbb{Z}_+$ such that $n\neq p^2$, for all $p\in\mathbb{Z}_+$, we have

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,n) \\ L(1,m^2)\ L(1,n)\end{pmatrix} = 1, \tag{4.13}$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k) \\ L(1,m^2)\ L(1,n)\end{pmatrix} = 0, \tag{4.14}$$

for $k\in\mathbb{Z}_+$ such that $k\neq n$.

Proof. By Lemma 4.4, for $k_1,k_2,k_3\in\mathbb{Z}_+$, $\dim I_{L(1,0)}\begin{pmatrix} L(1,k_3) \\ L(1,k_1)\ L(1,k_2)\end{pmatrix}$ is less than or equal to

$$\dim Hom_{A(L(1,0))}(A(L(1,k_1))\otimes_{A(L(1,0))} L(1,k_2)(0), L(1,k_3)(0)),$$

where $L(1,h)(0)=\mathbb{C}\mathbf{1}_h$ is the one-dimensional lowest weight space of the irreducible $L(1,0)$-module $L(1,h)$ such that

$$L(0)\mathbf{1}_h = h\mathbf{1}_h,\ L(n)\mathbf{1}_h=0,\ 1\le n\in\mathbb{Z}_+.$$

That is, x in $\mathbb{C}[x]=A(L(1,0))$ acts on $L(1,h)(0)$ as h.

Let $m,n,k\in\mathbb{Z}_+$ such that $|m-n|\le k\le m+n$. It is easy to see that

$$A(L(1,m^2))\otimes_{A(L(1,0))} L(1,n^2)(0)\cong \mathbb{C}[x]/<(x-n^2)\prod_{i=1}^{m}[(x-n^2)^2 -2i^2(x+n^2)+i^4]> .$$

Denote the ideal $<(x-n^2)\prod_{i=1}^{m}[(x-n^2)^2-2i^2(x+n^2)+i^4]>$ by $\bar{I}_n$. For $0\neq\phi\in Hom_{A(L(1,0))}(A(L(1,m^2))\otimes_{A(L(1,0))} L(1,n^2)(0), L(1,k^2)(0))$, we have

$$x\cdot\phi(1+\bar{I}_n)\mathbf{1}_{k^2} = k^2\mathbf{1}_{k^2} = \phi(x+\bar{I}_n)\mathbf{1}_{k^2},$$

since $x\cdot\mathbf{1}_{k^2}=k^2\mathbf{1}_{k^2}$. So

$$\phi(p(x)+\bar{I})\mathbf{1}_{k^2} = p(k^2)\mathbf{1}_{k^2},$$

for $p(x)\in\mathbb{C}[x]$. This means that

$$\dim Hom_{A(L(1,0))}(A(L(1,m^2))\otimes_{A(L(1,0))} L(1,n^2)(0), L(1,k^2)(0)) = 1.$$

On the other hand, by Theorem 4.6, we have

$$I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\ L(1,n^2)\end{pmatrix}\neq 0.$$

So (4.11) holds.

For $n, k \in \mathbb{Z}_+$ such that $k < |n-m|$ or $k > n+m$, let $x = k^2$, $y = n^2$, then we have

$$f(k^2, n^2) = (k^2 - n^2)\prod_{i=1}^{m}[(k^2-n^2)^2 - 2i^2(k^2+n^2) + i^4]$$
$$= (k^2 - n^2)\prod_{i=1}^{m}[k^2 - (n-i)^2][k^2 - (n+i)^2] \neq 0.$$

This proves that

$$\dim Hom_{A(L(1,0))}(A(L(1,m^2)) \otimes_{A(L(1,0))} L(1,n^2)(0), L(1,k^2)(0)) = 0.$$

So (4.12) is true. For (4.14), we have

$$f(k,n) = (k-n)\prod_{i=1}^{m}[(k-n)^2 - 2i^2(k+n) + i^4]$$
$$= (k-n)\prod_{i=1}^{m}[(k-n-i)^2 - 4i^2 n] \neq 0,$$

since $n \neq k$ and $n \neq p^2$, for all $p \in \mathbb{Z}_+$. Therefore (4.14) holds. By Theorem 4.6, we have

$$\dim Hom_{A(L(1,0))}(A(L(1,m^2)) \otimes_{A(L(1,0))} L(1,n)(0), L(1,n)(0)) = 1.$$

Since for $n \in \mathbb{Z}_+$ such that $n \neq p^2$, for all $p \in \mathbb{Z}_+$, $L(1,n) = V(1,n) \cong L(1,n)'$, (4.13) then follows from Lemma 4.4. □

The following corollary is not used in this paper. But it is an interesting result.

Corollary 4.8. *Let U be a highest weight module for the Virasoro algebra generated by the highest weight vector $u^{(r)}$ such that*

$$L(0)u^{(r)} = r^2 u^{(r)}, \ L(k)u^{(r)} = 0, \ r \in \mathbb{Z}_+ \backslash \{0\}.$$

Let $m, n \in \mathbb{Z}_+ \backslash \{0\}$ be such that $m \neq n$ and m, n are not perfect squares. Then

$$I_{L(1,0)}\begin{pmatrix} U \\ L(1,m)\ L(1,n) \end{pmatrix} = 0.$$

Proof. If U is irreducible, the lemma immediately follows from Proposition 4.2 and Theorem 4.7. Otherwise, let U' be the graded dual of U. Then U' contains an irreducible submodule $W^{(r)}$ which is isomorphic to $L(1, r^2)$. By Theorem 4.7,

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ W^{(r)}\ \ L(1,m) \end{pmatrix} = 0.$$

U' contains a submodule $W^{(r+1)}$ such that $\bar{W}^{(r+1)} = W^{(r+1)}/W^{(r)}$ is an irreducible $L(1,0)$-module isomorphic to $L(1,(r+1)^2)$. Again by Theorem 4.7, we have

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ \bar{W}^{(r+1)}\ \ L(1,m) \end{pmatrix} = 0.$$

This implies

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ W^{(r+1)}\ L(1,m)\end{pmatrix}=0.$$

Continuing the above steps, we deduce that

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ W\ L(1,m)\end{pmatrix}=0$$

for any proper submodule W of U'.

We now claim that

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ U'\ L(1,m)\end{pmatrix}=0.$$

Let $\mathcal{Y}\in I_{L(1,0)}\begin{pmatrix} L(1,n) \\ U'\ L(1,m)\end{pmatrix}$ be a nonzero intertwining operator. Then $\mathcal{Y}(u,z)\neq 0$ for some $u\in U'$. Since U is a highest weight module for the Virasoro algebra, there exists a proper submodule W of U' such that $u\in W$. This shows that

$$I_{L(1,0)}\begin{pmatrix} L(1,n) \\ W\ L(1,m)\end{pmatrix}\neq 0,$$

a contradiction.

Using Proposition 4.2 we conclude that

$$\dim I_{L(1,0)}\begin{pmatrix} U \\ L(1,m)\ L(1,n)\end{pmatrix}=\dim I_{L(1,0)}\begin{pmatrix} L(1,n) \\ U'\ L(1,m)\end{pmatrix}=0,$$

as desired. □

5. Uniqueness of $L(1/2,0)\otimes L(1/2,0)$

In this section we prove the main theorem in this paper:

Theorem 5.1. *If V is a simple, rational and C_2-cofinite vertex operator algebra such that $V_1=0$, $c=\tilde{c}=1$, V is a sum of highest weight modules for the Virasoro algebra and* $\dim V_2\geq 2$, *then* $\dim V_2=2$ *and V is isomorphic to $L(1/2,0)\otimes L(1/2,0)$.*

From now on we assume that V satisfies all the assumptions given in Theorem 5.1. First we notice that $V_n=0$ if $n<0$ and $V_0=\mathbb{C}\mathbf{1}$ (see [DGL]). Also there is a unique symmetric, non-degenerate invariant bilinear from $(,)$ on V such that $(\mathbf{1},\mathbf{1})=1$ (see [L1]). Then for any $u,v,w\in V$,

$$(u,v)\mathbf{1}=\mathrm{Res}_z z^{-1}Y(e^{L(1)z}(-z^{-2})^{L(0)}u,z^{-1})v.$$

In particular, the restriction of the form to each homogeneous subspace V_n is non-degenerate and

$$(u_{n+1}v,w)=(v,u_{-n+1}w)$$

for all $u, v \in V_2$ and $w \in V$. V_2 is a commutative non-associative algebra with the product $ab = a_1 b$ for $a, b \in V_2$ and the identity $\frac{\omega}{2}$ (cf. [FLM]). For $a, b \in V_2$ we have $(a, b)\mathbf{1} = a_3 b$. Moreover, the form on V_2 is associative. That is, $(ab, c) = (a, bc)$ for $a, b, c \in V_2$.

By [R], either there is a nontrivial nilpotent element $x \in V_2$ or V_2 is spanned by idempotent elements.

Lemma 5.2. *If V_2 is spanned by the idempotent elements, then V is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$.*

Proof. Let $x \in V_2$ be a nontrivial idempotent element. Set $\omega_1 = 2x$ and $\omega_2 = \omega - 2x$. Then ω_i are Virasoro elements [M1]. It follows from the proof of Theorem 3.1 of [ZD] that V contains $L(c_1, 0) \otimes L(c_2, 0)$ as a subalgebra for some complex numbers c_1, c_2 such that $c_1 + c_2 = 1$. In fact, $L(c_i, 0)$ is isomorphic to the subalgebra generated by ω_i. It then follows from the proof of Lemmas 4.5 and 4.6 of [ZD] that both c_1 and c_2 are $1/2$. That is, V contains rational vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$ (see [DMZ] and [W]) as a subalgebra and V is a completely reducible $L(1/2, 0) \otimes L(1/2, 0)$-module. Since the irreducible modules of $L(1/2, 0) \otimes L(1/2, 0)$ are $L(1/2, h_1) \otimes L(1/2, h_2)$ for $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$ and $\dim V_0 = 1$, $\dim V_1 = 0$, we immediately see that $V = L(1/2, 0) \otimes L(1/2, 0)$. In particular, $\dim V_2 = 2$. $\square$

We now deal with the case that there exists $0 \neq x \in V_2$ such that $x^2 = 0$. There are two cases: (1) $(\omega, x) \neq 0$; (2) $(\omega, x) = 0$.

Lemma 5.3. *We must have $(\omega, x) = 0$.*

Proof. If $(\omega, x) \neq 0$, we can assume that $(\omega, x) = 1$. Then the component operators $W(n)$ of $Y(x, z) = \sum_{n\in\mathbb{Z}} W(n) z^{-n-2}$ and the component operators $L(n)$ of the $Y(\omega, z)$ generate a copy of the W-algebra $W(2, 2)$ with central charge 1, where $W(2, 2)$ is an infinite dimensional Lie algebra with basis L_m, W_m, C for $m \in \mathbb{Z}$ and Lie brackets,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C,$$

$$[L_m, W_n] = (m - n)W_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C,$$

$$[W_m, W_n] = 0$$

for $m, n \in \mathbb{Z}$, where C is a central element(see [ZD]).

Let $c, h_1, h_2 \in \mathbb{C}$ and denote by $V(c, h_1, h_2)$ the Verma module for $W(2, 2)$ with central charge c and highest weight (h_1, h_2). Then $V(c, h_1, h_2) = U(W(2, 2))/I_{c,h_1,h_2}$, where I_{c,h_1,h_2} is the left ideal of the universal enveloping algebra $U(W(2, 2))$ generated by $L_m, W_m, C - c, L_0 - h_1$ and $W_0 - h_2$ for positive m. By PBW theorem $V(c, h_1, h_2)$ has basis

$$\{W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t} \mathbf{1}_{(h_1,h_2)} | m_1 \geq \cdots \geq m_s \geq 1, n_1 \geq \cdots \geq n_t \geq 1\},$$

where $\mathbf{1}_{(h_1,h_2)} = 1 + I_{c,h_1,h_2}$. It is standard that $V(c, h_1, h_2)$ has a unique maximal submodule $J(c, h_1, h_2)$ so that $L(c, h_1, h_2) = V(c, h_1, h_2)/J(c, h_1, h_2)$ is an irreducible highest weight module of $W(2, 2)$. By Theorem 2.1 of [ZD], if $c \neq 0$ then $J(c, 0, 0) = U(W(2, 2))L_{-1}\mathbf{1}_{(0,0)} + U(W(2, 2))W_{-1}\mathbf{1}_{(0,0)}$ and $L(c, 0, 0)$ has a basis

$$\{W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t} \mathbf{1}_0 | m_1 \geq \cdots \geq m_s > 1, n_1 \geq \cdots \geq n_t > 1\}, \tag{5.1}$$

where $\mathbf{1}_0$ is the canonical highest weight vector of $L(c, 0, 0)$.

Let U be the vertex operator subalgebra generated by ω, x. Then U is a highest weight $W(2,2)$-module with highest weight vector $\mathbf{1}$ such that W_n acts as $W(n)$ and L_n acts as $L(n)$ for all $n\in\mathbb{Z}$. Since $L(-1)\mathbf{1}=W(-1)\mathbf{1}=0$, we see that U is isomorphic to $L(1,0,0)$. By (5.1), U has q-character

$$\mathrm{ch}_q U=\frac{q^{-1/24}}{\prod_{n>1}(1-q^n)^2}.$$

By Proposition 4.2 of [ZD], the coefficients of $\eta(q)\mathrm{ch}_q U=\frac{1-q}{\prod_{n>1}(1-q^n)}$ grow faster than any polynomial in n. But this is a contradiction as the coefficients of $\eta(q)\mathrm{ch}_q V$ satisfy the polynomial growth condition by Lemma 2.6. □

So we can now assume that $(\omega,x)=0$. Since $L(1)x\in V_1$ and $(\omega,x)=(L(2)x,\mathbf{1})$ we see that x is a highest weight vector for the Virasoro algebra. By the fact that the bilinear form $(\cdot,\cdot)$ on V is non-degenerate and $(\omega,\omega)=\frac{1}{2}$, there exists $y\in V_2$ such that $(x,y)=1$, $(y,\omega)=0$. So $(L(2)y,\mathbf{1})=0$. This means that $L(2)y=0$. Since $L(1)y\in V_1=0$, we deduce that y is a highest weight vector for the Virasoro algebra. Assume that

$$xy=a\omega+\alpha x+\beta y+u,$$

where $\alpha,\beta\in\mathbb{C}$, and $u\in V_2$ such that $(u,x)=(u,y)=(u,\omega)=0$. Note that

$$(x,y)=\frac{1}{2}(x,y\omega)=\frac{1}{2}(xy,\omega)$$

and $(\omega,\omega)=\frac{1}{2}$. We have $a=4$. Since $(y,xx)=(xy,x)=\beta(x,y)=0$, it follows that $\beta=0$. Therefore

$$xy=4\omega+\alpha x+u.$$

It is obvious that u is a highest weight vector for the Virasoro algebra.

The following lemma is an immediate consequence of the commutator formula in vertex operator algebras.

Lemma 5.4. *Let v be a highest weight vector for the Virasoro algebra with highest weight 2. Then*

$$[L(m),v_n]=(m-n+1)v_{n+m}$$

for all $m,n\in\mathbb{Z}$.

Lemma 5.5. *Assume that $x_{-1}x=0$. Then we have*

(1) $u_1x=-10x$,
(2) $u_0x=-5x_{-2}\mathbf{1}$.

Proof. Since $V_n=0$ for $n<0$, we have $x_nx=0$, for $n\geq 4$. By the fact that $x_1x=x^2=0$, we have $(x,x)=(x_3x,\mathbf{1})=(\omega/2,x^2)=0$. So $x_3x=0$. Using the skew symmetry $Y(x,z)x=e^{L(-1)z}Y(x,-z)x$ we see that

$$x_0x=-x_0x+L(-1)x_1x=-x_0x+L(-1)x^2=-x_0x.$$

This proves that $x_0x = 0$. Note that $x_2x = 0$, since $V_1 = 0$. So we have $x_nx = 0$ for $n \geq 0$. Thus

$$Y(x, z_1)Y(x, z_2) = Y(x, z_2)Y(x, z_1)$$

and $Y(x_{-1}x, z) = Y(x, z)Y(x, z) = 0$. In particular,

$$x_1x_1 + 2\sum_{i\geq 1} x_{1-i}x_{1+i} = 0$$

and

$$(x_1x_1 + 2\sum_{i\geq 1} x_{1-i}x_{1+i})y = x_1x_1y + 2x = 10x + x_1u = 0.$$

This proves (1).

For (2), we apply the zero operator $\sum_{i\geq 0} x_{-i}x_{i+1}$ to y to obtain

$$0 = x_0x_1y + x_{-2}x_3y = x_0(4\omega + \alpha x + u) + x_{-2}\mathbf{1} = 5x_{-2}\mathbf{1} + x_0u,$$

where we have used Lemma 5.4. Thus, $x_0u = -5x_{-2}\mathbf{1}$. Using the skew symmetry we see that

$$u_0x = -x_0u + L(-1)x_1u = 5x_{-2}\mathbf{1} - 10x_{-2}\mathbf{1} = -5x_{-2}\mathbf{1},$$

as desired. □

From now on we redefine y as $y = y + \frac{\alpha}{10}u$. It follows from Lemma 5.5 that $x_1y = y_1x = 4\omega + u$. Although this new y is again a highest weight vector for the Virasoro algebra, we cannot assume $(y, u) = 0$ any more.

Corollary 5.6. (1) $[u_m, x_n] = 5(n - m)x_{m+n-1}$ *for* $m, n \in \mathbb{Z}$.
(2) $(u, u) = -10$.

Proof. (1) follows from Lemma 5.5 and the commutator formula

$$[u_m, x_n] = \sum_{i\geq 0} \binom{m}{i} (u_ix)_{m+n-i}.$$

For (2) we compute $(x_1y, x_1y) = (4\omega + u, 4\omega + u) = 8 + (u, u)$. On the other hand,

$$(x_1y, x_1y) = (y, x_1(4\omega + u)) = (y, 8x - 10x) = -2.$$

That is, $(u, u) = -10$. □

Lemma 5.7. *Assume that* $x_{-1}x = 0$. *Then there exist* $a, b \in \mathbb{C}$ *such that* $v = u_{-1}x + ax_{-3}\mathbf{1} + bL(-2)x$ *is a nonzero highest weight vector of weight* 4 *for the Virasoro algebra.*

Proof. We first use the conditions $L(1)v = L(2)v = 0$ to determine a, b. Using Lemmas 5.4 and 5.5 we have

$$\begin{aligned} L(1)v &= L(1)u_{-1}x + aL(1)x_{-3}\mathbf{1} + bL(1)L(-2)x \\ &= 3u_0x + 5ax_{-2}\mathbf{1} + 3bx_{-2}\mathbf{1} \\ &= (-15 + 5a + 3b)x_{-2}\mathbf{1} \end{aligned}$$

and

$$\begin{aligned} L(2)v &= L(2)u_{-1}x + aL(2)x_{-3}\mathbf{1} + bL(2)L(-2)x \\ &= 4u_1x + 6ax + b(4L(0) + \frac{1}{2})x \\ &= (-40 + 6a + b(8 + \frac{1}{2}))x. \end{aligned}$$

So $a = \dfrac{15}{49}, b = \dfrac{220}{49}$ are uniquely determined by the linear system

$$5a + 3b = 15, \quad 12a + 17b = 80.$$

It is clear that $L(n)v = 0$ for $n > 2$.

We now prove that v is nonzero. It is enough to prove that $y_3v \neq 0$. We have the following computation:

$$\begin{aligned} y_3v &= \sum_{i=0}^{3}\binom{3}{i}(y_iu)_{2-i}x + u + a\sum_{i=0}^{3}\binom{3}{i}(y_ix)_{-i}\mathbf{1} + b(4y_1 + L(-2)y_3)x \\ &= (y_0u)_2x + 3(y_1u)_1x + (y,u)x + u + 3ay_1x + 4by_1x + b\omega \\ &= (-u_0y + L(-1)u_1y)_2x + 3(y_1u)_1x + (y,u)x + u + (3a + 4b)(4\omega + u) + b\omega \\ &= -u_0y_2x + y_2u_0x - 2(u_1y)_1x + 3(y_1u)_1x + (y,u)x + (12a + 17b)\omega + (3a + 4b + 1)u \\ &= -5y_2x_{-2}\mathbf{1} + (u_1y)_1x + (y,u)x + (12a + 17b)\omega + (3a + 4b + 1)u. \end{aligned}$$

Thus we have

$$\begin{aligned} (y_3v, u) &= (-5y_2x_{-2}\mathbf{1} + (u_1y)_1x + (y,u)x + (12a + 17b)\omega + (3a + 4b + 1)u, u) \\ &= -5(x_{-2}\mathbf{1}, y_0u) + (u_1y, x_1u) + (3a + 4b + 1)(u,u) \\ &= -5(x_{-2}\mathbf{1}, -u_0y + L(-1)u_1y) - 10(u_1y, x) - 10(3a + 4b + 1) \\ &= 5(u_2x_{-2}\mathbf{1}, y) - 5(L(1)x_{-2}\mathbf{1}, u_1y) + 100 - 10(3a + 4b + 1) \\ &= -100(x,y) - 20(x, u_1y) + 100 - 10(3a + 4b + 1) \\ &= 200 - 10(3a + 4b + 1) = \frac{60}{49} \neq 0. \end{aligned}$$

The proof is complete. □

Lemma 5.8. *Assume that $x_{-1}x = 0$. Let $v = u_{-1}x + ax_{-3}\mathbf{1} + bL(-2)x$ be the nonzero highest weight vector given in Lemma 5.7. Then $x_iv = 0$ for all $i \geq 0$.*

Proof. Since $x_{-1}x = 0$, it follows that $x_{-2}x = \frac{1}{2}L(-1)x_{-1}x = 0$. So for $i \geq 0$, we have

$$\begin{aligned} x_i v &= x_i u_{-1}x + a x_i x_{-3}\mathbf{1} + b x_i L(-2)x \\ &= 5(-1-i)x_{i-2}x + u_{-1}x_i x + b(i+1)x_{i-2}x + bL(-2)x_i x = 0, \end{aligned}$$

as desired. □

Lemma 5.9. *V is a completely reducible module for the Virasoro algebra.*

Proof. By the assumption, V is a sum of highest weight modules for the Virasoro algebra. We claim that any highest weight module for the Virasoro algebra generated by a highest weight vector $w \in V$ with highest weight n is isomorphic to $L(1, n)$. If not, let U be the highest weight module generated by w for the Virasoro algebra. Then U has a unique maximal submodule M generated by a highest weight vector f. Then we can write f as a linear combination of $L(-n_1)\cdots L(-n_k)w$ for $n_1 \geq \cdots \geq n_k \geq 1$. Let X be a highest weight module in V for the Virasoro algebra generated by a highest weight vector g. It is clear that

$$(L(-n_1)\cdots L(-n_k)w, g) = (w, L(n_k)\cdots L(n_1)g) = 0,$$

and so $(f, g) = 0$. Let $L(-m_1)\cdots L(-m_p)g \in X$ such that $m_i > 0$ and $p \geq 1$. Then

$$(f, L(-m_1)\cdots L(-m_p)g) = (L(m_p)\cdots L(m_1)f, g) = 0.$$

This shows that $(f, V) = 0$. Since the form is non-degenerate, this is impossible. As a result, V is a completely reducible module for the Virasoro algebra. □

We now can complete the proof of Theorem 5.1. Let v be the vector given in Lemma 5.7 if $x_{-1}x = 0$, otherwise let $v = x_{-1}x$. Then v is a nonzero highest weight vector for the Virasoro algebra with highest weight 4 such that $x_i v = 0$ for all $i \geq 0$. It follows from Lemma 5.9 that highest weight modules generated by x and v are isomorphic to $L(1, 2)$ and $L(1, 4)$ respectively. By Proposition 11.9 of [DL], $Y(x, z)v \neq 0$ as V is simple. Thus there exists $n > 0$ such that $x_{-n}v \neq 0$ and $x_{-m}v = 0$ for all $m < n$. Then $x_{-n}v$ is a highest weight vector for the Virasoro algebra with highest weight $n+5$ and generates an irreducible highest weight module isomorphic to $L(1, n+5)$. As a result we have a nonzero intertwining operator of type $\binom{L(1, n+5)}{L(1,4),\, L(1,2)}$. This is a contradiction by Theorem 4.7. Hence there is no nontrivial nilpotent element in V_2 and Theorem 5.1 holds by Lemma 5.2. □

Remark 5.10. As we pointed out in [ZD] the assumption $c = \tilde{c}$ in Theorem 5.1 is necessary. We believe that the assumption that V is a sum of highest weight modules for the Virasoro algebra is unnecessary. But we do not know how to prove the main result without this assumption in this paper.

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