

# Infraparticle Scattering States in Non-Relativistic QED: I. The Bloch-Nordsieck Paradigm

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**Abstract:** We construct infraparticle scattering states for Compton scattering in the standard model of non-relativistic QED. In our construction, an infrared cutoff initially introduced to regularize the model is removed completely. We rigorously establish the properties of infraparticle scattering theory predicted in the classic work of Bloch and Nordsieck from the 1930's, Faddeev and Kulish, and others. Our results represent a basic step towards solving the infrared problem in (non-relativistic) QED.

## I. Introduction

The construction of scattering states in Quantum Electrodynamics (QED) is an old open problem. The main difficulties in solving this problem are linked to the infamous *infrared catastrophe* in QED: It became clear very early in the development of QED that, at the level of perturbation theory (e.g., for Compton scattering), the transition amplitudes between formal scattering states with charges and a *finite* number of photons are *ill-defined*, because, typically, Feynman amplitudes containing vertex or electron self-energy corrections exhibit logarithmic infrared divergences; [14,22].

A pragmatic approach proposed by Jauch and Rohrlich, [21,27], and by Yennie, Frautschi, and Suura, [31], is to circumvent this difficulty by considering *inclusive cross sections*: One sums over all possible final states that include photons whose total energy lies below an arbitrary threshold energy  $\epsilon > 0$ . Then the infrared divergences due to soft virtual photons are *formally* canceled by those corresponding to the emission of *soft* photons of total energy below  $\epsilon$ , order by order in perturbation theory in powers of the finestructure constant  $\alpha$ . A drawback of this approach becomes apparent when one tries to formulate a scattering theory that is  $\epsilon$ -independent: Because the transition probability  $P^\epsilon$  for an inclusive process is estimated to be  $\mathcal{O}(\epsilon^{\text{const.}\alpha})$ , the threshold energy  $\epsilon$  cannot be allowed to approach zero, unless “Bremsstrahlungs” processes (emission of photons) are properly incorporated in the calculation.

An alternative approach to solving the infrared problem is to go beyond inclusive cross sections and to define  $\alpha$ -dependent scattering states containing an infinite number of photons (so-called *soft-photon clouds*), which are expected to yield finite transition amplitudes, order by order in perturbation theory. The works of Chung [12], Kibble [23], and Faddeev and Kulish [13], between 1965 and 1970, represent promising, albeit incomplete progress in this direction. Their approaches are guided by an *ansatz* identified in the analysis of certain solvable models introduced in early work by Bloch and Nordsieck, [2], and extended by Pauli and Fierz, [14], in the late 1930's. In a seminal paper [2] by Bloch and Nordsieck, it was shown (under certain approximations that render their model solvable) that, in the presence of asymptotic charged particles, the scattering representations of the asymptotic photon field are a *coherent non-Fock* representation, and that formal scattering states with a finite number of soft photons do not belong to the physical Hilbert space of a system of asymptotically freely moving electrons interacting with the quantized radiation field. These authors also showed that the coherent states describing the soft-photon cloud are parameterized by the asymptotic velocities of the electrons.

The perturbative recipes for the construction of scattering states did not remove some of the major conceptual problems. New puzzles appeared, some of which are related to the problem that Lorentz boosts cannot be unitarily implemented on charged scattering states; see [19]. This host of problems was addressed in a fundamental analysis of the structural properties of QED, and of the *infrared problem* in the framework of *general quantum field theory*; see [30]. Subsequent developments in *axiomatic quantum field theory* have led to results that are of great importance for the topics treated in the present paper:

- i) Absence of dressed one-electron states with a sharp mass; see [4,28].
- ii) Corrections to the asymptotic dynamics, as compared to the one in a theory with a positive mass gap; see [3].
- iii) Superselection rules pertaining to the space-like asymptotics of the quantized electromagnetic field, and connections to Gauss' law; see [4].

In the early 1970's, significant advances on the infrared problem were made for Nelson's model, which describes non-relativistic matter linearly coupled to a scalar field of massless bosons. In [15,16], the disappearance of a sharp mass shell for the charged particles was established for Nelson's model, in the limit where an infrared cut-off is removed. (An infrared cutoff is introduced, initially, with the purpose to eliminate the interactions between charged particles and soft boson modes). Techniques developed in [15,16] have become standard tools in more recent work on non-relativistic QED, and attempts made in [15,16] have stimulated a deeper understanding of the asymptotic dynamics of charged particles and photons. The analysis of spectral and dynamical aspects of non-relativistic QED and of Nelson's model constitutes an active branch of contemporary mathematical physics. In questions relating to the infrared problem, mathematical control of the removal of the infrared cutoff is a critical issue still unsolved in many situations.

The construction of an *infraparticle scattering theory* for Nelson's model, after removal of the infrared cutoff, has recently been achieved in [26] by introducing a suitable scattering scheme. This analysis involves spectral results substantially improving those in [16]. It is based on a new *multiscale technique* developed in [25].

While the interaction in Nelson's model is linear in the creation- and annihilation operators of the boson field, it is non-linear and of vector type in non-relativistic QED. For this reason, the methods developed in [25,26] do not directly apply to the latter.

The main goal of the present work is to construct an infraparticle scattering theory for non-relativistic QED inspired by the methods of [25,26]. In a companion paper, [11], we derive those spectral properties of QED that are crucial for our analysis of scattering theory and determine the mass shell structure in the infrared limit. We will follow ideas developed in [25]. Bogoliubov transformations, proven in [10] to characterize the soft photon clouds in non-relativistic QED, represent an important element in our construction. The proof in [10] uses the uniform bounds on the renormalized electron mass previously established in [9].

We present a detailed definition of the model of non-relativistic QED in Sect. II. Aspects of *infraparticle scattering theory*, developed in this paper, are described in Sect. III.

To understand why free radiation parametrized by the asymptotic velocities of the charged particles must be expected to be present in *all* the scattering states, we recall a useful point of view based on *classical* electrodynamics that was brought to our attention by Morchio and Strocchi.

We consider a single, classical charged point-particle, e.g., an electron, moving along a world line  $(t, \vec{x}(t))$  in Minkowski space, with  $\vec{x}(0) = \vec{0}$ . We suppose that, for  $t \leq 0$ , it moves at a constant velocity  $\vec{v}_{in}$ , and, for  $t > \bar{t} > 0$ , at a constant velocity  $\vec{v}_{out} \neq \vec{v}_{in}$ ,  $|\vec{v}_{out}|, |\vec{v}_{in}| < c$ , where  $c$  is the speed of light that we set equal to 1. Thus,

$$\vec{x}(t) = \vec{v}_{in} \cdot t, \quad \text{for } t \leq 0, \tag{I.1}$$

and

$$\vec{x}(t) = \vec{x}_* + \vec{v}_{out} \cdot t, \quad \text{for } t \geq \bar{t}, \tag{I.2}$$

for some  $\vec{x}_*$ .

For times  $t \in [0, \bar{t}]$ , the particle performs an accelerated motion. We propose to analyze the behavior of the electromagnetic field in the vicinity of the particle and the properties of the free electromagnetic radiation at very early times ( $t \rightarrow -\infty$ , “in”) and very late times ( $t \rightarrow +\infty$ , “out”). For this purpose, we must solve Maxwell’s equations for the electromagnetic field tensor,  $F^{\mu\nu}(t, \vec{y})$ , given the 4-current density corresponding to the trajectory of the particle; (back reaction of the electromagnetic field on the motion of the charged particle is neglected):

$$\partial_\mu F^{\mu\nu}(t, \vec{y}) = J^\nu(t, \vec{y}) \tag{I.3}$$

with

$$J^\nu(t, \vec{y}) := -q ( \delta^{(3)}(\vec{y} - \vec{x}(t)), \dot{\vec{x}}(t) \delta^{(3)}(\vec{y} - \vec{x}(t)) ), \tag{I.4}$$

where, in the units used in our paper,  $q = 2(2\pi)^3 \alpha^{1/2}$  ( $\alpha$  is the finestructure constant). We solve Eq. (I.3) with prescribed spatial asymptotics ( $|\vec{y}| \rightarrow \infty$ ): Let  $F_{[\vec{v}_{L.W.}] }^{\mu\nu}(t, \vec{y})$  be a solution of (I.3) that, to leading order in  $|\vec{y}|^{-1}$  ( $|\vec{y}| \rightarrow \infty$ ), approaches the Liénard-Wiechert field tensor for a point-particle with charge  $-q$  and a *constant* velocity  $\vec{v}_{L.W.}$  at *all* times.

Let us denote the Liénard-Wiechert field tensor of a point-particle with charge  $-q$  moving along a trajectory  $(t, \vec{x}(t))$  in Minkowski space with  $\vec{x}(0) =: \vec{x}$  and  $\dot{\vec{x}}(t) \equiv \vec{v}$ , for *all*  $t$ , by  $F_{[\vec{x}, \vec{v}]}^{\mu\nu}(t, \vec{y})$ . Apparently, we are looking for solutions,  $F_{[\vec{v}_{L.W.}] }^{\mu\nu}(t, \vec{y})$ , of (I.3) with the property that, for all times  $t$ ,

$$|F_{[\vec{v}_{L.W.}] }^{\mu\nu}(t, \vec{y}) - F_{[\vec{x}, \vec{v}_{L.W.}] }^{\mu\nu}(t, \vec{y})| = o(|\vec{y}|^{-2}), \tag{I.5}$$

as  $|\vec{y}| \rightarrow \infty$ , for any  $\vec{x}$ . This class of solutions of (I.3) is denoted by  $\mathcal{C}_{\vec{v}_{L.W.}}$ . It is important to observe that, by causality, the class  $\mathcal{C}_{\vec{v}_{L.W.}}$  is *non-empty*, for any  $\vec{v}_{L.W.}$ , with  $|\vec{v}_{L.W.}| < 1 (= c)$ . This can be seen by choosing Cauchy data for the solution of (I.3) satisfying (I.5) at some time  $t_0$ , e.g.,  $t_0 = 0$ .

Let us now consider a specific solution,  $F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y})$ , of Eq. (I.3) in the class  $\mathcal{C}_{\vec{v}_{L.W.}}$ . We are interested in the behavior of this solution at very early times ( $t \ll 0$ ). We expect that, for  $|\vec{y} - \vec{x}(t)| = o(|t|)$ ,

$$F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) \simeq F_{0, \vec{v}_{in}}^{\mu\nu}(t, \vec{y}) \tag{I.6}$$

(here the symbol  $\simeq$  means: up to a solution of the homogeneous Maxwell equation decaying at least like  $\frac{1}{r^2}$ ). However, for  $|\vec{y} - \vec{x}(t)| \rightarrow \infty$ ,

$$F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) \simeq F_{0, \vec{v}_{L.W.}}^{\mu\nu}(t, \vec{y}), \tag{I.7}$$

as quantified in (I.5).

We note that, by (I.1),  $F_{0, \vec{v}_{in}}^{\mu\nu}(t, \vec{y})$  solves Eq. (I.3), for times  $t < 0$ . Thus,

$$\phi_{in}^{\mu\nu}(t, \vec{y}) := F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) - F_{0, \vec{v}_{in}}^{\mu\nu}(t, \vec{y}) \quad t < 0 \tag{I.8}$$

solves the *homogenous* Maxwell equation, i.e., Eq. (I.3) with  $J^\nu \equiv 0$ . For  $t \gg \bar{t}$ , we expect that, for  $|\vec{y} - \vec{x}(t)| = o(t)$ ,

$$F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) \simeq F_{\vec{x}^*, \vec{v}_{out}}^{\mu\nu}(t, \vec{y}) \tag{I.9}$$

(here the symbol  $\simeq$  means: up to a solution of the *homogeneous* Maxwell equation decaying at least like  $\frac{1}{r^2}$ ). But, for  $|\vec{y} - \vec{x}(t)| \rightarrow \infty$ ,

$$F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) \simeq F_{0, \vec{v}_{L.W.}}^{\mu\nu}(t, \vec{y}), \tag{I.10}$$

as quantified in (I.5). We note that, by (I.2),  $F_{\vec{x}^*, \vec{v}_{out}}^{\mu\nu}(t, \vec{y})$  solves Eq. (I.3), for times  $t > \bar{t}$ . Thus,

$$\phi_{out}^{\mu\nu}(t, \vec{y}) := F_{[\vec{v}_{L.W.}]}^{\mu\nu}(t, \vec{y}) - F_{\vec{x}^*, \vec{v}_{out}}^{\mu\nu}(t, \vec{y}) \quad t > \bar{t} \tag{I.11}$$

solves the homogenous Maxwell equation.

Next, we recall that  $\phi_{as}^{\mu\nu}(t, \vec{y})$ , with  $as = in/out$ , can be derived from an electromagnetic vector potential,  $A_{as}^\mu$ , by

$$\phi_{as}^{\mu\nu}(t, \vec{y}) = \partial^\mu A_{as}^\nu(t, \vec{y}) - \partial^\nu A_{as}^\mu(t, \vec{y}). \tag{I.12}$$

We can impose the *Coulomb gauge condition* on  $A_{as}^\mu$ :  $A_{as}^\mu = (0, \vec{A}_{as}(t, \vec{y}))$ , with  $\vec{\nabla} \cdot \vec{A}_{as}(t, \vec{y}) \equiv 0$ . It turns out (and this can be derived from formulae one finds, e.g., in [20]), that, to leading order in  $|\vec{y}|^{-1}$  ( $|\vec{y}| \rightarrow \infty$ ),  $\vec{A}_{as}(t, \vec{y})$  is given by

$$\begin{aligned} \vec{A}_{as}(t, \vec{y}) := & \alpha^{\frac{1}{2}} \sum_{\lambda} \int \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \frac{\vec{v}_{as} \cdot \vec{\epsilon}_{\vec{k}, \lambda}^*}{|\vec{k}|^{\frac{3}{2}}(1 - \vec{v}_{as} \cdot \hat{\vec{k}})} \vec{\epsilon}_{\vec{k}, \lambda} e^{-i\vec{k} \cdot \vec{y} + i|\vec{k}|t} + c.c. \right\} \\ & - \alpha^{\frac{1}{2}} \sum_{\lambda} \int \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \frac{\vec{v}_{L.W.} \cdot \vec{\epsilon}_{\vec{k}, \lambda}^*}{|\vec{k}|^{\frac{3}{2}}(1 - \vec{v}_{L.W.} \cdot \hat{\vec{k}})} \vec{\epsilon}_{\vec{k}, \lambda} e^{-i\vec{k} \cdot \vec{y} + i|\vec{k}|t} + c.c. \right\}, \tag{I.13} \end{aligned}$$

where  $\hat{k}$  is the unit vector in the direction of  $\vec{k}$ , and  $\vec{\varepsilon}_{\vec{k},+}, \vec{\varepsilon}_{\vec{k},-}$  are transverse polarization vectors with  $\hat{k} \cdot \vec{\varepsilon}_{\vec{k},\lambda} = 0, \lambda = +, -$ , and  $\vec{\varepsilon}_{\vec{k},\lambda}^* \cdot \vec{\varepsilon}_{\vec{k},\lambda'} = \delta_{\lambda,\lambda'}$ .

The free field

$$\phi^{\mu\nu}(t, \vec{y}) = \phi_{out}^{\mu\nu}(t, \vec{y}) - \phi_{in}^{\mu\nu}(t, \vec{y}) \tag{I.14}$$

is the radiation emitted by the particle due to its accelerated motion, as  $t \rightarrow \infty$ . It is well known that Eqs. (I.6)-(I.13) can be made precise within classical electrodynamics under some standard assumptions on the Cauchy data for the solutions in addition to the condition in Eq. (I.5). We will see that analogous statements also hold in our model of quantum electrodynamics with non-relativistic matter.

In this paper, we treat the quantum theory of a system consisting of a nonrelativistic charged particle only interacting with the quantized e.m. field. The motion of the quantum particle depends on the back-reaction of the field, and the asymptotic *in*- and *out*-velocities of this particle are not attained at finite times. However, the infrared features of the asymptotic radiation in the classical model, described above for a given current, are reproduced in this interacting quantum model.

In fact, the set of classes  $\mathcal{C}_{\vec{v}_{L.W.}}$ , associated with different currents but at fixed  $\vec{v}_{L.W.}$ , corresponds to one of the superselection sectors of the quantized theory; see e.g. [3]. In particular, the Fock representation, which is the usual (but not the only possible) choice for the representation of the algebra of photon creation- and annihilation operators, corresponds to  $\vec{v}_{L.W.} = 0$ . This implies that, in the Fock representation of the interpolating photon creation- and annihilation operators, an infrared-singular asymptotic electromagnetic-field configuration must be present for all values of the asymptotic velocity of the electron different from zero. In particular, after replacing the classical velocities with the spectral values of the quantum operators  $\vec{v}_{out/in}$ , the background field with  $\vec{v}_{L.W.} = 0$  (given by (I.13)) corresponds to the background radiation described by the coherent non-Fock representations of the algebra of asymptotic photon creation- and annihilation operators labeled by  $\vec{v}_{out/in}$ ; see also Sect. III.5.

## II. Definition of the Model

The Hilbert space of pure state vectors of the system consisting of one non-relativistic electron interacting with the quantized electromagnetic field is given by

$$\mathcal{H} := \mathcal{H}_{el} \otimes \mathcal{F}, \tag{II.1}$$

where  $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$  is the Hilbert space for a single electron; (for expository convenience, we neglect the spin of the electron). The Fock space used to describe the states of the transverse modes of the quantized electromagnetic field (the *photons*) in the Coulomb gauge is given by

$$\mathcal{F} := \bigoplus_{N=0}^{\infty} \mathcal{F}^{(N)}, \quad \mathcal{F}^{(0)} = \mathbb{C} \Omega, \tag{II.2}$$

where  $\Omega$  is the vacuum vector (the state of the electromagnetic field without any excited modes), and

$$\mathcal{F}^{(N)} := \mathcal{S}_N \bigotimes_{j=1}^N \mathfrak{h}, \quad N \geq 1, \tag{II.3}$$

where the Hilbert space  $\mathfrak{h}$  of a single photon is

$$\mathfrak{h} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2). \tag{II.4}$$

Here,  $\mathbb{R}^3$  is momentum space, and  $\mathbb{Z}_2$  accounts for the two independent transverse polarizations (or helicities) of a photon. In (II.3),  $\mathcal{S}_N$  denotes the orthogonal projection onto the subspace of  $\bigotimes_{j=1}^N \mathfrak{h}$  of totally symmetric  $N$ -photon wave functions, to account for the fact that photons satisfy Bose-Einstein statistics. Thus,  $\mathcal{F}^{(N)}$  is the subspace of  $\mathcal{F}$  of state vectors for configurations of exactly  $N$  photons. In this paper, we use units such that Planck’s constant  $\hbar$ , the speed of light  $c$ , and the mass of the electron are equal to unity. The dynamics of the system is generated by the Hamiltonian

$$H := \frac{\left(-i\vec{\nabla}_{\vec{x}} + \alpha^{1/2}\vec{A}(\vec{x})\right)^2}{2} + H^f. \tag{II.5}$$

The multiplication operator  $\vec{x} \in \mathbb{R}^3$  corresponds to the position of the electron. The electron momentum operator is given by  $\vec{p} = -i\vec{\nabla}_{\vec{x}}$ ;  $\alpha \cong 1/137$  is the finestructure constant (which, in this paper, is treated as a small parameter),  $\vec{A}(\vec{x})$  denotes the (ultraviolet regularized) vector potential of the transverse modes of the quantized electromagnetic field at the point  $\vec{x}$  (the electron position) in the *Coulomb gauge*,

$$\vec{\nabla}_{\vec{x}} \cdot \vec{A}(\vec{x}) = 0. \tag{II.6}$$

$H^f$  is the Hamiltonian of the quantized, free electromagnetic field, given by

$$H^f := \sum_{\lambda=\pm} \int d^3k |\vec{k}| a_{\vec{k},\lambda}^* a_{\vec{k},\lambda}, \tag{II.7}$$

where  $a_{\vec{k},\lambda}^*$  and  $a_{\vec{k},\lambda}$  are the usual photon creation- and annihilation operators, which satisfy the canonical commutation relations

$$[a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}^*] = \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}'), \tag{II.8}$$

$$[a_{\vec{k},\lambda}^\#, a_{\vec{k}',\lambda'}^\#] = 0, \tag{II.9}$$

with  $a^\# = a$  or  $a^*$ . The vacuum vector  $\Omega$  obeys the condition

$$a_{\vec{k},\lambda} \Omega = 0, \tag{II.10}$$

for all  $\vec{k} \in \mathbb{R}^3$  and  $\lambda \in \mathbb{Z}_2 \equiv \{+, -\}$ .

The quantized electromagnetic vector potential is given by

$$\vec{A}(\vec{y}) := \sum_{\lambda=\pm} \int_{\mathcal{B}_\Lambda} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \vec{\varepsilon}_{\vec{k},\lambda}^- e^{-i\vec{k}\cdot\vec{y}} a_{\vec{k},\lambda}^* + \vec{\varepsilon}_{\vec{k},\lambda}^* e^{i\vec{k}\cdot\vec{y}} a_{\vec{k},\lambda} \right\}, \tag{II.11}$$

where  $\vec{\varepsilon}_{\vec{k},-}$ ,  $\vec{\varepsilon}_{\vec{k},+}$  are photon polarization vectors, i.e., two unit vectors in  $\mathbb{R}^3 \otimes \mathbb{C}$  satisfying

$$\vec{\varepsilon}_{\vec{k},\lambda}^* \cdot \vec{\varepsilon}_{\vec{k},\mu} = \delta_{\lambda\mu}, \quad \vec{k} \cdot \vec{\varepsilon}_{\vec{k},\lambda} = 0, \tag{II.12}$$

for  $\lambda, \mu = \pm$ . The equation  $\vec{k} \cdot \vec{\varepsilon}_{\vec{k},\lambda} = 0$  expresses the Coulomb gauge condition. Moreover,  $\mathcal{B}_\Lambda$  is a ball of radius  $\Lambda$  centered at the origin in momentum space;  $\Lambda$  represents an *ultraviolet cutoff* that will be kept fixed throughout our analysis. The vector potential defined in (II.11) is thus regularized in the ultraviolet.

Throughout this paper, it will be assumed that  $\Lambda \approx 1$  (the rest energy of an electron), and that  $\alpha$  is sufficiently small. Under these assumptions, the Hamiltonian  $H$  is selfadjoint on  $D(H_0)$ , the domain of definition of the operator

$$H_0 := \frac{(-i\vec{\nabla}_{\vec{x}})^2}{2} + H^f. \tag{II.13}$$

The perturbation  $H - H_0$  is small in the sense of Kato.

The operator measuring the total momentum of a state of the system consisting of the electron and the electromagnetic field is given by

$$\vec{P} := \vec{p} + \vec{P}^f, \tag{II.14}$$

where  $\vec{p} = -i\vec{\nabla}_{\vec{x}}$  is the momentum operator for the electron, and

$$\vec{P}^f := \sum_{\lambda=\pm} \int d^3k \vec{k} a_{\vec{k},\lambda}^* a_{\vec{k},\lambda} \tag{II.15}$$

is the momentum operator for the radiation field.

The operators  $H$  and  $\vec{P}$  are essentially selfadjoint on the domain  $D(H_0)$ , and since the dynamics is invariant under translations, they commute:  $[H, \vec{P}] = 0$ . The Hilbert space  $\mathcal{H}$  can be decomposed on the joint spectrum,  $\mathbb{R}^3$ , of the component-operators of  $\vec{P}$ . Their spectral measure is absolutely continuous with respect to Lebesgue measure,

$$\mathcal{H} := \int^\oplus \mathcal{H}_{\vec{p}} d^3P, \tag{II.16}$$

where each fiber space  $\mathcal{H}_{\vec{p}}$  is a copy of Fock space  $\mathcal{F}$ .

*Remark.* Throughout this paper, the symbol  $\vec{P}$  stands for both a variable in  $\mathbb{R}^3$  and a vector operator in  $\mathcal{H}$ , depending on the context. Similarly, a double meaning is also associated with functions of the total momentum operator. (E.g.: In Eq. (III.1)  $E_{\vec{P}}^c$  is an operator on the Hilbert space  $\mathcal{H}$ , while in Eq. (III.3) it is a function of  $\vec{P} \in \mathbb{R}^3$ .)

To each fiber space  $\mathcal{H}_{\vec{p}}$  there corresponds an isomorphism

$$I_{\vec{p}} : \mathcal{H}_{\vec{p}} \longrightarrow \mathcal{F}^b, \tag{II.17}$$

where  $\mathcal{F}^b$  is the Fock space corresponding to the annihilation- and creation operators  $b_{\vec{k},\lambda}, b_{\vec{k},\lambda}^*$ , where  $b_{\vec{k},\lambda}$  is given by  $e^{i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda}$ , and  $b_{\vec{k},\lambda}^*$  by  $e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda}^*$ , with vacuum  $\Omega_f = I_{\vec{p}}(e^{i\vec{P}\cdot\vec{x}})$ , where  $\vec{x}$  is the electron position. To define  $I_{\vec{p}}$  more precisely, we consider an (improper) vector  $\psi_{(f^{(n)}, \vec{p})} \in \mathcal{H}_{\vec{p}}$  with a definite total momentum, which describes an electron and  $n$  photons. Its wave function, in the variables  $(\vec{x}; \vec{k}_1, \dots, \vec{k}_n; \lambda_1, \dots, \lambda_n)$ , is given by

$$e^{i(\vec{P}-\vec{k}_1-\dots-\vec{k}_n)\cdot\vec{x}} f^{(n)}(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n), \tag{II.18}$$

where  $f^{(n)}$  is totally symmetric in its  $n$  arguments. The isomorphism  $I_{\vec{P}}$  acts by way of

$$I_{\vec{P}} \left( e^{i(\vec{P}-\vec{k}_1-\dots-\vec{k}_n)\cdot\vec{x}} f^{(n)}(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n) \right) \tag{II.19}$$

$$= \frac{1}{\sqrt{n!}} \sum_{\lambda_1, \dots, \lambda_n} \int d^3k_1 \dots d^3k_n f^{(n)}(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n) b_{\vec{k}_1, \lambda_1}^* \dots b_{\vec{k}_n, \lambda_n}^* \Omega_f. \tag{II.20}$$

The Hamiltonian  $H$  maps each fiber space  $\mathcal{H}_{\vec{P}}$  into itself, i.e., it can be written as

$$H = \int^{\otimes} H_{\vec{P}} d^3P, \tag{II.21}$$

where

$$H_{\vec{P}} : \mathcal{H}_{\vec{P}} \longrightarrow \mathcal{H}_{\vec{P}}. \tag{II.22}$$

Written in terms of the operators  $b_{\vec{k}, \lambda}^*$ ,  $b_{\vec{k}, \lambda}$ , and of the variable  $\vec{P}$ , the fiber Hamiltonian  $H_{\vec{P}}$  has the form

$$H_{\vec{P}} := \frac{(\vec{P} - \vec{P}^f + \alpha^{1/2} \vec{A})^2}{2} + H^f, \tag{II.23}$$

where

$$\vec{P}^f = \sum_{\lambda} \int d^3k \vec{k} b_{\vec{k}, \lambda}^* b_{\vec{k}, \lambda}, \tag{II.24}$$

$$H^f = \sum_{\lambda} \int d^3k |\vec{k}| b_{\vec{k}, \lambda}^* b_{\vec{k}, \lambda}, \tag{II.25}$$

and

$$\vec{A} := \sum_{\lambda} \int_{\mathcal{B}_{\lambda}} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ b_{\vec{k}, \lambda}^* \vec{\varepsilon}_{\vec{k}, \lambda} + \vec{\varepsilon}_{\vec{k}, \lambda}^* b_{\vec{k}, \lambda} \right\}. \tag{II.26}$$

Let

$$\mathcal{S} := \{ \vec{P} \in \mathbb{R}^3 : |\vec{P}| < \frac{1}{3} \}. \tag{II.27}$$

In order to give precise meaning to the constructions used in this work, we restrict the total momentum  $\vec{P}$  to the set  $\mathcal{S}$ , and we introduce an infrared cut-off at an energy  $\sigma > 0$  in the vector potential. *The removal of the infrared cutoff in the construction of scattering states is the main problem solved in this paper.* The restriction of  $\vec{P}$  to  $\mathcal{S}$  guarantees that the propagation speed of a dressed electron is strictly smaller than the speed of light. However, our results can be extended to a region  $\mathcal{S}$  (inside the unit ball) of radius larger than  $\frac{1}{3}$ .

We start by studying a regularized fiber Hamiltonian given by

$$H_{\vec{P}}^{\sigma} := \frac{(\vec{P} - \vec{P}^f + \alpha^{1/2} \vec{A}^{\sigma})^2}{2} + H^f \tag{II.28}$$



acting on the fiber space  $\mathcal{H}_{\vec{P}}$ , for  $\vec{P} \in \mathcal{S}$ , where

$$\bar{A}^\sigma := \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ b_{\vec{k},\lambda}^* \bar{\varepsilon}_{\vec{k},\lambda} + \bar{\varepsilon}_{\vec{k},\lambda}^* b_{\vec{k},\lambda} \right\}, \tag{II.29}$$

and where  $\mathcal{B}_\sigma$  is a ball of radius  $\sigma$ .

*Remark.* In a companion paper [11], we construct dressed one-electron states of fixed momentum given by the ground state vectors  $\Psi_{\vec{P}}^{\sigma_j}$  of the Hamiltonians  $H_{\vec{P}}^{\sigma_j}$ , and we compare ground state vectors  $\Psi_{\vec{P}}^{\sigma_j}, \Psi_{\vec{P}'}^{\sigma_{j'}}$  corresponding to different fiber Hamiltonians  $H_{\vec{P}}^{\sigma_j}, H_{\vec{P}'}^{\sigma_{j'}}$  with  $\vec{P} \neq \vec{P}'$ . We compare these ground state vectors as vectors in the Fock space  $\mathcal{F}^b$ . In the sequel, we use the expression

$$\|\Psi_{\vec{P}}^{\sigma_j} - \Psi_{\vec{P}'}^{\sigma_{j'}}\|_{\mathcal{F}} \tag{II.30}$$

as an abbreviation for

$$\|I_{\vec{P}}(\Psi_{\vec{P}}^{\sigma_j}) - I_{\vec{P}'}(\Psi_{\vec{P}'}^{\sigma_{j'}})\|_{\mathcal{F}}; \tag{II.31}$$

$\|\cdot\|_{\mathcal{F}}$  stands for the Fock norm. Hölder continuity properties of  $\Psi_{\vec{P}}^\sigma$  in  $\sigma$  and in  $\vec{P}$  are proven in [11]. These properties play a crucial role in the present paper.

*II.1. Summary of contents.* In Sect. III, time-dependent vectors  $\psi_{h,\kappa}(t)$  approximating scattering states are constructed, and the main results of this paper are described, along with an outline of infraparticle scattering theory. In Sects. IV and V,  $\psi_{h,\kappa}(t)$  is shown to converge to a scattering state  $\psi_{h,\kappa}^{out/in}$  in the Hilbert space  $\mathcal{H}$ , as time  $t$  tends to infinity. This result is based on mathematical techniques introduced in [26]. The vector  $\psi_{h,\kappa}^{out/in}$  represents a dressed electron with a wave function  $h$  on momentum space whose support is contained in the set  $\mathcal{S}$  (see details in Sect. III.1), accompanied by a cloud of soft photons described by a *Bloch-Nordsieck operator*, and with an upper cutoff  $\kappa$  imposed on photon frequencies. This cutoff can be chosen arbitrarily.

In Sect. VI, we construct the scattering subspaces  $\mathcal{H}^{out/in}$ . Vectors in these subspaces are obtained from certain subspaces,  $\mathcal{H}_\circ^{out/in}$ , by applying “hard” asymptotic photon creation operators. These spaces carry representations of the algebras  $\mathcal{A}_{ph}^{out/in}$  and  $\mathcal{A}_{el}^{out/in}$  of asymptotic photon creation- and annihilation operators and asymptotic electron observables, respectively, which commute with each other. The latter property proves asymptotic decoupling of the electron and photon dynamics. We rigorously establish the coherent nature and the infrared properties of the representation of  $\mathcal{A}_{ph}^{out/in}$  identified by Bloch and Nordsieck in their classic paper, [2].

In a companion paper [11], we establish the main spectral ingredients for the construction and convergence of the vectors  $\{\Psi_{\vec{P}}^\sigma\}$ , as  $\sigma$  tends to 0. These results are obtained with the help of a new multiscale method introduced in [25], to which we refer the reader for some details of the proofs.

In the Appendix, we prove some technical results used in the proofs.

### III. Infraparticle Scattering States

*Infraparticle scattering theory* is concerned with the asymptotic dynamics in QFT models of infraparticles and massless fields. Contrary to theories with a non-vanishing mass gap, the picture of asymptotically freely moving particles in the Fock representation is not valid, due to the inseparability of the dynamics of charged massive particles and the soft modes of the massless asymptotic fields.

Our starting point is to study the (dressed one-particle) states of a (non-relativistic) electron when the interactions with the soft modes of the photon field are turned off. We then analyze their limiting behavior when this infrared cut-off is removed. This amounts to studying vectors  $\psi^\sigma$ ,  $\sigma > 0$ , in the Hilbert space  $\mathcal{H}$  that are solutions to the equation

$$H^\sigma \psi^\sigma = E_{\vec{P}}^\sigma \psi^\sigma, \quad (\text{III.1})$$

where  $H^\sigma = \int^\oplus H_{\vec{P}}^\sigma d^3P$ , and  $E_{\vec{P}}^\sigma$  is a function of the vector operator  $\vec{P}$ ;  $E_{\vec{P}}^\sigma$  is the electron energy function defined more precisely in Sect. III.1. Since in our model non-relativistic matter is coupled to a relativistic field, the form of  $E_{\vec{P}}^\sigma$  is not fixed by symmetry, except for rotation invariance. Furthermore, the solutions of (III.1) give rise to vectors in the physical Hilbert space describing wave packets of dressed electrons of the form

$$\psi^\sigma(h) = \int h(\vec{P}) \Psi_{\vec{P}}^\sigma d^3P, \quad (\text{III.2})$$

where the support of  $h$  is contained in a ball centered at  $\vec{P} = 0$ , chosen such that  $|\vec{\nabla} E_{\vec{P}}^\sigma| < 1$ , as a function of  $\vec{P}$ , i.e., we must impose the condition that the maximal group velocity of the electron which, a priori, is not bounded from above in our non-relativistic model, is bounded by the speed of light. (For group velocities larger than the velocity of light, the one-electron states decay by emission of Cerenkov radiation.)

The guiding principle motivating our analysis of limiting or improper one-particle states,  $\psi^\sigma(h)$  for  $\sigma \rightarrow 0$ , is that refined control of the infrared singularities, which push these vectors out of the space  $\mathcal{H}$ , as  $\sigma \rightarrow 0$ , should enable one to characterize the soft photon cloud encountered in the scattering states. The analysis of Bloch and Nordsieck, [2], suggests that the infrared behavior of the state describing the soft photons accompanying an electron should be singular (i.e., not square-integrable at the origin in photon momentum space), and that it should be determined by the momentum of the asymptotic electron. In mathematical terms, this means that the asymptotic electron velocity is expected to determine an asymptotic Weyl operator (creating a cloud of asymptotic photons), which when applied to a dressed one-electron state  $\psi^{\sigma=0}(h)$  yields a well defined vector in the Hilbert space  $\mathcal{H}$ . This vector is expected to describe an asymptotic electron with wave function  $h$  surrounded by a cloud of infinitely many asymptotic free photons, in accordance with the observations sketched in (I.6)–(I.13).

Our goal in this paper is to translate this physical picture into rigorous mathematics, following suggestions made in [15] and methods developed in [9, 10, 25, 26].

*III.1. Key spectral properties.* In our construction of scattering states, we make extensive use of a number of spectral properties of our model proven in [11], and summarized in Theorem III.1 below; (they are analogous to those used in the analysis of Nelson's model in [26]).

We define the energy of a dressed one-electron state of momentum  $\vec{P}$  by

$$E_{\vec{P}}^\sigma = \inf \text{spec} H_{\vec{P}}^\sigma, \quad E_{\vec{P}} = \inf \text{spec} H_{\vec{P}} = E_{\vec{P}}^{\sigma=0}. \quad (\text{III.3})$$

We refer to  $E_{\vec{P}}^\sigma$  as the *ground state energy* of the fiber Hamiltonian  $H_{\vec{P}}^\sigma$ . We assume that the finestructure constant  $\alpha$  is so small that

$$|\vec{\nabla} E_{\vec{P}}^\sigma| < v_{max} < 1 \quad (\text{III.4})$$

for all  $\vec{P} \in \mathcal{S} := \{\vec{P} \in \mathbb{R}^3 : |\vec{P}| < \frac{1}{3}\}$ , for some constant  $v_{max} < 1$ , uniformly in  $\sigma$ .

Corresponding to  $\vec{\nabla} E_{\vec{P}}^\sigma$ , we introduce a Weyl operator

$$W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) := \exp \left( \alpha^{\frac{1}{2}} \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} d^3k \frac{\vec{\nabla} E_{\vec{P}}^\sigma}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P},\sigma}(\vec{k})} \cdot (\vec{\varepsilon}_{\vec{k},\lambda} b_{\vec{k},\lambda}^* - h.c.) \right), \quad (\text{III.5})$$

where

$$\delta_{\vec{P},\sigma}(\vec{k}) := 1 - \vec{\nabla} E_{\vec{P}}^\sigma \cdot \frac{\vec{k}}{|\vec{k}|}, \quad (\text{III.6})$$

acting on  $\mathcal{H}_{\vec{P}}$ , which is unitary for  $\sigma > 0$ . We consider the transformed fiber Hamiltonian

$$K_{\vec{P}}^\sigma := W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) H_{\vec{P}}^\sigma W_\sigma^*(\vec{\nabla} E_{\vec{P}}^\sigma). \quad (\text{III.7})$$

We note that conjugation by  $W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma)$  acts on the creation- and annihilation operators as a linear Bogoliubov transformation (translation)

$$W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) b_{\vec{k},\lambda}^\# W_\sigma^*(\vec{\nabla} E_{\vec{P}}^\sigma) = b_{\vec{k},\lambda}^\# - \alpha^{1/2} \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P},\sigma}(\vec{k})} \vec{\nabla} E_{\vec{P}}^\sigma \cdot \vec{\varepsilon}_{\vec{k},\lambda}^\#, \quad (\text{III.8})$$

where  $\mathbf{1}_{\sigma,\Lambda}(\vec{k})$  stands for the characteristic function of the set  $\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma$ . Our methods rely on proving regularity properties in  $\sigma$  and  $\vec{P}$  of the ground state vector,  $\Phi_{\vec{P}}^\sigma$ , and of the ground state energy,  $E_{\vec{P}}^\sigma$ , of  $K_{\vec{P}}^\sigma$ . These regularity properties are summarized in the following theorem, which is the main result of the companion paper [11].

**Theorem III.1.** *For  $\vec{P} \in \mathcal{S}$  and for  $\alpha > 0$  sufficiently small, the following statements hold.*

- ( $\mathcal{S}1$ ) *The energy  $E_{\vec{P}}^\sigma$  is a simple eigenvalue of the operator  $K_{\vec{P}}^\sigma$  on  $\mathcal{F}^b$ . Let  $\mathcal{B}_\sigma := \{\vec{k} \in \mathbb{R}^3 \mid |\vec{k}| \leq \sigma\}$ , and let  $\mathcal{F}_\sigma$  denote the Fock space over  $L^2((\mathbb{R}^3 \setminus \mathcal{B}_\sigma) \times \mathbb{Z}_2)$ . Likewise, we define  $\mathcal{F}_0^\sigma$  to be the Fock space over  $L^2(\mathcal{B}_\sigma \times \mathbb{Z}_2)$ ; hence  $\mathcal{F}^b = \mathcal{F}_\sigma \otimes \mathcal{F}_0^\sigma$ . On  $\mathcal{F}_\sigma$ , the operator  $K_{\vec{P}}^\sigma$  has a spectral gap of size  $\rho^- \sigma$  or larger, separating  $E_{\vec{P}}^\sigma$  from the rest of its spectrum, for some constant  $\rho^-$ , with  $0 < \rho^- < 1$ . The contour*

$$\gamma := \{z \in \mathbb{C} \mid |z - E_{\vec{P}}^\sigma| = \frac{\rho^- \sigma}{2}\}, \quad \sigma > 0 \quad (\text{III.9})$$

bounds a disc which intersects the spectrum of  $K_{\vec{p}}^\sigma|_{\mathcal{F}_\sigma}$  in only one point,  $\{E_{\vec{p}}^\sigma\}$ . The ground state vectors of the operators  $K_{\vec{p}}^\sigma$  are given by

$$\Phi_{\vec{p}}^\sigma := \frac{\frac{1}{2\pi i} \int_{\mathcal{Y}} \frac{1}{K_{\vec{p}}^\sigma - z} dz \Omega_f}{\left\| \frac{1}{2\pi i} \int_{\mathcal{Y}} \frac{1}{K_{\vec{p}}^\sigma - z} dz \Omega_f \right\|_{\mathcal{F}}} \tag{III.10}$$

and converge strongly to a non-zero vector  $\Phi_{\vec{p}} \in \mathcal{F}^b$ , in the limit  $\sigma \rightarrow 0$ . The rate of convergence is at least of order  $\sigma^{\frac{1}{2}(1-\delta)}$ , for any  $0 < \delta < 1$ . (Although it is not relevant for the purposes of this paper, we note that the results in [17] imply the uniformity in  $\delta$  of the range of values of  $\alpha$ , where the rate estimate  $\sigma^{\frac{1}{2}(1-\delta)}$  holds; analogous conclusions follow for the rate estimates below.) The dependence of the ground state energies  $E_{\vec{p}}^\sigma$  of the fiber Hamiltonians  $K_{\vec{p}}^\sigma$  on the infrared cutoff  $\sigma$  is characterized by the following estimates:

$$|E_{\vec{p}}^\sigma - E_{\vec{p}}^{\sigma'}| \leq \mathcal{O}(\sigma), \tag{III.11}$$

and

$$|\vec{\nabla} E_{\vec{p}}^\sigma - \vec{\nabla} E_{\vec{p}}^{\sigma'}| \leq \mathcal{O}(\sigma^{\frac{1}{2}(1-\delta)}), \tag{III.12}$$

for any  $0 < \delta < 1$ , with  $\sigma > \sigma' > 0$ .

( $\mathcal{J}2$ ) The following Hölder regularity properties in  $\vec{P} \in \mathcal{S}$  hold uniformly in  $\sigma \geq 0$ :

$$\|\Phi_{\vec{p}}^\sigma - \Phi_{\vec{p}+\Delta\vec{p}}^\sigma\|_{\mathcal{F}} \leq C_{\delta'} |\Delta\vec{P}|^{\frac{1}{4}-\delta'} \tag{III.13}$$

and

$$|\vec{\nabla} E_{\vec{p}}^\sigma - \vec{\nabla} E_{\vec{p}+\Delta\vec{p}}^\sigma| \leq C_{\delta''} |\Delta\vec{P}|^{\frac{1}{4}-\delta''}, \tag{III.14}$$

for any  $0 < \delta'' < \delta' < \frac{1}{4}$ , with  $\vec{P}, \vec{P} + \Delta\vec{P} \in \mathcal{S}$ , where  $C_{\delta'}$  and  $C_{\delta''}$  are finite constants depending on  $\delta'$  and  $\delta''$ , respectively.

( $\mathcal{J}3$ ) Given a positive number  $v_{min}$ , there are numbers  $r_\alpha = v_{min} + \mathcal{O}(\alpha) > 0$  and  $v_{max} < 1$  such that, for  $\vec{P} \in \mathcal{S} \setminus \mathcal{B}_{r_\alpha}$  and for  $\alpha$  sufficiently small,

$$1 > v_{max} > |\vec{\nabla} E_{\vec{p}}^\sigma| > v_{min} > 0, \tag{III.15}$$

uniformly in  $\sigma$ . (We also notice that the control on the second derivative of  $E_{\vec{p}}^\sigma$  in  $\vec{P}$  uniformly in the sharp infrared cut-off  $\sigma \geq 0$  (see [17]) would allow us to take  $v_{min} \equiv 0, r_\alpha \equiv 0$ , and to include electron velocities  $\vec{\nabla} E_{\vec{p}}^\sigma$  arbitrarily close to 0, but we prefer to work with an assumption self-contained in the paper).

( $\mathcal{J}4$ ) For  $\vec{P} \in \mathcal{S}$  and for any  $\vec{k} \neq 0$ , the following inequality holds uniformly in  $\sigma$ , for  $\alpha$  small enough:

$$E_{\vec{p}-\vec{k}}^\sigma > E_{\vec{p}}^\sigma - C_\alpha |\vec{k}|, \tag{III.16}$$

where  $E_{\vec{p}-\vec{k}}^\sigma := \inf \text{spec } H_{\vec{p}-\vec{k}}^\sigma$  and  $\frac{1}{3} < C_\alpha < 1$ , with  $C_\alpha \rightarrow \frac{1}{3}$  as  $\alpha \rightarrow 0$ .

( $\mathcal{I}5$ ) Let  $\Psi_{\vec{p}}^\sigma \in \mathcal{F}$  denote the ground state vector of the fiber Hamiltonian  $H_{\vec{p}}^\sigma$ , so that

$$\Phi_{\vec{p}}^\sigma = \zeta W_\sigma(\vec{\nabla} E_{\vec{p}}^\sigma) \frac{\Psi_{\vec{p}}^\sigma}{\|\Psi_{\vec{p}}^\sigma\|_{\mathcal{F}}} \quad \zeta \in \mathbb{C}, \quad |\zeta| = 1. \tag{III.17}$$

For  $\vec{P} \in \mathcal{S}$ , one has that

$$\|b_{\vec{k},\lambda} \frac{\Psi_{\vec{P}}^\sigma}{\|\Psi_{\vec{P}}^\sigma\|_{\mathcal{F}}}\|_{\mathcal{F}} \leq C \alpha^{1/2} \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{3/2}}, \tag{III.18}$$

where  $\Psi_{\vec{P}}^\sigma$  is the ground state of  $H_{\vec{P}}^\sigma$  and  $C$  is a positive constant; see Lemma 6.1 of [10] which can be extended to  $\vec{k} \in \mathbb{R}^3$  using ( $\mathcal{I}4$ ).

Detailed proofs of Theorem III.1 based on results in [25, 10] are given in [11].

*III.2. Definition of the approximating vector  $\Psi_{h,\kappa}(t)$ .* We construct infraparticle scattering states by using a *time-dependent approach to scattering theory*. We define a time-dependent approximating vector  $\psi_{h,\kappa}(t)$  that converges to an asymptotic vector, as  $t \rightarrow \infty$ . It describes an electron with wave function  $h$  (whose momentum space support is contained in  $\mathcal{S}$ ), and a cloud of asymptotic free photons with an upper photon frequency cutoff  $0 < \kappa \leq \Lambda$ . This interpretation will be justified a posteriori.

We closely follow an approach to infraparticle scattering theory developed for Nelson’s model in [26], (see also [15]). In the context of the present paper, our task is to give a mathematically rigorous meaning to the formal expression

$$\Phi_\kappa^{out}(h) := \lim_{t \rightarrow \infty} \lim_{\sigma \rightarrow 0} e^{iHt} \mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t) e^{-iH^\sigma t} \psi^\sigma(h), \tag{III.19}$$

where

$$\mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t) := \exp \left( \alpha^{1/2} \sum_\lambda \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_\sigma} \frac{d^3k}{\sqrt{|\vec{k}|}} \frac{\vec{v}(t) \cdot \{\vec{\varepsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda}^* e^{-i|\vec{k}|t} - \vec{\varepsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda} e^{i|\vec{k}|t}\}}{|\vec{k}|(1 - \hat{k} \cdot \vec{v}(t))} \right).$$

The operator  $\vec{v}(t)$  is not known a priori; but, in the limit  $t \rightarrow \infty$ , it must converge to the asymptotic velocity operator of the electron. The latter is determined by the operator  $\vec{\nabla} E_{\vec{p}}$ , applied to the (non-Fock) vectors  $\Psi_{\vec{p}}$ . This can be seen by first considering the infrared regularized model, with  $\sigma > 0$ , which has dressed one-electron states  $\psi^\sigma(h)$  in  $\mathcal{H}$ , and by subsequently passing to the limit  $\sigma \rightarrow 0$ . Formally, for  $\sigma \rightarrow 0$ , the Weyl operator

$$e^{iHt} \mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t) e^{-iH^\sigma t} \tag{III.20}$$

is an interpolating operator used in the L.S.Z. (Lehmann-Symanzik-Zimmermann) approach to scattering theory for the electromagnetic field, where the photon test functions (in the operator  $\mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t)$ ) are evolved backwards in time with the free evolution, and the photon creation- and annihilation operators are evolved forward in time with the interacting time evolution. Moreover, the photon test functions in (III.20) coincide with the test functions in the Weyl operator  $W_\sigma(\vec{\nabla} E_{\vec{p}}^\sigma)$  defined in (III.5), after replacing

the operator  $\vec{\nabla} E_{\vec{P}}^\sigma$  by the operator  $\vec{v}(t)$ . We stress that, while the Weyl operator  $W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma)$  leaves the fiber spaces  $\mathcal{H}_{\vec{P}}$  invariant, the Weyl operator  $\mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t)$  is expressed in terms of the operators  $\{a, a^*\}$ , as it must be when describing real photons in a scattering process, and hence does *not* preserve the fiber spaces.

Guided by the expected relation between  $\vec{v}(t)$  and  $\vec{\nabla} E_{\vec{P}}^\sigma$ , as  $t \rightarrow \infty$  and  $\sigma \rightarrow 0$ , two key ideas used to make (III.19) precise are to render the infrared cut-off *time-dependent*, with  $\sigma_t \rightarrow 0$ , as  $t \rightarrow \infty$ , and to discretize the ball  $\mathcal{S} = \{\vec{P} \in \mathbb{R}^3 \mid |\vec{P}| < \frac{1}{3}\}$ , with a grid size decreasing in time  $t$ . This discretization also applies to the velocity operator  $\vec{v}(t)$  in expression (III.19).

The existence of infraparticle scattering states in  $\mathcal{H}$  is established by proving that the corresponding sequence of time-dependent approximating scattering states, which depend on the cutoff  $\sigma_t$  and on the discretization, defines a strongly convergent sequence of vectors in  $\mathcal{H}$ . This is accomplished by appropriately tuning the convergence rates of  $\sigma_t$  and of the discretization of  $\mathcal{S}$ . Our sequence of approximate infraparticle scattering states is defined as follows (for  $t \gg 1$ ):

- i) We consider a wave function  $h$  with support in a region  $\mathcal{R}$  which is a union of cubes contained in  $\mathcal{S} \setminus \mathcal{B}_{r_\alpha}$ ; (see condition (S3) in Theorem III.1). We introduce a time-dependent cell partition  $\mathcal{G}^{(t)}$  of  $\mathcal{R}$ . This partition is constructed as follows: At time  $t$ , the linear dimension of each cell is  $\frac{L}{2^n}$ , where  $L$  is the diameter of  $\mathcal{R}$ , and  $n \in \mathbb{N}$  is such that

$$(2^n)^{\frac{1}{\epsilon}} \leq t < (2^{n+1})^{\frac{1}{\epsilon}}, \tag{III.21}$$

for some  $\epsilon > 0$  to be fixed later. Thus, the total number of cells in  $\mathcal{G}^{(t)}$  is  $N(t) = 2^{3n}$ , where  $n = \lfloor \log_2 t^\epsilon \rfloor$ ; ( $\lfloor x \rfloor$  extracts the integer part of  $x$ ). By  $\mathcal{G}_j^{(t)}$ , we denote the  $j$ th cell of the partition  $\mathcal{G}^{(t)}$ .

- ii) For each cell, we consider a one-particle state of the Hamiltonian  $H^{\sigma_t}$ ,

$$\psi_{j,\sigma_t}^{(t)} := \int_{\mathcal{G}_j^{(t)}} h(\vec{P}) \Psi_{\vec{P}}^{\sigma_t} d^3 P, \tag{III.22}$$

where

- $h(\vec{P}) \in C_0^1(\mathcal{S} \setminus \mathcal{B}_{r_\alpha})$ , with  $\text{supp } h \subseteq \mathcal{R}$ ;
- $\sigma_t := t^{-\beta}$ , for some exponent  $\beta (> 1)$  to be fixed later;
- in (III.22), the ground state vector,  $\Psi_{\vec{P}}^{\sigma_t}$ , of  $H_{\vec{P}}^{\sigma_t}$  is defined by

$$\Psi_{\vec{P}}^{\sigma_t} := W_{\sigma_t}^*(\nabla E_{\vec{P}}^{\sigma_t}) \Phi_{\vec{P}}^{\sigma_t}, \tag{III.23}$$

where  $\Phi_{\vec{P}}^{\sigma_t}$  is the ground state of  $K_{\vec{P}}^{\sigma_t}$ ; (see Theorem III.1).

- iii) With each cell  $\mathcal{G}_j^{(t)}$  we associate a soft-photon cloud described by the following ‘‘LSZ (Lehmann-Symanzik-Zimmermann) Weyl operator’’:

$$e^{iHt} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, t) e^{-iH\sigma_t t}, \tag{III.24}$$

where

$$\mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, t) := \exp \left( \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \frac{d^3 k}{\sqrt{|\vec{k}|}} \frac{\vec{v}_j \cdot \{\vec{\varepsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda}^* e^{-i|\vec{k}|t} - \vec{\varepsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda} e^{i|\vec{k}|t}\}}{|\vec{k}|(1 - \widehat{\vec{k}} \cdot \vec{v}_j)} \right). \tag{III.25}$$

- Here  $\kappa$ , with  $0 < \kappa \leq \Lambda$ , is an arbitrary (but fixed) photon energy threshold or counter threshold.
- $\vec{v}_j \equiv \vec{\nabla} E^{\sigma_t}(\vec{P}_j^*)$  is the c-number vector corresponding to the value of the “velocity”  $\vec{\nabla} E^{\sigma_t}(\vec{P})$  in the center,  $\vec{P}_j^*$ , of the cell  $\mathcal{G}_j^{(t)}$ .

iv) For each cell, we consider a time-dependent phase factor

$$e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, t)}, \quad (\text{III.26})$$

with

$$\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, t) := -\alpha \int_1^t \vec{\nabla} E_{\vec{P}}^{\sigma_t} \cdot \int_{\mathcal{B}_{\sigma_t^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} \tau - |\vec{k}| \tau) d^3 k d\tau, \quad (\text{III.27})$$

and

$$\Sigma_{\vec{v}_j}^l(\vec{k}) := 2 \sum_{l'} (\delta_{l,l'} - \frac{k^l k^{l'}}{|\vec{k}|^2}) v_j^{l'} \frac{1}{|\vec{k}|^2 (1 - \hat{k} \cdot \vec{v}_j)}. \quad (\text{III.28})$$

Here,  $\sigma_t^S := \tau^{-\theta}$ , and the exponent  $0 < \theta < 1$  will be chosen later. Note that, in (III.26), (III.27),  $\vec{\nabla} E_{\vec{P}}^{\sigma_t}$  is interpreted as an *operator*.

v) The approximate scattering state at time  $t$  is given by the expression

$$\psi_{h,\kappa}(t) := e^{iHt} \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, t) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, t)} e^{-iE_{\vec{P}}^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)}, \quad (\text{III.29})$$

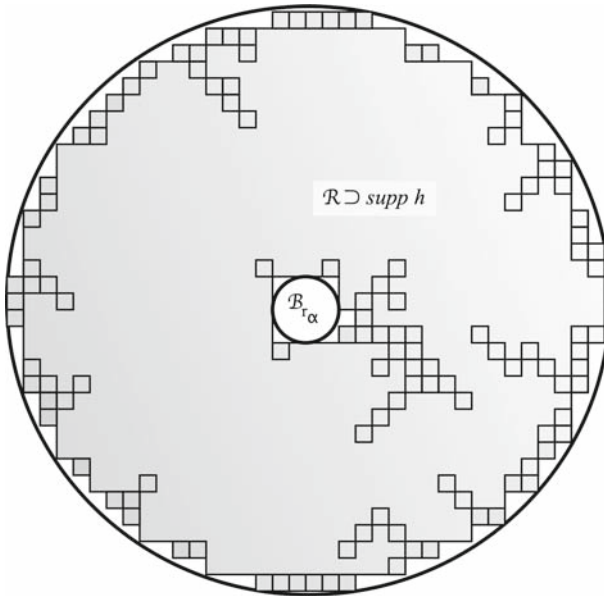
where  $N(t)$  is the number of cells in  $\mathcal{G}^{(t)}$ .

The role played by the phase factor  $e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, t)}$  is similar to that of the Coulomb phase in Coulomb scattering. However, in the present case, the phase has a *limit*, as  $t \rightarrow \infty$ , and is introduced to control an oscillatory term in the Cook argument which is not absolutely convergent (see Sect. III.3).

**III.3. Statement of the main result.** The main result of this paper is Theorem III.2, below, from which the asymptotic picture described in Sect. III.5, below, emerges. It relies on the assumptions summarized in the following hypothesis.

**Main Assumption III.1.** *The following assumptions hold throughout this paper:*

- (1) *The conserved momentum  $\vec{P}$  takes values in  $\mathcal{S}$ ; see (II.27).*
- (2) *The finestructure constant  $\alpha$  satisfies  $\alpha < \alpha_c$ , for some small constant  $\alpha_c \ll 1$  independent of the infrared cutoff.*
- (3) *The wave function  $h$  is supported in a set  $\mathcal{R}$  and is of class  $C^1$ , where  $\mathcal{R}$  is contained in  $\mathcal{S} \setminus \mathcal{B}_{r_\alpha}$ , as indicated in Fig. 1, and  $r_\alpha$  is introduced in (I.3).*



**Fig. 1.**  $\mathcal{R}$  can be described as a union of cubes with sides of length  $\frac{L}{2^{n_0}}$ , for some  $n_0 < \infty$

**Theorem III.2.** *Given the Main Assumption III.1, the following holds: There exist positive real numbers  $\beta > 1$ ,  $\theta < 1$  and  $\epsilon > 0$  such that the limit*

$$s - \lim_{t \rightarrow +\infty} \psi_{h,\kappa}(t) =: \psi_{h,\kappa}^{(out)} \tag{III.30}$$

(where  $\psi_{h,\kappa}(t)$  is defined in Eq. (III.29) and  $\kappa$ , see (III.25), is the threshold frequency) exists as a vector in  $\mathcal{H}$ , and  $\|\psi_{h,\kappa}^{(out)}\|^2 = \int |h(\vec{P})|^2 d^3 P$ . Furthermore, the rate of convergence is at least of order  $t^{-\rho'}$ , for some  $\rho' > 0$ .

We note that this result corresponds to Theorem 3.1 of [26] for Nelson’s model.

The limiting state is the desired infraparticle scattering state without infrared cut-offs. We shall verify that  $\{\psi_{h,\kappa}(t)\}$  is a Cauchy sequence in  $\mathcal{H}$ , as  $t \rightarrow +\infty$ ; (or  $t \rightarrow -\infty$ ).

In Sect. III.4, we outline the key mechanisms responsible for the convergence of the approximating vectors  $\psi_{h,\kappa}(t)$ , as  $t \rightarrow \infty$ . We note that, in (III.30), three different convergence rates are involved:

- The rate  $t^{-\beta}$  related to the fast infrared cut-off  $\sigma_t$ ;
- the rate  $t^{-\theta}$ , related to the slow infrared cut-off  $\sigma_t^S$  (see (III.27));
- the rate  $t^{-\epsilon}$  of the grid size of the cell partition.

We anticipate that, in order to control the interaction,

- $\beta$  has to be larger than 1, due to the time-energy uncertainty principle.
- The exponent  $\theta$  has to be smaller than 1, in order to ensure the cancelation of some “infrared tails” discussed in Sect. IV.
- The exponent  $\epsilon$ , which controls the rate of refinement of the cell decomposition, will have to be chosen small enough to be able to prove certain decay estimates.



*III.4. Strategy of convergence proof.* Here we outline the key mechanisms used to prove that the approximating vectors  $\psi_{h,\kappa}(t)$  converge to a nonzero vector in  $\mathcal{H}$ , as  $t \rightarrow \pm\infty$ .

Among other things, we will prove that

$$\lim_{t \rightarrow \infty} \|\psi_{h,\kappa}(t)\| = \|h\|_2 := \left( \int |h(\vec{P})|^2 d^3 P \right)^{\frac{1}{2}}. \tag{III.31}$$

From its definition, see (III.29), one sees that the square of the norm of the vector  $\psi_{h,\kappa}(t)$  involves a double sum over cells of the partitions  $\mathcal{G}^{(t)}$ , i.e.,

$$\begin{aligned} \|\psi_{h,\kappa}(t)\|^2 &= \sum_{l,j=1}^{N(t)} \left\langle e^{i\gamma_{\sigma_l}(\vec{v}_l, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{l,\sigma_l}^{(t)}, \mathcal{W}_{\kappa,\sigma_l}^*(\vec{v}_l, t) \right. \\ &\quad \left. \times \mathcal{W}_{\kappa,\sigma_l}(\vec{v}_j, t) e^{i\gamma_{\sigma_l}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j,\sigma_l}^{(t)} \right\rangle, \end{aligned} \tag{III.32}$$

where the individual terms, labeled by  $(l, j)$ , are inner products between vectors labeled by cells  $\mathcal{G}_l^{(t)}$  and  $\mathcal{G}_j^{(t)}$  of  $\mathcal{G}^{(t)}$ .

A heuristic argument to see where (III.31) comes from is as follows. Assuming that

- the vectors  $\psi_{h,\kappa}(t)$  converge to an asymptotic vector of the form

$$\lim_{t \rightarrow \pm\infty} \lim_{\sigma \rightarrow 0} e^{iHt} \mathcal{W}_{\kappa,\sigma}(\vec{v}(t), t) e^{-iH^\sigma t} \psi^\sigma(h) = \mathcal{W}_{\kappa,\sigma=0}^{out/in}(\vec{v}(\pm\infty)) \psi^{\sigma=0}(h), \tag{III.33}$$

where

$$\mathcal{W}_{\kappa,\sigma=0}^{out/in}(\vec{v}(\pm\infty)) := \exp \left( \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_\kappa} \frac{d^3 k}{\sqrt{|\vec{k}|}} \frac{\vec{v}(\pm\infty) \cdot \{ \vec{\varepsilon}_{\vec{k},\lambda}^{out/in*} a_{\vec{k},\lambda}^{out/in*} - \vec{\varepsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda}^{out/in} \}}{|\vec{k}|(1 - \widehat{k} \cdot \vec{v}(\pm\infty))} \right),$$

and  $a_{\vec{k},\lambda}^{out/in*}$ ,  $a_{\vec{k},\lambda}^{out/in}$  are the creation- and annihilation operators of the asymptotic photons;

- the operators  $\vec{v}(\pm\infty)$  commute with the algebra of asymptotic creation- and annihilation operators  $\{a_{\vec{k},\lambda}^{out/in*}, a_{\vec{k},\lambda}^{out/in}\}$ ; (this can be expected to be a consequence of asymptotic decoupling of the photon dynamics from the dynamics of the electron);
- the restriction of the asymptotic velocity operators,  $\vec{v}(\pm\infty)$ , to the improper dressed one-electron state is given by the operator  $\vec{\nabla} E_{\vec{P}}$ , i.e.,

$$\vec{v}(\pm\infty)\Psi_{\vec{P}} \equiv \vec{\nabla} E_{\vec{P}}\Psi_{\vec{P}}; \tag{III.34}$$

then, the two vectors

$$\mathcal{W}_{\kappa,\sigma_l}(\vec{v}_j, t) e^{i\gamma_{\sigma_l}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j,\sigma_l}^{(t)} \quad \text{and} \quad \mathcal{W}_{\kappa,\sigma_l}(\vec{v}_l, t) e^{i\gamma_{\sigma_l}(\vec{v}_l, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{l,\sigma_l}^{(t)} \tag{III.35}$$

corresponding to two different cells of  $\mathcal{G}^{(t)}$  (i.e.,  $j \neq l$ ) turn out to be orthogonal in the limit  $t \rightarrow \pm\infty$ . One can then show that the diagonal terms in the sum (III.32) are the only ones that survive in the limit  $t \rightarrow \infty$ . The fact that their sum converges to  $\|h\|_2^2$  is comparatively easy to prove.

A mathematically precise formulation of this mechanism is presented in Sect. IV. In Sect. IV.1, part A., the analysis of the scalar products between the cell vectors in (III.35) is reduced to the study of an ODE. To prove (III.31), we invoke the following properties of the one-particle states  $\psi_{j,\sigma_t}^{(t)}$  and  $\psi_{l,\sigma_t}^{(t)}$  located in the  $j$ th and  $l$ th cell, respectively:

- Their spectral supports with respect to the momentum operator  $\vec{P}$  are disjoint up to sets of measure zero.
- They are vacua for asymptotic annihilation operators, as long as an infrared cut-off  $\sigma_t$  for a fixed time  $t$  is imposed: For Schwartz test functions  $g^\lambda$ , we define

$$a_{\sigma_t}^{out/in}(g) := \lim_{s \rightarrow \pm\infty} e^{iH^{\sigma_t}s} \sum_{\lambda} \int a_{\vec{k},\lambda} \overline{g^\lambda(\vec{k})} e^{i|\vec{k}|s} e^{-iH^{\sigma_t}s} d^3k, \quad (\text{III.36})$$

on the domain of  $H^{\sigma_t}$ .

An important step in the proof of (III.31) is to control the decay in time of the off-diagonal terms. After completion of this step, one can choose the rate,  $t^{-\epsilon}$ , by which the diameter of the cells of the partition  $\mathcal{G}^{(t)}$  tends to 0 in such a way that the sum of the off-diagonal terms vanishes, as  $t \rightarrow \infty$ . Precise control is achieved in Sect. IV.1, part B., where we invoke Cook's argument and analyze the decay in time  $s$  of

$$\frac{d}{ds} \left( e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t}s} \psi_{j,\sigma_t}^{(t)} \right) \quad (\text{III.37})$$

$$= i e^{iH^{\sigma_t}s} [H_I^{\sigma_t}, \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s)] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t}s} \psi_{j,\sigma_t}^{(t)} \quad (\text{III.38})$$

$$+ i e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t}s} \psi_{j,\sigma_t}^{(t)}, \quad (\text{III.39})$$

for a fixed infrared cut-off  $\sigma_t$ , and a fixed partition. As we will show, the term in (III.38) can be written (up to a unitary operator) as

$$\alpha^{\frac{1}{2}} \int d^3y \left\{ \vec{J}_{\sigma_t}(s, \vec{y}) \cdot \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{y} - |\vec{q}|s) d^3q \right\} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j,\sigma_t}^{(t)} \quad (\text{III.40})$$

plus subleading terms, where  $\vec{J}_{\sigma_t}(s, \vec{y})$  is essentially the electron current at time  $s$ , which is proportional to the velocity operator

$$i[H^{\sigma_t}, \vec{x}] = \vec{p} + \alpha^{\frac{1}{2}} \vec{A}^{\sigma_t}(\vec{x}). \quad (\text{III.41})$$

In (III.40), the electron current is smeared out with the vector function

$$\vec{g}_t(s, \vec{y}) := \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{y} - |\vec{q}|s) d^3q, \quad (\text{III.42})$$

which solves the wave equation

$$\square_{s,\vec{y}} \vec{g}_t(s, \vec{y}) = 0, \quad (\text{III.43})$$

and is then applied to the one-particle state  $e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j,\sigma_t}^{(t)}$ . Because of the dispersive properties of the dynamics of the system, the resulting vector is expected to converge to 0 in norm at an integrable rate, as  $s \rightarrow \infty$ . An intuitive explanation proceeds as follows:

- i) A vector function  $\vec{g}_t(s, \vec{y})$  that solves (III.43) propagates along the light cone, and  $\sup_{\vec{y} \in \mathbb{R}^3} |\vec{g}_t(s, \vec{y})|$  decays in time like  $s^{-1}$ , while a much faster decay is observed when  $\vec{y}$  is restricted to the interior of the light cone (i.e.,  $|\frac{|\vec{y}|}{s}| < 1$ ).
- ii) Because of the support in  $\vec{P}$  of the vector  $\psi_{j, \sigma_t}^{(t)}$ , the propagation of the electron current in (III.38) is limited to the interior of the light cone, up to subleading tails.

Combination of i) and ii) is expected to suffice to exhibit decay of the vector norm of (III.38) and to complete our argument. An important refinement of this reasoning process, involving the term (III.39), is, however, necessary:

A mathematically precise version of statement ii) is as follows: Let  $\chi_h$  be a smooth, approximate characteristic function of the support of  $h$ . We will prove a *propagation estimate*

$$\begin{aligned} & \left\| \chi_h\left(\frac{\vec{x}}{s}\right) e^{i\gamma\sigma_t(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \right. \\ & \quad \left. - \chi_h(\vec{\nabla} E_{\vec{P}}^{\sigma_t}) e^{i\gamma\sigma_t(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \right\| \\ & \leq \frac{1}{s^\nu} \frac{1}{t^{\frac{3\epsilon}{2}}} |\ln(\sigma_t)|, \end{aligned} \tag{III.44}$$

as  $s \rightarrow \infty$ , where  $\nu > 0$  is independent of  $\epsilon$ . Using result ( $\mathcal{J}3$ ) of Theorem III.1, and our assumption on the support of  $h$  formulated in point ii) of Sect. III.2, this estimate provides sufficient control of the asymptotic dynamics of the electron.

An important modification of the argument above is necessary because of the dependence of

$$\vec{g}_t(s, \vec{y}) := \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{y} - |\vec{q}|s) d^3q, \tag{III.45}$$

on  $t$ , which cannot be neglected even if  $\vec{y}$  is in the interior of the light cone. In order to exhibit the desired decay, it is necessary to split  $\vec{g}_t(s, \vec{y})$  into two pieces,

$$\int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_s^S}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{y} - |\vec{q}|s) d^3q \tag{III.46}$$

and

$$\int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{y} - |\vec{q}|s) d^3q \tag{III.47}$$

for  $s$  such that  $\sigma_s^S > \sigma_t$ , where  $\sigma_s^S = s^{-\theta}$ , with  $0 < \theta < 1$ . (The same procedure will also be used in (III.50), below.) The function (III.46) has good decay properties inside the light cone. Expression (III.40), with  $\vec{g}_t(s, \vec{y})$  replaced by (III.46), can be controlled by standard dispersive estimates. The other contribution, proportional to (III.47), is in principle singular in the infrared region, but is canceled by (III.39). This can be seen by using a propagation estimate similar to (III.44). This strategy has been designed in [26]. However, because of the vector nature of the interaction in non-relativistic QED, the cancelation in our proof is technically more subtle than the one in [26].

After having proven the uniform boundedness of the norms of the approximating vectors  $\psi_{h, \kappa}(t)$ , one must prove that they define a Cauchy sequence in  $\mathcal{H}$ . To this end,

we compare these vectors at two different times,  $t_2 > t_1$  (for the limit  $t \rightarrow +\infty$ ), and split their difference into

$$\begin{aligned} \psi_{h,\kappa}(t_1) - \psi_{h,\kappa}(t_2) &= \Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}) + \Delta\psi(t_2 \rightarrow t_1, \sigma_{t_2}, \mathcal{G}^{(t_1)}) \\ &\quad + \Delta\psi(t_1, \sigma_{t_2} \rightarrow \sigma_{t_1}, \mathcal{G}^{(t_1)}), \end{aligned} \tag{III.48}$$

where the three terms on the r.h.s. correspond to

I) changing the partition  $\mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}$  in  $\psi_{h,\kappa}(t_2)$ :

$$\begin{aligned} &\Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}) \\ &:= e^{iHt_2} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{j, \sigma_{t_2}}^{(t_1)} \\ &\quad - e^{iHt_2} \sum_{j=1}^{N(t_1)} \sum_{l(j)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) e^{i\gamma\sigma_{t_2}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{l(j), \sigma_{t_2}}^{(t_2)}, \end{aligned} \tag{III.49}$$

where  $l(j)$  labels all cells of  $\mathcal{G}^{(t_2)}$  contained in the  $j$ th cell  $\mathcal{G}_j^{(t_1)}$  of  $\mathcal{G}^{(t_1)}$ . Moreover,

$$\vec{v}_{l(j)} \equiv \vec{\nabla} E_{\vec{p}_{l(j)}^*}^{\sigma_{t_2}} \quad \text{and} \quad \vec{v}_j \equiv \vec{\nabla} E_{\vec{p}_j^*}^{\sigma_{t_1}};$$

II) subsequently changing the time,  $t_2 \rightarrow t_1$ , for the fixed partition  $\mathcal{G}^{(t_1)}$ , and the fixed infrared cut-off  $\sigma_{t_2}$ :

$$\begin{aligned} &\Delta\psi(t_2 \rightarrow t_1, \sigma_{t_2}, \mathcal{G}^{(t_1)}) \\ &:= e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)} \\ &\quad - e^{iHt_2} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{j, \sigma_{t_2}}^{(t_1)}; \end{aligned} \tag{III.50}$$

and, finally,

III) shifting the infrared cut-off,  $\sigma_{t_2} \rightarrow \sigma_{t_1}$ :

$$\begin{aligned} &\Delta\psi(t_1, \sigma_{t_2} \rightarrow \sigma_{t_1}, \mathcal{G}^{(t_1)}) \\ &:= e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) e^{i\gamma\sigma_{t_1}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \psi_{j, \sigma_{t_1}}^{(t_1)} \\ &\quad - e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)}. \end{aligned} \tag{III.51}$$

It is important to take these three steps in the order indicated above.

In Step I), the size of  $\|\Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)})\|^2$  in (III.49) is controlled as follows: The sum of off-diagonal terms yields a subleading contribution. The diagonal terms are shown to tend to 0 by controlling the differences

$$\vec{v}_{l(j)} - \vec{v}_j.$$

In Step II), Cook’s argument, combined with the cancelation of an infrared tail (as in the mechanism described above), yields the desired decay in  $t_1$ .

Step III) is more involved. But the basic idea is quite simple to grasp: It consists in rewriting

$$e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)} \quad (\text{III.52})$$

as

$$e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) W_{\sigma_{t_2}}(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)}. \quad (\text{III.53})$$

The term in (III.53) corresponding to the cell  $\mathcal{G}_j^{(t_1)}$  of  $\mathcal{G}^{(t_1)}$  can then be obtained by acting with the “*dressing operator*”

$$e^{iHt_1} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_1}, \quad (\text{III.54})$$

on the “*infrared-regular*” vector

$$e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} \Phi_{j, \sigma_{t_2}}^{(t_1)} := e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} \int_{\mathcal{G}_j^{(t_1)}} h(\vec{P}) \Phi_{\vec{P}}^{\sigma_{t_2}} d^3 P \quad (\text{III.55})$$

corresponding to the vectors  $\Phi_{\vec{p}}^{\sigma} = W_{\sigma}(\vec{\nabla} E_{\vec{p}}^{\sigma}) \Psi_{\vec{p}}^{\sigma}$  (see (III.23)), for all  $j$ . The advantage of (III.53) over (III.52) is that the vector  $\Phi_{j, \sigma_{t_2}}^{(t_1)}$  inherits the regularity properties of  $\Phi_{\vec{p}}^{\sigma}$  described in Theorem III.1. In particular, the vectors  $\Phi_{j, \sigma_{t_2}}^{(t_1)}$  converge strongly, as  $\sigma_{t_2} \rightarrow 0$ , and the vector

$$e^{-i\vec{q} \cdot \vec{x}} \Phi_{j, \sigma_{t_2}}^{(t_1)} \quad (\text{III.56})$$

depends on  $\vec{q}$  in a Hölder continuous manner, uniformly in  $\sigma_{t_2}$ . This last property entails enough decay to offset various logarithmic divergences appearing in the removal of the infrared cut-off in the dressing operator (III.54).

Our analysis of the strong convergence of the sequence of approximating vectors culminates in the estimate

$$\|\psi_{h, \kappa}(t_2) - \psi_{h, \kappa}(t_1)\| \leq \mathcal{O}\left((\ln(t_2))^2 / t_1^{\rho}\right), \quad (\text{III.57})$$

for some  $\rho > 0$ . By telescoping, this bound suffices to prove Theorem III.2. Indeed, to estimate the difference between the two vectors at times  $t_2$  and  $t_1$ , respectively, where  $t_2 > t_1 > 1$ , we may consider a sequence of times  $\{t_1^2, \dots, t_1^n\}$ , such that  $t_1^n \leq t_2 < t_1^{n+1}$ , and use Estimate (III.57) for each difference

$$\psi_{h, \kappa}(t_2) - \psi_{h, \kappa}(t_1^n), \quad (\text{III.58})$$

$$\psi_{h, \kappa}(t_1^m) - \psi_{h, \kappa}(t_1^{m-1}), \quad 2 \leq m \leq n. \quad (\text{III.59})$$

Then, one can show that there exists a constant  $\rho' > 0$  such that the rate of convergence of the time-dependent vector is at least of order  $t^{-\rho'}$ , as stated in Theorem III.2.

III.5. *A space of scattering states.* We use the asymptotic states  $\psi_{h,\kappa}^{(out/in)}$  to construct a subspace,  $\mathcal{H}_\kappa^{out/in}$ , of scattering states invariant under space-time translations, and with a photon energy threshold  $\kappa$ ,

$$\mathcal{H}_\kappa^{out/in} := \overline{\left\{ \bigvee \psi_{h,\kappa}^{out/in}(\tau, \vec{a}) : h(\vec{P}) \in C_0^1(\mathcal{S} \setminus \mathcal{B}_{r_\alpha}), \tau \in \mathbb{R}, \vec{a} \in \mathbb{R}^3 \right\}}, \tag{III.60}$$

where

$$\psi_{h,\kappa}^{out/in}(\tau, \vec{a}) \equiv e^{-i\vec{a} \cdot \vec{P}} e^{-iH\tau} \psi_{h,\kappa}^{out}. \tag{III.61}$$

This space contains states describing an asymptotically freely moving electron, accompanied by asymptotic free photons with energy smaller than  $\kappa$ .

Spaces of scattering states are obtained from the space  $\mathcal{H}_\kappa^{out/in}$  by adding (and subtracting) “hard” photons, i.e.,

$$\mathcal{H}^{out/in} := \overline{\left\{ \bigvee \psi_{h,\vec{F}}^{out/in} : h(\vec{P}) \in C_0^1(\mathcal{S} \setminus \mathcal{B}_{r_\alpha}), \hat{\vec{F}} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^3) \right\}}, \tag{III.62}$$

where

$$\psi_{h,\vec{F}}^{out/in} := s - \lim_{t \rightarrow +/\infty} e^{i(\vec{A}[\vec{F}_t, t] - \vec{A}[\vec{F}_t, t])} \psi_{h,\kappa}(t), \tag{III.63}$$

and

$$\vec{A}[\vec{F}_t, t] := i \int \left( \vec{A}(t, \vec{y}) \cdot \frac{\partial \vec{F}_t(\vec{y})}{\partial t} - \frac{\partial \vec{A}(t, \vec{y})}{\partial t} \cdot \vec{F}_t(\vec{y}) \right) d^3y \tag{III.64}$$

is the L.S.Z. photon field smeared out with the vector test function

$$\vec{F}_t(\vec{y}) := \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3 2\sqrt{|k|}} \vec{\varepsilon}_{\vec{k},\lambda}^* \hat{F}^\lambda(\vec{k}) e^{-i|k|t+i\vec{k} \cdot \vec{y}} \tag{III.65}$$

with

$$\hat{F}(\vec{k}) := \sum_{\lambda} \vec{\varepsilon}_{\vec{k},\lambda}^* \hat{F}^\lambda(\vec{k}) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^3). \tag{III.66}$$

An *a posteriori* physical interpretation of the scattering states constructed here emerges by studying how certain algebras of asymptotic operators are represented on the spaces of scattering states:

- The Weyl algebra,  $\mathcal{A}_{ph}^{out/in}$ , associated with the asymptotic electromagnetic field.
- The algebra  $\mathcal{A}_{el}^{out/in}$  generated by smooth functions of compact support of the asymptotic velocity of the electron.

These algebras will be defined in terms of the limits (III.67) and (III.69), below, whose existence is established in Sect. VI.2.

**Theorem III.3.** *Functions  $f \in C_0^\infty(\mathbb{R}^3)$ , of the variable  $e^{iHt} \frac{\vec{x}}{t} e^{-iHt}$ , have strong limits, as  $t \rightarrow \pm\infty$ , as operators acting on  $\mathcal{H}^{out/in}$ ,*

$$s - \lim_{t \rightarrow \pm/\infty} e^{iHt} f\left(\frac{\vec{x}}{t}\right) e^{-iHt} \psi_{h, \vec{F}}^{out/in} =: \psi_{f_{\vec{\nabla}E} h, \vec{F}}^{out/in}, \quad (\text{III.67})$$

where  $f_{\vec{\nabla}E}(\vec{P}) := \lim_{\sigma \rightarrow 0} f(\vec{\nabla}E_\sigma^\sigma)$ .

**Theorem III.4.** *The LSZ Weyl operators*

$$\left\{ e^{i(\vec{A}[\vec{G}_t, t] - \vec{A}[\vec{G}_t, t])} : \hat{G}^\lambda(\vec{k}) \in L^2(\mathbb{R}^3, (1 + |\vec{k}|^{-1})d^3k), \lambda = \pm \right\}, \quad (\text{III.68})$$

have strong limits in  $\mathcal{H}^{out/in}$ ; i.e.,

$$\mathcal{W}^{out/in}(\vec{G}) := s - \lim_{t \rightarrow \pm/\infty} e^{i(\vec{A}[\vec{G}_t, t] - \vec{A}[\vec{G}_t, t])} \quad (\text{III.69})$$

exists.

The limiting operators are unitary and satisfy the following properties:

i)

$$\mathcal{W}^{out/in}(\vec{G}) \mathcal{W}^{out/in}(\vec{G}') = \mathcal{W}^{out/in}(\vec{G} + \vec{G}') e^{-\frac{\rho(\vec{G}, \vec{G}')}{2}}, \quad (\text{III.70})$$

where

$$\rho(\vec{G}, \vec{G}') = 2i \operatorname{Im} \left( \sum_\lambda \int \hat{G}^\lambda(\vec{k}) \overline{\hat{G}'^\lambda(\vec{k})} d^3k \right). \quad (\text{III.71})$$

ii) The mapping  $\mathbb{R} \ni s \rightarrow \mathcal{W}^{out/in}(s \vec{G})$  defines a strongly continuous one-parameter group of unitary operators.

iii)

$$e^{iH\tau} \mathcal{W}^{out/in}(\vec{G}) e^{-iH\tau} = \mathcal{W}^{out/in}(\vec{G}_{-\tau}), \quad (\text{III.72})$$

where  $\vec{G}_{-\tau}$  is the freely time-evolved (vector) test function at time  $-\tau$ .

The two algebras,  $\mathcal{A}_{ph}^{out/in}$  and  $\mathcal{A}_{el}^{out/in}$ , commute. This is the precise mathematical expression of the asymptotic decoupling of the dynamics of photons from the one of the electron. The proof is non-trivial, because non-Fock representations of the asymptotic photon creation- and annihilation operators appear. (For the representation of  $\mathcal{A}_{ph}^{out/in}$ , which is *non-Fock* but *locally Fock* see Sect. VI.2.) We will show that

$$\langle \psi_{h, \kappa}^{out/in}, \mathcal{W}^{out}(\vec{G}) \psi_{h, \kappa}^{out/in} \rangle = \int e^{-\frac{\|\vec{G}\|_2^2}{2}} e^{Q_{\vec{\nabla}E_{\vec{P}}}(\vec{G})} |h(\vec{P})|^2 d^3P, \quad (\text{III.73})$$

where

$$\|\vec{G}\|_2 = \left( \int |\vec{G}(\vec{k})|^2 d^3k \right)^{1/2}, \quad (\text{III.74})$$

and

$$\varrho_{\vec{u}}(\vec{G}) := 2i \operatorname{Re} \left( \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\kappa}} \hat{G}^{\lambda}(\vec{k}) \frac{\vec{u} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} (1 - \vec{u} \cdot \hat{\vec{k}})} d^3k \right). \tag{III.75}$$

More precisely, the representation of  $\mathcal{A}_{ph}^{out/in}$  on the space of scattering states can be decomposed in a direct integral of inequivalent irreducible representations labelled by the asymptotic velocity of the electron. For different values of the asymptotic velocity, these representations turn out to be inequivalent. Only for a vanishing electron velocity, the representation is Fock; for non-zero velocity, it is a coherent non-Fock representation. The coherent photon cloud, labeled by the asymptotic velocity, is the one anticipated by Bloch and Nordsieck in the non-relativistic approximation.

These results can be interpreted as follows: *In every scattering state, an asymptotically freely moving electron is observed (with an asymptotic velocity whose size is strictly smaller than the speed of light, by construction) accompanied by a cloud of asymptotic photons propagating along the light cone.*

*Remark.* We point out that, in our definition of scattering states, we can directly accommodate an arbitrarily large number of “hard” photons without energy restriction, i.e., we can construct the limiting vector

$$\vec{A}^{out}[\vec{F}^{(m)}] \dots \vec{A}^{out}[\vec{F}^{(1)}] \psi_{h,\kappa}^{out} := s - \lim_{t \rightarrow +\infty} \vec{A}[\vec{F}_t^{(m)}, t] \dots \vec{A}[\vec{F}_t^{(1)}, t] \psi_{h,\kappa}(t) \tag{III.76}$$

which represents the state  $\psi_{h,\kappa}^{out}$  plus  $m$  asymptotic photons with wave functions  $\vec{F}^{(m)}, \dots, \vec{F}^{(1)}$ , respectively. Analogously, we define

$$\vec{A}^{in}[\vec{G}^{(m')}] \dots \vec{A}^{in}[\vec{G}^{(1)}] \psi_{h,\kappa}^{in} := s - \lim_{t \rightarrow -\infty} \vec{A}[\vec{G}_t^{(m')}, t] \dots \vec{A}[\vec{G}_t^{(1)}, t] \psi_{h,\kappa}(t). \tag{III.77}$$

This is possible because, apart from some *higher order estimates* to control the commutator  $i[H, \vec{x}]$ , and the photon creation operators in (III.76)–(III.77) (see for example [18]), we use the propagation estimate (III.44), which only limits the asymptotic velocity of the electron. This fact is very important for estimating scattering amplitudes involving an arbitrary number of “hard” photons.

In particular, for any  $m, m' \in \mathbb{N}$ , we can define the S-matrix element

$$S_{\alpha}^{m,m'}(\{\vec{F}_j\}, \{\vec{G}_j\}) = \left( \vec{A}^{out}[\vec{F}^{(m)}] \dots \vec{A}^{out}[\vec{F}^{(1)}] \psi_{h^{out}, \kappa^{out}}^{out}, \vec{A}^{in}[\vec{G}^{(m')}] \dots \vec{A}^{in}[\vec{G}^{(1)}] \psi_{h^{in}, \kappa^{in}}^{in} \right) \tag{III.78}$$

which corresponds to the transition amplitude between two states describing an incoming electron with wave function  $h^{in}$ , accompanied by a soft photon cloud of free photons of energy smaller than  $\kappa^{in}$ , plus  $m'$  hard photons (with wave functions  $\vec{G}^{(1)}, \dots, \vec{G}^{(m')}$ ), and an outgoing electron with wave function  $h^{out}$  and soft photon energy threshold  $\kappa^{out}$ , plus  $m$  hard photons, respectively.

The expansion of  $S_{\alpha}^{m,m'}(\{\vec{F}_j\}, \{\vec{G}_j\})$  in the finestructure constant  $\alpha$  can be carried out, at least to leading order, along the lines of [1]. This yields a rigorous proof of the



transition amplitudes for Compton scattering in leading order, and in the non-relativistic approximation, that one can find in textbooks.

Moreover, as expected from classical electromagnetism, “close” to the electron a Liénard-Wiechert electromagnetic field is observed. The precise mathematical statement is

$$\lim_{|\vec{d}| \rightarrow \infty} \lim_{t \rightarrow \pm\infty} |\vec{d}|^2 \left\{ \left\langle \psi_{h, \vec{F}}^{out/in}, e^{iHt} \int d^3y F_{\mu\nu}(0, \vec{y}) \tilde{\delta}_\Lambda(\vec{y} - \vec{x} - \vec{d}) e^{-iHt} \psi_{h, \vec{F}}^{out/in} \right\rangle - \int F_{\mu\nu}^{\vec{\nabla} E_{\vec{P}}} (0, \vec{d}) \langle \psi_{\vec{P}}^{\sigma_t}, \psi_{\vec{P}}^{\sigma_t} \rangle |h(\vec{P})|^2 d^3P \right\} = 0, \quad (\text{III.79})$$

where  $\tilde{\delta}_\Lambda$  is a smooth,  $\Lambda$ -dependent delta function which has the property that its Fourier transform is supported in  $B_\Lambda$ ,  $\vec{x}$  is the electron position,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (\text{III.80})$$

with

$$A_0(0, \vec{y}) := - \frac{(2\pi)^2 \alpha^{1/2}}{|\vec{y} - \vec{x}|} \quad (\text{III.81})$$

$$A_i(0, \vec{y}) := - \sum_\lambda \int \frac{d^3k}{\sqrt{|\vec{k}|}} \left( (\vec{\epsilon}_{\vec{k}, \lambda}^-)^i e^{-i\vec{k} \cdot \vec{y}} a_{\vec{k}, \lambda}^* + (\vec{\epsilon}_{\vec{k}, \lambda}^*)^i e^{i\vec{k} \cdot \vec{y}} a_{\vec{k}, \lambda} \right), \quad (\text{III.82})$$

and  $F_{\mu\nu}^{\vec{\nabla} E_{\vec{P}}}$  is the electromagnetic field tensor corresponding to a Liénard-Wiechert solution for the current

$$J_\mu(t, \vec{y}) := \left( -2(2\pi)^3 \alpha^{\frac{1}{2}} \delta^{(3)}(\vec{y} - \vec{\nabla} E_{\vec{P}} t), 2(2\pi)^3 \alpha^{\frac{1}{2}} \vec{\nabla} E_{\vec{P}} \delta^{(3)}(\vec{y} - \vec{\nabla} E_{\vec{P}} t) \right), \quad |\vec{\nabla} E_{\vec{P}}| < 1; \quad (\text{III.83})$$

see the discussion in Sect. I.

#### IV. Uniform Boundedness of the Limiting Norm

Our first aim is to prove the *uniform boundedness* of  $\|\psi_{h, \kappa}(t)\|$ , as  $t \rightarrow \infty$ ; more precisely, that

$$\lim_{t \rightarrow \infty} \langle \psi_{h, \kappa}(t), \psi_{h, \kappa}(t) \rangle = \int |h(\vec{P})|^2 d^3P. \quad (\text{IV.1})$$

The sum of the diagonal terms – with respect to the partition  $\mathcal{G}^{(t)}$  introduced above – is easily seen to yield  $\int |h(\vec{P})|^2 d^3P$  in the limit  $t \rightarrow \infty$ , as one expects. Thus, our main task is to show that the sum of the off-diagonal terms vanishes in this limit.

In Sect. V, we prove that the norm-bounded sequence  $\{\psi_{h, \kappa}(t)\}$  is, in fact, Cauchy.

We recall that the definition of the vector  $\psi_{h, \kappa}(t)$  involves three different rates:

- The rate  $t^{-\beta}$  related to the fast infrared cut-off  $\sigma_t$ ;
- the rate  $t^{-\theta}$  of the slow infrared cut-off  $\sigma_t^S$  (see (III.27));
- the rate  $t^{-\epsilon}$  of refinement of the cell partitions  $\mathcal{G}^{(t)}$ .

IV.1. *Control of the off-diagonal terms.* We denote the off-diagonal term labeled by the pair  $(l, j)$  of cell indices  $l \neq j$  contributing to the l.h.s. of (IV.1) by

$$M_{l,j}(t) := \left\langle e^{i\gamma_{\sigma_l}(\vec{v}_l, \nabla E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{l, \sigma_l}^{(t)}, \mathcal{W}_{\kappa, \sigma_l, l, j}(t) e^{i\gamma_{\sigma_l}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j, \sigma_l}^{(t)} \right\rangle, \tag{IV.2}$$

where we use the notation

$$\mathcal{W}_{\kappa, \sigma_l, l, j}(t) := \exp \left( \sum_{\lambda} \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_l}} \vec{\eta}_{l,j}(\vec{k}) \cdot \left\{ \vec{\varepsilon}_{\vec{k}, \lambda}^* a_{\vec{k}, \lambda}^* e^{-i|\vec{k}|t} - \vec{\varepsilon}_{\vec{k}, \lambda}^* a_{\vec{k}, \lambda} e^{i|\vec{k}|t} \right\} d^3k \right), \tag{IV.3}$$

where  $\vec{v}_j \equiv \vec{\nabla} E_{\vec{P}_j^*}^{\sigma_l}$ ,  $\vec{P}_j^*$  being the center of the cell  $\mathcal{C}_j^{(t)}$ , and

$$\vec{\eta}_{l,j}(\vec{k}) := \alpha^{\frac{1}{2}} \frac{\vec{v}_j}{|\vec{k}|^{\frac{3}{2}} (1 - \widehat{k} \cdot \vec{v}_j)} - \alpha^{\frac{1}{2}} \frac{\vec{v}_l}{|\vec{k}|^{\frac{3}{2}} (1 - \widehat{k} \cdot \vec{v}_l)}. \tag{IV.4}$$

We study the limit  $t \rightarrow +\infty$ ; the case  $t \rightarrow -\infty$  is analogous.

A. *Asymptotic orthogonality.* In order to prove that the off-diagonal terms (IV.2) vanish in the limit  $t \rightarrow +\infty$ , we separate the role played by the time variable  $t$  as the parameter determining the dynamical cell decomposition and infrared cutoffs, from its usual role as the conjugate variable to the energy. For the latter, we introduce an auxiliary variable  $s \geq t$ . Then, for fixed  $t$  (such that the cell decomposition and the cutoffs are constant), we interpret the terms (IV.2) as special values  $M_{l,j}(t) = M_{l,j}^1(t, t)$  of families  $M_{l,j}^{\mu}(t, s)$  introduced below, which depend on  $t, s$ , and an additional auxiliary parameter  $\mu \in \mathbb{R}$ . Our strategy will be based on proving that the dispersive properties of  $M_{l,j}^{\mu}(t, s)$  as a function of  $s \geq t$  alone, for fixed  $t$  and  $\mu$ , imply that  $M_{l,j}(t)$  has a sufficiently fast decay in  $t$  such that our desired result of asymptotic orthogonality follows.

More precisely, we introduce a family of operators

$$\mathcal{W}_{\kappa, \sigma_l, l, j}^{\mu}(s) := \exp \left( \mu \sum_{\lambda} \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_l}} \vec{\eta}_{l,j}(\vec{k}) \cdot \left\{ \vec{\varepsilon}_{\vec{k}, \lambda}^* a_{\vec{k}, \lambda}^* e^{-i|\vec{k}|s} - \vec{\varepsilon}_{\vec{k}, \lambda}^* a_{\vec{k}, \lambda} e^{i|\vec{k}|s} \right\} d^3k \right) \tag{IV.5}$$

depending on a parameter  $\mu \in \mathbb{R}$ , and define

$$\widehat{M}_{l,j}^{\mu}(t, s) := \left\langle e^{i\gamma_{\sigma_l}(\vec{v}_l, \nabla E_{\vec{P}}^{\sigma_l}, s)} e^{-iE_{\vec{P}}^{\sigma_l} s} \psi_{l, \sigma_l}^{(t)}, \mathcal{W}_{\kappa, \sigma_l, l, j}^{\mu}(s) e^{i\gamma_{\sigma_l}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_l}, s)} e^{-iE_{\vec{P}}^{\sigma_l} s} \psi_{j, \sigma_l}^{(t)} \right\rangle \tag{IV.6}$$

for  $s \geq t (\gg 1)$ . Obviously,  $\widehat{M}_{l,j}^{\mu=1}(t, t) = M_{l,j}(t)$ .

The phase factor  $\gamma_{\sigma_l}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_l}, s)$  is chosen as follows:

$$\gamma_{\sigma_l}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_l}, s) := -\alpha \int_1^s \vec{\nabla} E_{\vec{P}}^{\sigma_l} \cdot \int_{\mathcal{B}_{\sigma_l} \setminus \mathcal{B}_{\sigma_l}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_l} \tau - |\vec{k}| \tau) d^3k d\tau \tag{IV.7}$$

for  $s^{-\theta} \geq \sigma_t$ , and

$$\gamma_{\sigma_t}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_t}, s) := -\alpha \int_1^{\sigma_t^{-\frac{1}{\theta}}} \vec{\nabla} E_{\vec{P}}^{\sigma_t} \cdot \int_{\mathcal{B}_{\sigma_t^{\frac{1}{\theta}}} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} \tau - |\vec{k}| \tau) d^3 k d\tau, \tag{IV.8}$$

for  $s^{-\theta} < \sigma_t$ . As a function of  $\mu$ , the scalar product in (IV.6) satisfies the ordinary differential equation

$$\frac{d\widehat{M}_{l,j}^{\mu}(t, s)}{d\mu} = -\mu C_{l,j,\sigma_t} \widehat{M}_{l,j}^{\mu}(t, s) + r_{\sigma_t}^{\mu}(t, s), \tag{IV.9}$$

where

$$C_{l,j,\sigma_t} := \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} |\vec{\eta}_{l,j}(\vec{k})|^2 d^3 k, \tag{IV.10}$$

and

$$\begin{aligned} r_{\sigma_t}^{\mu}(t, s) := & - \left\langle e^{i\gamma_{\sigma_t}(\vec{v}_l, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{l,\sigma_t}^{(t)}, \right. \\ & \left. \mathcal{W}_{\kappa,\sigma_t,l,j}^{\mu}(s) \vec{a}_{\sigma_t}(\vec{\eta}_{l,j})(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)} \right\rangle \\ & + \left\langle \vec{a}_{\sigma_t}(\vec{\eta}_{l,j})(s) e^{i\gamma_{\sigma_t}(\vec{v}_l, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{l,\sigma_t}^{(t)}, \right. \\ & \left. \mathcal{W}_{\kappa,\sigma_t,l,j}^{\mu}(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)} \right\rangle, \end{aligned}$$

with

$$\vec{a}_{\sigma_t}(\vec{\eta}_{l,j})(s) := \sum_{\lambda} \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} \vec{\eta}_{l,j}(\vec{k}) \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda} e^{i|\vec{k}|s} d^3 k. \tag{IV.11}$$

The solution of the ODE (IV.9) is given by

$$\widehat{M}_{l,j}^{\mu}(t, s) = e^{-\frac{C_{l,j,\sigma_t}}{2} \mu^2} \widehat{M}_{l,j}^0(t, s) + \int_0^{\mu} r_{\sigma_t}^{\mu'}(t, s) e^{-\frac{C_{l,j,\sigma_t}}{2} (\mu^2 - \mu'^2)} d\mu', \tag{IV.12}$$

where the initial condition at  $\mu = 0$  is given by

$$\widehat{M}_{l,j}^0(t, s) = 0, \tag{IV.13}$$

since the supports in  $\vec{P}$  of the two vectors  $\psi_{l,\sigma_t}^{(t)}, \psi_{j,\sigma_t}^{(t)}$  are disjoint (up to sets of measure 0), for arbitrary  $t$  and  $s$ .

Furthermore, condition (S4) in Theorem III.1 implies that the vectors  $\psi_{l,\sigma_t}^{(t)}, \psi_{j,\sigma_t}^{(t)}$  are vacua for the annihilation part of the asymptotic photon field<sup>1</sup> under the dynamics generated by the Hamiltonian  $H^{\sigma_t}$ . As a consequence, we find that

$$\lim_{s \rightarrow +\infty} r_{\sigma_t}^{\mu}(t, s) = 0, \tag{IV.14}$$

<sup>1</sup> The existence of the asymptotic field operator for a fixed cut-off dynamics is derived as explained in part B. of this section.

for fixed  $\mu$  and  $t$ . To arrive at this conclusion, the following is used: The one-particle state, multiplied by the phase  $e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)}$  continues to be a one-particle state for the Hamiltonian  $H^{\sigma_t}$ ; for large  $s$  (see (IV.8)) the phase is  $s$ -independent; the operator  $E_{\vec{P}}^{\sigma_t}$  coincides with the operator  $H^{\sigma_t}$  when applied to one-particle states of the Hamiltonian  $H^{\sigma_t}$ .

Therefore, by dominated convergence, it follows that

$$\lim_{s \rightarrow +\infty} \widehat{M}_{l,j}^1(t, s) = 0. \tag{IV.15}$$

Since  $\widehat{M}_{l,j}^1(t, t) \equiv M_{l,j}(t)$ , we have

$$\begin{aligned} & |M_{l,j}(t)| \\ &= |M_{l,j}(t) - \widehat{M}_{l,j}^1(t, +\infty)| \\ &\leq 2 \sup_l (\|\psi_{l,\sigma_t}^{(t)}\|) \sup_j \left( \left\| \int_t^{+\infty} \frac{d}{ds} \left( e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t}s} \psi_{j,\sigma_t}^{(t)} \right) ds \right\| \right). \end{aligned} \tag{IV.16}$$

To estimate the r.h.s. of (IV.16), we proceed as follows.

Since we are interested in the limit  $t \rightarrow +\infty$ , and the integration domain on the r.h.s. of (IV.16) is  $[t, +\infty)$ , our aim is to show that

$$\frac{d}{ds} \left( e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \nabla E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t}s} \psi_{j,\sigma_t}^{(t)} \right) \tag{IV.17}$$

is integrable in  $s$ , and that the rate at which the r.h.s. of (VI.16) converges to zero offsets the growth of the number of cells in the partition. This allows us to conclude that

$$\sum_{l,j (l \neq j)}^{N(t)} M_{l,j}(t) \longrightarrow 0 \tag{IV.18}$$

in the limit  $t \rightarrow +\infty$ , and, as a corollary,

$$\lim_{t \rightarrow +\infty} \sum_{l,j}^{N(t)} M_{l,j}(t) = \int |h(\vec{P})|^2 d^3 P, \tag{IV.19}$$

as asserted in Theorem III.2. The convergence (IV.18) follows from the following theorem.

**Theorem IV.1.** *The off-diagonal terms  $M_{l,j}(t)$ ,  $l \neq j$ , satisfy*

$$|M_{l,j}(t)| \leq C \frac{1}{t^\eta} |\ln \sigma_t|^2 t^{-3\epsilon}, \tag{IV.20}$$

for some constants  $C < \infty$  and  $\eta > 0$ , both independent of  $l, j$ , and  $\epsilon > 0$ . In particular,  $M_{l,j}(t) \rightarrow 0$ , as  $t \rightarrow +\infty$ .

As a corollary, we find

$$\sum_{1 \leq l \neq j \leq N(t)} |M_{l,j}(t)| \leq C N^2(t) \frac{1}{t^\eta} |\ln \sigma_t|^2 t^{-3\epsilon} \leq C' \frac{1}{t^\eta} |\ln \sigma_t|^2 t^{3\epsilon}, \quad (\text{IV.21})$$

since  $N(t) \approx t^{3\epsilon}$ . (Throughout the paper,  $C$ ,  $C'$ ,  $c$ , and  $c'$  denote positive constants.) We conclude that, for  $\epsilon < \frac{\eta}{4}$ , (IV.18) follows.

*B. Time derivative and infrared tail.* We now proceed to prove Theorem IV.1. The arguments developed here will also be relevant for the proof of the (strong) convergence of the vectors  $\psi_{h,\kappa}(t)$ , as  $t \rightarrow +\infty$ , which we discuss in Sect. V.

To control (IV.16), we focus on the derivative

$$\frac{d}{ds} \left( e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)} \right) \quad (\text{IV.22})$$

$$= i e^{iH^{\sigma_t} s} [H_I^{\sigma_t}, \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s)] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)} \quad (\text{IV.23})$$

$$+ i e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)}, \quad (\text{IV.24})$$

where

$$H_I^{\sigma_t} := \alpha^{\frac{1}{2}} \vec{p} \cdot \vec{A}^{\sigma_t}(\vec{x}) + \alpha \frac{\vec{A}^{\sigma_t}(\vec{x}) \cdot \vec{A}^{\sigma_t}(\vec{x})}{2}. \quad (\text{IV.25})$$

We have used that  $\mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) = e^{-isH_0} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, 0) e^{isH_0}$ , where  $H_0 := H^{\sigma_t} - H_I^{\sigma_t}$  is the free Hamiltonian, to obtain the commutator in (IV.23). We rewrite the latter in the form

$$[H_I^{\sigma_t}, \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s)] = \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \left( \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s)^* H_I^{\sigma_t} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) - H_I^{\sigma_t} \right), \quad (\text{IV.26})$$

and use that

$$\mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s)^* \vec{A}^{\sigma_t}(\vec{x}) \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) = \vec{A}^{\sigma_t}(\vec{x}) + \alpha^{\frac{1}{2}} \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \quad (\text{IV.27})$$

(see (III.8)), where  $\vec{\Sigma}_{\vec{v}_j}^l(\vec{k})$  is defined in (III.28). We can then write the term (IV.23) as

$$\begin{aligned} (\text{IV.23}) &= i e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \cdot \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \\ &\quad \times \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) e^{-iE_{\vec{p}}^{\sigma_t} s} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j,\sigma_t}^{(t)} \\ &\quad + i e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \frac{\alpha^2}{2} \left( \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \right)^2 \\ &\quad \times e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)}, \end{aligned} \quad (\text{IV.28})$$

where we recall that  $i[H^{\sigma_t}, \vec{x}] = \vec{p} + \alpha^{\frac{1}{2}} \vec{A}^{\sigma_t}(\vec{x})$ ; see (III.41).

From the decay estimates provided by Lemma A.2 in the Appendix one concludes that the norm of (IV.29) is integrable in  $s$ , and that

$$\int_t^{+\infty} ds \, \| \text{(IV.29)} \| \leq \frac{1}{t^\eta} |\ln \sigma_t|^2 t^{-\frac{3\epsilon}{2}} (t \gg 1) \tag{IV.30}$$

for some  $\eta > 0$  independent of  $\epsilon$ .

The analysis of (IV.28) is more involved. Our argument will eventually involve the derivative of the phase factor in (IV.24). To begin with, we write (IV.28) as

$$\begin{aligned} \text{(IV.28)} &= i e^{iH^{\sigma_t s}} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \\ &\times (H^{\sigma_t} + i) \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3 k \, \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) e^{-iE_{\vec{P}}^{\sigma_t} s} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)}. \end{aligned} \tag{IV.31}$$

Pulling the operator  $(H^{\sigma_t} + i)$  through to the right, the vector (IV.31) splits into the sum of a term involving the commutator  $[H^{\sigma_t}, \vec{x}]$ ,

$$\begin{aligned} &i e^{iH^{\sigma_t s}} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3 k \, \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \\ &\times [H^{\sigma_t}, \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s)] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)}, \end{aligned} \tag{IV.32}$$

and

$$\begin{aligned} &i e^{iH^{\sigma_t s}} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \cdot \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3 k \, \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \\ &\times \frac{1}{H^{\sigma_t} + i} \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)}. \end{aligned} \tag{IV.33}$$

Note that  $[H^{\sigma_t}, \vec{x}] \frac{1}{H^{\sigma_t} + i}$  is bounded in the operator norm, uniformly in  $\sigma_t$ . To control (IV.32) and (IV.33), we invoke a propagation estimate for the electron position operator as follows. Due to condition ( $\mathcal{S}$ 3) in Theorem III.1, we can introduce a  $C^\infty$ -function  $\chi_h(\vec{y})$ ,  $\vec{y} \in \mathbb{R}^3$ , such that

- $\chi_h(\vec{y}) = 1$  for  $v_{min} \leq |\vec{y}| \leq v_{max}$ .
- $\chi_h(\vec{y}) = 0$  for  $|\vec{y}| \leq \frac{1}{2}v_{min}$  and  $|\vec{y}| \geq \frac{1+v_{max}}{2}$ .

It is shown in Theorem A.3 of the Appendix that, for  $\theta < 1$  sufficiently close to 1 and  $s$  large, the propagation estimate

$$\begin{aligned} &\left\| \chi_h \left( \frac{\vec{x}}{s} \right) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} - \chi_h(\vec{\nabla} E_{\vec{P}}^{\sigma_t}) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \right\| \\ &\leq \frac{1}{s^\nu} \frac{1}{t^{\frac{3\epsilon}{2}}} |\ln(\sigma_t)|, \end{aligned} \tag{IV.34}$$

holds, where  $\nu > 0$  is independent of  $\epsilon$ . The argument uses the Hölder regularity of  $\vec{\nabla} E_{\vec{P}}^\sigma$  and of  $\Phi_{\vec{P}}^\sigma$  listed under properties ( $\mathcal{S}$ 2) in Theorem III.1, differentiability of  $h(\vec{P})$ , and (III.18).

We continue with the discussion of the expressions (IV.32) and (IV.33). We split (IV.33) into two parts:

$$i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \mathcal{J}_{\sigma_t^S}^\kappa(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \quad (\text{IV.35})$$

$$+ i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \mathcal{J}_{\sigma_t^S}^S(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)}, \quad (\text{IV.36})$$

using the definitions

$$\begin{aligned} \mathcal{J}_{\sigma_t^S}^\kappa(s) &:= \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t^S}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \quad \text{if } s^{-\theta} \geq \sigma_t \\ &:= \alpha i [H^{\sigma_t}, \vec{x}] \cdot \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \frac{1}{H^{\sigma_t} + i} \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) \quad \text{if } s^{-\theta} < \sigma_t, \end{aligned} \quad (\text{IV.37})$$

and

$$\begin{aligned} \mathcal{J}_{\sigma_t^S}^S(s) &:= \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \int_{\mathcal{B}_{\sigma_t^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \quad \text{if } s^{-\theta} \geq \sigma_t \\ &:= 0 \quad \text{if } s^{-\theta} < \sigma_t, \end{aligned} \quad (\text{IV.38})$$

where we refer to  $\sigma_t^S := s^{-\theta}$  as the *slow infrared cut-off*. (We consider  $s, t$  large enough such that  $\kappa > \sigma_t^S, \sigma_t$ .)

To control  $\mathcal{J}_{\sigma_t^S}^S(s)$  in (IV.38), we define the “infrared tail”

$$\begin{aligned} \frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} &:= \alpha e^{-iH\sigma_t s} \frac{1}{H^{\sigma_t} + i} \frac{d(e^{iH\sigma_t s} \vec{x}_h(s) e^{-iH\sigma_t s})}{ds} e^{iH\sigma_t s} \cdot \\ &\quad \cdot \int_{\mathcal{B}_{\sigma_t^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} - |\vec{k}|s) d^3k \quad \text{if } s^{-\theta} \geq \sigma_t, \\ &:= 0 \quad \text{if } s^{-\theta} < \sigma_t, \end{aligned} \quad (\text{IV.39})$$

where  $\vec{x}_h(s) := \vec{x} \chi_h(\frac{\vec{x}}{s})$ . Summarizing, we can write (IV.22) as

$$(\text{IV.22}) = (\text{IV.29})$$

$$+ i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \mathcal{J}_{\sigma_t^S}^\kappa(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \quad (\text{IV.40})$$

$$+ i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \mathcal{J}_{\sigma_t^S}^S(s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \quad (\text{IV.41})$$

$$+ i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \quad (\text{IV.42})$$

$$+ i e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \quad (\text{IV.43})$$

$$\times \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) [H^{\sigma_t}, \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s)] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)},$$

where we recall that (IV.29) satisfies (IV.30). We claim that

$$\left\| \int_t^{+\infty} [(\text{IV.40}) + (\text{IV.41}) + (\text{IV.42})] ds \right\| \leq \frac{1}{t^\eta} |\ln \sigma_t|^2 t^{-3\epsilon/2}, \quad (t \gg 1) \tag{IV.44}$$

for some  $\eta > 0$  depending on  $\theta$ , but independent of  $\epsilon$ . This is obtained from

$$\left\| \int_t^{+\infty} [(\text{IV.40}) + (\text{IV.41}) + (\text{IV.42})] ds \right\| \tag{IV.45}$$

$$\begin{aligned} &\leq \int_t^{+\infty} ds \left\| \alpha i [H^{\sigma_t}, \vec{x}] \cdot \frac{1}{H^{\sigma_t} + i} \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) \right. \\ &\quad \times \left. \left[ \chi_h(\vec{\nabla} E_{\vec{p}}^{\sigma_t}) - \chi_h\left(\frac{\vec{x}}{s}\right) \right] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\| \end{aligned} \tag{IV.46}$$

$$+ \int_t^{+\infty} ds \left\| \mathcal{J}|_{\sigma_s^S}(s) \chi_h\left(\frac{\vec{x}}{s}\right) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\| \tag{IV.47}$$

$$+ \left\| \int_t^{+\infty} ds e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \left[ \mathcal{J}|_{\sigma_t^S}(s) \chi_h\left(\frac{\vec{x}}{s}\right) - \frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} \right] \right\| \tag{IV.48}$$

$$\begin{aligned} &\times \left\| e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\| \\ &+ \left\| \int_t^{+\infty} ds e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \left[ \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}{ds} - \frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} \right] \right\| \tag{IV.49} \\ &\times \left\| e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\| \end{aligned}$$

using the following arguments:

- The term (IV.46) can be bounded from above by  $\frac{1}{t^\eta} |\ln \sigma_t|^2 t^{-3\epsilon/2}$ , for some  $\eta > 0$  independent of  $\epsilon$ , due to the propagation estimate for (III.44) and Lemma A.2, which show that the integrand has a sufficiently strong decay in  $s$ .
- In (IV.47), the slow cut-off  $\sigma_s^S$  and the function  $\chi_h(\frac{\vec{x}}{s})$  make the norm integrable in  $s$  with the desired rate (i.e., to get a bound as in (IV.30)), for a suitable choice of  $\theta < 1$ . In particular, we can exploit that

$$\sup_{\vec{x} \in \mathbb{R}^3} \left| \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_s^S}} d^3k \vec{\Sigma}(\vec{k}, \vec{v}_j) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) \chi_h\left(\frac{\vec{x}}{s}\right) \right| \leq \mathcal{O}\left(\frac{s^\theta}{s^2}\right), \tag{IV.50}$$

see Lemma A.2 in the Appendix.

- In (IV.48), only terms integrable in  $s$  and decaying fast enough to satisfy the bound (IV.30) are left after subtracting

$$\frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} \tag{IV.51}$$

from  $\mathcal{J}|_{\sigma_t^S}(s)$ . This is explained in detail in the proof of Theorem A.4 in the Appendix.



- To bound (IV.49), we use the electron propagation estimate, combined with an integration by parts, to show that the derivative of the phase factor tends to the “infrared tail” for large  $s$ , at an integrable rate that provides a bound as in (IV.30). We note that due to the vector interaction in non-relativistic QED, this argument is more complicated here than in the Nelson model treated in [26] where the interaction term is scalar. Here, we have to show (see Theorem A.4) that, in the integral with respect to  $s$ , the pointwise velocity  $e^{iH^{\sigma_t} s} i[\vec{x}, H^{\sigma_t}] e^{-iH^{\sigma_t} s}$  can be replaced by the (asymptotic) mean velocity  $\vec{\nabla} E_{\vec{p}}^{\sigma_t}$  at asymptotic times.

Finally, to control (IV.43), we observe that the commutator introduces additional decay in  $s$  into the integrand when  $\frac{\vec{x}}{s}$  is restricted to the support of  $\chi_h$ . It then follows that the propagation estimate suffices (without infrared tail) to control the norm, by the same arguments that were used to estimate (IV.46), (IV.47).

Combining the above arguments, the proof of Theorem IV.1 is completed.

### V. Proof of Convergence of $\psi_{h,\kappa}(t)$

In this section, we prove that  $\psi_{h,\kappa}(t)$  defines a bounded Cauchy sequence in  $\mathcal{H}$ , as  $t \rightarrow +\infty$ . To this end, it is necessary to control the norm difference between vectors  $\psi_{h,\kappa}(t_i)$ ,  $i = 1, 2$ , at times  $t_2 > t_1 \gg 1$ .

*V.1. Three key steps.* As anticipated in Sect. III.4, we decompose the difference of  $\psi_{h,\kappa}(t_1)$  and  $\psi_{h,\kappa}(t_2)$  into three terms

$$\begin{aligned} \psi_{h,\kappa}(t_1) - \psi_{h,\kappa}(t_2) &= \Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}) + \Delta\psi(t_2 \rightarrow t_1, \sigma_{t_2}, \mathcal{G}^{(t_1)}) \\ &\quad + \Delta\psi(t_1, \sigma_{t_2} \rightarrow \sigma_{t_1}, \mathcal{G}^{(t_1)}), \end{aligned} \tag{V.1}$$

where we recall from (III.49) – (III.51):

I) The term

$$\begin{aligned} &\Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}) \\ &= e^{iHt_2} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{j, \sigma_{t_2}}^{(t_1)} \\ &\quad - e^{iHt_2} \sum_{j=1}^{N(t_1)} \sum_{l(j)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) e^{i\gamma\sigma_{t_2}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{l(j), \sigma_{t_2}}^{(t_2)}, \end{aligned} \tag{V.2}$$

accounts for the change of the partition  $\mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}$  in  $\psi_{h,\kappa}(t_2)$ , where  $l(j)$  labels the sub-cells belonging to the sub-partition  $\mathcal{G}^{(t_2)} \cap \mathcal{G}_j^{(t_1)}$  of  $\mathcal{G}_j^{(t_1)}$ , and where we define

$$\vec{v}_{l(j)} \equiv \vec{\nabla} E_{\vec{p}_{l(j)}^*}^{\sigma_{t_2}} \quad \text{and} \quad \vec{v}_j \equiv \vec{\nabla} E_{\vec{p}_j^*}^{\sigma_{t_1}};$$

II) the term

$$\begin{aligned} &\Delta\psi(t_2 \rightarrow t_1, \sigma_{t_2}, \mathcal{G}^{(t_1)}) \\ &= e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)} \\ &\quad - e^{iHt_2} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_2)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_2} \psi_{j, \sigma_{t_2}}^{(t_1)}, \end{aligned} \tag{V.3}$$

accounts for the subsequent change of the time variable,  $t_2 \rightarrow t_1$ , for the fixed partition  $\mathcal{G}^{(t_1)}$ , and the fixed infrared cut-off  $\sigma_{t_2}$ ; and finally,

III) the term

$$\begin{aligned} &\Delta\psi(t_1, \sigma_{t_2} \rightarrow \sigma_{t_1}, \mathcal{G}^{(t_1)}) \\ &= e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \psi_{j, \sigma_{t_1}}^{(t_1)} \\ &\quad - e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)} \end{aligned} \tag{V.4}$$

accounts for the change of the infrared cut-off,  $\sigma_{t_2} \rightarrow \sigma_{t_1}$ .

Our goal is to prove

$$\| \psi_{h, \kappa}(t_2) - \psi_{h, \kappa}(t_1) \| \leq \mathcal{O} \left( (\ln(t_2))^2 / t_1^\rho \right), \tag{V.5}$$

for some  $\rho > 0$ . To this end, it is necessary to perform the three steps in the order displayed above. As a corollary of the bound (V.5), we obtain Theorem III.2 by telescoping (see the comment after Eq. (III.57)).

The arguments in our proof are very similar to those in [26], but a number of modifications are necessary because of the vector nature of the QED interaction. For these modifications, we provide detailed explanations.

*V.2. Refining the cell partition.* In this section, we discuss step (V.2) in which the momentum space cell partition is modified. It is possible to apply the methods developed in [26], up to some minor modifications.

We will prove that

$$\| \Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)}) \| \leq \mathcal{O} \left( (\ln(t_2))^2 / t_1^\rho \right) \tag{V.6}$$

for some  $\rho > 0$ . The contributions from the off-diagonal terms with respect to the subpartition  $\mathcal{G}^{(t_2)}$  of  $\mathcal{G}^{(t_1)}$  can be estimated by the same arguments that have culminated in the proof of Theorem IV.1. That is, we first express  $\psi_{j, \sigma_{t_2}}^{(t_1)}$  as

$$\psi_{j, \sigma_{t_2}}^{(t_1)} = \int_{\mathcal{G}_j^{(t_1)}} h(\vec{P}) \Psi_{\vec{P}}^{\sigma_{t_2}} d^3 P = \sum_{l(j)} \int_{\mathcal{G}_{l(j)}^{(t_2)}} h(\vec{P}) \Psi_{\vec{P}}^{\sigma_{t_2}} d^3 P = \sum_{l(j)} \psi_{l(j), \sigma_{t_2}}^{(t_2)}. \tag{V.7}$$

Then,

$$\begin{aligned} & \|\Delta\psi(t_2, \sigma_{t_2}, \mathcal{G}^{(t_2)} \rightarrow \mathcal{G}^{(t_1)})\|^2 \\ &= \sum_{j=1}^{N(t_1)} \sum_{l(j), l'(j')} \left\langle \left[ \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) - \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) \right] e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{l(j), \sigma_{t_2}}^{(t_2)}, \right. \\ & \quad \left. \left[ \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_{l'(j')}, t_2) - \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_{j'}, t_2) \right] e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{l'(j'), \sigma_{t_2}}^{(t_2)} \right\rangle, \end{aligned} \quad (\text{V.8})$$

where we define

$$\widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) := \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)}, \quad (\text{V.9})$$

$$\widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) := \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_2)}. \quad (\text{V.10})$$

Following the analysis in Sect. IV.1, one finds that the sum over pairs  $(l'(j'), l(j))$  with either  $l \neq l'$  or  $j \neq j'$  can be bounded by  $\mathcal{O}(t_2^{-\epsilon})$ , provided that  $\epsilon < \frac{\eta}{4}$ , as in (IV.21).

Let  $\langle \cdot \rangle_{\tilde{\Psi}}$  stand for the expectation value with respect to the vector  $\tilde{\Psi}$ . Then, we are left with the diagonal terms

$$\sum_{j=1}^{N(t_1)} \sum_{l(j)} \left\langle \left[ 2 - \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^*(\vec{v}_{l(j)}, t_2) \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) - \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^*(\vec{v}_j, t_2) \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_{l(j)}, t_2) \right] \right\rangle_{\tilde{\Psi}}, \quad (\text{V.11})$$

labeled by pairs  $(l(j), l(j))$ , where  $\tilde{\Psi} \equiv e^{-iE_{\vec{p}}^{\sigma_{t_2}} t_2} \psi_{l(j), \sigma_{t_2}}^{(t_2)}$  in the case considered here. For each term

$$\left\langle \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^*(\vec{v}_{l(j)}, t_2) \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_2) \right\rangle_{\tilde{\Psi}}, \quad (\text{V.12})$$

we can again invoke the arguments developed for off-diagonal elements indexed by  $(l, j)$  (where  $l \neq j$ ) from Sect. IV.1.

In particular, we define for  $s > t_2$ ,

$$\begin{aligned} \widehat{\mathcal{M}}_{[l(j), \vec{v}_j], [l(j), \vec{v}_{l(j)}]}^{\mu}(t_2, s) &:= \left\langle e^{-iE_{\vec{p}}^{\sigma_{t_2}} s} \psi_{l(j), \sigma_{t_2}}^{(t_2)}, \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^{\mu*}(\vec{v}_j, s) \right. \\ & \quad \left. \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^{\mu}(\vec{v}_{l(j)}, s) e^{-iE_{\vec{p}}^{\sigma_{t_2}} s} \psi_{l(j), \sigma_{t_2}}^{(t_2)} \right\rangle, \end{aligned} \quad (\text{V.13})$$

where

$$\begin{aligned} \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^{\mu*}(\vec{v}_j, s) \widehat{\mathcal{W}}_{\kappa, \sigma_{t_2}}^{\mu}(\vec{v}_{l(j)}, s) &= e^{-i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} \mathcal{W}_{\kappa, \sigma_{t_2}}^{\mu*}(\vec{v}_j, s) \\ & \quad \times \mathcal{W}_{\kappa, \sigma_{t_2}}^{\mu}(\vec{v}_{l(j)}, s) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)}, \end{aligned} \quad (\text{V.14})$$

and

$$\begin{aligned} \mathcal{W}_{\kappa, \sigma_{t_2}}^{\mu*}(\vec{v}_j, s) \mathcal{W}_{\kappa, \sigma_{t_2}}^{\mu}(\vec{v}_{l(j)}, s) &= \exp \left( \mu \sum_{\lambda} \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_{t_2}}} \vec{\eta}_{j, l(j)}(\vec{k}) \cdot \left\{ \vec{\varepsilon}_{\vec{k}, \lambda} a_{\vec{k}, \lambda}^* e^{-i|\vec{k}|s} \right. \right. \\ & \quad \left. \left. - \vec{\varepsilon}_{\vec{k}, \lambda}^* a_{\vec{k}, \lambda} e^{i|\vec{k}|s} \right\} d^3k \right), \end{aligned} \quad (\text{V.15})$$

with  $\mu$  a real parameter.

Proceeding similarly as in (IV.12), the solution of the ODE analogous to (IV.9) for  $\widehat{M}_{[l(j), \vec{v}_j], [l(j), \vec{v}_{l(j)}]}^\mu(t_2, s)$  at  $\mu = 1$  consists of a contribution at  $\mu = 0$ , which remains non-zero as  $s \rightarrow \infty$ , and a remainder term that vanishes in the limit  $s \rightarrow \infty$ . In fact,

$$\lim_{s \rightarrow +\infty} \widehat{M}_{[l(j), \vec{v}_j], [l(j), \vec{v}_{l(j)}]}^1(t_2, s) = e^{-\frac{C_{j,l(j)} \sigma_{t_2}}{2}} \left\langle e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, \sigma_{t_2}^{-\frac{1}{\theta}}})} \psi_{l(j), \sigma_{t_2}}^{(t_2)}, e^{i\gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, \sigma_{t_2}^{-\frac{1}{\theta}}})} \psi_{l(j), \sigma_{t_2}}^{(t_2)} \right\rangle, \tag{V.16}$$

where

$$C_{j,l(j), \sigma_{t_2}} := \int_{\mathcal{B}_k \setminus \mathcal{B}_{\sigma_{t_2}}} |\vec{\eta}_{j,l(j)}(\vec{k})|^2 d^3k, \tag{V.17}$$

as in (IV.10), with  $\vec{\eta}_{j,l(j)}(\vec{k})$  defined in (IV.4). Hence, (V.11) is given by the sum of

$$\begin{aligned} & - \sum_{j=1}^{N(t_2)} \sum_{\ell(j)} \int_{t_2}^{+\infty} \frac{d}{ds} \widehat{M}_{[\ell(j), \vec{v}_j], [\ell(j), \vec{v}_{\ell(j)}]}^1(t_2, s) ds \\ & - \sum_{j=1}^{N(t_2)} \sum_{\ell(j)} \int_{t_2}^{+\infty} \frac{d}{ds} \widehat{M}_{[\ell(j), \vec{v}_{\ell(j)}], [\ell(j), \vec{v}_j]}^1(t_2, s) ds \end{aligned} \tag{V.18}$$

and

$$\sum_{j=1}^{N(t_2)} \sum_{\ell(j)} \left\langle \psi_{l(j), \sigma_{t_2}}^{(t_2)}, \left[ 2 - 2 \cos \left( \Delta \gamma_{\sigma_{t_2}}(\vec{v}_j - \vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, t_2}) \right) e^{-\frac{C_{l(j),j, \sigma_{t_2}}}{2}} \right] \psi_{l(j), \sigma_{t_2}}^{(t_2)} \right\rangle, \tag{V.19}$$

where

$$\Delta \gamma_{\sigma_{t_2}}(\vec{v}_j - \vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, t_2}) := \gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, \sigma_{t_2}^{-\frac{1}{\theta}}}) - \gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, \sigma_{t_2}^{-\frac{1}{\theta}}}). \tag{V.20}$$

The arguments that have culminated in Theorem IV.1 also imply that the sum (V.18) can be bounded by  $\mathcal{O}(t_2^{-4\epsilon})$ , for  $\eta > 4\epsilon$ . The leading contribution in (V.6) is represented by the sum (V.19) of diagonal terms (with respect to  $\mathcal{G}^{(t_2)}$ ), which can now be bounded from above. It suffices to show that

$$\sup_{\vec{p} \in \mathcal{S}} \left| 2 - 2 \cos \left( \Delta \gamma_{\sigma_{t_2}}(\vec{v}_j - \vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, t_2}) \right) e^{-\frac{C_{l(j),j, \sigma_{t_2}}}{2}} \right| \leq \frac{1}{t_1^{\eta'}} \ln t_2 \tag{V.21}$$

for some  $\eta' > 0$  that depends on  $\epsilon$ . To see this, we note that the lower integration bound in the integral (V.17) contributes a factor to (V.21) proportional to  $\ln t_2$ . In Lemma A.1, it is proven that

$$\left| \gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, (\sigma_{t_2})^{-\frac{1}{\theta}}}) - \gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}, (\sigma_{t_2})^{-\frac{1}{\theta}}}) \right| \leq \mathcal{O}(|\vec{v}_j - \vec{v}_{l(j)}|). \tag{V.22}$$

We can estimate the difference  $\vec{v}_j - \vec{v}_{l(j)} = \vec{\nabla} E_{\vec{p}_j^*}^{\sigma_{t_1}} - \vec{\nabla} E_{\vec{p}_{l(j)}^*}^{\sigma_{t_2}}$ , which also appears in  $\vec{\eta}_{l(j),j}(\vec{k})$ , using condition ( $\mathcal{S}$ 2) of Theorem III.1. This yields the  $\epsilon$ -dependent negative power of  $t_1$  in (V.21).

V.3. *Shifting the time variable for a fixed cell partition and infrared cut-off.* In this subsection, we prove that

$$\left\| \Delta\psi(t_2 \rightarrow t_1, \sigma_{t_2}, \mathcal{G}^{(t_1)}) \right\| \leq \mathcal{O}\left((\ln(t_2))^2/t_1^\rho\right) \tag{V.23}$$

for some  $\rho > 0$ ; see (V.3). This accounts for the change of the time variable, while both the cell partition and the infrared cutoff are kept fixed. It can be controlled by a standard Cook argument, and methods similar to those used in the discussion of (IV.22).

For  $t_1 \leq s \leq t_2$ , we define

$$\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s) := - \int_1^{t_2} \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}} \cdot \int_{\mathcal{B}_{\sigma_{t_2}} \setminus \mathcal{B}_{\sigma_{t_1}}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}} \tau - |\vec{k}| \tau) d^3k d\tau. \tag{V.24}$$

Then, we estimate

$$\int_{t_1}^{t_2} ds \frac{d}{ds} \left( e^{iH^{\sigma_{t_2}}s} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, s) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_2}}s} \psi_{j, \sigma_{t_2}}^{(t_1)} \right) \tag{V.25}$$

cell by cell. To this end, we can essentially apply the same arguments that entered the treatment of the time derivative in (IV.22), see also the remark after Theorem A.3, by defining an *infrared tail* in a similar fashion. The only modification to be added is that, apart from two terms analogous to (IV.23), (IV.24), we now also have to consider

$$i e^{iHs} (H - H^{\sigma_{t_2}}) \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, s) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_2}}s} \psi_{j, \sigma_{t_2}}^{(t_1)}, \tag{V.26}$$

which enters from the derivative in  $s$  of the operator underlined in

$$\underline{e^{iHs} e^{-iH^{\sigma_{t_2}}s}} e^{iH^{\sigma_{t_2}}s} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, s) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_2}}s} \psi_{j, \sigma_{t_2}}^{(t_1)}. \tag{V.27}$$

To control the norm of (V.26), we observe that

$$H - H^{\sigma_{t_2}} = \alpha^{\frac{1}{2}} i[H, \vec{x}] \cdot \vec{A}_{<\sigma_{t_2}} - \alpha \frac{\vec{A}_{<\sigma_{t_2}} \cdot \vec{A}_{<\sigma_{t_2}}}{2}, \tag{V.28}$$

where

$$\vec{A}_{<\sigma_{t_2}} := \sum_{\lambda=\pm} \int_{\mathcal{B}_{\sigma_{t_2}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \vec{\varepsilon}_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^* + \vec{\varepsilon}_{\vec{k}, \lambda}^* b_{\vec{k}, \lambda} \right\}, \tag{V.29}$$

and we note that

$$[H, \vec{x}] \cdot \vec{A}_{<\sigma_{t_2}} = \vec{A}_{<\sigma_{t_2}} \cdot [H, \vec{x}],$$

because of the Coulomb gauge condition. Moreover,

$$\mathcal{W}_{\kappa, \sigma_{t_2}}^*(\vec{v}_j, s) i[H, \vec{x}] \mathcal{W}_{\sigma_{t_2}}(\vec{v}_j, s) = i[H, \vec{x}] + \vec{h}_s(\vec{x}) \tag{V.30}$$

with  $\|\vec{h}_s(\vec{x})\| < \mathcal{O}(1)$  and

$$[b_{\vec{k}, \lambda}^*, \vec{h}_s(\vec{x})] = [\vec{h}_s(\vec{x}), \vec{A}_{<\sigma_{t_2}}] = 0.$$

Furthermore, we have

$$b_{\vec{k},\lambda} \psi_{j,\sigma_{t_2}}^{(t_1)} = 0$$

for  $\vec{k} \in \mathcal{B}_{\sigma_{t_2}}$ , and

$$\|\vec{A}_{<\sigma_{t_2}} \psi_{j,\sigma_{t_2}}^{(t_1)}\|, \|\vec{A}_{<\sigma_{t_2}} \cdot \vec{A}_{<\sigma_{t_2}} \psi_{j,\sigma_{t_2}}^{(t_1)}\| \leq \mathcal{O}(\sigma_{t_2} \|\psi_{j,\sigma_{t_2}}^{(t_1)}\|). \tag{V.31}$$

The estimate

$$\|\vec{A}_{<\sigma_{t_2}} \cdot [H, \vec{x}] \psi_{j,\sigma_{t_2}}^{(t_1)}\| \leq \mathcal{O}\left(\sigma_{t_2} \left\{ \|[H, \vec{x}] \psi_{j,\sigma_{t_2}}^{(t_1)}\| + \|\psi_{j,\sigma_{t_2}}^{(t_1)}\| \right\}\right), \tag{V.32}$$

holds, where

$$\|[H, \vec{x}] \psi_{j,\sigma_{t_2}}^{(t_1)}\| \leq \mathcal{O}(t_1^{-\frac{3\epsilon}{2}}), \tag{V.33}$$

because

$$\|[H, \vec{x}] \psi_{j,\sigma_{t_2}}^{(t_1)}\| \leq c_1 \|(H^{\sigma_{t_2}} + i) \psi_{j,\sigma_{t_2}}^{(t_1)}\|$$

for some constant  $c_1$ , and

$$\|\psi_{j,\sigma_{t_2}}^{(t_1)}\| = \mathcal{O}(t_1^{-\frac{3\epsilon}{2}}).$$

Consequently, we obtain that

$$\begin{aligned} & \left\| (H - H^{\sigma_{t_2}}) \mathcal{W}_{\kappa,\sigma_{t_2}}(\vec{v}_j, s) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} s} \psi_{j,\sigma_{t_2}}^{(t_1)} \right\| \\ & \leq \mathcal{O}(\sigma_{t_2} \left\{ \|[H, \vec{x}] \psi_{j,\sigma_{t_2}}^{(t_1)}\| + \|\psi_{j,\sigma_{t_2}}^{(t_1)}\| \right\}) \leq \mathcal{O}(\sigma_{t_2} t^{-3/2\epsilon}). \end{aligned} \tag{V.34}$$

Following the procedure in Sect. II.2.1, B, one can also check that

$$\begin{aligned} & \left\| \int_{t_1}^{t_2} e^{iHs} e^{-iH^{\sigma_{t_2}} s} \frac{d}{ds} \left( e^{iH^{\sigma_{t_2}} s} \mathcal{W}_{\kappa,\sigma_{t_2}}(\vec{v}_j, s) e^{i\gamma\sigma_{t_2}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_2}} s} \psi_{j,\sigma_{t_2}}^{(t_1)} \right) ds \right\| \\ & \leq \mathcal{O}\left(\frac{1}{t_1^\eta} |\ln \sigma_{t_2}|^2 t_1^{-\frac{3\epsilon}{2}}\right), \end{aligned} \tag{V.35}$$

for some  $\eta > 0$  independent of  $\epsilon$ . Similarly as in (IV.21), we choose  $\epsilon$  small enough such that  $\frac{\eta}{4} > \epsilon$ .

The number of cells in the partition  $\mathcal{G}^{(t_1)}$  is  $N(t_1) \approx t_1^{3\epsilon}$ . Therefore, summing over all cells, we get

$$\mathcal{O}\left(N(t_1) t_1^{-\frac{3\epsilon}{2}} \sigma_{t_2} t_2\right) + \mathcal{O}\left(N(t_1) \frac{1}{t_1^\eta} |\ln \sigma_{t_2}|^2 t_1^{-\frac{3\epsilon}{2}}\right), \tag{V.36}$$

as an upper bound on the norm of the term in (V.3).

The parameter  $\beta$  in the definition of  $\sigma_{t_2} = t_2^{-\beta}$  can be chosen arbitrarily large, independently of  $\epsilon$ . Hereby, we arrive at the upper bound claimed in (V.5).

V.4. *Shifting the infrared cut-off.* In this section, we prove that

$$\| \Delta \psi(t_1, \sigma_{t_2} \rightarrow \sigma_{t_1}, \mathcal{G}^{(t_1)}) \| \leq \mathcal{O} \left( (\ln(t_2))^2 / t_1^\rho \right) \tag{V.37}$$

for some  $\rho > 0$ ; see (V.4). The analysis of this last step is the most involved one, and will require extensive use of our previous results.

The starting idea is to rewrite the last term in (V.4),

$$e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)}, \tag{V.38}$$

as

$$e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) W_{\sigma_{t_2}} (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \psi_{j, \sigma_{t_2}}^{(t_1)}, \tag{V.39}$$

and to group the terms appearing in (V.39) in such a way that, cell by cell, we consider the new *dressing operator*

$$e^{iHt_1} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1}, \tag{V.40}$$

which acts on

$$\Phi_{j, \sigma_{t_2}}^{(t_1)} := \int_{\mathcal{G}_j^{(t_1)}} h(\vec{P}) \Phi_{\vec{P}}^{\sigma_{t_2}} d^3 P, \tag{V.41}$$

where  $\Phi_{\vec{P}}^\sigma = W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) \Psi_{\vec{P}}^\sigma$ , see (III.23). The key advantage is that the vector  $\Phi_{j, \sigma_{t_2}}^{(t_1)}$  inherits the Hölder regularity of  $\Phi_{\vec{P}}^\sigma$ ; see (III.13) in condition ( $\mathcal{I}2$ ) of Theorem III.1. We will refer to (V.41) as an *infrared-regular vector*.

Accordingly, (V.39) now reads

$$e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_2}}^{(t_1)}, \tag{V.42}$$

and we proceed as follows.

A. *Shifting the IR cutoff in the infrared-regular vector.* First, we substitute

$$\begin{aligned} & e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_2}}^{(t_1)} \\ & \longrightarrow e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}, \end{aligned} \tag{V.43}$$

where  $\sigma_{t_2}$  is replaced by  $\sigma_{t_1}$  in the underlined terms. We prove that the norm difference of these two vectors is bounded by the r.h.s. of (V.37). The necessary ingredients are:

- 1) Condition ( $\mathcal{S}1$ ) in Theorem III.1.
- 2) The estimate

$$|\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1) - \gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)| \leq \mathcal{O}\left(\sigma_{t_1}^{\frac{1}{2}(1-\delta)} t_1^{1-\theta}\right) + \mathcal{O}(t_1 \sigma_{t_1}),$$

for  $t_2 > t_1 \gg 1$ , proven in Lemma A.1. The parameter  $0 < \theta < 1$  is the same as the one in (III.26).

- 3) The cell partition  $\mathcal{G}^{(t_1)}$  depends on  $t_1 < t_2$ .
- 4) The parameter  $\beta$  can be chosen arbitrarily large, independently of  $\epsilon$ , so that the infrared cutoff  $\sigma_{t_1} = t_1^{-\beta}$  can be made as small as one wishes.

First of all, it is clear that the norm difference of the two vectors in (V.43) is bounded by the norm difference of the two underlined vectors, summed over all  $N(t_1)$  cells. Using 1) and 2), one straightforwardly derives that the norm difference between the two underlined vectors in (V.43) is bounded from above by

$$\mathcal{O}(t_1 \sigma_{t_1}^{\frac{1}{2}(1-\delta)} t_1^{-\frac{3}{2}\epsilon}), \tag{V.44}$$

where the last factor,  $t_1^{-\frac{3}{2}\epsilon}$ , accounts for the volume of an individual cell in  $\mathcal{G}^{(t_1)}$ , by 3). The sum over all cells in  $\mathcal{G}^{(t_1)}$  yields a bound

$$\mathcal{O}(N(t_1) \sigma_{t_1}^{\frac{1}{2}(1-\delta)} t_1^{1-\frac{3}{2}\epsilon}), \tag{V.45}$$

where  $N(t_1) \approx t_1^{3\epsilon}$ , by 3). Picking  $\beta$  sufficiently large, by 4), we find that the norm difference of the two vectors in (V.43) is bounded by  $t_1^{-\eta}$ , for some  $\eta > 0$ . This agrees with the bound stated in (V.37).

*B. Shifting the IR cutoff in the dressing operator.* Subsequently to (V.43), we substitute

$$\begin{aligned} & e^{iHt_1} \sum_{j=1}^{N(t_1)} \frac{\mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}}{ } \\ & \longrightarrow e^{iHt_1} \sum_{j=1}^{N(t_1)} \frac{\mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}}{ } \end{aligned} \tag{V.46}$$

where  $\sigma_{t_2} \rightarrow \sigma_{t_1}$  in the underlined operators. A crucial point in our argument is that when  $\sigma_{t_1} (> \sigma_{t_2})$  tends to 0, the Hölder continuity of  $\Phi_{\vec{P}}^{\sigma_{t_1}}$  in  $\vec{P}$  offsets the (logarithmic) divergence in  $t_2$  which arises from the dressing operator.

We subdivide the shift  $\sigma_{t_2} \rightarrow \sigma_{t_1}$  in

$$\mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) \longrightarrow \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) \tag{V.47}$$

into the following three intermediate steps, where the operators modified in each step are underlined:

*Step a)*

$$\begin{aligned} & \frac{\mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}})}{ } \\ & \longrightarrow \frac{\mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}})}{ } \end{aligned} \tag{V.48}$$



Step b)

$$\begin{aligned} & \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) \underline{W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}})} \\ & \longrightarrow \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) \underline{W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}})}, \end{aligned} \quad (\text{V.49})$$

Step c)

$$\begin{aligned} & \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) \underline{W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}})} \\ & \longrightarrow \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) \underline{W_{\sigma_{t_1}}(\vec{v}_j) W_{\sigma_{t_1}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}})}. \end{aligned} \quad (\text{V.50})$$

*Analysis of Step a).* In Step a), we analyze the difference between the vectors

$$e^{iHt_1} \underline{\mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}})} e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \quad (\text{V.51})$$

and

$$e^{iHt_1} \underline{\mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}})} e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}, \quad (\text{V.52})$$

for each cell in  $\mathcal{G}^{(t_1)}$ . Our goal is to prove that

$$\| (\text{V.51}) - (\text{V.52}) \| \leq \text{const} \ln t_2 P(t_1, t_2), \quad (\text{V.53})$$

where

$$\begin{aligned} P(t_1, t_2) & := \sup_{\vec{k} \in \mathcal{B}_{\sigma_{t_1}}} \left\| (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - 1) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) \right. \\ & \quad \left. \times e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \right\| \leq \mathcal{O}(t_1^{-\eta} \ln t_2) \end{aligned} \quad (\text{V.54})$$

as  $t_1 \rightarrow +\infty$ , for some  $\eta > 0$  independent of  $\epsilon$  and for  $\beta$  large enough.

Using the identity

$$\begin{aligned} & \mathcal{W}_{\kappa, \sigma_{t_2}}(\vec{v}_j, t_1) W_{\sigma_{t_2}}^*(\vec{v}_j) = \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) \\ & \quad \times \exp\left(\frac{i\alpha}{2} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} d^3k \frac{\vec{v}_j \cdot \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \sin(\vec{k} \cdot \vec{x} - |\vec{k}|t_1)}{|\vec{k}|(1 - \widehat{\vec{k}} \cdot \vec{v}_j)}\right) \\ & \quad \times \exp\left(\alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \frac{\vec{v}_j \cdot \{\vec{\epsilon}_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^* (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - 1) - h.c.\}}{|\vec{k}|(1 - \widehat{\vec{k}} \cdot \vec{v}_j)}\right), \end{aligned} \quad (\text{V.55})$$

the difference between (V.51) and (V.52) is given by

$$\begin{aligned}
 & e^{iHt_1} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) \exp\left(\frac{i\alpha}{2} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} d^3k \frac{\vec{v}_j \cdot \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \sin(\vec{k} \cdot \vec{x} - |\vec{k}|t_1)}{|\vec{k}|(1 - \widehat{k} \cdot \vec{v}_j)}\right) \\
 & \times \left[ \exp\left(\alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \frac{\vec{v}_j \cdot \{\vec{\varepsilon}_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^*\} (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - 1) - h.c.}{|\vec{k}|(1 - \widehat{k} \cdot \vec{v}_j)}\right) - \mathcal{I} \right] \\
 & \times W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \tag{V.56}
 \end{aligned}$$

$$\begin{aligned}
 & + e^{iHt_1} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) W_{\sigma_{t_1}}^*(\vec{v}_j) \left[ \exp\left(\frac{i\alpha}{2} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} d^3k \frac{\vec{v}_j \cdot \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \sin(\vec{k} \cdot \vec{x} - |\vec{k}|t_1)}{|\vec{k}|(1 - \widehat{k} \cdot \vec{v}_j)}\right) - \mathcal{I} \right] \\
 & \times W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}, \tag{V.57}
 \end{aligned}$$

where  $\mathcal{I}$  is the identity operator in  $\mathcal{H}$ .

The norm of the vector (V.56) equals

$$\begin{aligned}
 & \left\| \left[ \exp\left(\alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \frac{\vec{v}_j \cdot \{\vec{\varepsilon}_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^*\} (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - \mathcal{I}) - h.c.}{|\vec{k}|(1 - \widehat{k} \cdot \vec{v}_j)}\right) - \mathcal{I} \right] \right. \\
 & \left. \times W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \right\|. \tag{V.58}
 \end{aligned}$$

We now observe that

- for  $\vec{k} \in \mathcal{B}_{\sigma_{t_1}}$ ,

$$b_{\vec{k}, \lambda} W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) \tag{V.59}$$

$$= W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) b_{\vec{k}, \lambda} \tag{V.60}$$

$$+ W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) f_{\vec{k}, \lambda}(\vec{v}_j, \vec{P}), \tag{V.61}$$

where

$$\int_{\mathcal{B}_{\sigma_{t_1}} \setminus \mathcal{B}_{\sigma_{t_2}}} d^3k |f_{\vec{k}, \lambda}(\vec{v}_j, \vec{P})|^2 \leq \mathcal{O}(|\ln \sigma_{t_2}|) \tag{V.62}$$

uniformly in  $\vec{v}_j$ , and in  $\vec{P} \in \mathcal{S}$ , and where  $j$  enumerates the cells.

- for  $\vec{k} \in \mathcal{B}_{\sigma_{t_1}}$ ,

$$b_{\vec{k}, \lambda} e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} = 0, \tag{V.63}$$

because of the infrared properties of  $\Phi_{j, \sigma_{t_1}}^{(t_1)}$ .

From the Schwarz inequality, we therefore get

$$\text{(V.58)} \leq c |\ln \sigma_{t_2}| P(t_1, t_2), \tag{V.64}$$

for some finite constant  $c$  as claimed in (V.53), where

$$P(t_1, t_2) = \sup_{\vec{k} \in \mathcal{B}_{\sigma_{t_1}}} \left\| (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - 1) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) \right. \\ \left. \times e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \right\|, \quad (\text{V.65})$$

as defined in (V.54). To estimate  $P(t_1, t_2)$ , we regroup the terms inside the norm into

$$(e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - 1) W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \\ = W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}-\vec{k}}^{\sigma_{t_2}}) (e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - \mathcal{I}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \quad (\text{V.66})$$

$$+ W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}-\vec{k}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \quad (\text{V.67})$$

$$- W_{\sigma_{t_2}}(\vec{v}_j) W_{\sigma_{t_2}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}. \quad (\text{V.68})$$

We next prove that

$$\|(\text{V.66})\|, \quad \|(\text{V.67}) - (\text{V.68})\| \leq \mathcal{O}((\sigma_{t_1})^\rho t_1 \ln t_2) \quad (\text{V.69})$$

for some  $\rho > 0$  independent of  $\epsilon$ . To this end, we use:

- i) The Hölder regularity of  $\Phi_{\vec{P}}^\sigma$  and  $\vec{\nabla} E_{\vec{P}}^\sigma$  described under condition ( $\mathcal{J}2$ ) in Theorem III.1.
- ii) The regularity of the phase function

$$\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1) \quad (\text{V.70})$$

with respect to  $\vec{P} \in \text{supph} \subset \mathcal{S}$  expressed in the following estimate, which is similar to (A.3) in Lemma A.1: For  $\vec{k} \in \mathcal{B}_{\sigma_{t_1}}$  and  $t_1$  large enough,

$$\left| \gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1) - \gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}-\vec{k}}^{\sigma_{t_2}}, t_1) \right| \leq \mathcal{O}(|\vec{k}|^{\frac{1}{4}(1-\delta'')} t_1^{(1-\theta)}), \quad (\text{V.71})$$

where  $0 < \theta (< 1)$  can be chosen arbitrarily close to 1.

- iii) The estimate

$$\|b_{\vec{k}, \lambda}^\sigma \Psi_{\vec{P}}^\sigma\| \leq C \frac{\mathbf{1}_{\sigma, \Lambda}(\vec{k})}{|\vec{k}|^{3/2}} \quad (\text{V.72})$$

from ( $\mathcal{J}5$ ) in Theorem III.1 for  $\vec{P} \in \mathcal{S}$ , which implies

$$\|N_f^{1/2} \Psi_{\vec{P}}^\sigma\| = \left( \sum_\lambda \int d^3k \|b_{\vec{k}, \lambda}^\sigma \Psi_{\vec{P}}^\sigma\|^2 \right)^{1/2} \leq C |\ln \sigma|^{1/2}. \quad (\text{V.73})$$

Likewise,

$$\begin{aligned} \| N_f^{1/2} \Phi_{\vec{p}}^\sigma \| &= \left( \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} d^3k \left\| \left( b_{\vec{k},\lambda} + \mathcal{O}(|\vec{k}|^{-3/2}) \right) \Psi_{\vec{p}}^\sigma \right\|^2 \right)^{1/2} \\ &\leq C |\ln \sigma|^{1/2}, \end{aligned} \tag{V.74}$$

which controls the expected photon number in the states  $\{\Phi_{\vec{p}}^{\sigma_{t_1}}\}$ . As a side remark, we note that the true size is in fact  $\mathcal{O}(1)$ , uniformly in  $\sigma$ , but the logarithmically divergent bound here is sufficient for our purposes.

- iv) The cell decomposition  $\mathcal{G}^{(t_1)}$  is determined by  $t_1 < t_2$ . Moreover, since  $\beta (> 1)$  can be chosen arbitrarily large and independent of  $\epsilon$ ,  $\sigma_{t_1} = t_1^{-\beta}$  can be made as small as desired.

We first prove the bound on  $\|(\text{V.66})\|$  stated in (V.69). To this end, we use

$$\left( e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - \mathcal{I} \right) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \tag{V.75}$$

$$= e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}-\vec{k}}^{\sigma_{t_1}} t_1} \left( e^{-i(|\vec{k}|t_1 - \vec{k} \cdot \vec{x})} - \mathcal{I} \right) \Phi_{j, \sigma_{t_1}}^{(t_1)} \tag{V.76}$$

$$+ e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}-\vec{k}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)} \tag{V.77}$$

$$- e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}. \tag{V.78}$$

The Hölder regularity of  $\Phi_{\vec{p}}^{\sigma_{t_1}}$  from i) yields

$$\|(\text{V.76})\| \leq \mathcal{O}(t_1 \sigma_{t_1}^{(\frac{1}{4}-\delta')} t_1^{-\frac{3\epsilon}{2}}), \tag{V.79}$$

where  $\delta'$  can be chosen arbitrarily small, and independently of  $\epsilon$ . The derivation of a similar estimate is given in the proof of Theorem A.3 in the Appendix, starting from (A.27), to which we refer for details. The Hölder continuity of  $E_{\vec{p}}^{\sigma_{t_1}}$  and  $\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}$ , again from i), combined with ii), with  $\theta$  sufficiently close to 1, implies that, with  $\vec{k} \in \mathcal{B}_{\sigma_{t_1}}$ ,

$$\|(\text{V.77}) - (\text{V.78})\| \leq \mathcal{O}(t_1 \sigma_{t_1}^{\frac{1}{5}} t_1^{-\frac{3\epsilon}{2}}), \tag{V.80}$$

as desired.

To prove the bound on  $\|(\text{V.67}) - (\text{V.68})\|$  asserted in (V.69), we write

$$W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}}) - W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) = W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) (W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}}; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) - \mathcal{I}), \tag{V.81}$$

where

$$W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}}; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) := W_{\sigma_{t_2}} (\vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}}),$$

and apply the Schwarz inequality in the form

$$\begin{aligned} & \left\| (W_{\sigma_{t_2}}^* (\vec{\nabla} E_{\vec{p}+\vec{k}}^{\sigma_{t_2}}; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}}) - \mathcal{I}) \tilde{\Phi} \right\| \\ & \leq C \left( \int_{\mathcal{B}_{\sigma_{t_2}}} \frac{d^3 q}{|\vec{q}|^3} \right)^{1/2} \sup_{\vec{p} \in \text{supp } h, \vec{k} \in \mathcal{B}_{\sigma_{t_1}}} \left| \vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}} - \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}} \right| \| N_f^{1/2} \tilde{\Phi} \|, \end{aligned} \quad (\text{V.82})$$

where in our case,  $\tilde{\Phi} \equiv e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \Phi_{j, \sigma_{t_1}}^{(t_1)}$ . We have

$$\| N_f^{1/2} \Phi_{\vec{p}}^{\sigma_{t_1}} \| \leq c |\ln \sigma_{t_1}|^{1/2} \leq c' (\ln t_1)^{1/2}, \quad (\text{V.83})$$

as a consequence of iii). Due to i),

$$\sup_{\vec{p} \in \text{supp } h, \vec{k} \in \mathcal{B}_{\sigma_{t_1}}} \left| \vec{\nabla} E_{\vec{p}-\vec{k}}^{\sigma_{t_2}} - \vec{\nabla} E_{\vec{p}}^{\sigma_{t_2}} \right| \leq \mathcal{O}(\sigma_{t_1}^{\frac{1}{4}-\delta''}), \quad (\text{V.84})$$

where  $\delta'' > 0$  is arbitrarily small, and independent of  $\epsilon$  (see (III.14)). Therefore,

$$\sup_{\vec{k} \in \mathcal{B}_{\sigma_{t_1}}} \| (\text{V.67}) - (\text{V.68}) \| \leq \mathcal{O}((\ln t_2) (\sigma_{t_1})^{\rho'})$$

for some  $\rho' > 0$  which does not depend on  $\epsilon$  (recalling that  $t_1 < t_2$ ).

We may now return to (V.53). From iv), and the fact that the number of cells is  $N(t_1) \approx t_1^{3\epsilon}$ , summation over all cells yields

$$\sum_{j=1}^{N(t_1)} \| (\text{V.51}) - (\text{V.52}) \| \leq \mathcal{O}\left(\frac{\ln(t_2)}{t_1^\rho}\right) \quad (\text{V.85})$$

for some  $\rho > 0$ , provided that  $\beta$  is sufficiently large. This agrees with (V.37).

The sum  $\sum_{j=1}^{N(t_1)} \| (\text{V.57}) \|$  can be treated in a similar way.

*Analysis of Step b).* To show that the norm difference of the two vectors corresponding to the change (V.49) in (V.46) is bounded by the r.h.s. of (V.5), we argue similarly as for Step a), and we shall not reiterate the details. One again uses properties i) – iv) as in Step a).

*Analysis of Step c).* Finally, we prove that the difference of the vectors corresponding to (V.50) satisfies

$$\begin{aligned} & \left\| e^{iHt_1} \sum_{j=1}^{N(t_1)} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, t_1) \left[ W_{|\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j) W_{|\sigma_{t_2}}^*{}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) - \mathcal{I} \right] e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}, t_1)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} t_1} \psi_{j, \sigma_{t_1}}^{(t_1)} \right\|^2 \\ & \leq \mathcal{O}\left((\ln(t_2))^2 / t_1^\rho\right) \end{aligned} \quad (\text{V.86})$$

for some  $\rho > 0$ , where we define

$$W_{|\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j) := W_{\sigma_{t_1}}^*(\vec{v}_j) W_{\sigma_{t_2}}(\vec{v}_j), \quad (\text{V.87})$$

and likewise,

$$W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) := W^*_{\sigma_{t_2}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) W_{\sigma_{t_1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}). \tag{V.88}$$

We separately discuss the diagonal and off-diagonal contributions to (V.86) from the sum over cells in  $\mathcal{G}^{(t_1)}$ .

• *The diagonal terms in (V.86).* To bound the diagonal terms in (V.86), we use that, with  $\vec{v}_j \equiv \vec{\nabla} E^{\sigma_{t_1}}|_{\vec{P}=\vec{P}_j^*}$ ,

$$W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) := W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j) W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) \tag{V.89}$$

allows for an estimate similar to (V.82), where we now use that

$$\sup_{\vec{P} \in \mathcal{G}_j^{(t_1)}} |\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}} - \vec{v}_j| \leq \mathcal{O}(t_1^{-\epsilon(\frac{1}{4}-\delta'')}). \tag{V.90}$$

The latter follows from the Hölder regularity of  $\vec{\nabla} E_{\vec{P}}^\sigma$ , due to condition ( $\mathcal{J}2$ ) in Theorem III.1; see (III.14). Moreover, we use (V.83) to bound the expected photon number in the states  $\{\Psi_{\vec{P}}^{\sigma_{t_1}}\}$ .

Hereby we find that the sum of diagonal terms can be bounded by

$$\mathcal{O}(N(t_1) \|\psi_{j,\sigma_{t_1}}^{(t_1)}\|^2 (\ln t_2) t_1^{-\epsilon(\frac{1}{4}-\delta'')}) \leq \mathcal{O}(t_1^{-\rho} \ln t_2) \tag{V.91}$$

for some  $\rho > 0$ , using  $N(t_1) = \mathcal{O}(t_1^{3\epsilon})$ , and  $\|\psi_{j,\sigma_{t_1}}^{(t_1)}\|^2 = \mathcal{O}(t_1^{-3\epsilon})$ .

• *The off-diagonal terms in (V.86).* Next, we bound the off-diagonal terms in (V.86), corresponding to the inner product of vectors supported on cells  $j \neq l$  of the partition  $\mathcal{G}^{(t_1)}$ . Those are similar to the off-diagonal terms  $M_{l,j}^1(t, s)$  in (IV.6) that were discussed in detail previously. Correspondingly, we can apply the methods developed in Sect. IV.1, up to some modifications which we explain now.

Our goal is to prove the asymptotic orthogonality of the off-diagonal terms in (V.86).

We first of all prove the auxiliary result

$$\lim_{s \rightarrow +\infty} \|\vec{a}_{\sigma_{t_1}}(\vec{\eta}_{l,j})(s) W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) e^{-iH^{\sigma_{t_1}}s} \psi_{j,\sigma_{t_1}}^{(t_1)}\| = 0. \tag{V.92}$$

To this end, we compare

$$W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}) \mathbf{1}_{\mathcal{G}_j^{(t_1)}}(\vec{P}) \tag{V.93}$$

(where  $\mathbf{1}_{\mathcal{G}_j^{(t_1)}}$  is the characteristic function of the cell  $\mathcal{G}_j^{(t_1)}$ ) to its discretization:

1. We pick  $\bar{t}$  large enough such that  $\mathcal{G}^{(\bar{t})}$  is a sub-partition of  $\mathcal{G}^{(t)}$ ; in particular,  $\mathcal{G}_j^{(t)} = \sum_{m(j)=1}^M \mathcal{G}_{m(j)}^{(\bar{t})}$ , where  $M = \frac{N(\bar{t})}{N(t)}$ .
2. Furthermore, defining  $\vec{u}_{m(j)} := \vec{\nabla} E_{\vec{P}_{m(j)}^*}^{\sigma_{t_1}}$ , where  $\vec{P}_{m(j)}^*$  is the center of the cell  $\mathcal{G}_{m(j)}^{(\bar{t})}$ , we have, for  $\vec{P} \in \mathcal{G}_{m(j)}^{(\bar{t})}$ ,

$$|\vec{u}_{m(j)} - \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}| \leq C \left(\frac{1}{\bar{t}}\right)^{\epsilon(\frac{1}{4}-\delta'')}, \tag{V.94}$$

where  $C$  is uniform in  $t_1$ .

3. We define

$$\mathbb{W}_{\sigma_2}^{\sigma_1}(M) := \sum_{m(j)=1}^M W^*|_{\sigma_2}^{\sigma_1}(\vec{u}_{m(j)}) \mathbf{1}_{\mathcal{G}_{m(j)}^{(\vec{r})}}(\vec{P}) \quad (\text{V.95})$$

and rewrite the vector

$$\vec{a}_{\sigma_1}(\overline{\vec{\eta}_{l,j}})(s) W^*|_{\sigma_2}^{\sigma_1}(\vec{\nabla} E_{\vec{P}}^{\sigma_1}) e^{-iH^{\sigma_1} s} \psi_{j,\sigma_1}^{(t_1)} \quad (\text{V.96})$$

in (V.92) as

$$\begin{aligned} & \sum_{\lambda} \int_{\mathcal{B}_k \setminus \mathcal{B}_{\sigma_1}} d^3 k e^{-i\vec{k} \cdot \vec{x}} \left[ W^*|_{\sigma_2}^{\sigma_1}(\vec{\nabla} E_{\vec{P}}^{\sigma_1}) - \mathbb{W}_{\sigma_2}^{\sigma_1}(M) \right] \\ & \times \vec{\eta}_{l,j}(\vec{k}) \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* b_{\vec{k},\lambda} e^{i|\vec{k}|s} e^{-iE_{\vec{P}}^{\sigma_1} s} \psi_{j,\sigma_1}^{(t_1)} \end{aligned} \quad (\text{V.97})$$

$$\begin{aligned} & + \sum_{m(j)=1}^M W^*|_{\sigma_2}^{\sigma_1}(\vec{u}_{m(j)}) \sum_{\lambda} \int_{\mathcal{B}_k \setminus \mathcal{B}_{\sigma_1}} d^3 k \vec{\eta}_{l,j}(\vec{k}) \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda} e^{i|\vec{k}|s} \\ & \times e^{-iE_{\vec{P}}^{\sigma_1} s} \psi_{m(j),\sigma_1}^{(\vec{r})}. \end{aligned} \quad (\text{V.98})$$

We now observe that, at fixed  $t_1, t_2$ , (III.18) and the bound (V.94) imply that the vector in (V.97) converges to the zero vector as  $\vec{r} \rightarrow +\infty$ , uniformly in  $s$ . Moreover, the norm of the vector in (V.98) tends to zero, as  $s \rightarrow +\infty$ , at fixed  $\vec{r}$ . This proves (V.92).

The main difference between the vector corresponding to the  $j$ th cell in (V.86) and the similar expression in (IV.22) that is differentiated in  $s$  is the term proportional to the operator

$$W|_{\sigma_2}^{\sigma_1}(\vec{v}_j) W^*|_{\sigma_2}^{\sigma_1}(\vec{\nabla} E_{\vec{P}}^{\sigma_1}), \quad (\text{V.99})$$

which is absent in (IV.22). To control it, we first note that the Hamiltonian

$$\widehat{H}_{\vec{P}}^{\sigma_1} := \int^{\oplus} \widehat{H}_{\vec{P}}^{\sigma_1} d^3 P, \quad (\text{V.100})$$

where

$$\widehat{H}_{\vec{P}}^{\sigma_1} := \frac{\left( \vec{P} - \vec{P}_{>\sigma_1}^f + \alpha^{1/2} \vec{A}^{\sigma_1} \right)^2}{2} + H_{>\sigma_1}^f \quad (\text{V.101})$$

with

$$\vec{P}_{>\sigma_1}^f := \int_{\mathbb{R}^3 \setminus \mathcal{B}_{\sigma_1}} \vec{k} b_{\vec{k},\lambda}^* b_{\vec{k},\lambda} d^3 k, \quad (\text{V.102})$$

and

$$H_{>\sigma_1}^f := \int_{\mathbb{R}^3 \setminus \mathcal{B}_{\sigma_1}} |\vec{k}| b_{\vec{k},\lambda}^* b_{\vec{k},\lambda} d^3 k, \quad (\text{V.103})$$

satisfies

$$\widehat{H}_{\vec{p}}^{\sigma_{t_1}} \Psi_{\vec{p}}^{\sigma_{t_1}} = E_{\vec{p}}^{\sigma_{t_1}} \Psi_{\vec{p}}^{\sigma_{t_1}}, \quad (\text{V.104})$$

and

$$[W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j) W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}), \widehat{H}^{\sigma_{t_1}}] = 0. \quad (\text{V.105})$$

Using (V.105), the vector in (V.86) corresponding to the  $j$ th cell can be replaced by

$$e^{i\widehat{H}^{\sigma_{t_1}} s} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, s) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} e^{-i\widehat{H}^{\sigma_{t_1}} s} (W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) - \mathcal{I}) \psi_{j, \sigma_{t_1}}^{(t)}, \quad \text{for } s = t_1, \quad (\text{V.106})$$

where we recall that

$$W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) = W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j) W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}).$$

Similarly to our strategy in Sect. IV.1, we control  $\widehat{M}_{l,j}^1(t, s)$ ; see (IV.16). The derivative in  $s$  of the term proportional to  $W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}})$  in the  $j$ th cell vector has the form

$$\begin{aligned} & \frac{d}{ds} \left( e^{i\widehat{H}^{\sigma_{t_1}} s} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, s) e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} s} W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) \psi_{j, \sigma_{t_1}}^{(t_1)} \right) \\ &= i e^{i\widehat{H}^{\sigma_{t_1}} s} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, s) \alpha i [\widehat{H}^{\sigma_{t_1}}, \vec{x}] \cdot \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_{t_1}}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3 k \\ & \quad \times e^{-iE_{\vec{p}}^{\sigma_{t_1}} s} e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) \psi_{j, \sigma_{t_1}}^{(t)} \\ & + i e^{i\widehat{H}^{\sigma_{t_1}} s} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, s) \alpha^2 \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_{t_1}}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3 k \\ & \quad \cdot \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_{t_1}}} \vec{\Sigma}_{\vec{v}_j}(\vec{q}) \cos(\vec{q} \cdot \vec{x} - |\vec{q}|s) d^3 q \\ & \quad \times e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} s} W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) \psi_{j, \sigma_{t_1}}^{(t)} \\ & + i e^{i\widehat{H}^{\sigma_{t_1}} s} \mathcal{W}_{\kappa, \sigma_{t_1}}(\vec{v}_j, s) \frac{d\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)}{ds} \\ & \quad \times e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} s} W|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{v}_j; \vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) \psi_{j, \sigma_{t_1}}^{(t_1)}. \end{aligned} \quad (\text{V.107})$$

Due to the similarity of this expression with (IV.22) – (IV.29), we can essentially adopt the analysis presented in Sect. IV.1. The only difference here is the operator  $\widehat{H}^{\sigma_{t_1}}$  instead of  $H^{\sigma_{t_1}}$ , and the additional term involving the commutator

$$\left[ \chi h \left( \frac{\vec{x}}{s} \right), W^*|_{\sigma_{t_2}}^{\sigma_{t_1}}(\vec{\nabla} E_{\vec{p}}^{\sigma_{t_1}}) \right] \quad (\text{V.108})$$

applied to the one-particle state

$$e^{i\gamma_{\sigma_{t_1}}(\vec{v}_j, \nabla E_{\vec{p}}^{\sigma_{t_1}}, s)} e^{-iE_{\vec{p}}^{\sigma_{t_1}} s} \psi_{j, \sigma_{t_1}}^{(t_1)}. \quad (\text{V.109})$$



However, the latter tends to zero as  $s \rightarrow +\infty$ , at a rate  $\mathcal{O}(\frac{1}{s^\eta})$ , for some  $\epsilon$ -independent  $\eta > 0$ . This follows from the Hölder regularity of  $\vec{\nabla} E_{\vec{p}}^\sigma$  (condition  $(\mathcal{S}2)$  in Theorem III.1), and (III.18). Similarly, we treat the commutator (V.108) with the infrared tail (IV.39) in place of  $\chi_h(\frac{\vec{z}}{s})$  (and with  $\widehat{H}^{\sigma t_1}$  replacing  $H^{\sigma t_1}$ ). It is then straightforward to see that we arrive at (V.5).  $\square$

## VI. Scattering Subspaces and Asymptotic Observables

This section is dedicated to the following key constructions in the scattering theory for an infraparticle with the quantized electromagnetic field:

- i) We define scattering subspaces  $\mathcal{H}^{out/in}$  which are invariant under space-time translations, built from vectors  $\{\Psi_{h,\kappa}^{out/in}\}$ .

To this end, we first define a subspace,  $\mathcal{H}_\kappa^{out/in}$ , depending on the choice of a threshold frequency  $\kappa$  with the following purpose: Apart from photons with energy smaller than  $\kappa$ , this subspace contains states describing only a freely moving (asymptotic) electron.

Adding asymptotic photons to the states in  $\mathcal{H}_\kappa^{out/in}$ , we define spaces of scattering states of the system, where the asymptotic electron velocity is restricted to the region  $\{\vec{\nabla} E_{\vec{p}} : |\vec{p}| < \frac{1}{3}\}$ .

We note that the choice of  $\mathcal{H}_\kappa^{out/in}$  is not unique, except for the behavior of the dressing photon cloud in the infrared limit. It is useful because

- in the construction of the spaces of scattering states, we can separate “hard photons” from the photon cloud present in the states in  $\mathcal{H}_\kappa^{out/in}$ , which is not completely removable – each state in the scattering spaces contains an *infinite number* of asymptotic photons.
  - from the physical point of view, every experimental setup is limited by a threshold energy  $\kappa$  below which photons cannot be measured.
- ii) The construction of asymptotic algebras of observables,  $\mathcal{A}_{ph}^{out/in}$  and  $\mathcal{A}_{el}^{out/in}$ , related to the electromagnetic field and to the electron, respectively.

The asymptotic algebras are

- the Weyl algebra,  $\mathcal{A}_{ph}^{out/in}$ , associated to the asymptotic electromagnetic field;
- the algebra  $\mathcal{A}_{el}^{out/in}$  generated by smooth functions of compact support of the asymptotic velocity of the electron.

The two algebras  $\mathcal{A}_{ph}^{out/in}$  and  $\mathcal{A}_{el}^{out/in}$  commute. This is the mathematical counterpart of the asymptotic decoupling between the photons and the electron. This decoupling is, however, far from trivial: In fact, in contrast to a theory with a mass gap or a theory where the interaction with the soft modes of the field is turned off, the system is characterized by the emission of soft photons for arbitrarily long times.

In this respect, the asymptotic convergence of the electron velocity is a new conceptual result, obtained from the solution of the infraparticle problem in a concrete model, here non-relativistic QED. Furthermore, the emission of soft photons for arbitrarily long times is reflected in the representation of the asymptotic electromagnetic algebra, which is non-Fock but only locally Fock (see Sect. VI.2). More precisely, the representation can be decomposed on the spectrum of the asymptotic

velocity of the electron; for different values of the asymptotic velocity, the representations turn out to be inequivalent. Only for  $\vec{\nabla} E_{\vec{p}} = 0$ , the representation is Fock, otherwise they are coherent non-Fock. The coherent photon cloud, labeled by the asymptotic velocity, is the well known Bloch-Nordsieck cloud.

All the results and definition clearly hold for both the *out* and the *in*-states. We shall restrict ourselves to the discussion of *out*-states.

*VI.1. Scattering subspaces and “One-particle” subspaces with counter threshold  $\kappa$ .* In Sect. III, we have constructed a scattering state with electron wave function  $h$ , and a dressing cloud exhibiting the correct behavior in the limit  $\vec{k} \rightarrow 0$ , with maximal photon frequency  $\kappa$ .

To construct a space which is invariant under space-time translations, we may either focus on the vectors

$$e^{-i\vec{a}\cdot\vec{P}} e^{-iH\tau} \psi_{h,\kappa}^{out}, \quad (\text{VI.1})$$

or on the vectors obtained from

$$s - \lim_{t \rightarrow +\infty} e^{iHt} \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}^{\tau,\vec{a}}(\vec{v}_j, t) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, t)} e^{-iE_{\vec{p}}^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)}(\tau, \vec{a}), \quad (\text{VI.2})$$

where

$$\mathcal{W}_{\kappa,\sigma_t}^{\tau,\vec{a}}(\vec{v}_j, t) := \exp \left( \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\kappa} \setminus \mathcal{B}_{\sigma_t}} \frac{d^3 k}{\sqrt{|\vec{k}|}} \frac{\vec{v}_j \cdot \{\vec{\varepsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda}^* e^{-i|\vec{k}|(t+\tau)} e^{-i\vec{k}\cdot\vec{a}} - h.c.\}}{|\vec{k}|(1 - \hat{k} \cdot \vec{v}_j)} \right), \quad (\text{VI.3})$$

and

$$\psi_{j,\sigma_t}^{(t)}(\tau, \vec{a}) := \int_{\mathcal{G}_j^{(t)}} e^{-i\vec{a}\cdot\vec{P}} e^{-iE_{\vec{p}}^{\sigma_t} \tau} h(\vec{P}) \psi_{\vec{p}}^{\sigma_t} d^3 P. \quad (\text{VI.4})$$

Using Theorem III.2, one straightforwardly finds that

$$e^{-i\vec{a}\cdot\vec{P}} e^{-iH\tau} \psi_{h,\kappa}^{out} \quad (\text{VI.5})$$

$$= s - \lim_{t \rightarrow +\infty} e^{-i\vec{a}\cdot\vec{P}} e^{-iH\tau} e^{iHt} \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}^{\tau,\vec{a}}(\vec{v}_j, t) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, t)} e^{-iE_{\vec{p}}^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)} \quad (\text{VI.6})$$

$$= s - \lim_{t \rightarrow +\infty} e^{iHt} \sum_{j=1}^{N(t+\tau)} \mathcal{W}_{\kappa,\sigma_{t+\tau}}^{\tau,\vec{a}}(\vec{v}_j, t) e^{i\gamma_{\sigma_{t+\tau}}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_{t+\tau}}, t+\tau)} e^{-iE_{\vec{p}}^{\sigma_{t+\tau}} t} \psi_{j,\sigma_{t+\tau}}^{(t+\tau)}(\tau, \vec{a}). \quad (\text{VI.7})$$

The two limits (VI.2) and (VI.7) coincide; this follows straightforwardly from the line of analysis presented in the previous section.

Therefore, we can define the “one-particle” space corresponding to the frequency threshold  $\kappa$  as

$$\mathcal{H}_{\kappa}^{out/in} := \overline{\left\{ \bigvee \psi_{h,\kappa}^{out/in}(\tau, \vec{a}) : h(\vec{P}) \in C_0^1(\mathcal{S} \setminus \mathcal{B}_{r_a}), \tau \in \mathbb{R}, \vec{a} \in \mathbb{R}^3 \right\}}. \quad (\text{VI.8})$$

By construction,  $\mathcal{H}_{\kappa}^{out/in}$  is invariant under space-time translations.

General scattering states of the system can contain an arbitrarily large number of “hard” photons, i.e., photons with an energy above a frequency threshold, say for instance  $\kappa$ . One can construct such states based on  $\mathcal{H}_\kappa^{out/in}$  according to the following procedure.

We consider positive energy solutions of the form

$$F_t(\vec{y}) := \int \frac{d^3k}{2(2\pi)^3\sqrt{|k|}} \widehat{F}(\vec{k}) e^{-i|k|t+i\vec{k}\cdot\vec{y}} \quad (\text{VI.9})$$

of the free wave equation

$$\vec{\nabla}_{\vec{y}} \cdot \vec{\nabla}_{\vec{y}} F_t(\vec{y}) - \frac{\partial^2 F_t(\vec{y})}{\partial t^2} = 0, \quad (\text{VI.10})$$

which exhibit fast decay in  $|\vec{y}|$  for arbitrary fixed  $t$ , and where  $\widehat{F}(\vec{k}) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ .

We then construct vector-valued test functions

$$\vec{F}_t(\vec{y}) := \sum_{\lambda=\pm} \int \frac{d^3k}{2(2\pi)^3\sqrt{|k|}} \vec{\varepsilon}_{\vec{k},\lambda}^* \widehat{F}^\lambda(\vec{k}) e^{-i|k|t+i\vec{k}\cdot\vec{y}} \quad (\text{VI.11})$$

satisfying the wave equation (VI.10), with

$$\widehat{F}(\vec{k}) := \sum_{\lambda} \vec{\varepsilon}_{\vec{k},\lambda}^* \widehat{F}^\lambda(\vec{k}) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^3). \quad (\text{VI.12})$$

We set

$$\vec{A}(t, \vec{y}) := e^{iHt} \vec{A}(\vec{y}) e^{-iHt}; \quad (\text{VI.13})$$

here  $\vec{A}(\vec{y})$  is the expression in (II.11) with  $\Lambda = \infty$ . An asymptotic vector potential is constructed starting from LSZ ( $t \rightarrow \pm\infty$ ) limits of interpolating field operators,

$$\vec{A}[\vec{F}_t, t] := i \int \left( \vec{A}(t, \vec{y}) \cdot \frac{\partial \vec{F}_t(\vec{y})}{\partial t} - \frac{\partial \vec{A}(t, \vec{y})}{\partial t} \cdot \vec{F}_t(\vec{y}) \right) d^3y, \quad (\text{VI.14})$$

with  $\vec{F}_t$  as in (VI.11) for the negative-energy component, and with  $-\vec{F}_t$  for the positive-energy component. We define

$$\psi_{h,\vec{F}}^{out/in} := s - \lim_{t \rightarrow +/ -\infty} \psi_{h,\vec{F}}(t), \quad (\text{VI.15})$$

where

$$\psi_{h,\vec{F}}(t) := e^{i(\vec{A}[\vec{F}_t, t] - \vec{A}[\vec{F}_t, t])} \psi_{h,\kappa}(t). \quad (\text{VI.16})$$

Here,  $\psi_{h,\kappa}(t)$  approximates a vector  $\psi_{h,\kappa}^{out}$  in  $\mathcal{H}_\kappa^{out/in}$  (we temporarily drop the dependence on  $(\tau, \vec{a})$  in our notation). The existence of the limit in (VI.15) is a straightforward consequence of standard decay estimates for oscillating integrals under the assumption in (VI.12), combined with the propagation estimate (III.44).

Finally, we can define the scattering subspaces as

$$\mathcal{H}^{out/in} := \left\{ \sqrt{\int \psi_{h,\vec{F}}^{out/in}} : h(\vec{P}) \in C_0^1(\mathcal{S} \setminus \mathcal{B}_{r_\alpha}), \widehat{F} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^3) \right\}. \quad (\text{VI.17})$$

VI.2. *Asymptotic algebras and Bloch-Nordsieck coherent factor.* We now state some theorems concerning the construction of the asymptotic algebras. The proofs can be easily derived using the arguments developed in Sects. IV and V; for further details we refer to [26].

**Theorem VI.1.** *The functions  $f \in C_0^\infty(\mathbb{R}^3)$ , of the variable  $e^{iHt} \frac{\vec{x}}{t} e^{-iHt}$ , have strong limits for  $t \rightarrow \infty$  in  $\mathcal{H}^{out/in}$ , namely:*

$$s - \lim_{t \rightarrow +/\infty} e^{iHt} f\left(\frac{\vec{x}}{t}\right) e^{-iHt} \psi_{h, \vec{F}}^{out/in} =: \psi_{h, f_{\vec{V}_E, \vec{F}}}^{out/in}, \quad (\text{VI.18})$$

where  $f_{\vec{V}_E}(\vec{P}) := \lim_{\sigma \rightarrow 0} f(\vec{V} E_\sigma^{\vec{P}})$ .

The proof is obtained from an adaptation of the proof of Theorem A.3 in the Appendix. For the radiation field, we have the following result.

**Theorem VI.2.** *The LSZ Weyl operators*

$$\left\{ e^{i(\vec{A}[\vec{G}_t, t] - \vec{A}[\vec{G}_t, t])} : \widehat{G}^\lambda(\vec{k}) \in L^2(\mathbb{R}^3, (1 + |\vec{k}|^{-1}) d^3k), \lambda = \pm \right\} \quad (\text{VI.19})$$

have strong limits in  $\mathcal{H}^{out/in}$ :

$$\mathcal{W}^{out/in}(\vec{G}) := s - \lim_{t \rightarrow +/\infty} e^{i(\vec{A}[\vec{G}_t, t] - \vec{A}[\vec{G}_t, t])}. \quad (\text{VI.20})$$

The limiting operators are unitary, and have the following properties:

i)

$$\mathcal{W}^{out/in}(\vec{G}) \mathcal{W}^{out/in}(\vec{G}') = \mathcal{W}^{out/in}(\vec{G} + \vec{G}') e^{-\frac{\rho(\vec{G}, \vec{G}')}{2}}, \quad (\text{VI.21})$$

where

$$\rho(\vec{G}, \vec{G}') = 2i \operatorname{Im} \left( \sum_\lambda \int \widehat{G}^\lambda(\vec{k}) \overline{\widehat{G}'^\lambda(\vec{k})} d^3k \right). \quad (\text{VI.22})$$

ii) The mapping  $\mathbb{R} \ni s \rightarrow \mathcal{W}^{out/in}(s \vec{G})$  defines a strongly continuous, one parameter group of unitary operators.

iii)

$$e^{iH\tau} \mathcal{W}^{out/in}(\vec{G}) e^{-iH\tau} = \mathcal{W}^{out/in}(\vec{G}_{-\tau}), \quad (\text{VI.23})$$

where  $\vec{G}_{-\tau}$  is a freely evolved, vector-valued test function in the time  $-\tau$ .

Next, we define

- $\mathcal{A}_{el}^{out/in}$  as the norm closure of the (abelian) \*algebra generated by the limits in (VI.18).
- $\mathcal{A}_{ph}^{out/in}$  as the norm closure of the \*algebra generated by the unitary operators in (VI.20).

From (VI.21) and (VI.23), we conclude that  $\mathcal{A}_{ph}^{out/in}$  is the Weyl algebra associated to a free radiation field. Moreover, from straightforward approximation arguments applied to the generators, we can prove that the two algebras,  $\mathcal{A}_{el}^{out/in}$  and  $\mathcal{A}_{ph}^{out/in}$ , commute.

Moreover, we can next establish key properties of the representation  $\Pi$  of the algebras  $\mathcal{A}_{ph}^{out/in}$  for the concrete model at hand that confirm structural features derived in [19] under general assumptions.

To study the infrared features of the representation of  $\mathcal{A}_{ph}^{out/in}$ , it suffices to analyze the expectation of the generators  $\{\mathcal{W}^{out}(\vec{G})\}$  of the algebra with respect to arbitrary states of the form  $\psi_{h,\kappa}^{out}$ ,

$$\langle \psi_{h,\kappa}^{out}, \mathcal{W}^{out}(\vec{G}) \psi_{h,\kappa}^{out} \rangle \tag{VI.24}$$

$$= \lim_{t \rightarrow +\infty} \sum_{j=1, l=1}^{N(t)} \left\langle e^{i\gamma_{\sigma_l}(\vec{v}_l, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{l, \sigma_l}^{(t)}, \right. \tag{VI.25}$$

$$\left. \mathcal{W}_{\kappa, \sigma_l}^*(\vec{v}_l, t) e^{i(\vec{A}[\vec{G}_t, 0] - \vec{A}[\vec{G}_t, 0])} \mathcal{W}_{\kappa, \sigma_l}(\vec{v}_j, t) e^{i\gamma_{\sigma_l}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j, \sigma_l}^{(t)} \right\rangle.$$

In the step passing from (VI.24) to (VI.25), we use Theorem III.2. One infers from the arguments developed in Sect. IV.1 that the sum of the off-diagonal terms,  $l \neq j$ , vanishes in the limit. Therefore,

$$\begin{aligned} \text{(VI.25)} = \lim_{t \rightarrow +\infty} \sum_{j=1}^{N(t)} & \left\langle e^{i\gamma_{\sigma_l}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j, \sigma_l}^{(t)}, \right. \\ & \left. e^{i(\vec{A}[\vec{G}_t, 0] - \vec{A}[\vec{G}_t, 0])} e^{Q_{\vec{v}_j}(\vec{G})} e^{i\gamma_{\sigma_l}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_l}, t)} e^{-iE_{\vec{P}}^{\sigma_l} t} \psi_{j, \sigma_l}^{(t)} \right\rangle, \end{aligned} \tag{VI.26}$$

where

$$Q_{\vec{u}}(\vec{G}) := 2i \operatorname{Re} \left( \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\kappa}} \widehat{G}^{\lambda}(\vec{k}) \frac{\vec{u} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}^*}{|\vec{k}|^{\frac{3}{2}} (1 - \vec{u} \cdot \vec{k})} d^3 k \right). \tag{VI.27}$$

After solving an ODE analogous to (IV.9), we find that the diagonal terms yield

$$\langle \psi_{h,\kappa}^{out}, \mathcal{W}^{out}(\vec{G}) \psi_{h,\kappa}^{out} \rangle = \int e^{-\frac{C_{\vec{G}}}{2}} e^{Q_{\vec{v}_E}(\vec{G})} |h(\vec{P})|^2 d^3 P, \tag{VI.28}$$

where

$$C_{\vec{G}} := \int |\widehat{G}(\vec{k})|^2 d^3 k. \tag{VI.29}$$

Here, we also use that  $\vec{v}_j \equiv \vec{\nabla} E_{\vec{P}_j}^{\sigma_j}$ , combined with the convergence  $\vec{\nabla} E_{\vec{P}}^{\sigma_l} \rightarrow \vec{\nabla} E_{\vec{P}}$  (as  $t \rightarrow \infty$  and  $\vec{P} \in S$ ).

Now, we can reproduce the following results in [19]: The representation  $\Pi(\mathcal{A}_{ph}^{out/in})$  is given by a direct integral on the spectrum of the operator  $\vec{v}_{as}^{out/in}$  in  $\mathcal{H}^{out/in}$ , defined by

$$f(\vec{v}_{as}^{out/in}) := s - \lim_{t \rightarrow +/\infty} e^{iHt} f\left(\frac{\vec{x}}{t}\right) e^{-iHt} \tag{VI.30}$$

for any  $f \in C_0^\infty(\mathbb{R}^3)$ , of mutually inequivalent, irreducible representations. These representations are coherent non-Fock for values  $\vec{v}_{as}^{out/in} \neq 0$ . The coherent factors, labeled by  $\vec{v}_{as}^{out/in}$ , are

$$\alpha^{\frac{1}{2}} \frac{\vec{v}_{as}^{out/in} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}(1 - \vec{v}_{as}^{out/in} \cdot \widehat{\vec{k}})} \quad \text{and} \quad \alpha^{\frac{1}{2}} \frac{\vec{v}_{as}^{out/in} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}(1 - \vec{v}_{as}^{out/in} \cdot \widehat{\vec{k}})}, \tag{VI.31}$$

for the annihilation and the creation part,  $a_{\vec{k},\lambda}^{out/in}$  and  $a_{\vec{k},\lambda}^{out/in*}$ , respectively.

The representation  $\Pi(\mathcal{A}_{ph}^{out/in})$  is locally Fock in momentum space. This property is equivalent to the following one:

For any  $\kappa > 0$ , and  $\widehat{G}^\kappa \in C_0^\infty(\mathbb{R}^3 \setminus \mathcal{B}_\kappa; \mathbb{C}^3)$ , the operator

$$\vec{A}[-\overline{G}_t^\kappa, t] \tag{VI.32}$$

annihilates vectors of the type  $\psi_{h,\kappa}^{out}$  in the limit  $t \rightarrow +\infty$ , i.e.,

$$\lim_{t \rightarrow +\infty} \vec{A}[-\overline{G}_t^\kappa, t] \psi_{h,\kappa}^{out} = 0. \tag{VI.33}$$

To prove this, we first consider Theorem III.2, then

$$\lim_{t \rightarrow +\infty} \vec{A}[-\overline{G}_t^\kappa, t] \psi_{h,\kappa}^{out} = \lim_{t \rightarrow +\infty} \vec{A}[-\overline{G}_t^\kappa, t] \psi_{h,\kappa}(t). \tag{VI.34}$$

Next, we rewrite the vector

$$\vec{A}[-\overline{G}_t^\kappa, t] \psi_{h,\kappa}(t) = e^{iHt} \vec{A}[-\overline{G}_t^\kappa, 0] \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, t) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, t)} e^{-iE_{\vec{p}}^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)} \tag{VI.35}$$

as

$$- \int_t^{+\infty} \frac{d}{ds} \{ e^{iHs} \vec{A}[-\overline{G}_s^\kappa, 0] \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)} \} ds \tag{VI.36}$$

$$+ \lim_{s \rightarrow +\infty} e^{iHs} \sum_{j=1}^{N(t)} \mathcal{W}_{\kappa,\sigma_t}(\vec{v}_j, s) \vec{A}[-\overline{G}_s^\kappa, 0] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j,\sigma_t}^{(t)}. \tag{VI.37}$$

The integral in (VI.36), and the limit in (VI.37) exist. To see this, it is enough to follow the procedure in Sect. IV.1, taking into account that the operator

$$\vec{A}[-\overline{G}_s^\kappa, 0] \frac{1}{(H^{\sigma_t} + i)} [H^{\sigma_t}, \vec{x}] \tag{VI.38}$$

is bounded, uniformly in  $t$  and  $s$ . The limit (VI.37) vanishes at fixed  $t$  because of condition (S4) in Theorem III.1. Therefore we finally conclude that the limit (VI.34) vanishes.

*Liénard-Wiechert fields generated by the charge.* Now we briefly explain how to obtain the result stated in (III.79). The assertion is obvious for the longitudinal degrees of

freedom; see the definition of  $F_{\mu\nu}$  in (III.80). For the transverse degrees of freedom, we argue as follows. Similarly to the treatment of (VI.24), we arrive at a sum over the diagonal terms,

$$\begin{aligned} & \lim_{t \rightarrow \pm\infty} \left\langle \psi_{h, \vec{F}}^{out/in}, e^{iHt} \int d^3y F_{\mu\nu}^{tr}(0, \vec{y}) \tilde{\delta}_\Lambda(\vec{y} - \vec{x} - \vec{d}) e^{-iHt} \psi_{h, \vec{F}}^{out/in} \right\rangle \\ &= \lim_{t \rightarrow \pm\infty} \sum_{j=1}^{N(t)} \left\langle \psi_{j, \sigma_t}^{\sigma_t}, \int d^3y F_{\mu\nu}^{tr}(0, \vec{y}) \tilde{\delta}_\Lambda(\vec{y} - \vec{x} - \vec{d}) \psi_{j, \sigma_t}^{\sigma_t} \right\rangle. \end{aligned}$$

Then, one uses the pull-through formula as in Lemma 6.1 in [10], and Proposition 5.1 in [10] which identifies the infrared coherent factor by showing that

$$\left| \left\langle \Psi_{\vec{P}}^\sigma, b_{\vec{k}, \lambda} \Psi_{\vec{P}}^\sigma \right\rangle + \alpha^{\frac{1}{2}} \frac{\mathbf{1}_{\sigma, \Lambda}(\vec{k})}{|\vec{k}|^{\frac{1}{2}}} \frac{1}{|\vec{k}| - \vec{k} \cdot \vec{\nabla} E_{\vec{P}}^\sigma} \vec{\varepsilon}_{\vec{k}, \lambda} \cdot \vec{\nabla} E_{\vec{P}}^\sigma \right| \leq \alpha^{1/2} C |\vec{k}|^{-1} \quad (\text{VI.39})$$

for  $\vec{k} \rightarrow 0$ . These ingredients imply that

$$\begin{aligned} |\vec{d}|^2 \left| \lim_{t \rightarrow \pm\infty} \left\{ \sum_{j=1}^{N(t)} \left\langle \psi_{j, \sigma_t}^{\sigma_t}, \int d^3y F_{\mu\nu}^{tr}(0, \vec{y}) \tilde{\delta}_\Lambda(\vec{y} - \vec{x} - \vec{d}) \psi_{j, \sigma_t}^{\sigma_t} \right\rangle \right. \right. \\ \left. \left. - \sum_{j=1}^{N(t)} \int_{\mathcal{G}_j^{(t)}} |h(\vec{P})|^2 \left\langle \psi_{\vec{P}}^{\sigma_t}, \psi_{\vec{P}}^{\sigma_t} \right\rangle \left( F_{\mu\nu}^{\vec{\nabla} E_{\vec{P}}^{\sigma_t}} \right)^{tr} (0, \vec{d}) d^3P \right\} \right| \leq \mathcal{O}(|\vec{d}|^{-1/2}), \end{aligned} \quad (\text{VI.40})$$

which vanishes in the limit  $|\vec{d}| \rightarrow \infty$ , as asserted in (III.79).

## Appendix A.

In the Appendix, we present detailed proofs of auxiliary results used in Sect. III.

**Lemma A.1.** *The following estimates hold for  $\vec{P} \in \mathcal{S}$ :*

(i) For  $t_2 > t_1 \gg 1$ ,

$$|\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, (\sigma_{t_2})^{-\frac{1}{\theta}}) - \gamma_{\sigma_{t_2}}(\vec{v}_{l(j)}, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, (\sigma_{t_2})^{-\frac{1}{\theta}})| \leq \mathcal{O}(|\vec{v}_j - \vec{v}_{l(j)}|), \quad (\text{A.1})$$

where  $\vec{v}_j \equiv \vec{\nabla} E_{\vec{P}_j^{\sigma_{t_1}}}^{\sigma_{t_1}}$  and  $\vec{v}_{l(j)} \equiv \vec{\nabla} E_{\vec{P}_{l(j)}^{\sigma_{t_2}}}^{\sigma_{t_2}}$ .

(ii) For  $t_2 > t_1 \gg 1$ ,

$$|\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, t_1) - \gamma_{\sigma_{t_1}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_1}}, t_1)| \leq \mathcal{O} \left( [(\sigma_{t_1})^{\frac{1}{2}(1-\delta)} t_1^{1-\theta} + t_1 \sigma_{t_1}] \right). \quad (\text{A.2})$$

(iii) For  $s, t \gg 1$  and  $\vec{q} \in \{\vec{q} : |\vec{q}| < s^{(1-\theta)}\}$ ,

$$|\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s) - \gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p} + \frac{\vec{q}}{s}}^{\sigma_t}, s)| \leq \mathcal{O}(s^{-\frac{\theta}{4}(1-\delta'')} s^{(1-\theta)}), \tag{A.3}$$

whenever  $\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)$  is defined.

*Proof.* The proofs only require the definition of the phase factor, and some elementary integral estimates, using conditions (S1) and (S2) in Theorem III.1.  $\square$

**Lemma A.2.** For  $s \geq t \gg 1$ , the estimates

$$\sup_{\vec{x} \in \mathbb{R}^3} \left| \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \Sigma_{\vec{v}_j}^l(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \right| \leq \mathcal{O}\left(\frac{|\ln \sigma_t|}{s}\right), \tag{A.4}$$

$$\sup_{\vec{x} \in \mathbb{R}^3} \left| \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t^S}} \Sigma_{\vec{v}_j}^l(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \chi_h\left(\frac{\vec{x}}{s}\right) \right| \leq \mathcal{O}\left(\frac{t^\theta}{s^2}\right), \tag{A.5}$$

hold, where

$$\Sigma_{\vec{v}_j}^l(\vec{k}) := 2 \sum_{l'} (\delta_{l,l'} - \frac{k^l k^{l'}}{|\vec{k}|^2}) v_j^{l'} \frac{1}{|\vec{k}|^2 (1 - \widehat{k} \cdot \vec{v}_j)}, \tag{A.6}$$

and where  $\sigma_t := t^{-\beta}$ ,  $\sigma_t^S := \tau^{-\theta}$ , with  $\beta > 1$ ,  $0 < \theta < 1$ . Moreover,  $\chi_h(\vec{y}) = 0$  for  $|\vec{y}| \leq \frac{1}{2} \nu_{min}$  and  $|\vec{y}| \geq \frac{1+\nu_{max}}{2}$  with  $0 < \nu_{min} < \nu_{max} < 1$  (see (III.15)).

*Proof.* To prove the estimate (A.4), we consider the variable  $\vec{x}$  first in the set

$$\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < (1 - \rho)s, \quad 0 < \rho < 1\}.$$

We denote by  $\theta_{\vec{k}}$  the angle between  $\vec{x}$  and  $\vec{k}$ . Integration with respect to  $|\vec{k}|$  yields

$$\left| \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \Sigma_{\vec{v}_j}^l(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \right| \tag{A.7}$$

$$= \left| \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k}) \frac{\sin(\kappa \widehat{k} \cdot \vec{x} - \kappa s) - \sin(t^{-\beta} \widehat{k} \cdot \vec{x} - t^{-\beta} s)}{\widehat{k} \cdot \vec{x} - s} d\Omega_{\widehat{k}} \right| \tag{A.8}$$

$$\leq \frac{2}{\rho s} \int |\widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})| d\Omega_{\widehat{k}}, \tag{A.9}$$

where  $\widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k}) := |\vec{k}|^2 \Sigma_{\vec{v}_j}^l(\vec{k})$ .

For  $\vec{x}$  in the set

$$\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| > (1 - \rho)s, \quad 0 < \rho < 1\},$$

we integrate by parts with respect to  $\cos \theta_{\vec{k}}$ , and observe that the two functions

$$\widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k}) \quad \text{and} \quad \frac{d(\widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k}))}{d \cos \theta_{\vec{k}}} \tag{A.10}$$



belong to  $L^1(S^2; d\Omega_{\vec{k}})$ . This yields

$$\int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \Sigma_{\vec{v}_j}^l(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k \tag{A.11}$$

$$= - \int_{\sigma_t}^\kappa \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})|_{\theta_{\vec{k}}=\pi} \frac{\sin(|\vec{k}| |\vec{x}| + |\vec{k}|s)}{|\vec{k}| |\vec{x}|} d|\vec{k}| d\phi_{\vec{k}} \tag{A.12}$$

$$- \int_{\sigma_t}^\kappa \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})|_{\theta_{\vec{k}}=0} \frac{\sin(|\vec{k}| |\vec{x}| - |\vec{k}|s)}{|\vec{k}| |\vec{x}|} d|\vec{k}| d\phi_{\vec{k}} \tag{A.13}$$

$$- \int_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\sigma_t}} \frac{d(\widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k}))}{d \cos \theta_{\vec{k}}} \frac{\sin(\vec{k} \cdot \vec{x} - |\vec{k}|s)}{|\vec{k}| |\vec{x}|} \frac{d^3k}{|\vec{k}|^2}. \tag{A.14}$$

The absolute values of (A.12), (A.13), and (A.14), are all bounded above by  $\mathcal{O}\left(\frac{|\ln \sigma_t|}{s - \rho s}\right)$ , as one easily verifies. This establishes (A.4), uniformly in  $\vec{x} \in \mathbb{R}^3$ .

To prove (A.5), we consider  $\vec{x}$  in a set of the form

$$\{\vec{x} \in \mathbb{R}^3 : (1 - \rho')s > |\vec{x}| > (1 - \rho)s, \quad 0 < \rho' < \rho < 1\}. \tag{A.15}$$

We apply integration by parts with respect to  $|\vec{k}|$  in (A.12), (A.13), and (A.14) in the case  $\sigma_t^S$ . As an example, we get for (A.12),

$$(A.12) = \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})|_{\theta_{\vec{k}}=\pi} \frac{\cos(\kappa (|\vec{x}| + s))}{\kappa (|\vec{x}| + s) |\vec{x}|} d\phi_{\vec{k}} \tag{A.16}$$

$$- \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})|_{\theta_{\vec{k}}=\pi} \frac{\cos(t^{-\theta} (|\vec{x}| + s))}{t^{-\theta} (|\vec{x}| + s) |\vec{x}|} d\phi_{\vec{k}} \tag{A.17}$$

$$+ \int_{\sigma_t^S}^\kappa \int \widehat{\Sigma}_{\vec{v}_j}^l(\widehat{k})|_{\theta_{\vec{k}}=\pi} \frac{\cos(|\vec{k}| |\vec{x}| + |\vec{k}|s)}{|\vec{k}|^2 |\vec{x}| (|\vec{x}| + s)} d\phi_{\vec{k}} d|\vec{k}|. \tag{A.18}$$

Since  $\vec{x}$  is assumed to be an element of (A.15), it follows that the bound (A.5) holds for (A.12). In the same manner, one obtains a similar bound for (A.13) and (A.14).  $\square$

**Theorem A.3.** *For  $\theta < 1$  sufficiently close to 1, and  $s \geq t$ , the propagation estimate*

$$\left\| \chi_h \left( \frac{\vec{x}}{s} \right) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t, s})} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \right. \tag{A.19}$$

$$\left. - \chi_h(\vec{\nabla} E_{\vec{p}}^{\sigma_t}) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t, s})} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \right\|$$

$$\leq c \frac{1}{s^\nu} \frac{1}{t^{\frac{3\epsilon}{2}}} |\ln(\sigma_t)| \tag{A.20}$$

holds, where  $\nu > 0$  is independent of  $\epsilon$ .

*Proof.* Since the detailed proof of a closely related result is given in Theorem A2 of [26], we only sketch the argument.

Expressing  $\chi_h$  (which we assume to be real) in terms of its Fourier transform  $\widehat{\chi}_h$ , start from the bound

$$\| \int d^3q \widehat{\chi}_h(\vec{q}) (e^{-i\vec{q}\cdot\vec{\nabla} E_{\vec{p}}^{\sigma_t}} - e^{-i\vec{q}\cdot\frac{\vec{x}}{s}}) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-i E_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \| \quad (\text{A.21})$$

$$\leq \| \int d^3q \widehat{\chi}_h(\vec{q}) (e^{-i\vec{q}\cdot\vec{\nabla} E_{\vec{p}}^{\sigma_t}} - e^{i(E_{\vec{p}}^{\sigma_t} - E_{\vec{p}+\frac{\vec{q}}{s}}^{\sigma_t})s}) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)} \| \quad (\text{A.22})$$

$$+ \| \int d^3q \widehat{\chi}_h(\vec{q}) e^{i(E_{\vec{p}}^{\sigma_t} - E_{\vec{p}+\frac{\vec{q}}{s}}^{\sigma_t})s} (e^{-i\vec{q}\cdot\frac{\vec{x}}{s}} - 1) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)} \| . \quad (\text{A.23})$$

We split the integration domains of (A.22) and (A.23) into the two regions

$$I_+ := \{\vec{q} : |\vec{q}| > s^{1-\theta}\} \quad \text{and} \quad I_- := \{\vec{q} : |\vec{q}| \leq s^{1-\theta}\}. \quad (\text{A.24})$$

In both (A.22) and (A.23), the contribution to the integral from  $I_+$  is controlled by the decay properties of  $\widehat{\chi}(\vec{q})$ , and one easily derives the bound in (A.20). For the contributions to (A.22) from the integral over  $I_-$ , the existence of the gradient of the energy, the Hölder property in  $\vec{P}$  of the gradient, and the decay properties of  $\widehat{\chi}(\vec{q})$  are enough to produce the bound in (A.20).

To control (A.23), we note that the two vectors

$$\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} \quad \text{and} \quad \widehat{\Psi}_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} := e^{-i\vec{q}\cdot\frac{\vec{x}}{s}} \Psi_{\vec{p}}^{\sigma_t} \quad (\text{A.25})$$

belong to the same fiber space  $\mathcal{H}_{\vec{p}-\frac{\vec{q}}{s}}$ , and that, as vectors in Fock space,  $\widehat{\Psi}_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t}$  and  $\Psi_{\vec{p}}^{\sigma_t}$  coincide, i.e.,

$$I_{\vec{p}-\frac{\vec{q}}{s}}(e^{-i\vec{q}\cdot\frac{\vec{x}}{s}} \Psi_{\vec{p}}^{\sigma_t}) \equiv I_{\vec{p}}(\Psi_{\vec{p}}^{\sigma_t}). \quad (\text{A.26})$$

We split and estimate (A.23)| $I_-$ , i.e., (A.23) where the integration domain is restricted to  $I_-$ , by

$$(\text{A.23})|_{I_-} = \| \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} - E_{\vec{p}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \widehat{\Psi}_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \| \quad (\text{A.27})$$

$$- \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}}^{\sigma_t} - E_{\vec{p}+\frac{\vec{q}}{s}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \Psi_{\vec{p}}^{\sigma_t} d^3 P d^3 q \| \quad (\text{A.28})$$

$$\leq \| \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} - E_{\vec{p}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \widehat{\Psi}_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \| \quad (\text{A.29})$$

$$- \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} - E_{\vec{p}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \|$$

$$+ \| \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} - E_{\vec{p}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \| \quad (\text{A.30})$$

$$- \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}}^{\sigma_t} - E_{\vec{p}+\frac{\vec{q}}{s}}^{\sigma_t})s} h_{\vec{p}-\frac{\vec{q}}{s}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t}, s)} \Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \|$$

$$+ \| \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} - E_{\vec{p}}^{\sigma_t})s} h_{\vec{p}-\frac{\vec{q}}{s}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t}, s)} \Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t} d^3 P d^3 q \|$$

$$- \int_{I_-} \widehat{\chi}_h(\vec{q}) \int_{\Gamma_j^{(t)}} e^{i(E_{\vec{p}}^{\sigma_t} - E_{\vec{p}+\frac{\vec{q}}{s}}^{\sigma_t})s} h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \Psi_{\vec{p}}^{\sigma_t} d^3 P d^3 q \| . \quad (\text{A.31})$$

The terms (A.29), (A.30), and (A.31) can be bounded by

$$(A.29) \leq \int_{I_-} |\widehat{\chi}_h(\vec{q})| \left[ \int_{\Gamma_j^{(t)}} |h_{\vec{p}}|^2 \|I_{\vec{p}}(\Psi_{\vec{p}}^{\sigma_t}) - I_{\vec{p}-\frac{\vec{q}}{s}}(\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\|_{\mathcal{F}}^2 d^3 P \right]^{\frac{1}{2}} d^3 q, \quad (A.32)$$

$$(A.30) \leq \int_{I_-} |\widehat{\chi}_h(\vec{q})| \left[ \int_{\Gamma_j^{(t)}} |\Delta_{\frac{\vec{q}}{s}}(h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)})|^2 \|I_{\vec{p}-\frac{\vec{q}}{s}}(\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\|_{\mathcal{F}}^2 d^3 P \right]^{\frac{1}{2}} d^3 q, \quad (A.33)$$

and

$$(A.31) \leq \int_{I_-} |\widehat{\chi}_h(\vec{q})| \left[ \int_{O_{\frac{\vec{q}}{s}}^j} |h_{\vec{p}}|^2 \|I_{\vec{p}}(\Psi_{\vec{p}}^{\sigma_t})\|_{\mathcal{F}}^2 d^3 P \right]^{\frac{1}{2}} d^3 q, \quad (A.34)$$

where

$$\Delta_{\frac{\vec{q}}{s}}(h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}) := h_{\vec{p}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} - h_{\vec{p}-\frac{\vec{q}}{s}} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t}, s)}, \quad (A.35)$$

and  $O_{\frac{\vec{q}}{s}}^j := (\Gamma_j^{(t)} \cup \Gamma_j^{(t), \frac{\vec{q}}{s}}) \setminus (\Gamma_j^{(t)} \cap \Gamma_j^{(t), \frac{\vec{q}}{s}})$ , where  $\Gamma_j^{(t), \frac{\vec{q}}{s}}$  is the translate by  $\frac{\vec{q}}{s}$  of the cell  $\Gamma_j^{(t)}$ .

Using (A.3), the  $C^1$ -regularity of  $h_{\vec{p}}$ , and the definition of  $I_-$ , one readily shows that the terms (A.33), (A.34) satisfy the bound (A.20), as desired.

To estimate (A.32), we use the inequality

$$\|I_{\vec{p}}(\Psi_{\vec{p}}^{\sigma_t}) - I_{\vec{p}-\frac{\vec{q}}{s}}(\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\|_{\mathcal{F}} \quad (A.36)$$

$$\leq \|I_{\vec{p}}(W_{\sigma_t}(\vec{\nabla} E_{\vec{p}}^{\sigma_t})\Psi_{\vec{p}}^{\sigma_t}) - I_{\vec{p}-\frac{\vec{q}}{s}}(W_{\sigma_t}(\vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\|_{\mathcal{F}} \quad (A.37)$$

$$+ \|I_{\vec{p}-\frac{\vec{q}}{s}}\left((W_{\sigma_t}^*(\vec{\nabla} E_{\vec{p}}^{\sigma_t}) - W_{\sigma_t}^*(\vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})) W_{\sigma_t}(\vec{\nabla} E_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t})\Psi_{\vec{p}-\frac{\vec{q}}{s}}^{\sigma_t}\right)\|_{\mathcal{F}}, \quad (A.38)$$

where it is clear that

$$W_{\sigma_t}(\vec{\nabla} E_{\vec{p}}^{\sigma_t})\Psi_{\vec{p}}^{\sigma_t} = \Phi_{\vec{p}}^{\sigma_t}. \quad (A.39)$$

Moreover, we use properties (S2), (S5) in Theorem III.1, where we recall that

(S2) Hölder regularity in  $\vec{P} \in \mathcal{S}$  uniformly in  $\sigma \geq 0$  holds in the sense of

$$\|\Phi_{\vec{P}}^{\sigma} - \Phi_{\vec{P}+\Delta\vec{P}}^{\sigma}\|_{\mathcal{F}} \leq C_{\delta'} |\Delta\vec{P}|^{\frac{1}{4}-\delta'} \quad (A.40)$$

and

$$|\vec{\nabla} E_{\vec{P}}^{\sigma} - \vec{\nabla} E_{\vec{P}+\Delta\vec{P}}^{\sigma}| \leq C_{\delta''} |\Delta\vec{P}|^{\frac{1}{4}-\delta''}, \quad (A.41)$$

for any  $0 < \delta'' < \delta' < \frac{1}{4}$ , with  $\vec{P}, \vec{P} + \Delta\vec{P} \in \mathcal{S}$ , and where  $C_{\delta'}$  and  $C_{\delta''}$  are finite constants depending on  $\delta'$  and  $\delta''$ , respectively.

We can bound (A.37) by use of (A.40).

In order to bound (A.38), we recall the definition of the Weyl operator

$$W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) := \exp \left( \alpha^{\frac{1}{2}} \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} d^3k \frac{\vec{\nabla} E_{\vec{P}}^\sigma}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}, \sigma}(\vec{k})} \cdot (\vec{\varepsilon}_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^* - h.c.) \right), \quad (\text{A.42})$$

and we note that

$$(\text{A.38}) \leq c \left| \vec{\nabla} E_{\vec{P}}^{\sigma_t} - \vec{\nabla} E_{\vec{P} - \frac{\vec{q}}{s}}^{\sigma_t} \right| \mathcal{R}_t \left( \mathcal{R}_t + \left( \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_t}} d^3k \| b_{\vec{k}, \lambda} \Phi_{\vec{P} - \frac{\vec{q}}{s}}^\sigma \|_{\mathcal{F}}^2 \right)^{\frac{1}{2}} \right) \quad (\text{A.43})$$

from a simple application of the Schwarz inequality, where

$$\mathcal{R}_t := \left( \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_t}} \frac{d^3k}{|\vec{k}|^3} \right)^{\frac{1}{2}} = \mathcal{O}(|\ln \sigma_t|^{\frac{1}{2}}). \quad (\text{A.44})$$

Moreover, we have

$$\sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_t}} d^3k \| b_{\vec{k}, \lambda} \Phi_{\vec{P} - \frac{\vec{q}}{s}}^\sigma \|_{\mathcal{F}}^2 \leq c |\ln \sigma_t|, \quad (\text{A.45})$$

which is derived similarly as (V.74).

From Hölder continuity of  $\vec{\nabla} E_{\vec{P}}^\sigma$  in  $\vec{P}$ , (A.41), we obtain a contribution to the upper bound on (A.38) which exhibits a power law decay in  $s$ .

We conclude that (A.32) is bounded by (A.20), as claimed.  $\square$

*Remark.* By a similar procedure, one finds that for  $t_2 \geq s \geq t_1$ ,

$$\begin{aligned} & \left\| \chi_h \left( \frac{\vec{x}}{s} \right) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} s} \psi_{j, \sigma_{t_2}}^{(t_1)} \right. \\ & \left. - \chi_h(\vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_2}}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_{t_2}}, s)} e^{-iE_{\vec{P}}^{\sigma_{t_2}} s} \psi_{j, \sigma_{t_2}}^{(t_1)} \right\| \leq c \frac{1}{s^v} \frac{|\ln(\sigma_{t_2})|}{t_1^{3/2}}. \end{aligned} \quad (\text{A.46})$$

Analogous extensions hold for the estimates in the next theorem.

**Theorem A.4.** *Both*

$$\begin{aligned} & \left\| \int_t^{+\infty} e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \left[ \mathcal{J}_{|\sigma_t}^{\sigma_t s}(s) \chi_h \left( \frac{\vec{x}}{s} \right) - \frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} \right] \right. \\ & \left. \times e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} ds \right\| \end{aligned} \quad (\text{A.47})$$

and

$$\begin{aligned} & \left\| \int_t^{+\infty} e^{iH\sigma_t s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \left\{ \frac{d\widehat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right. \right. \\ & \left. \left. - \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} e^{-iE_{\vec{P}}^{\sigma_t} s} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\} ds \right\| \end{aligned} \quad (\text{A.48})$$

are bounded by

$$c \frac{1}{t^\eta} |\ln(\sigma_t)|^2 t^{-\frac{3\epsilon}{2}}, \quad (\text{A.49})$$

where  $\eta > 0$  is  $\epsilon$ -independent.  $\mathcal{J}_{\sigma_t^S}^S(s)$ ,  $\frac{d\hat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds}$ , and  $\frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)}{ds}$  are defined in (IV.38), (IV.39), and (IV.7) – (IV.8), respectively.

*Proof.* We recall from (IV.38) that for  $\sigma_s^S \geq \sigma_t$ ,

$$\mathcal{J}_{\sigma_t^S}^S(s) = \alpha i [H^{\sigma_t}, \vec{x}] \cdot \int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \frac{1}{H^{\sigma_t} + i} \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k,$$

where  $\sigma_s^S := \frac{1}{s^\theta}$  is the slow cut-off, and from (IV.39),

$$\begin{aligned} \frac{d\hat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} &:= \alpha e^{-iH^{\sigma_t}s} \frac{1}{H^{\sigma_t} + i} \frac{d\{e^{iH^{\sigma_t}s} \vec{x}_h(s) e^{-iH^{\sigma_t}s}\}}{ds} e^{iH^{\sigma_t}s} \\ &\cdot \int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) d^3k. \end{aligned} \quad (\text{A.50})$$

For  $s$  such that  $\sigma_s^S \leq \sigma_t$  the expressions (A.47) and (A.48) are identically zero. By unitarity of  $e^{iH^{\sigma_t}s}$  and  $\mathcal{W}_{\sigma_t}(\vec{v}_j, s)$ , we can replace the part in the integrand of (A.47) proportional to  $\mathcal{J}_{\sigma_t^S}^S(s)$  by

$$\begin{aligned} e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha i [H^{\sigma_t}, \vec{x}] \frac{1}{H^{\sigma_t} + i} \chi_h\left(\frac{\vec{x}}{s}\right) \cdot \int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) \\ \times e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t}s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)}, \end{aligned} \quad (\text{A.51})$$

up to a term which yields an integral bounded in norm by

$$\frac{1}{t^\eta} |\ln(\sigma_t)|^2 t^{-\frac{3\epsilon}{2}}, \quad (\text{A.52})$$

for some  $\eta > 0$  and independent of  $\epsilon$ .

To justify this step, we exploit the fact that the operator

$$i[H^{\sigma_t}, \vec{x}] \frac{1}{H^{\sigma_t} + i} \quad (\text{A.53})$$

is bounded. Moreover, we are applying the propagation estimate

$$\begin{aligned} \left\| \left\{ \int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{x} - |\vec{k}|s) d^3k - \int_{\mathcal{B}_{\sigma_s^S} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) d^3k \right\} \right. \\ \left. \cdot e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t}s} (E_{\vec{p}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \right\| \leq c \frac{1}{s^{1+\nu}} \frac{1}{t^{\frac{3\epsilon}{2}}} |\ln(\sigma_t)|, \end{aligned} \quad (\text{A.54})$$

for some  $\nu > 0$ , which is similar to (A.19). To obtain the upper bound, we exploit the fact that due to the slow cut-off  $\sigma_s^S = s^{-\theta}$ ,  $\theta > 0$ , in  $\mathcal{J}_{\sigma_t^S}^S(s)$ , the upper integration bound in the radial part of the momentum variables vanishes in the limit  $s \rightarrow \infty$ .

We note that we have to assume  $\theta < 1$  as required in (IV.37), in order to use the result in Lemma A.2.

Next, we approximate (A.51) by

$$e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha e^{-iH^{\sigma_t}s} \frac{1}{H^{\sigma_t} + i} \frac{d\vec{x}(s)}{ds} \chi_h\left(\frac{\vec{x}(s)}{s}\right) \cdot \int_{\mathcal{B}_{\sigma_t^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) \cdot (E_{\vec{P}}^{\sigma_t} + i) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)}, \tag{A.55}$$

where  $\vec{x}(s) := e^{iH^{\sigma_t}s} \vec{x} e^{-iH^{\sigma_t}s}$ . To pass from (A.51) to (A.55), we have used

$$\frac{d\vec{x}(s)}{ds} \frac{1}{H^{\sigma_t} + i} = \frac{1}{H^{\sigma_t} + i} \frac{d\vec{x}(s)}{ds} - \frac{1}{H^{\sigma_t} + i} \frac{d[\vec{x}(s), H^{\sigma_t}]}{ds} \frac{1}{H^{\sigma_t} + i}, \tag{A.56}$$

and we have noticed that the term containing

$$\frac{1}{H^{\sigma_t} + i} \frac{d[\vec{x}(s), H^{\sigma_t}]}{ds} \frac{1}{H^{\sigma_t} + i} \tag{A.57}$$

can be neglected because an integration by parts shows that the corresponding integral is bounded in norm by  $\frac{1}{\nu} |\ln(\sigma_t)|^2 t^{-\frac{3\epsilon}{2}}$  for some  $\nu > 0$  and  $\epsilon$ -independent. This uses

$$\sup_{\vec{P} \in \mathcal{S}} \left| \int_{\mathcal{B}_{\sigma_t^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) \right| \leq \mathcal{O}\left(\frac{|\ln(\sigma_t)|}{s}\right) \tag{A.58}$$

and

$$\sup_{\vec{P} \in \mathcal{S}} \left| \frac{d}{ds} \int_{\mathcal{B}_{\sigma_t^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) \right| \leq \mathcal{O}\left(\frac{1}{s^{1+\theta}}\right), \tag{A.59}$$

which can be derived as in Lemma A.2.

To bound the integral corresponding to (A.55), we note that up to a term whose integral is bounded in norm by (A.49), one can replace  $\frac{d\vec{x}(s)}{ds} \chi_h\left(\frac{\vec{x}(s)}{s}\right)$  by

$$\frac{d}{ds} \left( e^{iH^{\sigma_t}s} \vec{x}_h(s) e^{-iH^{\sigma_t}s} \right), \tag{A.60}$$

where  $\vec{x}_h(s) := \vec{x} \chi_h\left(\frac{\vec{x}}{s}\right)$ , with  $\chi_h(\vec{y})$  defined as in Sect. IV.1. This is possible because

$$\frac{d}{ds} \left( e^{iH^{\sigma_t}s} \vec{x}_h(s) e^{-iH^{\sigma_t}s} \right) \tag{A.61}$$

$$= -e^{iH^{\sigma_t}s} \vec{x} \left[ \frac{\vec{x}}{s^2} \cdot \vec{\nabla} \chi_h\left(\frac{\vec{x}}{s}\right) \right] e^{-iH^{\sigma_t}s} \tag{A.62}$$

$$+ e^{iH^{\sigma_t}s} i[H^{\sigma_t}, \vec{x}] \chi_h\left(\frac{\vec{x}}{s}\right) e^{-iH^{\sigma_t}s} \tag{A.63}$$

$$+ e^{iH^{\sigma_t}s} \frac{\vec{x}}{s} \left[ \vec{\nabla} \chi_h\left(\frac{\vec{x}}{s}\right) \cdot \frac{i[H^{\sigma_t}, \vec{x}]}{2} \right] e^{-iH^{\sigma_t}s} \tag{A.64}$$

$$+ e^{iH^{\sigma_t}s} \frac{\vec{x}}{s} \left[ \frac{i[H^{\sigma_t}, \vec{x}]}{2} \cdot \vec{\nabla} \chi_h\left(\frac{\vec{x}}{s}\right) \right] e^{-iH^{\sigma_t}s}, \tag{A.65}$$

where (A.63) corresponds to  $\frac{d\vec{x}(s)}{ds} \chi_h(\frac{\vec{x}(s)}{s})$ . Moreover, we use the fact that the vector operator  $\frac{1}{H^{\sigma_t+i}} i[H^{\sigma_t}, \vec{x}]$  is bounded, and apply the propagation estimate (A.19) to  $\frac{\vec{x}^i \vec{x}^j}{s^2} \nabla^j \chi_h(\frac{\vec{x}}{s})$  and to  $\frac{\vec{x}^i}{s} \nabla^j \chi_h(\frac{\vec{x}}{s})$  with appropriate modifications (see (A.72) and recall that  $\vec{\nabla} \chi_h(\vec{\nabla} E_{\vec{P}}^{\sigma_t}) = 0$  for  $\vec{P} \in \text{supp } h$ ).

We observe that

$$e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha e^{-iH^{\sigma_t}s} \frac{1}{H^{\sigma_t+i}} \frac{d(e^{iH^{\sigma_t}s} \vec{x}_h(s) e^{-iH^{\sigma_t}s})}{ds} \cdot \int_{\mathcal{B}_{\sigma_s^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cdot \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) (E_{\vec{P}}^{\sigma_t} + i) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)} \quad (\text{A.66})$$

corresponds to

$$e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \left[ \frac{d\hat{\gamma}_{\sigma_t}(\vec{v}_j, \frac{\vec{x}}{s}, s)}{ds} \right] e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)}. \quad (\text{A.67})$$

This immediately implies (A.47).

To prove (A.48), we need to control the integral

$$\int_I^{\bar{s}} e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) e^{-iH^{\sigma_t}s} \frac{\alpha}{H^{\sigma_t+i}} \frac{d(e^{iH^{\sigma_t}s} \vec{x}_h(s) e^{-iH^{\sigma_t}s})}{ds} \cdot \int_{\mathcal{B}_{\sigma_s^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cdot \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} ds \quad (\text{A.68})$$

for  $\bar{s} \rightarrow +\infty$ . An integration by parts with respect to  $s$  yields

$$e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \frac{\alpha}{H^{\sigma_t+i}} \vec{x}_h(s) \cdot \int_{\mathcal{B}_{\sigma_s^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \times \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) e^{-iE_{\vec{P}}^{\sigma_t}s} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} \Big|_{\bar{s}} \quad (\text{A.69})$$

$$- \int_I^{\bar{s}} \left\{ \frac{d}{ds} (e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) e^{-iH^{\sigma_t}s}) \right\} e^{iH^{\sigma_t}s} \frac{\alpha}{H^{\sigma_t+i}} \times \vec{x}_h(s) \cdot \int_{\mathcal{B}_{\sigma_s^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) \times e^{-iE_{\vec{P}}^{\sigma_t}s} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} ds \quad (\text{A.70})$$

$$- \int_I^{\bar{s}} e^{iH^{\sigma_t}s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \frac{\alpha}{H^{\sigma_t+i}} \times \vec{x}_h(s) \cdot \left\{ \frac{d}{ds} \left[ \int_{\mathcal{B}_{\sigma_s^s} \setminus \mathcal{B}_{\sigma_t}} d^3k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_t} s - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{P}}^{\sigma_t}, s)} \right] \right\} \times e^{-iE_{\vec{P}}^{\sigma_t}s} (E_{\vec{P}}^{\sigma_t} + i) \psi_{j, \sigma_t}^{(t)} ds \quad (\text{A.71})$$

Here, we notice that

$$\vec{x}_h(s) = \vec{x} \chi_h\left(\frac{\vec{x}}{s}\right) = -is \int \vec{\nabla} \widehat{\chi}_h(\vec{q}) e^{-i\vec{q} \cdot \frac{\vec{x}}{s}} d^3q. \quad (\text{A.72})$$

Furthermore, the operator

$$-i \int \vec{\nabla} \widehat{\chi}_h(\vec{q}) e^{-i\vec{q} \cdot \frac{\vec{x}}{s}} d^3 q \quad (\text{A.73})$$

tends to

$$-i \int \vec{\nabla} \widehat{\chi}_h(\vec{q}) e^{-i\vec{q} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t}} d^3 q \quad (\text{A.74})$$

for  $s \rightarrow \infty$ , if it is applied to the vectors

$$e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)}, \quad (\text{A.75})$$

or

$$\int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} d^3 k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)}, \quad (\text{A.76})$$

or

$$\left\{ \frac{d}{ds} \left[ \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} d^3 k \vec{\Sigma}(\vec{k}, \vec{v}_j) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \right] \right\} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)}. \quad (\text{A.77})$$

The rate of convergence of the corresponding expression in (A.48) is bounded by (A.49).

Therefore, we can replace expressions (A.69), (A.70), and (A.71) by

$$e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha s \vec{\nabla} E_{\vec{p}}^{\sigma_t} \quad (\text{A.78})$$

$$\cdot \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} d^3 k \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} \Big|_t^{\bar{s}} \\ - \int_t^{\bar{s}} ds \left\{ \frac{d}{ds} (e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) e^{-iH^{\sigma_t} s}) \right\} \alpha s \vec{\nabla} E_{\vec{p}}^{\sigma_t} \quad (\text{A.79})$$

$$\cdot \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) d^3 k e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \psi_{j, \sigma_t}^{(t)} \\ - \int_t^{\bar{s}} e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \alpha s \vec{\nabla} E_{\vec{p}}^{\sigma_t} \quad (\text{A.80})$$

$$\cdot \left\{ \frac{d}{ds} \left[ \int_{\mathcal{B}_{\sigma_t} \setminus \mathcal{B}_{\sigma_t}} \vec{\Sigma}_{\vec{v}_j}(\vec{k}) \cos(\vec{k} \cdot \vec{\nabla} E_{\vec{p}}^{\sigma_t} s - |\vec{k}|s) d^3 k e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} \right] \right\} \\ \times e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)} ds.$$

Recalling the definition of the phase factor in (IV.7)-(IV.8), the sum of the expressions (A.78), (A.79), and (A.80) can be written compactly as

$$\int_t^{\bar{s}} ds e^{iH^{\sigma_t} s} \mathcal{W}_{\kappa, \sigma_t}(\vec{v}_j, s) \frac{d\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}, s)}{ds} e^{i\gamma_{\sigma_t}(\vec{v}_j, \vec{\nabla} E_{\vec{p}}^{\sigma_t}, s)} e^{-iE_{\vec{p}}^{\sigma_t} s} \psi_{j, \sigma_t}^{(t)}, \quad (\text{A.81})$$

after an integration by parts.

This implies the asserted bound for (A.47).  $\square$



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