Commun. Math. Phys. 293, 37–83 (2010)
Digital Object Identifier (DOI) 10.1007/s00220-009-0940-z

Communications in
Mathematical Physics

Strichartz Estimates on Schwarzschild Black Hole Backgrounds★

Jeremy Marzuola[1], **Jason Metcalfe**[2], **Daniel Tataru**[3], **Mihai Tohaneanu**[4]

[1] Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA
[2] Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
[3] Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
[4] Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067, USA

Received: 9 April 2008 / Accepted: 13 August 2009
Published online: 4 November 2009 –

Abstract: We study dispersive properties for the wave equation in the Schwarzschild space-time. The first result we obtain is a local energy estimate. This is then used, following the spirit of [29], to establish global-in-time Strichartz estimates. A considerable part of the paper is devoted to a precise analysis of solutions near the trapping region, namely the photon sphere.

1. Introduction

The aim of this article is to contribute to the understanding of the global-in-time dispersive properties of solutions to wave equations on Schwarzschild black hole backgrounds. Precisely, we consider two robust ways to measure dispersion, namely the local energy estimates and the Strichartz estimates.

Let us begin with the local energy estimates. For solutions to the constant coefficient wave equation in 3 + 1 dimensions,

$$\Box u = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1,$$

we have the original estimates of Morawetz [33], [1]

$$\int_0^t \int_{\mathbb{R}^3} \frac{1}{|x|} |\nabla\!\!\!/ u|^2(t,x)\,dt\,dx \lesssim \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2, \tag{1.1}$$

where $\nabla\!\!\!/$ denotes the angular derivative. To prove this one multiplies the wave equation by the multiplier $(\partial_r + \frac{1}{r})u$ and integrates by parts. Within dyadic spatial regions one can also control u, $\partial_t u$ and $\partial_r u$. Precisely, we have the local energy estimates

★ The authors were supported in part by the NSF grants DMS0354539 and DMS0301122.

[1] There is another estimate commonly referred to as a Morawetz estimate. This corresponds to using the multiplier $(t^2 + r^2)\partial_t + 2tr\partial_r$. We will reserve the term Morawetz estimate for (1.1) and shall call the latter estimate the Morawetz conformal estimate.

$$R^{-\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}\times B(0,R))} + R^{-\frac{3}{2}}\|u\|_{L^2(\mathbb{R}\times B(0,R))} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}. \tag{1.2}$$

See for instance [20,22,40–42].

One can also consider the inhomogeneous problem,

$$\Box u = f, \qquad u(0) = u_0, \quad u_t(0) = u_1. \tag{1.3}$$

In view of (1.2) we define the local energy space LE_M for the solution u by

$$\|u\|_{LE_M} = \sup_{j\in\mathbb{Z}} \left[2^{-\frac{j}{2}}\|\nabla u\|_{L^2(A_j)} + 2^{-\frac{3j}{2}}\|u\|_{L^2(A_j)}\right], \tag{1.4}$$

where

$$A_j = \mathbb{R} \times \{2^j \le |x| \le 2^{j+1}\}.$$

For the inhomogeneous term f we introduce a dual type norm

$$\|f\|_{LE_M^*} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|f\|_{L^2(A_j)}.$$

Then we have:

Theorem 1.1. *The solution u to* (1.3) *satisfies the following estimate:*

$$\|u\|_{LE_M} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \|f\|_{LE_M^*}. \tag{1.5}$$

One may ask whether similar bounds also hold for perturbations of the Minkowski space-time. Indeed, in the case of small long range perturbations the same bounds as above were established very recently by two of the authors, see [30, Prop. 2.2] or [28, (2.23)] (with no obstacle, $\Omega = \emptyset$). See also [1,27] for related local energy estimates for small perturbations of the d'Alembertian. For large perturbations one faces additional difficulties, due on one hand to trapping for large frequencies and on the other hand to eigenvalues and resonances for low frequencies. The Schwarzschild space-time, considered in the present paper, is a very interesting example of a large perturbation of the Minkowski space-time, where trapping causes significant difficulties.

The Schwarzschild space-time $\mathcal{M}$ is a spherically symmetric solution to Einstein's equations with an additional Killing vector field K, which models the exterior of a massive spherically symmetric body. Factoring out the $\mathbb{S}^2$ component it can be represented via the Penrose diagram in Fig. 1. The radius r of the $\mathbb{S}^2$ spheres is intrinsically determined and is a smooth function on $\mathcal{M}$ which has a single critical point at the center. The regions I and I' represent the exterior of the black hole, respectively its symmetric twin, and are characterized by the relation $r > 2M$. We can represent I as

$$I = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$$

with a metric whose line element is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2, \tag{1.6}$$

where $d\omega^2$ is the measure on the sphere $\mathbb{S}^2$. The Killing vector field K is given by $K = \partial_t$, which is time-like within I. The differential dt is intrinsic, but the function t is only defined up to translations on I.

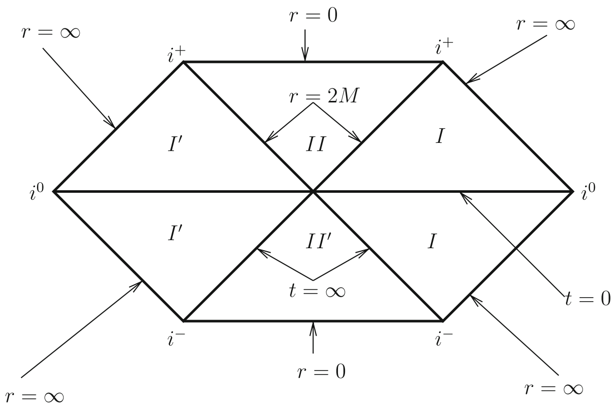

Fig. 1. The Penrose diagram for the Kruskal extension of the Schwarzschild solution

The regions II and II' represent the black hole, respectively its symmetric twin, the white hole, and are characterized by the relation $r < 2M$. The same metric as in (1.6) can be used. The Killing vector field K is still given by $K = \partial_t$, which is now space-like. Light rays can enter the black hole but not leave it. By symmetry light rays can leave the white hole but not enter it.

The surface $r = 2M$ is called the event horizon. While the singularity at $r = 0$ is a true metric singularity, we note that the apparent singularity at $r = 2M$ is merely a coordinate singularity. Indeed, denote

$$r^* = r + 2M \log(r - 2M) - 3M - 2M \log M,$$

so that

$$dr^* = \left(1 - \frac{2M}{r}\right)^{-1} dr, \qquad r^*(3M) = 0$$

and set $v = t + r^*$. Then in the (r, v, ω) coordinates the metric in region I is expressed in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\omega^2,$$

which extends analytically into the black hole region $I + II$. In particular, given a choice of the function t in region I, this uniquely determines the function t in the region II via the same change of coordinates.

In a symmetric fashion we set $w = t - r^*$. Then in the (r, w, ω) coordinates the metric is expressed in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dw^2 - 2dwdr + r^2 d\omega^2,$$

which extends analytically into the white hole region $I + II'$.

One can also introduce global nonsingular coordinates by rewriting the metric in the Kruskal-Szekeres coordinate system,

$$v' = e^{\frac{v}{4M}}, \qquad w' = -e^{-\frac{w}{4M}}.$$

However, this is of less interest for our purposes here. Further information on the Schwarzschild space can be found in a number of excellent texts. We refer the interested reader to, e.g., [18,31 and 51].

As far as the results in this paper are concerned, for large r the Schwarzschild space-time can be viewed as a small perturbation of the Minkowski space-time. The difficulties in our analysis are caused by the dynamics for small r, where trapping occurs. The presence of trapped rays, i.e. rays which do not escape either to infinity or to the singularity $r = 0$, are known to be a significant obstacle to proving local energy, dispersive, and Strichartz estimates and, in some cases, are known to necessitate a loss of regularity. See, e.g., [10 and 37].

There are two places where trapping occurs on the Schwarzschild manifold. The first is the surface $r = 3M$ which is called the photon sphere. Null geodesics which are initially tangent to the photon sphere will remain on the surface for all times. Microlocally the energy is preserved near such periodic orbits. However what allows for local energy estimates near the photon sphere is the fact that these periodic orbits are hyperbolic. The second is at the event horizon $r = 2M$, where the trapped geodesics are the vertical ones in the (r, v, ω) coordinates. However, this second family of trapped rays turns out to cause no difficulty in the decay estimates since in the high frequency limit the energy decays exponentially along it as $v \to \infty$. This is due to the fact that the frequency decays exponentially along the Hamilton flow, and in the physics literature it is well-known as the red shift effect.

To describe the decay properties of solutions to the wave equation in the Schwarzschild space, it is convenient to use coordinates which make good use of the Killing vector field and are nonsingular along the event horizon. The (r, v, ω) coordinates would satisfy these requirements. However the level sets of v are null surfaces, which would cause some minor difficulties. This is why in $I + II$ we introduce the function $\tilde{v}$ defined by

$$\tilde{v} = v - \mu(r),$$

where μ is a smooth function of r. In the $(\tilde{v}, r, \omega)$ coordinates the metric has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{v}^2 + 2\left(1 - \left(1 - \frac{2M}{r}\right)\mu'(r)\right) d\tilde{v}dr + \left(2\mu'(r) - \left(1 - \frac{2M}{r}\right)(\mu'(r))^2\right) dr^2 + r^2 d\omega^2.$$

On the function μ we impose the following two conditions:

(i) $\mu(r) \geq r^*$ for $r > 2M$, with equality for $r > 5M/2$.
(ii) The surfaces $\tilde{v} = const$ are space-like, i.e.

$$\mu'(r) > 0, \qquad 2 - \left(1 - \frac{2M}{r}\right)\mu'(r) > 0.$$

The first condition (i) insures that the $(r, \tilde{v}, \omega)$ coordinates coincide with the (r, t, ω) coordinates in $r > 5M/2$. This is convenient but not required for any of our results. What is important is that in these coordinates the metric is asymptotically flat as $r \to \infty$. In the proof of the Strichartz estimates, it is also required that $\mu'(r) = \left(1 - \frac{2M}{r}\right)^{-1}$ near $r = 3M$, which in other words says that we can work in the (r, t) coordinates near the photon sphere. However, this may be merely an artifact of our method.

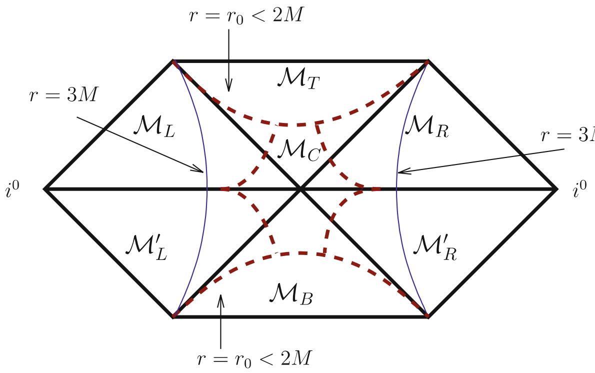

Fig. 2. The Schwarzschild space partition represented on the Penrose diagram

We introduce a symmetric function $\tilde{v}_1$ in $I' + II$, as well as the functions $\tilde{w}$ and $\tilde{w}_1$ in $I + II'$, respectively $I' + II'$. Given a parameter $0 < r_0 < 2M$ we partition the Schwarzschild space into seven regions

$$\mathcal{M} = \mathcal{M}_R \cup \mathcal{M}_L \cup \mathcal{M}'_R \cup \mathcal{M}'_L \cup \mathcal{M}_T \cup \mathcal{M}_C \cup \mathcal{M}_B$$

as in Fig. 2. The right/left top/bottom regions are

$$\mathcal{M}_R = \{\tilde{v} \geq 0,\ r \geq r_0\} \subset I + II, \qquad \mathcal{M}_L = \{\tilde{v}_1 \geq 0,\ r \geq r_0\} \subset I' + II,$$
$$\mathcal{M}'_R = \{\tilde{w} \leq 0,\ r \geq r_0\} \subset I + II', \quad \mathcal{M}'_L = \{\tilde{w}_1 \leq 0,\ r \geq r_0\} \subset I' + II',$$

the top and bottom regions are

$$\mathcal{M}_T = \{r < r_0\} \cap II, \qquad \mathcal{M}_B = \{r < r_0\} \cap II',$$

and the central region $\mathcal{M}_C$ is the remainder of $\mathcal{M}$. Moreover, define

$$\Sigma_R^- = \mathcal{M}_R \cap \{\tilde{v} = 0\},$$
$$\Sigma_R^+ = \mathcal{M}_R \cap \{r = r_0\}.$$

and similarly for the other regions.

In what follows we consider the Cauchy problem

$$\Box_g \phi = f, \qquad \phi_{|\Sigma_0} = \phi_0, \qquad \tilde{K}\phi_{|\Sigma_0} = \phi_1, \tag{1.7}$$

where for convenience we choose the initial surface Σ_0 to be the horizontal surface of symmetry

$$\Sigma_0 = \{t = 0\} \cap (I + I')$$

and $\tilde{K}$ is smooth, everywhere timelike and equals K on Σ_0 outside $\mathcal{M}_C$. Observe that we cannot use K on all of Σ_0 since it is degenerate at the center (i.e. on the bifurcate sphere).

Equation (1.7) can be solved as follows:

(i) Solve the equation in $\mathcal{M}_C$ with Cauchy data on Σ_0. Since $\mathcal{M}_C$ is compact and has forward and backward space-like boundaries, this is a purely local problem.

(ii) Solve the equation in $\mathcal{M}_R$ with Cauchy data on Σ_R^-. The forward boundary of $\mathcal{M}_R$ is Σ_R^+, which is space-like. This is the most interesting part, where we are interested in the decay properties as $\tilde{v} \to \infty$. In a similar manner solve the equation in $\mathcal{M}_L$, $\mathcal{M}'_R$ and $\mathcal{M}'_L$.
(iii) Solve the equation in $\mathcal{M}_T$ with initial data on the space-like surface $\Sigma_T = \{r = r_0\} \cap II$. Here one can track the solution up to the singularity and encounter a mix of local and global features. This part of the analysis is not pursued in the present article.

A significant role in our analysis is played by the Killing vector field K, which in the $(r, \tilde{v})$ coordinates equals $\partial_{\tilde{v}}$. This is time-like outside the black hole but space-like inside it. Furthermore, it is degenerate at the center. Using the Killing vector field outside the black hole we obtain a conserved energy $E_0[\phi]$ for solutions ϕ to the homogeneous equation $\Box_g \phi = 0$. On surfaces $t = const$ in the (r, t) coordinates the energy $E_0[\phi](t)$ has the form

$$E_0[\phi] = \int_{S^2} \int_{2M}^{\infty} \left[\left(1 - \frac{2M}{r}\right)^{-1} (\partial_t \phi)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r \phi)^2 + |\nabla\!\!\!/ \phi|^2 \right] r^2 dr d\omega. \quad (1.8)$$

Since the vector field K is degenerate at the center, so is the corresponding energy E_0 at $r = 2M$. Hence it would be natural to replace it with a nondegenerate energy, which on the initial surface Σ_0 can be expressed as

$$E[\phi](\Sigma_0) = \int_{S^2} \int_{2M}^{\infty} \left[\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}} (\partial_t \phi)^2 + \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} (\partial_r \phi)^2 + |\nabla\!\!\!/ \phi|^2 \right] r^2 dr d\omega. \quad (1.9)$$

Unfortunately this is no longer conserved, and this is one of the difficulties which we face in our analysis. We remark that a related form of a nondegenerate energy expression was introduced in [14] and proved to be bounded in the exterior region on surfaces $t = const$.

Part of the novelty of our approach is to prove bounds not only in the exterior region, but also inside the event horizon. This is natural if one considers the fact that the singularity at $r = 2M$ is merely a removable coordinate singularity. In order to do this, it is no longer suitable to measure the evolution of the energy on the surfaces $t = const$ (see below). Thus the above energy $E[\phi](\Sigma_0)$ is relegated to a secondary role here and is used only to measure the size of the initial data.

A priori the energy $E[\phi](t)$ of ϕ only determines its Cauchy data at time t modulo constants. However, in what follows we implicitly assume that ϕ decays at ∞, in which case ϕ can also be estimated via a Hardy-type inequality,

$$\int \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} r^{-2} \phi^2 \, r^2 dr d\omega \lesssim \int \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} (\partial_r \phi)^2 \, r^2 dr d\omega. \quad (1.10)$$

This is proved in a standard manner; the details are left to the reader.

We shall now further describe our main estimates in the region $\mathcal{M}_R$: the local energy decay, the WKB analysis which yields a local energy decay with only a logarithmic loss, and finally the Strichartz estimates.

For the initial energy on Σ_R^- we use

$$E[\phi](\Sigma_R^-) = \int_{\Sigma_R^-} \left(|\partial_r \phi|^2 + |\partial_{\tilde{v}} \phi|^2 + |\nabla\!\!\!/ \phi|^2 \right) r^2 dr d\omega.$$

For the final energy on Σ_R^+ we set

$$E[\phi](\Sigma_R^+) = \int_{\Sigma_R^+} \left(|\partial_r \phi|^2 + |\partial_{\tilde{v}} \phi|^2 + |\nabla\!\!\!/ \phi|^2 \right) r_0^2 d\tilde{v} d\omega.$$

We also track the energy on the space-like slices $\tilde{v} = const$,

$$E[\phi](\tilde{v}_0) = \int_{\mathcal{M}_R \cap \{\tilde{v} = \tilde{v}_0\}} \left(|\partial_r \phi|^2 + |\partial_{\tilde{v}} \phi|^2 + |\nabla\!\!\!/ \phi|^2 \right) r^2 dr d\omega.$$

Thus $E[\phi](\Sigma_R^-) = E[\phi](0)$.

For the local energy estimates one may first consider a direct analogue of the Minkowski bound (1.5). Unfortunately such a bound is hopeless due to the trapping which occurs at $r = 3M$. Instead, for our first result we define a weaker preliminary local energy space LE_0 with norm

$$\|\phi\|_{LE_0}^2 = \int_{\mathcal{M}_R} \left(\frac{1}{r^2} |\partial_r \phi|^2 + \left(1 - \frac{3M}{r}\right)^2 \left(\frac{1}{r^2} |\partial_{\tilde{v}} \phi|^2 + \frac{1}{r} |\nabla\!\!\!/ \phi|^2 \right) + \frac{1}{r^4} \phi^2 \right) r^2 dr d\tilde{v} d\omega. \tag{1.11}$$

Compared to the LE_M norm we note the power loss in the angular and $\tilde{v}$ derivatives at $r = 3M$. The LE_0 norm is also weaker than LE_M as $r \to \infty$, but this is merely for convenience.

At the same time we would like to also consider the inhomogeneous problem $\Box_g \phi = f$. To measure the inhomogeneous term f, we introduce the norm LE_0^*, which is stronger than LE_M^*:

$$\|f\|_{LE_0^*}^2 = \int_{\mathcal{M}_R} \left(1 - \frac{3M}{r}\right)^{-2} r^2 f^2 \, r^2 dr d\tilde{v} d\omega. \tag{1.12}$$

Again the important difference is at $r = 3M$. Our first local energy estimate is the following:

Theorem 1.2. *Let ϕ solve the inhomogeneous wave equation $\Box_g \phi = f$ on the Schwarzschild manifold. Then we have*

$$E[\phi](\Sigma_R^+) + \sup_{\tilde{v} \geq 0} E[\phi](\tilde{v}) + \|\phi\|_{LE_0}^2 \lesssim E[\phi](\Sigma_R^-) + \|f\|_{LE_0^*}^2. \tag{1.13}$$

Here we made no effort to optimize the weights at $r = 3M$ and $r = \infty$. This is done later in the paper. On the other hand the above estimate follows from a relatively simple application of the classical positive commutator method. The advantage of having even such a weaker estimate is that it is sufficient in order to allow localization near the interesting regions $r = 3M$ and $r = \infty$, which can then be studied in greater detail using specific tools.

The first related results regarding the solution of the wave equation on Schwarzschild backgrounds were obtained in [50 and 24] which proved uniform boundedness in region I (including the event horizon). The first pointwise decay result (without, however, a rate of decay) was obtained in [49]. Heuristics from [36] suggest that solutions to the wave equation in the Schwarzschild case should locally decay like v^{-3}. For spherically symmetric data a $v^{-3+\epsilon}$ decay rate was obtained in [16], and under the additional assumption of the initial data vanishing near the event horizon, the v^{-3} decay rate was proved in [23]. In general the best known decay rate, proved in [14], is v^{-1} (see also [7]). We also refer the reader to [38], where optimal pointwise decay rates for each spherical harmonic are established for a closely related problem.

Estimates related to (1.13) were first proved in [25] for radially symmetric Schrödinger equations on Schwarzschild backgrounds. In [2–4], those estimates are extended to allow for general data for the wave equation. The same authors, in [5,6], have provided studies that give improved estimates near the photon sphere $r = 3M$.

Moreover, we note that variants of these bounds have played an important role in the works [7 and 14] which prove analogues of the Morawetz conformal estimates on Schwarzschild backgrounds. This allows one to deduce a uniform decay rate for the local energy away from the event horizon, though there is necessarily a loss of regularity due to the trapping that occurs at the photon sphere. Instead in this paper we restrict ourselves to time translation invariant estimates, and we aim to clarify/streamline these as much as possible.

All of the above articles use the conserved (degenerate) energy $E_0[\phi]$ on time slices, obtained using the Killing vector field ∂_t. As such, their estimates are degenerate near the event horizon. Further progress was made in [14], where an additional vector field was introduced near the event horizon, in connection to the red shift effect. This led to bounds in the exterior region involving a nondegenerate form of the energy related to (1.9).

The approach of [2,7,14 and 25] is to write the equation using the Regge-Wheeler tortoise coordinate and to expand in spherical harmonics. For the equation corresponding to each spherical harmonic, one uses a multiplier which changes sign at the critical point of the effective potential.

Here we work in the coordinates $(r, \tilde{v}, \omega)$, though this is not of particular significance, and we do not expand into spherical harmonics. We prove (1.13) using a positive commutator argument which requires a single differential multiplier. We hope that this makes the methods more robust for other potential applications.

During final preparations of this article, localized energy estimates proved without using the spherical harmonic decomposition also appeared in [15]. The methods contained therein are somewhat different from ours.

Compared to the stronger norms LE_M, LE_M^* the weights in (1.13) have a polynomial singularity at $r = 3M$, which corresponds to the family of trapped geodesics on the photon sphere. As a consequence of the results we prove later, see Theorem 3.2, the latter fact can be remedied to produce a stronger estimate.

Theorem 1.3. *Let ϕ solve the inhomogeneous wave equation $\Box_g \phi = f$ on the Schwarzschild manifold. Then (1.13) still holds if the coefficient $(1 - 3M/r)^2$ in the LE_0 and the LE_0^* norms is replaced by*

$$\left(1 - \ln\left|1 - \frac{3M}{r}\right|\right)^{-2}.$$

Now we have only a logarithmic singularity at $r = 3M$. The result above is only stated in this form for the reader's convenience. The full result in Theorem 3.2 is stronger but also more complicated to state since it provides a more precise microlocal local energy estimate.

The logarithmic loss is not surprising, since it is characteristic of geometries with trapped hyperbolic orbits (see for instance [9,12,34]). Indeed, a similar estimate in the semiclassical setting is obtained in [13] using entirely different techniques. Note, however, that the aforementioned estimate only involves logarithmic loss of the frequency; our result is stronger since it also implies bounds for $\|(\ln|r^*|)^{-1}u\|_{L^2}$, which are necessary in order to prove Strichartz estimates.

There are two regions on which the analysis is distinct. The metric is asymptotically flat, and thus, near infinity, one can retrieve the classical Morawetz type estimate. On the other hand, around the photon sphere $r = 3M$ we take an expansion into spherical harmonics as well as a time Fourier transform. Then it remains to study an ordinary differential equation which is essentially similar to

$$(\partial_x^2 - \lambda^2(x^2 + \epsilon))u = f, \qquad |\epsilon| \ll 1, \quad |x| \lesssim 1.$$

For this we use a rough WKB approximation in the hyperbolic region combined with energy estimates in the elliptic region. Airy type dynamics occur near the zeroes of the potential.

Even though it is weaker, the initial bound in Theorem 1.2 plays a key role in the analysis. Precisely, it allows us to glue together the estimates in the two regions described above.

We next consider the Strichartz estimates. For solutions to the constant coefficient wave equation on $\mathbb{R} \times \mathbb{R}^3$, the well-known Strichartz estimates state that

$$\| |D_x|^{-\rho_1} \nabla u\|_{L_t^{p_1} L_x^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \| |D_x|^{\rho_2} f\|_{L_t^{p_2'} L_x^{q_2'}}. \tag{1.14}$$

Here the exponents (ρ_i, p_i, q_i) are subject to the scaling relation

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \rho \tag{1.15}$$

and the dispersion relation

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad 2 < p \leq \infty. \tag{1.16}$$

All pairs (ρ, p, q) satisfying (1.15) and (1.16) are called Strichartz pairs. Those for which the equality holds in (1.16) are called sharp Strichartz pairs. Such estimates first appeared in the seminal works [8,43,44] and as stated include contributions from, e.g., [17,19,26,35 and 21].

If one allows variable coefficients, such estimates are well-understood locally-in-time. For smooth coefficients, this was first shown in [32] and later for C^2 coefficients in [39 and 45–47].

Globally-in-time, the problem is more delicate. Even a small, smooth, compactly supported perturbation of the flat metric may refocus a group of rays and produce caustics. Thus, constructing a parametrix for incoming rays proves to be quite difficult. At the same time, one needs to contend with the possibility of trapped rays at high frequencies and with eigenfunctions/resonances at low frequencies.

Global-in-time estimates were shown for small, long range perturbations of the metric in [29] using an outgoing parametrix. In order to keep the parametrix outgoing one must allow evolution both forward and backward in time. This construction is based on an earlier argument in [48] for the Schrödinger equation. The smallness assumption, however, precludes trapping and does not permit a direct application to the current setup.

On the other hand, a second result of [29] asserts that even for large, long range perturbations of the metric one can still establish global-in-time Strichartz estimates provided that a strong form of the local energy estimates holds. This switches the burden to the question of proving local energy estimates.

The result in [29] cannot be applied directly to the present problem due to the logarithmic losses in the local energy estimates near the trapped rays. However, it can be applied for the near infinity part of the solution. In a bounded spatial region, on the other hand, we take advantage of the local energy estimates to localize the problem to bounded sets, in which estimates are shown using the local-in-time Strichartz estimates of [39,45]. Thus we obtain

Theorem 1.4. *If ϕ solves $\Box_g \phi = f$ in $\mathcal{M}_R$ then for all nonsharp Strichartz pairs (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) we have*

$$E[\phi](\Sigma_R^+) + \sup_{\tilde{v}\geq 0} E[\phi](\tilde{v}) + \|\nabla\phi\|^2_{L^{p_1}_{\tilde{v}} \dot{H}^{-\rho_1,q_1}_x} \lesssim E[\phi](\Sigma_R^-) + \|f\|^2_{L^{p_2'}_{\tilde{v}} \dot{H}^{\rho_2,q_2'}_x}. \quad (1.17)$$

Here the Sobolev-type spaces $\dot{H}^{s,p}$ coincide with the usual $\dot{H}^{s,p}$ homogeneous spaces in $\mathbb{R}^3$ expressed in polar coordinates (r, ω).

As a corollary of this result one can consider the global solvability question for the energy critical semilinear wave equation in the Schwarzschild space,

$$\begin{cases} \Box_g \phi = \pm\phi^5 & \text{in } \mathcal{M} \\ \phi = \phi_0, \ \tilde{K}\phi = \phi_1 & \text{in } \Sigma_0. \end{cases} \quad (1.18)$$

Theorem 1.5. *Let $r_0 > 0$. Then there exists $\epsilon > 0$ so that for each initial data (ϕ_0, ϕ_1) which satisfies*

$$E[\phi](\Sigma_0) \leq \epsilon,$$

Eq. (1.18) admits an unique solution ϕ in the region $\{r > r_0\}$ which satisfies the bound

$$E[\phi](\Sigma_{r_0}) + \|\phi\|_{\dot{H}^{s,p}(\{r>r_0\})} \lesssim E[\phi](\Sigma_0)$$

for all indices s, p satisfying

$$\frac{4}{p} = s + \frac{1}{2}, \qquad 0 \leq s < \frac{1}{2}.$$

Furthermore, the solution has a Lipschitz dependence on the initial data in the above topology.

Some further clarification is needed for the function space $\dot{H}^{s,p}(\{r > r_0\})$ appearing above, in view of the ambiguity due to the choice of coordinates. In a compact neighbourhood of the center region $\mathcal{M}_C$ this is nothing but the classical $H^{s,p}$ norm. By compactness, different choices of coordinates lead to equivalent norms. Consider now the upper exterior region $\mathcal{M}_R$ (as well as its three other mirror images). Using the coordinates $(\tilde{v}, x)$ with $x = \omega r$, we define $\dot{H}^{s,p}(\mathcal{M}_R)$ as the restrictions to $\mathbb{R}^+ \times \{|x| > r_0\}$ of functions in the homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R} \times \mathbb{R}^3)$.

2. The Morawetz-Type Estimate

In this section, we shall prove Theorem 1.2. We note that the estimate (1.13) is trivial over a finite $\tilde{v}$ interval by energy estimates for the wave equation; the difficulty consists in proving a global bound in $\tilde{v}$. By the same token, once we prove (1.13) for some choice of $r_0 < 2M$, we can trivially make the transition to any $r_0 < 2M$ due to the local theory. Thus in the arguments which follow we reserve the right to take r_0 sufficiently close to $2M$.

We consider solutions to the inhomogeneous wave equation on the Schwarzschild manifold in $\mathcal{M}_R$, which is given by

$$\Box_g \phi = \nabla^\alpha \partial_\alpha \phi = f.$$

Here ∇ represents the metric connection. Associated to this equation is an energy-momentum tensor given by

$$Q_{\alpha\beta}[\phi] = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial^\gamma \phi \partial_\gamma \phi.$$

A simple calculation yields the most important property of $Q_{\alpha\beta}$, namely that if ϕ solves the homogeneous wave equation then $Q_{\alpha\beta}[\phi]$ is divergence-free:

$$\nabla^\alpha Q_{\alpha\beta}[\phi] = 0, \quad \text{if } \nabla^\alpha \partial_\alpha \phi = 0.$$

More generally, we have

$$\nabla^\alpha Q_{\alpha\beta}[\phi] = \partial_\beta \phi \, \Box_g \phi.$$

In order to prove Theorem 1.2, we shall contract $Q_{\alpha\beta}$ with a vector field X to form the momentum density

$$P_\alpha[\phi, X] = Q_{\alpha\beta}[\phi] X^\beta.$$

Computing the divergence of this vector field, we have

$$\nabla^\alpha P_\alpha[\phi, X] = \Box_g \phi X \phi + Q_{\alpha\beta}[\phi] \pi^{\alpha\beta},$$

where

$$\pi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)$$

is the deformation tensor of X.

If X is the Killing vector field K then the above divergence vanishes,

$$\nabla^\alpha P_\alpha[\phi, K] = 0 \quad \text{if} \quad \Box_g \phi = 0. \tag{2.1}$$

This gives rise to the $E_0[\phi]$ conservation law outside the black hole.

Naively, one may seek vector fields X so that the quadratic form $Q_{\alpha\beta}[\phi]\pi^{\alpha\beta}$ is positive definite. However, this may not always be possible to achieve. Instead we note that it may be just as good to have the symbol of this quadratic form positive on the characteristic set of $\Box_g$. Then it would be possible to make the above quadratic form positive after adding a Lagrangian correction term of the form $q \partial^\gamma \phi \partial_\gamma \phi$. Such a term can be conveniently expressed in divergence form modulo lower order terms. Precisely, for a vector field X, a scalar function q and a 1-form m we define

$$P_\alpha[\phi, X, q, m] = P_\alpha[\phi, X] + q\phi\partial_\alpha\phi - \frac{1}{2}\partial_\alpha q\phi^2 + \frac{1}{2}m_\alpha\phi^2,$$

where m allows us to modify the lower order terms in the divergence formula. Then we obtain the modified divergence relation

$$\begin{aligned} \nabla^\alpha P_\alpha[\phi, X, q, m] &= \Box_g\phi\,(X\phi + q\phi) + Q[\phi, X, q, m], \\ Q[\phi, X, q, m] &= Q_{\alpha\beta}[\phi]\pi^{\alpha\beta} + q\partial^\alpha\phi\,\partial_\alpha\phi + m_\alpha\phi\,\partial^\alpha\phi + \frac{1}{2}(\nabla^\alpha m_\alpha - \nabla^\alpha\partial_\alpha q)\,\phi^2. \end{aligned} \tag{2.2}$$

Theorem 1.2 is proved by making appropriate choices for X, q and m so that the quadratic form $Q[\phi, X, q, m]$ defined by the divergence relation is positive definite. In what follows we assume that X, q and m are all spherically symmetric and invariant with respect to the Killing vector field K.

Lemma 2.1. *There exist smooth, spherically symmetric, K-invariant X, q, and m in $r \geq 2M$ satisfying the following properties:*

(i) *X is bounded[2], $|q(r)| \lesssim r^{-1}$, $|q'(r)| \lesssim r^{-2}$ and m has compact support in r.*

(ii) *The quadratic form $Q[\phi, X, q, m]$ is positive definite,*

$$Q[\phi, X, q, m] \gtrsim r^{-2}|\partial_r\phi|^2 + \left(1 - \frac{3M}{r}\right)^2 (r^{-2}|\partial_{\tilde{v}}\phi|^2 + r^{-1}|\not{\nabla}\phi|^2) + r^{-4}\phi^2.$$

(iii) *$X(2M)$ points toward the black hole, $X(dr)(2M) < 0$, and $\langle m, dr\rangle(2M) > 0$.*

We postpone the proof of the lemma and use it to conclude the proof of Theorem 1.2. Let X, q and m be as in the lemma. We extend them smoothly beyond the event horizon preserving the spherical symmetry and the K-invariance. By (2.1) we can modify the vector field X without changing the quadratic form Q in (2.3),

$$\nabla^\alpha P_\alpha[\phi, X + CK, q, m] = \Box_g\phi\,((X + CK)\phi + q\phi) + Q[\phi, X, q, m].$$

Here C is a large constant. We integrate this relation in the region

$$D = \{0 < \tilde{v} < \tilde{v}_0,\ r > r_0\}$$

using the $(r, \tilde{v}, \omega)$ coordinates. This yields

$$\begin{aligned} &\int_D \left(\Box_g\phi\,((X + CK)\phi + q\phi) + Q[\phi, X, q, m]\right) r^2 dr d\tilde{v} d\omega \\ &= \int \langle d\tilde{v}, P[\phi, X + CK, q, m]\rangle r^2 dr d\omega \Big|_{\tilde{v}=0}^{\tilde{v}=\tilde{v}_0} \\ &\quad - \int_{r=r_0} \langle dr, P[\phi, X + CK, q, m]\rangle r_0^2 d\tilde{v} d\omega. \end{aligned}$$

[2] In the $(r, \tilde{v})$ coordinates

We claim that if C is large enough and r_0 sufficiently close to $2M$ then the integrals on the right have the correct sign,

$$E[\phi](\tilde{v}_1) \lesssim -\int_{\tilde{v}=\tilde{v}_1} \langle d\tilde{v}, P[\phi, X+CK, q, m]\rangle r^2 dr d\omega \lesssim CE[\phi](\tilde{v}_1), \quad \tilde{v}_1 \geq 0, \quad (2.3)$$

$$\langle dr, P[\phi, X+CK, q, m]\rangle \gtrsim |\partial_r\phi|^2 + |\partial_{\tilde{v}}\phi|^2 + |\partial_\omega\phi|^2 + \phi^2, \qquad r = r_0. \quad (2.4)$$

If these bounds hold then the conclusion of the theorem follows by (ii) and Cauchy-Schwarz.

Indeed, a direct computation yields

$$\langle d\tilde{v}, P[\phi, \partial_{\tilde{v}}]\rangle = -\frac{1}{2}\left[\left(2\mu' - \left(1-\frac{2M}{r}\right)\mu'^2\right)|\partial_{\tilde{v}}\phi|^2 + \left(1-\frac{2M}{r}\right)|\partial_r\phi|^2 + r^{-2}|\partial_\omega\phi|^2\right],$$

respectively

$$\langle dr, P[\phi, \partial_{\tilde{v}}]\rangle = |\partial_{\tilde{v}}\phi|^2 + \left(1-\frac{2M}{r}\right)(\partial_r - \mu'\partial_{\tilde{v}})\phi\partial_{\tilde{v}}\phi.$$

On the other hand

$$\langle d\tilde{v}, P[\phi, \partial_r]\rangle = \left(1-\left(1-\frac{2M}{r}\right)\mu'\right)|\partial_r\phi|^2 - \left(2\mu' - \left(1-\frac{2M}{r}\right)\mu'^2\right)\partial_{\tilde{v}}\phi\partial_r\phi,$$

while

$$\langle dr, P[\phi, \partial_r]\rangle = -\frac{1}{2}\left[-\left(2\mu' - \left(1-\frac{2M}{r}\right)\mu'^2\right)|\partial_{\tilde{v}}\phi|^2 - \left(1-\frac{2M}{r}\right)|\partial_r\phi|^2 + r^{-2}|\partial_\omega\phi|^2\right].$$

We compute

$$\langle d\tilde{v}, P[\phi, X+CK]\rangle = (X(d\tilde{v}) + C)\langle d\tilde{v}, P[\phi, \partial_{\tilde{v}}]\rangle + X(dr)\langle d\tilde{v}, P[\phi, \partial_r]\rangle.$$

For large enough C we have $X(d\tilde{v}) + C \gtrsim C$. Therefore the first term on the right is negative definite for $r > 2M$. More precisely, it is only the coefficient of the $|\partial_r\phi|^2$ term which degenerates at $r = 2M$. However, due to condition (iii) in the lemma we have $X(dr)(2M) < 0$; therefore we pick up a negative $|\partial_r\phi|^2$ coefficient at $r = 2M$. Thus we obtain

$$-\langle d\tilde{v}, P[\phi, X+CK]\rangle \approx C\left[|\partial_{\tilde{v}}\phi|^2 + \left(1-\frac{2M}{r}\right)|\partial_r\phi|^2 + r^{-2}|\partial_\omega\phi|^2\right] + |\partial_r\phi|^2, \quad r > 2M.$$

Since all the coefficients in the quadratic form on the left are continuous, it follows that the above relation extends to $r > r_0$ for some $r_0 < 2M$ depending on C, namely

$$0 < 2M - r_0 \ll C^{-1}. \quad (2.5)$$

In order to prove (2.3) it remains to estimate the lower order terms $P[\phi, 0, q, m]$ in terms of the positive contribution above. Since $|q| \lesssim r^{-1}$ and m has compact support in r, we can bound

$$|\langle d\tilde{v}, P[\phi, 0, q, m]\rangle| \lesssim r^{-1}|\phi| \left[|\partial_{\tilde{v}}\phi|^2 + |\partial_r\phi|^2\right]^{\frac{1}{2}} + r^{-2}|\phi|^2.$$

Then by Cauchy-Schwarz it suffices to estimate

$$\int_{r_0}^{\infty} r^{-2}|\phi|^2 r^2 dr \lesssim C^{-\frac{1}{2}} \int_{r_0}^{\infty} \left[C\left(1 - \frac{2M}{r}\right) + 1\right] |\partial_r\phi|^2 r^2 dr$$

which is a routine Hardy-type inequality.

We next turn our attention to (2.4) and begin with the principal part

$$\langle dr, P[\phi, X + CK]\rangle = (X(d\tilde{v}) + C)\langle dr, P[\phi, \partial_{\tilde{v}}]\rangle + X(dr)\langle dr, P[\phi, \partial_r]\rangle.$$

Examining the expressions for the two terms above, we see that for r_0 subject to (2.5) we have

$$\langle dr, P[\phi, X + CK]\rangle \gtrsim C|\partial_{\tilde{v}}\phi|^2 + |\partial_\omega\phi|^2 - \left(1 - \frac{2M}{r_0}\right)|\partial_r\phi|^2, \qquad r = r_0.$$

Next we consider the lower order terms. The contribution of m is

$$\frac{1}{2}\langle m, dr\rangle\phi^2 \gtrsim \phi^2$$

due to condition (iii) in the lemma. The contribution of q is

$$q\phi\langle dr, d\phi\rangle - \frac{1}{2}\phi^2\langle dr, dq\rangle.$$

The coefficient of the second term is $\left(1 - \frac{2M}{r}\right)q'$, which is negligible for r_0 close to $2M$. In the first term we have

$$\langle dr, d\phi\rangle = \left(1 - \frac{2M}{r}\right)\partial_r\phi + \left(1 - \left(1 - \frac{2M}{r}\right)\mu'\right)\partial_{\tilde{v}}\phi.$$

All terms involving $\left(1 - \frac{2M}{r}\right)$ are negligible, and since q is bounded we get

$$q\phi\partial_{\tilde{v}}\phi \ll C|\partial_{\tilde{v}}\phi|^2 + \phi^2$$

for large enough C.

Proof of Lemma 2.1. It is convenient to look for X in the (r, t) coordinates, where we choose the vector field X of the form

$$X = X_1 + \delta X_2, \qquad \delta \ll 1,$$

with

$$X_1 = a(r)\left(1 - \frac{2M}{r}\right)\partial_r, \qquad X_2 = b(r)\left(1 - \frac{2M}{r}\right)\left(\partial_r - \left(1 - \frac{2M}{r}\right)^{-1}\partial_t\right)$$

and a and $b\big(1-\frac{2M}{r}\big)$ will be chosen to be smooth. Note that X is a smooth vector field in the nonsingular coordinates (r, v), since in these coordinates we have

$$X_1 = a(r)\left(\left(1-\frac{2M}{r}\right)\partial_r + \partial_v\right), \qquad X_2 = b(r)\left(1-\frac{2M}{r}\right)\partial_r.$$

We remark that the vector field X_2 is closely related to the vector field Y introduced earlier in [14] in order to take advantage of the red shift effect. However, in their construction Y is in a form which is nonsmooth near the event horizon and which is restricted to the exterior region.

The primary role played by X_2 here is to ensure that $X + CK$ is time-like near the event horizon. The red-shift effect largely takes care of the rest.

For convenience, we set

$$t_1(r) = \left(1-\frac{2M}{r}\right)\frac{1}{r^2}\partial_r\left(r^2 a(r)\right).$$

A direct computation yields

$$\begin{aligned}\nabla^\alpha P_\alpha[\phi, X_1] = &\left(1-\frac{2M}{r}\right)^2 a'(r)(\partial_r\phi)^2 + a(r)\frac{r-3M}{r^2}|\not\nabla\phi|^2 \\ &-\frac{1}{2}t_1(r)\partial^\gamma\phi\,\partial_\gamma\phi + X_1\phi\Box_g\phi,\end{aligned} \tag{2.6}$$

respectively

$$\begin{aligned}\nabla^\alpha P_\alpha[\phi, X_2] = &\frac{1}{2}b'(r)\left(\left(1-\frac{2M}{r}\right)\partial_r\phi - \partial_t\phi\right)^2 \\ &+\left(\frac{r-3M}{r^2}b(r) - \frac{1}{2}\left(1-\frac{2M}{r}\right)b'(r)\right)|\not\nabla\phi|^2 \\ &-\frac{1}{r}\left(1-\frac{2M}{r}\right)b(r)\partial^\gamma\phi\,\partial_\gamma\phi + X_2\phi\Box_g\phi,\end{aligned} \tag{2.7}$$

where

$$\partial^\gamma\phi\,\partial_\gamma\phi = -\left[\left(1-\frac{2M}{r}\right)^{-1}(\partial_t\phi)^2 - \left(1-\frac{2M}{r}\right)(\partial_r\phi)^2 - |\not\nabla\phi|^2\right].$$

We choose a so that the first line of the right side of (2.6) is positive. This requires that

$$a'(r) \gtrsim r^{-2}, \qquad a(3M) = 0. \tag{2.8}$$

We choose b so that the first line of the right-hand side of (2.7) is positive. Precisely, we take b supported in $r \le 3M$ with

$$b = -\frac{b_0(r)}{1-\frac{2M}{r}}, \qquad r \in [2M, 3M],$$

with b_0 smooth, decreasing in $[2M, 3M)$ and supported in $\{r \le 3M\}$. In particular this guarantees that $b_0(2M) > 0$, which is later used to verify the condition (iii) in the lemma.

The exact choice of b_0 is not important, and in effect b only plays a role very close to the event horizon $r = 2M$. Even though b is singular at $2M$, the second term of the coefficient of $|\nabla\!\!\!/\phi|^2$ in the second line of (2.7) is nonsingular. Hence if δ is sufficiently small this term is controlled by the first line in (2.6).

Taking the above choices into account, we have

$$Q[\phi, X, 0, 0] = \left(1 - \frac{2M}{r}\right)^2 a'(r)(\partial_r\phi)^2 + \delta\frac{1}{2}b'(r)\left(\left(1 - \frac{2M}{r}\right)\partial_r\phi - \partial_t\phi\right)^2 + O\left(\frac{(r-3M)^2}{r^3}\right)|\nabla\!\!\!/\phi|^2 - q_0\partial^\gamma\phi\,\partial_\gamma\phi \tag{2.9}$$

where

$$q_0(r) = \left(\frac{1}{2}t_1(r) + \delta\frac{1}{r}b(r)\left(1 - \frac{2M}{r}\right)\right).$$

The last term in (2.9) is a Lagrangian expression and is accounted for via the q term. The first three terms give a nonnegative quadratic form in $\nabla\phi$. This form is in effect positive definite for $r < 3M$, where $b' > 0$. However for larger r it controls $\partial_r\phi$ and $\nabla\!\!\!/\phi$ but not $\partial_t\phi$. This can be easily remedied with the Lagrangian term. Precisely, we choose q of the form

$$q = q_0 + \delta_1 q_1, \qquad q_1(r) = \chi_{\{r>5M/2\}}\frac{(r-3M)^2}{r^4},$$

where $\chi_{\{r>5M/2\}}$ is a smooth nonnegative cutoff which is supported in $\{r > 5M/2\}$ and equals 1 for $r > 3M$. The positive parameter δ_1 is chosen so that $\delta_1 \ll \delta$. Then the only nonnegligible contribution of $\delta_1 q_1$ is the one involving $\partial_t\phi$. We obtain

$$Q[\phi, X, q, 0] = \left(1 - \frac{2M}{r}\right)^2 O(r^{-2})(\partial_r\phi)^2 + \delta\frac{1}{2}b'(r)\left(\left(1 - \frac{2M}{r}\right)\partial_r\phi - \partial_t\phi\right)^2 + O\left(\frac{(r-3M)^2}{r^3}\right)|\nabla\!\!\!/\phi|^2 + \delta_1 q_1\left(1 - \frac{2M}{r}\right)^{-1}|\partial_t\phi|^2 - \frac{1}{2}\nabla^\alpha\partial_\alpha q\,\phi^2. \tag{2.10}$$

The contribution of q_1 can be made arbitrarily small by taking δ_1 small. Hence it will be neglected in the sequel. At this stage it would be convenient to be able to choose a so that $\nabla^\alpha\partial_\alpha t_1(r) < 0$. A direct computation yields

$$\nabla^\alpha\partial_\alpha t_1(r) = -La$$

with

$$La(r) = -\frac{1}{r^2}\partial_r\left[\left(1 - \frac{2M}{r}\right)r^2\partial_r\left\{\left(1 - \frac{2M}{r}\right)\frac{1}{2r^2}\partial_r\left(r^2 a(r)\right)\right\}\right].$$

Unfortunately it turns out that the condition $La > 0$ and (2.8) are incompatible, in the sense that there is no smooth a which satisfies both. However, one can find a with a logarithmic blow-up at $2M$ which satisfies both requirements. Such an example is

$$a(r) = r^{-2}\left((r-3M)(r+2M) + 6M^2\log\left(\frac{r-2M}{M}\right)\right).$$

This is in no way unique, it is merely the simplest we were able to produce. One verifies directly that

$$a'(r) \gtrsim r^{-2}, \qquad La(r) \gtrsim r^{-4}.$$

To eliminate the singularity of a above we replace it by

$$a_\epsilon(r) = \frac{1}{r^2} f_\epsilon(R),$$

where ϵ is a small parameter,

$$R = (r - 3M)(r + 2M) + 6M^2 \log\left(\frac{r - 2M}{M}\right),$$

and

$$f_\epsilon(R) = \epsilon^{-1} f(\epsilon R),$$

where f is a smooth nondecreasing function such that $f(R) = R$ on $[-1, \infty]$ and $f = -2$ on $(-\infty, -3]$. The condition (2.8) is satisfied uniformly with respect to small ϵ; therefore the choice of δ is independent of the choice of ϵ.

With this modification of a we recompute

$$La_\epsilon = f'(\epsilon R)La + O(\epsilon) f''(\epsilon R) + O\left(\epsilon^2\left(1 - \frac{2M}{r}\right)^{-1}\right) f'''(\epsilon R).$$

This is still positive except for the region $\{\epsilon R < -1\}$. To control it we introduce an m term in the divergence relation as follows: Let $\gamma(r)$ be a function to be chosen later. We set

$$m_t = \delta b'(r)\left(1 - \frac{2M}{r}\right)^2 \gamma, \qquad m_r = \delta b'(r)\left(1 - \frac{2M}{r}\right)\gamma, \qquad m_\omega = 0.$$

Then

$$m_\alpha \partial^\alpha \phi = \delta b'(r)\gamma(r)\left(1 - \frac{2M}{r}\right)\left(\left(1 - \frac{2M}{r}\right)\partial_r\phi - \partial_t\phi\right),$$

while

$$\nabla^\alpha m_\alpha = \delta r^{-2}\partial_r\left(\left(1 - \frac{2M}{r}\right)^2 r^2 b'(r)\gamma(r)\right).$$

Hence, completing the square we obtain

$$\begin{aligned} Q[\phi, X, q, m] = {} & \left(1 - \frac{2M}{r}\right)^2 O(r^{-2})(\partial_r\phi)^2 + O\left(\frac{(r - 3M)^2}{r^3}\right)|\not\nabla\phi|^2 \\ & + \delta_1 q_1\left(1 - \frac{2M}{r}\right)^{-1}|\partial_t\phi|^2 + n\phi^2 \\ & + \delta\frac{1}{2}b'(r)\left(\left(1 - \frac{2M}{r}\right)\partial_r\phi - \partial_t\phi + \left(1 - \frac{2M}{r}\right)\gamma\phi\right)^2, \end{aligned}$$

where the coefficient n is given by

$$n = La_\epsilon - \frac{1}{2}\delta r^{-2}\partial_r r^2\left(1-\frac{2M}{r}\right)\partial_r\left(r^{-1}b(r)\left(1-\frac{2M}{r}\right)\right) - \delta\frac{b'(r)}{2}\left(1-\frac{2M}{r}\right)^2\gamma(r)^2$$
$$+\frac{1}{2}\delta r^{-2}\gamma\partial_r\left(r^2\left(1-\frac{2M}{r}\right)^2 b'(r)\right) + \frac{1}{2}\delta\gamma'\left(1-\frac{2M}{r}\right)^2 b'(r).$$

We assume that γ is supported in $\{r < 3M\}$ and satisfies

$$0 \leq \gamma \leq 1, \qquad \gamma' > -1.$$

Then for $r > 3M$ we have

$$n = La_\epsilon \gtrsim r^{-4},$$

while for $r \leq 3M$ we can write

$$n = La_\epsilon + \delta\gamma'(r)\left(1-\frac{2M}{r}\right)^2 b'(r) + O(\delta).$$

If $\epsilon R > -1$ then, using the bound from below on γ', we further have

$$n \geq La + O(\delta),$$

which is positive provided that δ is sufficiently small. On the other hand in the region $\{\epsilon R \leq -1\}$, we have

$$n \geq \frac{1}{2}\delta\left(1-\frac{2M}{r}\right)^2 b'(r)\gamma'(r) + O(\delta) + O(\epsilon)f''(\epsilon R) + O\left(\epsilon^2\left(1-\frac{2M}{r}\right)^{-1}\right)f'''(\epsilon R).$$

The γ' term can be taken positive, while all the other terms may be negative so they must be controlled by it. The restriction we face in the choice of γ' comes from the fact that $0 \leq \gamma \leq 1$. Hence we need to verify that

$$I = \int_{\epsilon R \leq -1} \delta + \epsilon|f''(\epsilon R)| + \epsilon^2\left(1-\frac{2M}{r}\right)^{-1}|f'''(\epsilon R)| \ll \delta.$$

Indeed, the interval of integration has size $\leq e^{-c\epsilon^{-1}}$; therefore the above integral can be bounded by

$$I \lesssim e^{-c\epsilon^{-1}} + \epsilon,$$

which suffices provided that ϵ is small enough.

Finally, note that

$$X(dr)(2M) = \left(a(r)\left(1-\frac{2M}{r}\right) + \delta b(r)\left(1-\frac{2M}{r}\right)\right)(2M) < 0,$$
$$\langle m, dr\rangle(2M) = \left(\delta b'(r)\left(1-\frac{2M}{r}\right)^2\gamma\right)(2M) > 0.$$

So (iii) is also satisfied. □

3. Log-Loss Local Energy Estimates

The aim of this section is to prove a local energy estimate for solutions to the wave equation on the Schwarzschild space which is stronger than the one in Theorem 1.2. Consequently, we strengthen the norm LE_0 to a norm LE and we relax the norm LE_0^* to a norm LE^* which satisfy the following natural bounds:

$$\|\phi\|_{LE_0}^2 \lesssim \|\phi\|_{LE}^2 \lesssim \|\phi\|_{LE_M}^2, \tag{3.11}$$

respectively

$$\|f\|_{LE_M^*}^2 \lesssim \|f\|_{LE^*}^2 \lesssim \|f\|_{LE_0^*}^2. \tag{3.12}$$

We note that these bounds uniquely determine the topology of the LE and LE^* spaces away from the photon sphere and from infinity. This is due to the fact that the local energy estimates in Theorem 1.2 have no loss in any bounded region away from the photon sphere. To define the LE, respectively LE^*, norms we consider a smooth partition of unity

$$1 = \chi_{eh}(r) + \chi_{ps}(r) + \chi_\infty(r),$$

where χ_{eh} is supported in $\{r < 11M/4\}$, χ_{ps} is supported in $\{5M/2 < r < 5M\}$ and χ_∞ is supported in $\{r > 4M\}$. Then we set

$$\|\phi\|_{LE}^2 = \|\chi_{eh}\phi\|_{LE_M}^2 + \|\chi_{ps}\phi\|_{LE_{ps}}^2 + \|\chi_\infty\phi\|_{LE_M}^2, \tag{3.13}$$

respectively

$$\|\phi\|_{LE^*}^2 = \|\chi_{eh}\phi\|_{LE_M^*}^2 + \|\chi_{ps}\phi\|_{LE_{ps}^*}^2 + \|\chi_\infty\phi\|_{LE_M^*}^2. \tag{3.14}$$

The norms LE_{ps} and LE_{ps}^* near the photon sphere are defined in Sect. 3.1 below, see (3.20), respectively (3.21); their topologies coincide with LE_M, respectively LE_M^*, away from the photon sphere.

With these notations, the main result of this section can be phrased in a manner similar to Theorem 1.2:

Theorem 3.2. *For all functions ϕ which solve $\Box_g \phi = f$ in $\mathcal{M}_R$ we have*

$$\sup_{\tilde v > 0} E[\phi](\tilde v) + E[\phi](\Sigma_R^+) + \|\phi\|_{LE}^2 \lesssim E[\phi](\Sigma_R^-) + \|f\|_{LE^*}^2. \tag{3.15}$$

We continue with the setup and estimates near the photon sphere in Sect. 3.1, the setup and estimates near infinity in Sect. 3.2 and finally the proof of the theorem in Sect. 3.3.

3.1. The analysis near the photon sphere. Here it is convenient to work in the Regge-Wheeler coordinates given by

$$r^* = r + 2M\log(r - 2M) - 3M - 2M\log M.$$

Then $r = 3M$ corresponds to $r^* = 0$, and a neighbourhood of $r = 3M$ away from infinity and the event horizon corresponds to a compact set in r^*. In these coordinates the operator $\Box_g$ has the form

$$r\left(1 - \frac{2M}{r}\right)\Box_g r^{-1} = L_{RW} = \partial_t^2 - \partial_{r^*}^2 - \frac{r-2M}{r^3}\partial_\omega + V(r), \quad V(r) = r^{-1}\partial_{r^*}^2 r. \tag{3.16}$$

For r^* in a compact set the energy has the form

$$E[\phi] \approx \int (\partial_t\phi)^2 + (\partial_{r^*}\phi)^2 + (\partial_\omega\phi)^2 dr d\omega,$$

and the initial local smoothing norms are expressed as

$$\|\phi\|_{LE_0}^2 \approx \int (\partial_{r^*}\phi)^2 + r^{*2}((\partial_\omega\phi)^2 + (\partial_t\phi)^2) + \phi^2 dr d\omega dt,$$

respectively

$$\|f\|_{LE_0^*}^2 \approx \int r^{*-2} f^2 dr d\omega dt.$$

On the other hand

$$\|\phi\|_{LE_M}^2 \approx \int (\partial_{r^*}\phi)^2 + (\partial_\omega\phi)^2 + (\partial_t\phi)^2 + \phi^2 dr d\omega dt,$$

$$\|f\|_{LE_M^*}^2 \approx \int f^2 dr d\omega dt.$$

In the sequel we work with spatial spherically symmetric pseudodifferential operators in the (r^*, ω) coordinates where $\omega \in \mathbb{S}^2$. We denote by ξ the dual variable to r^*, and by λ the spectral parameter for $(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}$. Thus the role of the Fourier variable is played by the pair (ξ, λ), and all our symbols are of the form

$$a(r^*, \xi, \lambda).$$

To such a symbol we associate the corresponding Weyl operator A^w. Since there is no symbol dependence on ω, one can view this operator as a combination of a one dimensional Weyl operator and the spectral projectors Π_λ associated to the operator $(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}$, namely

$$A^w = \sum_\lambda a^w(\lambda)\Pi_\lambda.$$

All of our L^2 estimates admit orthogonal decompositions with respect to spherical harmonics, therefore in order to prove them it suffices to work with the fixed λ operators $a^w(\lambda)$, and treat λ as a parameter. However, in the proof of the Strichartz estimates later on we need kernel bounds for operators of the form A^w, which is why we think of λ as a second Fourier variable and track the symbol regularity with respect to λ as well. Of course, this is meaningless for λ in a compact set; only the asymptotic behavior as $\lambda \to \infty$ is relevant.

Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^+$ be a smooth increasing function so that

$$\gamma_0(y) = \begin{cases} 1 & y < 1, \\ y & y \geq 2. \end{cases}$$

Let $\gamma_1 : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth increasing function so that

$$\gamma_1(y) = \begin{cases} y^{\frac{1}{2}} & y < 1/2, \\ 1 & y \geq 1. \end{cases}$$

Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^+$ be a smooth function with the following properties:

$$\gamma(y,z) = \begin{cases} 1 & z < C, \\ \gamma_0(y) & y < \sqrt{z/2},\ z \geq C, \\ z^{\frac{1}{2}} \gamma_1(y^2/z) & y \geq \sqrt{z/2},\ z \geq C, \end{cases}$$

where C is a large constant. In the sequel z is a discrete parameter, so the lack of smoothness at $z = C$ is of no consequence.

Consider the symbol

$$a_{ps}(r^*, \xi, \lambda) = \gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln \lambda),$$

and its inverse

$$a_{ps}^{-1}(r^*, \xi, \lambda) = \frac{1}{\gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln \lambda)}.$$

We note that if λ is small then they both equal 1, while if λ is large then they satisfy the bounds

$$\begin{aligned} 1 \leq a_{ps}(r^*, \xi, \lambda) \leq a_{ps}(r^*, 0, \lambda) \leq (\ln \lambda)^{\frac{1}{2}}, \\ (\ln \lambda)^{-\frac{1}{2}} \leq a_{ps}^{-1}(r^*, 0, \lambda) \leq a_{ps}^{-1}(r^*, \xi, \lambda) \leq 1. \end{aligned} \tag{3.17}$$

We also observe that the region where $y^2 > z$ corresponds to $r^{*2} + \lambda^{-2}\xi^2 < e^{-\sqrt{\ln \lambda}}$. Thus differentiating the two symbols we obtain the following bounds

$$|\partial_{r^*}^{\alpha} \partial_{\xi}^{\beta} \partial_{\lambda}^{\nu} a_{ps}(r^*, \xi, \lambda)| \leq c_{\alpha,\beta,\nu} \lambda^{-\beta-\nu} (r^{*2} + \lambda^{-2}\xi^2 + e^{-\sqrt{\ln \lambda}})^{-\frac{\alpha+\beta}{2}}, \tag{3.18}$$

respectively

$$|\partial_{r^*}^{\alpha} \partial_{\xi}^{\beta} \partial_{\lambda}^{\nu} a_{ps}^{-1}(r^*, \xi, \lambda)| \leq c_{\alpha,\beta,\nu} a_{ps}^{-2}(r^*, \xi, \lambda) \lambda^{-\beta-\nu} (r^{*2} + \lambda^{-2}\xi^2 + e^{-\sqrt{\ln \lambda}})^{-\frac{\alpha+\beta}{2}}, \tag{3.19}$$

where $\alpha + \beta + \nu > 0$. These show that we have a good operator calculus for the corresponding pseudodifferential operators. In particular in terms of the classical symbol classes we have

$$a_{ps}, a_{ps}^{-1} \in S^{\delta}_{1,0}, \qquad \delta > 0.$$

Then we introduce the Weyl operators

$$A_{ps} = \sum_{\lambda} a_{ps}^{w}(\lambda)\Pi_{\lambda},$$

respectively

$$A_{ps}^{-1} = \sum_{\lambda} (a_{ps}^{-1})^{w}(\lambda)\Pi_{\lambda}.$$

By (3.18) and (3.19) one easily sees that these operators are approximate inverses. More precisely for small λ, $\ln\lambda < C$, they are both the identity, while for large λ,

$$\|a_{ps}^{w}(\lambda)(a_{ps}^{-1})^{w}(\lambda) - I\|_{L^2\to L^2} \lesssim \lambda^{-1}e^{\sqrt{\ln\lambda}}, \qquad \ln\lambda \geq C.$$

Choosing C large enough we insure that the bound above is always much smaller than 1.

We use these two operators in order to define the improved local smoothing norms

$$\|\phi\|_{LE_{ps}} = \|A_{ps}^{-1}\phi\|_{H^1_{t,x}} \approx \|A_{ps}^{-1}\nabla_{t,x}\phi\|_{L^2}, \tag{3.20}$$

$$\|f\|_{LE^*_{ps}} = \|A_{ps}f\|_{L^2}. \tag{3.21}$$

Due to the inequalities (3.17) we have a bound from above for $a_{ps}^{w}(\lambda)$,

$$\|a_{ps}^{w}(\lambda)f\|_{L^2} \lesssim \|a_{ps}(r^*,0,\lambda)f\|_{L^2} \lesssim \||\ln|r^*||f\|_{L^2},$$

respectively a bound from below for $(a_{ps}^{-1})^{w}(\lambda)$,

$$\|(a_{ps}^{-1})^{w}(\lambda)f\|_{L^2} \gtrsim \|a_{ps}^{-1}(r^*,0,\lambda)f\|_{L^2} \gtrsim \||\ln|r^*||^{-1}f\|_{L^2}$$

for f supported near $r^* = 0$. In particular this shows that for f supported near the photon sphere we have

$$\|\phi\|_{LE_{ps}} \gtrsim \||\ln|r^*||^{-1}\nabla\phi\|_{L^2}, \qquad \|f\|_{LE^*_{ps}} \lesssim \||\ln|r^*||f\|_{L^2}, \tag{3.22}$$

which makes Theorem 1.3 a direct consequence of Theorem 3.2.

Our main estimate near the photon sphere is

Proposition 3.3. a) *Let ϕ be a function supported in $\{5M/2 < r < 5M\}$ which solves $\Box_g\phi = f$. Then*

$$\|\phi\|^2_{LE_{ps}} \lesssim \|f\|^2_{LE^*_{ps}}. \tag{3.23}$$

b) *Let $f \in LE^*_{ps}$ be supported in $\{11M/4 < r < 4M\}$. Then there is a function ϕ supported in $\{5M/2 < r < 5M\}$ so that*

$$\sup_t E[\phi] + \|\phi\|^2_{LE_{ps}} + \|\Box_g\phi - f\|^2_{LE^*_0} \lesssim \|f\|^2_{LE^*_{ps}}. \tag{3.24}$$

Proof. Due to (3.16) we can recast the problem in Regge-Wheeler coordinates. Denoting $u = r\phi$, $g = \left(1 - \frac{2M}{r}\right)rf$, we have $L_{RW}u = g$. Also it is easy to verify that for ϕ and f supported in a fixed compact set in r^* we have

$$\|\phi\|_{LE_{ps}} \approx \|u\|_{LE_{ps}}, \qquad \|f\|_{LE^*_{ps}} \approx \|g\|_{LE^*_{ps}}.$$

Hence in the proposition we can replace ϕ and f by u and g, and $\Box_g$ by L_{RW}.

To prove part (a) we expand in spherical harmonics with respect to the angular variable and take a time Fourier transform. We are left with the ordinary differential equation

$$(\partial^2_{r^*} + V_{\lambda,\tau}(r^*))u = g, \tag{3.25}$$

where

$$V_{\lambda,\tau}(r^*) = \tau^2 - \frac{r-2M}{r^3}\lambda^2 + V.$$

Depending on the relative sizes of λ and τ we consider several cases. In the easier cases it suffices to replace the bound (3.23) with a simpler bound

$$\|\partial_{r^*}u\|_{L^2} + (|\tau| + |\lambda|)\|u\|_{L^2} \lesssim \|g\|_{L^2}. \tag{3.26}$$

Case 1. $\lambda, \tau \lesssim 1$. Then we solve (3.25) as a Cauchy problem with data on one side and obtain a pointwise bound,

$$|u| + |u_{r^*}| \lesssim \|g\|_{L^2},$$

which easily implies (3.26).

Case 2. $\lambda \ll \tau$. Then $V_{\lambda,\tau}(r^*) \approx \tau^2$ for r^* in a compact set; therefore (3.25) is hyperbolic in nature. Hence we can solve (3.25) as a Cauchy problem with data on one side and obtain

$$\tau|u| + |u_{r^*}| \lesssim \|g\|_{L^2},$$

which implies (3.26).

Case 3. $\lambda \gg \tau$. Then $V_{\lambda,\tau}(r^*) \approx -\lambda^2$ for r^* in a compact set; therefore (3.25) is elliptic. Then we solve (3.25) as an elliptic problem with Dirichlet boundary conditions on a compact interval and obtain

$$\lambda^{\frac{3}{2}}|u| + \lambda^{\frac{1}{2}}|u_{r^*}| \lesssim \|g\|_{L^2},$$

which again gives (3.26).

Case 4. $\lambda \approx \tau \gg 1$. In this case (3.26) is no longer true, and we need to prove (3.23), which in this case can be written in the form

$$\|\partial_{r^*}u\|_{L^2} + \lambda\|(a^{-1}_{ps})^w(\lambda)u\|_{L^2} \lesssim \|a^w_{ps}(\lambda)g\|_{L^2}, \tag{3.27}$$

where u, g are subject to (3.25). The $\partial_{r^*}u$ term above is present in order to estimate the high frequencies $|\xi| \gg \lambda$. For lower frequencies it is controlled by the second term on the left of (3.27).

The potential V in (3.25) can be treated perturbatively in (3.23) and is negligible. The remaining part of $V_{\lambda,\tau}(r^*)$ has a nondegenerate minimum at $r = 3M$ which corresponds to $r^* = 0$. Hence we express it in the form

$$V_{\lambda,\tau}(r^*) = \lambda^2(W(r^*) + \epsilon),$$

where W is smooth and has a nondegenerate zero minimum at $r^* = 0$ and $|\epsilon| \lesssim 1$.

We now prove the following:

Proposition 3.4. *Let W be a smooth function satisfying $W(0) = W'(0) = 0$, $W''(0) > 0$, and $|\epsilon| \lesssim 1$. Let w be a solution of the ordinary differential equation*

$$(\partial_{r^*}^2 + \lambda^2(W(r^*) + \epsilon))w(r^*) = g,$$

supported near $r^* = 0$. Then (3.27) holds.

It would be convenient to replace the norm on the right in (3.27) by $\|a_{ps}(r^*, 0, \lambda)g\|_{L^2}$. This is not entirely possible since this is a stronger norm. However, we can split g into a component g_1 with $a_{ps}(r^*, 0, \lambda)g_1 \in L^2$ plus a high frequency part:

Lemma 3.5. *Each function $g \in L^2$ supported near the photon sphere can be expressed in the form*

$$g = g_1 + \lambda^{-2}\partial_{r^*}^2 g_2$$

with g_1 and g_2 supported near the photon sphere so that

$$\|a_{ps}(r^*, 0, \lambda)g_1\|_{L^2} + \||r^{*2} + e^{-\sqrt{\ln\lambda}}|^{\frac{1}{8}} g_2\|_{L^2} + \lambda^{-2}\|\partial_{r^*} g_2\|_{L^2} \lesssim \|a_{ps}^w(\lambda)g\|_{L^2}. \tag{3.28}$$

Proof. The symbols $a_{ps}(r^*, 0, \lambda)$ and $a_{ps}(r^*, \xi, \lambda)$ are comparable provided that

$$\ln(r^{*2} + e^{-\sqrt{\ln\lambda}}) \approx \ln(r^{*2} + e^{-\sqrt{\ln\lambda}} + \lambda^{-2}\xi^2).$$

This includes a region of the form

$$D = \left\{\ln(\lambda^{-2}\xi^2) < \frac{1}{8}\ln(r^{*2} + e^{-\sqrt{\ln\lambda}})\right\}.$$

We note that the factor $\frac{1}{8}$, arising also in the exponent of the second term in (3.28), is somewhat arbitrary. A small choice leads to a better bound in (3.28).

If χ is a smooth function which is 1 in $(-\infty, -1]$ and 0 in $[0, \infty)$ then we define a smooth characteristic function χ_D of the domain D by

$$\chi_D(r^*, \xi, \lambda) = \chi(\ln(\lambda^{-2}\xi^2) - \frac{1}{8}\ln(r^{*2} + e^{-\sqrt{\ln\lambda}})).$$

One can directly compute the regularity of χ_D,

$$\chi_D \in S^0_{1,\delta}, \qquad \delta > 0.$$

To obtain the decomposition of g we set

$$g_2 = q^w g,$$

where the symbol of q is

$$q(r^*, \xi, \lambda) = \lambda^2 \xi^{-2}(1 - \chi_D).$$

Since $(a_{ps}^{-1})^w(\lambda)$ is an approximate inverse for $a_{ps}^w(\lambda)$, the estimate for g_2 in the lemma can be written in the form

$$\|(r^{*2} + e^{-\sqrt{\ln \lambda}})^{\frac{1}{8}} q^w (a_{ps}^{-1})^w(\lambda) f\|_{L^2} + \lambda^{-2}\|\partial_{r^*} q^w (a_{ps}^{-1})^w(\lambda) f\|_{L^2} \lesssim \|f\|_{L^2}. \quad (3.29)$$

In the first term it suffices to look at the principal symbol of the operator product since the remainder belongs to $OPS_{1,\delta}^{-1+\delta}$ for all $\delta > 0$. To verify that the product of the symbols is bounded we note that a_{ps}^{-1} is bounded. For the other two factors we consider two cases. If $|\xi| \gtrsim \lambda$ then both factors are bounded. On the other hand if $|\xi| \lesssim \lambda$ then in the support of q we have

$$\lambda^{-2}\xi^2 \gtrsim (r^{*2} + e^{-\sqrt{\ln \lambda}})^{\frac{1}{8}},$$

which gives

$$q \lesssim (r^{*2} + e^{-\sqrt{\ln \lambda}})^{-\frac{1}{8}}.$$

The estimate for the second term in (3.29) is similar but simpler.

It remains to consider the bound for g_1, which is given by

$$g_1 = (1 + \lambda^{-2} D_{r^*}^2 q^w) g, \qquad D_{r^*} = \frac{1}{i}\partial_{r^*}.$$

As above, the bound for g_1 can be written in the form

$$\|a_{ps}(r^*, 0, \lambda)(1 + \lambda^{-2} D_{r^*}^2 q^w)(a_{ps}^{-1})^w(\lambda) f\|_{L^2} \lesssim \|f\|_{L^2}.$$

The three operators above belong respectively to $S_{1,\delta}^{\delta}$, $S_{1,\delta}^{0}$, and $S_{1,\delta}^{\delta}$ for all $\delta > 0$. Hence the product belongs to $S_{1,\delta}^{\delta}$, and it suffices to show that its principal symbol is bounded. But the principal symbol of the product is given by

$$a_{ps}(r^*, 0, \lambda)\chi_D a_{ps}^{-1}(r^*, \xi, \lambda),$$

which is bounded due to the choice of D.

Finally we remark that as constructed the functions g_1 and g_2 are not necessarily supported near the photon sphere. This is easily rectified by replacing them with truncated versions,

$$g_1 := \chi_1(r^*) g_1, \qquad g_2 := \chi_1(r^*) g_2,$$

where χ_1 is a smooth compactly supported cutoff which equals 1 in the support of g. It is clear that the bound (3.28) is still valid after truncation. □

Using the above decomposition of g we write u in the form

$$u = \lambda^{-2} g_2 + \tilde{u}.$$

For the first term we use the above lemma to estimate

$$\lambda \|\lambda^{-2} g_2\|_{L^2} + \|\lambda^{-2} \partial_{r^*} g_2\|_{L^2} \lesssim \|a_{ps}^w(\lambda) g\|_{L^2},$$

which is stronger than what we need. For $\tilde{u}$ we write the equation

$$(\partial_{r^*}^2 + \lambda^2 (W + \epsilon))\tilde{u} = \tilde{g}, \qquad \tilde{g} = g_1 - (W + \epsilon) g_2. \tag{3.30}$$

For $\tilde{g}$ we only use a weighted L^1 bound,

$$\|(\lambda^{-1} + |W + \epsilon|)^{-\frac{1}{4}} \tilde{g}\|_{L^1} \lesssim \|a_{ps}^w(\lambda) g\|_{L^2},$$

which is obtained from the weighted L^2 bounds on g_1 and g_2 by Cauchy-Schwarz.
For $\tilde{u}$ on the other hand, it suffices to obtain a pointwise bound:

Lemma 3.6. *For each $\lambda^{-1} < \sigma < 1$ and each function $\tilde{u}$ with compact support, we have*

$$\lambda \|(a_{ps}^{-1})^w(\lambda)\tilde{u}\|_{L^2} \lesssim \|(\sigma + |W + \epsilon|)^{-\frac{1}{4}} \partial_{r^*} \tilde{u}\|_{L^\infty} + \lambda \|(\sigma + |W + \epsilon|)^{\frac{1}{4}} \tilde{u}\|_{L^\infty}.$$

Proof. Since W has a nondegenerate zero minimum at 0, if $\epsilon > -\sigma$ then $\sigma + |W + \epsilon| \approx \sigma + |\epsilon| + W$. Hence without any restriction in generality we can replace (ϵ, σ) by $(0, \sigma + |\epsilon|)$. Thus in the sequel we can assume that either $\epsilon = 0$ or $\epsilon < -\sigma$. We consider three cases:

Case 1. $|\epsilon|, \sigma < e^{-\sqrt{\ln \lambda}}$. We consider an almost orthogonal partition of $\tilde{u}$ in dyadic regions with respect to r^*:

$$\tilde{u} = \tilde{u}_{<s_0} + \sum_{s_0 \leq s < 1} \tilde{u}_s, \qquad s_0 = e^{-\frac{1}{2}\sqrt{\ln \lambda}}.$$

For each piece we can freeze r^* in the symbol of a_{ps}^{-1} and estimate in L^2,

$$\|(a_{ps}^{-1})^w(\lambda)\tilde{u}_s\|_{L^2} \approx \|a_{ps}^{-1}(s, D, \lambda)\tilde{u}_s\|_{L^2},$$
$$\|(a_{ps}^{-1})^w(\lambda)\tilde{u}_{<s_0}\|_{L^2} \approx \|a_{ps}^{-1}(0, D, \lambda)\tilde{u}_{<s_0}\|_{L^2}.$$

In the first case we use the symbol bound

$$\lambda a_{ps}^{-1}(s, \xi, \lambda) \lesssim \lambda |\ln s|^{-1} + s^{-\delta}|\xi|, \qquad \delta > 0,$$

where the second term accounts for the region where $|\xi| > \lambda s^\delta$. This yields

$$\lambda \|(a_{ps}^{-1})^w(\lambda)\tilde{u}_s\|_{L^2} \lesssim |\ln s|^{-1} \lambda \|\tilde{u}_s\|_{L^2} + s^{-\delta} \|\partial_{r^*} \tilde{u}_s\|_{L^2}.$$

In the support of $\tilde{u}_s$ we have $\sigma + |W + \epsilon| \approx s^2$; therefore by Cauchy-Schwarz we obtain

$$\lambda \|(a_{ps}^{-1})^w(\lambda)\tilde{u}_s\|_{L^2} \lesssim |\ln s|^{-1} \lambda \|(\sigma + |W + \epsilon|)^{\frac{1}{4}} \tilde{u}\|_{L^\infty} + s^{1-\delta} \|(\sigma + |W + \epsilon|)^{-\frac{1}{4}} \partial_{r^*} \tilde{u}\|_{L^\infty}.$$

The summation with respect to s follows due to the L^2 almost orthogonality of the functions $(a_{ps}^{-1})^w(\lambda)\tilde{u}_s$,

$$\lambda \left\| (a_{ps}^{-1})^w(\lambda) \sum_{s_0 \le s < 1} \tilde{u}_s \right\|_{L^2} \lesssim \lambda \left(\sum_{s_0 \le s < 1} \|(a_{ps}^{-1})^w(\lambda)\tilde{u}_s\|_{L^2}^2 \right)^{\frac{1}{2}}$$
$$\lesssim \lambda \|(\sigma + |W + \epsilon|)^{\frac{1}{4}} \tilde{u}\|_{L^\infty} + \|(\sigma + |W + \epsilon|)^{-\frac{1}{4}} \partial_{r^*} \tilde{u}\|_{L^\infty}.$$

This orthogonality is due to the fact that the kernel of $(a_{ps}^{-1})^w(\lambda)$ decays rapidly on the $\lambda^{-1+\delta}$ scale in r^*, therefore the overlapping of the two functions of the form $(a_{ps}^{-1})^w(\lambda)\tilde{u}_s$ is trivially small for nonconsecutive values of s.

On the other hand for the center piece $\tilde{u}_{<s_0}$ a similar computation yields

$$\lambda \|(a_{ps}^{-1})^w(\lambda)\tilde{u}_{<s_0}\|_{L^2} \lesssim |\ln s_0|^{-1} \lambda \|\tilde{u}_{<s_0}\|_{L^2} + s_0^{-\delta} \|\partial_{r^*} \tilde{u}_{<s_0}\|_{L^2}$$
$$\lesssim \lambda \|(\sigma + |W + \epsilon|)^{\frac{1}{4}} \tilde{u}\|_{L^\infty} + s_0^{1-\delta} \|(\sigma + |W + \epsilon|)^{-\frac{1}{4}} \partial_{r^*} \tilde{u}\|_{L^\infty},$$

where the weaker bound in the first term is due to the fact that

$$\int_{|r^*| < s_0} (\sigma + |W + \epsilon|)^{-\frac{1}{2}} dr^* \lesssim \ln \lambda.$$

Case 2. $\epsilon = 0$, $\sigma \ge e^{-\sqrt{\ln \lambda}}$. Then we select s_0 by $s_0^2 = \sigma$ and partition $\tilde{u}$ into

$$\tilde{u} = \tilde{u}_{<s_0} + \sum_{s_0 \le s \le 1} \tilde{u}_s.$$

The analysis proceeds as in the first case, with the simplification that there is no longer a singularity in the weight $(\sigma + |W + \epsilon|)^{-\frac{1}{2}}$ for $|r^*| < s_0$.

Case 3. $\sigma, e^{-\sqrt{\ln \lambda}} < -\epsilon$. Then we select s_0 by $s_0^2 = -\epsilon$ and partition $\tilde{u}$ into

$$\tilde{u} = \tilde{u}_{<s_0} + \tilde{u}_{s_0} + \sum_{s_0 < s \le 1} \tilde{u}_s.$$

Then all pieces are estimated as in Case 2 with the exception of $\tilde{u}_{s_0}$, where we have to contend with the singularity in the weight. However, compared to Case 1 we have a better integral bound

$$\int_{|r^*| \approx s_0} |W + \epsilon|^{-\frac{1}{2}} dr^* \lesssim 1$$

and the conclusion follows again.

□

Due to the above lemma, it suffices to prove that, given $|\epsilon| \lesssim 1$ and (u, g) supported near the photon sphere so that

$$(\partial_{r^*}^2 + \lambda^2(W + \epsilon))u = g, \tag{3.31}$$

then for some $\lambda^{-1} < \sigma < 1$ we have

$$\|(\sigma+|W+\epsilon|)^{-\frac{1}{4}}\partial_{r^*}u\|_{L^\infty}+\lambda\|(\sigma+|W+\epsilon|)^{\frac{1}{4}}u\|_{L^\infty}\lesssim\|(\lambda^{-1}+|W+\epsilon|)^{-\frac{1}{4}}g\|_{L^1}. \tag{3.32}$$

We remark that the first term on the left gives the L^2 bound for u_{r^*} in (3.27).

We consider three subcases depending on the choice of ϵ:

Case 4 (i). $\epsilon \gg \lambda^{-1}$. Then $|W + \epsilon| \approx r^{*2} + \epsilon$. Choosing $\sigma = \epsilon$, it suffices to prove that:

Lemma 3.7. *Suppose that $\epsilon \gg \lambda^{-1}$. Then for u with compact support solving* (3.31)*, we have*

$$\lambda(\epsilon + r^{*2})^{\frac{1}{4}}|u| + (\epsilon + r^{*2})^{-\frac{1}{4}}|u_{r^*}| \lesssim \|(\epsilon + r^{*2})^{-\frac{1}{4}}g\|_{L^1}.$$

Proof. We solve (3.31) as a Cauchy problem from both sides toward 0. For this we use an energy functional which is inspired by the classical WKB approximation,

$$E(u(r^*)) = \lambda^2(W + \epsilon)^{\frac{1}{2}}u^2 + (W + \epsilon)^{-\frac{1}{2}}u_{r^*}^2 + \frac{1}{2}W_{r^*}(W + \epsilon)^{-\frac{3}{2}}uu_{r^*}.$$

Since $|W_{r^*}| \lesssim W^{\frac{1}{2}}$, the condition $\epsilon \gg \lambda^{-1}$ guarantees that E is positive definite. Computing its derivative, we have

$$\begin{aligned}\frac{d}{dr^*}E(u(r^*)) = &\frac{1}{2}(W_{r^*r^*}(W + \epsilon)^{-\frac{3}{2}} - \frac{3}{2}W_{r^*}^2(W + \epsilon)^{-\frac{5}{2}})uu_{r^*}\\ &+(2(W + \epsilon)^{-\frac{1}{2}}u_{r^*} + \frac{1}{2}W_{r^*}(W + \epsilon)^{-\frac{3}{2}}u)g.\end{aligned}$$

This leads to the bound

$$\left|\frac{d}{dr^*}E(u(r^*))\right| \lesssim \lambda^{-1}(W + \epsilon)^{-\frac{3}{2}}E(u(r^*)) + E^{\frac{1}{2}}(u(r^*))(W + \epsilon)^{-\frac{1}{4}}g.$$

Since

$$\int \lambda^{-1}(W + \epsilon)^{-\frac{3}{2}}dr \lesssim 1,$$

the conclusion follows by Gronwall's inequality. □

Case 4 (ii). $|\epsilon| \lesssim \lambda^{-1}$. We choose $\sigma = \lambda^{-1}$. Then $\sigma + |W + \epsilon| \approx \lambda^{-1} + r^{*2}$. Hence it suffices to prove the pointwise bound:

Lemma 3.8. *Suppose that $|\epsilon| \lesssim \lambda^{-1}$. Then for u with compact support solving* (3.31)*, we have*

$$\lambda(\lambda^{-1} + r^{*2})^{\frac{1}{4}}|u| + (\lambda^{-1} + r^{*2})^{-\frac{1}{4}}|u_{r^*}| \lesssim \|(\lambda^{-1} + r^{*2})^{-\frac{1}{4}}g\|_{L^1}.$$

Proof. We use the same energy functional E as above. However, this time E is only positive definite when $W \gg \lambda^{-1}$ or equivalently $|r^*| \gg \lambda^{-\frac{1}{2}}$. Applying Gronwall as in the first case yields the conclusion of the lemma for $|r^*| \gg \lambda^{-\frac{1}{2}}$.

In the remaining interval $\{|r^*| \lesssim \lambda^{-\frac{1}{2}}\}$ we view (3.31) as a small perturbation of the equation $\partial_{r^*}^2 u = g$. Precisely, we can use the energy functional

$$E_1(u(r^*)) = \lambda^{\frac{3}{2}}|u|^2 + \lambda^{\frac{1}{2}}|u_{r^*}|^2,$$

which satisfies

$$\left|\frac{d}{dr^*} E_1(u(r^*))\right| \lesssim \lambda^{\frac{1}{2}} E_1(u(r^*)) + E_1^{\frac{1}{2}}(u(r^*))\lambda^{\frac{1}{4}} g.$$

This allows us to use Gronwall's inequality to estimate the remaining part of u. □

Case 4 (iii). $-\epsilon \gg \lambda^{-1}$. Then we choose $\sigma = |\epsilon|^{\frac{1}{3}}\lambda^{-\frac{2}{3}}$ and prove the pointwise bound:

Lemma 3.9. *Suppose that $-\epsilon \gg \lambda^{-1}$. Then for u with compact support solving* (3.31) *we have*

$$\lambda(|W+\epsilon| + |\epsilon|^{\frac{1}{3}}\lambda^{-\frac{2}{3}})^{\frac{1}{4}}|u| + (|W+\epsilon| + |\epsilon|^{\frac{1}{3}}\lambda^{-\frac{2}{3}})^{-\frac{1}{4}}|u_{r^*}| \lesssim \|(|W+\epsilon| + |\epsilon|^{\frac{1}{3}}\lambda^{-\frac{2}{3}})^{-\frac{1}{4}} g\|_{L^1}.$$

Proof. The energy E above is still useful for as long as it stays positive definite, i.e. if

$$|W_{r^*}| < 2\lambda|W+\epsilon|^{3/2}. \tag{3.33}$$

Given the quadratic behavior of W at 0, this amounts to

$$W + \epsilon \gg \lambda^{-\frac{2}{3}}|\epsilon|^{\frac{1}{3}}.$$

In this range, due to (3.33) we obtain, as in Case 4(i),

$$\left|\frac{d}{dr^*} E(u(r^*))\right| \lesssim \lambda^{-1}(W+\epsilon)^{-\frac{3}{2}} E(u(r^*)) + E^{\frac{1}{2}}(u(r^*))(W+\epsilon)^{-\frac{1}{4}} g,$$

which by Gronwall's inequality and Cauchy-Schwarz yields the desired bound.

In a symmetric region around the zeroes of $W + \epsilon$,

$$|W+\epsilon| \lesssim \lambda^{-\frac{2}{3}}|\epsilon|^{\frac{1}{3}},$$

the bounds for u and u_{r^*} remain unchanged and Eq. (3.31) behaves like a small perturbation of $\partial_{r^*}^2 u = g$, and we can use a straightforward modification of the argument above.

Finally, in the region

$$[r_-, r_+] = \{W + \epsilon < -C\lambda^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\},$$

we use an elliptic estimate. Denote

$$\omega = |W+\epsilon| + |\epsilon|^{\frac{1}{3}}\lambda^{-\frac{2}{3}}.$$

Then multiplying Eq. (3.31) by $-\lambda u$ and integrating by parts we obtain

$$\int_{r_-}^{r_+} \lambda|\partial_{r^*}u|^2 + \lambda^3|W+\epsilon||u|^2 dr^* = \int_{r_-}^{r_+} -\lambda ug dr^* + \lambda u u_{r^*}\Big|_{r_-}^{r^+}$$
$$\lesssim \lambda\|\omega^{\frac14}u\|_{L^\infty(r_-,r_+)}\|\omega^{-\frac14}g\|_{L^1} + \|\omega^{-\frac14}g\|_{L^1}^2, \tag{3.34}$$

where for the boundary terms at $r_\pm$ we have used the previously obtained pointwise bounds. On the other hand from the fundamental theorem of calculus one obtains

$$\lambda^2\|\omega^{\frac14}u\|_{L^\infty(r_-,r_+)}^2 \lesssim \int_{r_-}^{r_+} \lambda|\partial_{r^*}u|^2 + \lambda^3|W+\epsilon||u|^2 dr^*,$$

where the bound (3.33) is used for the derivative of W in $[r_-, r_+]$. Combining the last two inequalities gives the desired bound for u,

$$\lambda\|\omega^{\frac14}u\|_{L^\infty(r_-,r_+)} \lesssim \|\omega^{-\frac14}g\|_{L^1}. \tag{3.35}$$

Returning to (3.34), it also follows that

$$\int_{r_-}^{r_+} \lambda|\partial_{r^*}u|^2 + \lambda^3|W+\epsilon||u|^2 dr^* \lesssim \|\omega^{-\frac14}g\|_{L^1}^2. \tag{3.36}$$

It remains to obtain the pointwise bound for u_{r^*}. In $[r_-, r_+]$ we have $W < |\epsilon|$, therefore $W_{r^*} \lesssim |\epsilon|^{\frac12}$. Given $r_0^* \in [r_-, r_+]$ we consider an interval $r_0^* \in I \subset [r_-, r_+]$ of size $|I| = c\lambda^{-1}\omega(r_0^*)^{-\frac12}$ with a small c. In I the size of the weight ω is constant; indeed, ω can change at most by

$$|I||W_{r^*}| = c\lambda^{-1}\omega(r_0^*)^{-\frac12}\epsilon^{\frac12} \lesssim c\omega(r_0^*),$$

where at the last step we have used the bound $\omega(r_0^*) \geq |\epsilon|^{\frac13}\lambda^{-\frac23}$.

Within I we first use the L^2 bound (3.36) to estimate the average $u_{r^*}^I$ of u_{r^*} in I,

$$|u_{r^*}^I|^2 \lesssim |I|^{-1}\int_I |u_{r^*}|^2 dr^* \lesssim \omega(r_0^*)^{\frac12}\|\omega^{-\frac14}g\|_{L^1}^2.$$

It remains to compute the variation of u_{r^*} in I, which is estimated using the equation $(\partial_{r^*}^2 + \lambda^2(W+\epsilon))u = g$ and (3.35),

$$\int_I |\partial_{r^*}^2 u| dr^* \lesssim \int_I \lambda^2\omega|u| + |g| dr^* \lesssim \omega(r_0^*)^{\frac14}\|\omega^{-\frac14}g\|_{L^1}.$$

Together, the last two bounds show that

$$|u_{r^*}(r_0^*)| \lesssim \omega(r_0^*)^{\frac14}\|\omega^{-\frac14}g\|_{L^1}.$$

The proof of the lemma is concluded. □

We continue with part (b) of the proposition. We switch to the Regge-Wheeler coordinates. By taking a spherical harmonics expansion it suffices to prove the result at a fixed spherical frequency λ. Let g_λ be at spherical frequency λ with support in $\{11M/4 < r < 4M\}$. Using a time frequency multiplier with smooth symbol we can split g_λ into two components, one with high ($\gg \lambda$) time frequency and one with low time frequency. We consider the two cases separately.

Case I. g_λ is localized at time frequencies $\{|\tau| \gg (1+\lambda)\}$. This corresponds to Cases 1,2,3 in the proof of part (a). As a consequence of the results there we have the a-priori bound

$$(|\tau| + \lambda)\|u\|_{L^2} \lesssim \|(\partial_{r^*}^2 + V_{\lambda,\tau})u\|_{L^2}$$

for all u with support in $\{5M/2 < r < 5M\}$. By duality this implies that for each $g \in L^2$ with support in $\{5M/2 < r < 5M\}$ there exists a solution v to

$$(\partial_{r^*}^2 + V_{\lambda,\tau})v = g$$

in[3] $\{5M/2 < r < 5M\}$ with

$$(|\tau| + \lambda)\|v\|_{L^2} \lesssim \|g\|_{L^2}.$$

Applying this at all time frequencies $|\tau| \gg (1+\lambda)$ we find a solution u_λ to

$$L_{RW} u_\lambda = g_\lambda \tag{3.37}$$

in $\{5M/2 < r < 5M\}$ so that

$$(1+\lambda)\|u_\lambda\|_{L^2} + \|\partial_t u_\lambda\|_{L^2} \lesssim \|g_\lambda\|_{L^2}.$$

Multiplying Eq. (3.37) by $\chi_{ps}^2 u_\lambda$ and integrating by parts we obtain

$$\|\partial_{r^*}(\chi_{ps}u_\lambda)\|_{L^2}^2 \lesssim \lambda^2\|\chi_{ps}u_\lambda\|_{L^2}^2 + \|\chi_{ps}\partial_t u_\lambda\|_{L^2}^2 + \|u_\lambda\|_{L^2}^2 + \|g_\lambda\|_{L^2}^2.$$

Hence the function $v_\lambda = \chi_{ps}u_\lambda$ satisfies

$$\|\nabla v_\lambda\|_{L^2} \lesssim \|g_\lambda\|_{L^2}. \tag{3.38}$$

On the other hand, since g_λ is supported in the smaller interval $\{11M/4 < r < 4M\}$, it follows that v_λ solves the equation

$$L_{RW} v_\lambda - g_\lambda = [L_{RW}, \chi_{ps}]u_\lambda.$$

Here the right-hand side is supported in a region, away from the photon sphere, where the L^2 and LE_{ps}^* norms are equivalent. Then this is seen to satisfy

$$\|L_{RW} v_\lambda - g_\lambda\|_{L^2} \lesssim \|g_\lambda\|_{L^2}$$

by applying (3.38) with χ_{ps} replaced by a cutoff with slightly larger support.

Finally, the standard energy estimates for v_λ allow us to obtain uniform energy bounds for v_λ from the averaged energy bounds in (3.38), thus improving (3.38) to

$$\|\nabla v_\lambda\|_{L^2} + \|\nabla v_\lambda\|_{L^\infty L^2} \lesssim \|g_\lambda\|_{L^2}. \tag{3.39}$$

[3] No boundary condition is imposed on v.

Case II. g_λ is localized at time frequencies $\{|\tau| \lesssim (1+\lambda)\}$. This corresponds to Case 4 in the proof of part (a). We first observe that the result in part (a) can be strengthened to

$$\|\phi\|_{LE_{ps}}^2 \lesssim \|f\|_{LE_{ps}^*+L^1L^2}^2. \tag{3.40}$$

Indeed, suppose that $f = f_1 + f_2$ with $f_1 \in LE_{ps}^*$ and $f_2 \in L^1L^2$. We solve the forward problem

$$\Box_g \phi_2 = f_2.$$

By Theorem 1.2 and Duhamel's formula we have

$$\|\phi_2\|_{LE_0} \lesssim \|f_2\|_{L^1L^2}.$$

We truncate $\phi_2 \to \tilde{\chi}_{ps}(r)\phi_2$ in a slightly larger set than the support of f_2 and compute

$$\|\Box_g(\tilde{\chi}_{ps}\phi_2) - f_2\|_{LE_{ps}^*} = \|[\Box_g, \tilde{\chi}_{ps}]\phi_2\|_{LE_{ps}^*} \approx \|[\Box_g, \tilde{\chi}_{ps}]\phi_2\|_{L^2} \lesssim \|\phi_2\|_{LE_0},$$

since the above commutator is supported in a compact set in r away from the photon sphere.

From Duhamel's formula and part (a) of the proposition it follows that

$$\|\tilde{\chi}_{ps}\phi_2\|_{LE_{ps}} \lesssim \|f_2\|_{L^1L^2}.$$

On the other hand applying directly part (a) of the proposition to $\phi - \tilde{\chi}_{ps}\phi_2$ we obtain

$$\|\phi - \tilde{\chi}_{ps}\phi_2\|_{LE_{ps}} \lesssim \|\Box_g(\phi - \tilde{\chi}_{ps}\phi_2)\|_{LE_{ps}^*} \lesssim \|f_1\|_{LE_{ps}^*} + \|f_2\|_{L^1L^2}.$$

Hence (3.40) follows.

As a consequence of (3.40) we obtain

$$\lambda\|(a_{ps}^{-1})^w(\lambda)u_\lambda\|_{L^2} \lesssim \inf_{L_{RW}u_\lambda = g_1+g_2} \left(\|a_{ps}^w(\lambda)g_1\|_{L^2} + \|g_2\|_{L^1L^2}\right).$$

By duality, from this bound from below for L_{RW}, we obtain a local solvability result. Precisely, for each g_λ at spherical frequency λ with support in $\{5M/2 < r < 5M\}$ there is a function u_λ in the same set which solves

$$L_{RW}u_\lambda = g_\lambda \tag{3.41}$$

and satisfies the bound

$$\lambda(\|(a_{ps}^{-1})^w(\lambda)u_\lambda\|_{L^2} + \|u_\lambda\|_{L^\infty L^2}) \lesssim \|a_{ps}^w(\lambda)g_\lambda\|_{L^2}. \tag{3.42}$$

Since $(a_{ps}^{-1})^w$ has an inverse in $OPS_{1,0}^\delta$, from the first term above we also obtain an L^2 bound for u_λ, namely

$$\lambda^{1-\delta}\|u_\lambda\|_{L^2} \lesssim \|a_{ps}^w(\lambda)g_\lambda\|_{L^2}. \tag{3.43}$$

Since g_λ is localized at time frequencies $|\tau| \lesssim (1+\lambda)$, it follows that u_λ above can be assumed to have a similar time frequency localization. Hence (3.42) also gives

$$\|(a_{ps}^{-1})^w(\lambda)u_{\lambda t}\|_{L^2} + \|u_{\lambda t}\|_{L^\infty L^2} \lesssim \|a_{ps}^w(\lambda)g_\lambda\|_{L^2}. \tag{3.44}$$

We can also obtain a similar bound for the r^* derivative of u_λ. For the local energy part we multiply (3.41) by $\chi_{ps}((a_{ps}^{-1})^w(\lambda))^2\chi_{ps}u_\lambda$. After some commutations where all errors are bounded using the previous estimates we obtain

$$\|(a_{ps}^{-1})^w(\lambda)\partial_{r^*}(\chi_{ps}u_\lambda)\|_{L^2}^2 \lesssim \lambda^2\|(a_{ps}^{-1})^w(\lambda)\chi_{ps}u_\lambda\|_{L^2}^2 + \|(a_{ps}^{-1})^w(\lambda)\chi_{ps}u_{\lambda t}\|_{L^2}^2 + \lambda^{2-2\delta}\|u_\lambda\|_{L^2}^2 + \|g_\lambda\|_{L^2}^2.$$

For the $L^\infty L^2$ bound on $\partial_{r^*}(\chi_{ps}u_\lambda)$ we consider a smooth compactly supported function $\chi(t)$. Then multiplying (3.41) by $\chi^2\chi_{ps}^2u_\lambda$ and commuting we obtain

$$\|\chi\partial_{r^*}(\chi_{ps}u_\lambda)\|_{L^2}^2 \lesssim \lambda^2\|\chi\chi_{ps}u_\lambda\|_{L^2}^2 + \|\chi\chi_{ps}u_{\lambda t}\|_{L^2}^2 + \|u_\lambda\|_{L^2}^2 + \|g_\lambda\|_{L^2}^2.$$

Taking also (3.42) and (3.44) into account we have a bound on local averaged energy for $\chi\chi_{ps}u_\lambda$:

$$\|\partial_{r^*}(\chi\chi_{ps}u_\lambda)\|_{L^2}^2 + \lambda^2\|\chi\chi_{ps}u_\lambda\|_{L^2}^2 + \|\partial_t(\chi\chi_{ps}u_\lambda)\|_{L^2}^2 \lesssim \|a_{ps}^w(\lambda)g_\lambda\|_{L^2}^2.$$

By energy estimates applied to $\chi\chi_{ps}u_\lambda$ we can convert the averaged energy bound into a pointwise energy bound to obtain

$$\|\partial_{r^*}(\chi\chi_{ps}u_\lambda)\|_{L^\infty L^2}^2 + \lambda^2\|\chi\chi_{ps}u_\lambda\|_{L^\infty L^2}^2 + \|\partial_t(\chi\chi_{ps}u_\lambda)\|_{L^\infty L^2}^2 \lesssim \|a_{ps}^w(\lambda)g_\lambda\|_{L^2}^2.$$

Summing up (3.42), (3.44) and the similar bounds above for the r^* derivatives we finally obtain

$$\|(a_{ps}^{-1})^w(\lambda)\nabla(\chi_{ps}u_\lambda)\|_{L^2} + \|\nabla(\chi_{ps}u_\lambda)\|_{L^\infty L^2} \lesssim \|g_\lambda\|_{LE^*},$$

where $\nabla = (\partial_{r^*}, \partial_t, \lambda)$.

On the other hand if g_λ is supported in $\{11M/4 < r < 4M\}$, then u_λ solves the equation

$$L_{RW}\chi_{ps}u_\lambda - g_\lambda = [L_{RW}, \chi_{ps}]u_\lambda.$$

The right-hand side is supported away from the photon sphere, where the L^2 and LE_0^* norms are equivalent. Then, by applying Theorem 1.2 with χ_{ps} replaced by a cutoff with slightly larger support, this is seen to satisfy

$$\|L_{RW}\chi_{ps}u_\lambda - g_\lambda\|_{LE_0^*} \lesssim \|g_\lambda\|_{LE_{ps}^*}.$$

The proof of the proposition is concluded. □

3.2. The analysis at infinity. In the Schwarzschild space $\mathcal{M}$, if a function u in $\mathcal{M}$ is supported in $\{r > 4M\}$ we interpret it as a function in $\mathbb{R}\times\mathbb{R}^3$ by setting $u(t,x) = u(t,r,\omega)$ for $x = r\omega$. We now state the analogue of Proposition 3.3.

Proposition 3.10. a) *Let ϕ solve $\Box_g\phi = 0$ in $\{r > 4M\}$. Then*

$$\|\chi_\infty\phi\|_{LE_M}^2 \lesssim \|\phi\|_{LE_0}^2 + E[\phi](0).$$

b) *Let* $f \in LE_M^*$ *be supported in* $\{r > 4M\}$. *Then there is a function* ϕ *supported in* $\{r > 3M\}$ *which solves* $\Box_g \phi = f$ *in* $\{r \gg M\}$ *so that*

$$\sup_t E[\phi](t) + \|\phi\|_{LE_M}^2 + \|\Box_g \phi - f\|_{L^2}^2 \lesssim \|f\|_{LE_M^*}^2. \tag{3.45}$$

Proof. a) For $R > 0$ we denote by $\chi_{>R}$ a smooth cutoff function which is supported in $\{|x| > R\}$ and equals 1 in $\{|x| \geq 2R\}$. If $R > 4M$ then

$$\|(\chi_\infty - \chi_{>R})\phi\|_{LE_M}^2 \lesssim \|\phi\|_{LE_0}^2.$$

It remains to show that for a fixed sufficiently large R we have

$$\|\chi_{>R}\phi\|_{LE_M}^2 \lesssim \|\phi\|_{LE_0}^2 + E[\phi](0).$$

For this we notice that $\chi_{>R}\phi$ solves the equation

$$\Box_g(\chi_{>R}\phi) = f_1(x)\nabla\phi + f_2(x)\phi, \tag{3.46}$$

where f_1 and f_2 are supported in $\{R < |x| < 2R\}$. If R is sufficiently large then outside the ball $\{|x| \leq R\}$ the operator $\Box_g$ is a small long range perturbation of the d'Alembertian. Then the estimate (1.5) applies, see e.g. [30, Prop. 2.2] or [28, (2.23)] (with no obstacle, $\Omega = \emptyset$) and we have

$$\begin{aligned} \|\chi_{>R}\phi\|_{LE_M}^2 &\lesssim E[\chi_{>R}\phi](0) + \|\Box_g(\chi_{>R}\phi)\|_{LE_M^*}^2 \\ &\lesssim E[\phi](0) + \|[\Box_g, \chi_{>R}]\phi\|_{L^2}^2 \\ &\lesssim E[\phi](0) + \|\phi\|_{LE_0}^2, \end{aligned}$$

where in the last two steps we have used the compact support of $\Box_g(\chi_{>R}\phi) = [\Box_g, \chi_{>R}]\phi$.

b) Let R be large enough, as in part (a). For $|x| > R$ the Schwarzchild metric g is a small long range perturbation of the Minkowski metric, according to the definition in [29]. We consider a second metric $\tilde{g}$ in $\mathbb{R}^{3+1}$ which coincides with g in $\{|x| > R\}$ but which is globally a small long range perturbation of the Minkowski metric. Let ψ be the forward solution to $\Box_{\tilde{g}}\psi = f$. Then we set

$$\phi = \chi_{>R}\psi.$$

The estimate (1.5) holds for the metric $\tilde{g}$, therefore we obtain

$$\sup_t E[\psi](t) + \|\psi\|_{LE_M} \lesssim \|f\|_{LE_M^*}^2.$$

Then the same bound holds as well for ϕ. Furthermore, we can compute the error

$$\Box_g\phi - f = (\chi_{>R} - 1)f + [\Box_g, \chi_{>R}]\psi.$$

This has compact spatial support, and can be easily estimated in L^2 as in part (a). □

3.3. Proof of Theorem 3.2. Given $f \in LE^*$ we split it into

$$f = \chi_{eh} f + \chi_{ps} f + \chi_\infty f.$$

For the last two terms we use part (b) of Propositions 3.3, 3.10 to produce approximate solutions ϕ_{ps} and ϕ_∞ near the photon sphere, respectively near infinity. Adding them up we obtain an approximate solution

$$\phi_0 = \phi_{ps} + \phi_\infty$$

for the equation $\Box_g \phi = f$. Due to (3.24) and (3.45) we obtain for ϕ_0 the bound

$$\sup_{\tilde{v}} E[\phi_0](\tilde{v}) + \|\phi_0\|_{LE}^2 \lesssim \|f\|_{LE^*}^2, \tag{3.47}$$

while the error

$$f_1 = \Box_g(\phi_{ps} + \phi_\infty) - f$$

is supported away from $r = 3M$ and $r = \infty$ and satisfies

$$\|f_1\|_{LE_0^*} \approx \|f_1\|_{L^2} \lesssim \|f\|_{LE^*}.$$

Then we find $\phi = \phi_0 + \phi_1$ by solving

$$\Box_g \phi_1 = f_1 \in LE_0^*, \qquad \phi_1[0] = \phi[0] - \phi_0[0].$$

By Theorem 1.2 we obtain the LE_0 bound for ϕ_1. It remains to improve this to an LE bound for ϕ_1. By part (a) of Proposition 3.10 we can estimate $\|\chi_\infty \phi\|_{LE_M}$.

Near the photon sphere we would like to apply part (a) of Proposition 3.3 to $\chi_{ps}\phi$. However we cannot proceed in an identical manner because part (a) of Proposition 3.3 does not involve the Cauchy data of ϕ at $t = 0$, and instead applies to functions ϕ defined on the full real axis in t. To address this issue we extend ϕ_1 backward in t to the set $\mathcal{M}'_R$, by solving the homogeneous problem $\Box_g \phi_1 = 0$ in $\mathcal{M}'_R$, with matching Cauchy data on the common boundary of $\mathcal{M}_R$ and $\mathcal{M}'_R$. The extended function ϕ_1 belongs to both $LE(\mathcal{M}_R)$ and $LE(\mathcal{M}'_R)$, and now we can estimate $\chi_{ps}\phi_1$ via part (a) of Proposition 3.3.

4. Strichartz Estimates

In this section we prove Theorem 1.4. The theorem follows from the following two propositions. The first gives the result for the right-hand side, f, in the dual local energy space:

Proposition 4.11. *Let (ρ, p, q) be a nonsharp Strichartz pair. Then for each $\phi \in LE$ with $\Box_g \phi \in LE^* + L^1_{\tilde{v}} L^2$ we have*

$$\|\nabla \phi\|_{L^p_{\tilde{v}} \dot{H}^{-\rho,q}}^2 \lesssim E[\phi](0) + \|\phi\|_{LE}^2 + \|\Box_g \phi\|_{LE^* + L^1_{\tilde{v}} L^2}^2. \tag{4.48}$$

The second one allows us to use $L^{p_2'} L^{q_2'}$ in the right-hand side of the wave equation.

Proposition 4.12. *There is a parametrix K for $\Box_g$ so that for all nonsharp Strichartz pairs (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) we have*

$$\sup_{\tilde v} E[Kf](\tilde v) + E[Kf](\Sigma_R^+) + \|Kf\|_{LE}^2 + \|\nabla Kf\|^2_{L^{p_1}_{\tilde v}\dot H^{-\rho_1,q_1}} \lesssim \|f\|^2_{L^{p_2'}_{\tilde v}\dot H^{\rho_2,q_2'}}, \tag{4.49}$$

and the error estimate

$$\|\Box_g Kf - f\|_{LE^*+L^1_{\tilde v}L^2} \lesssim \|f\|_{L^{p_2'}_{\tilde v}\dot H^{\rho_2,q_2'}}. \tag{4.50}$$

We first show how to use the propositions in order to prove the theorem.

Proof of Theorem 1.4. Suppose that $\Box_g\phi = f$ with $f \in L^{p_2'}\dot H^{\rho_2,q_2'}$. We write ϕ as

$$\phi = \phi_1 + Kf$$

with K as in Proposition 4.12. By (4.49) the Kf term satisfies all the required estimates; therefore it remains to consider ϕ_1. Using also (4.50) we obtain

$$\|\Box_g\phi_1\|^2_{LE^*+L^1_{\tilde v}L^2} + E[\phi_1](0) \lesssim E[\phi](0) + \|f\|^2_{L^{p_2'}\dot H^{\rho_2,q_2'}}.$$

Then Theorem 3.2 combined with Duhamel's formula yields

$$\|\phi_1\|^2_{LE} + \|\Box_g\phi_1\|^2_{LE^*+L^1_{\tilde v}L^2} + \sup_{\tilde v} E[\phi_1](\tilde v) \lesssim E[\phi](0) + \|f\|^2_{L^{p_2'}\dot H^{\rho_2,q_2'}}.$$

Finally the $L^{p_1}\dot H^{-\rho_1,q_1}$ bound for $\nabla\phi_1$ follows by Proposition 4.11. □

We continue with the proofs of the two propositions.

Proof of Proposition 4.11. By Duhamel's formula and Theorem 3.2 we can neglect the L^1L^2 part of $\Box_g\phi$. Hence in the sequel we assume that $\Box_g\phi \in LE^*$.

We use cutoffs to split the space into three regions, namely near the event horizon, near the photon sphere and near infinity,

$$\phi = \chi_{eh}\phi + \chi_{ps}\phi + \chi_\infty\phi.$$

Due to the definition of the LE and LE^* norms we have

$$\begin{aligned} E[\phi](0) + \|\phi\|^2_{LE} + \|\Box_g\phi\|^2_{LE^*} \gtrsim\ & E[\chi_{eh}\phi](0) + \|\chi_{eh}\phi\|^2_{H^1} + \|\Box_g(\chi_{eh}\phi)\|^2_{L^2} \\ & + E[\chi_{ps}\phi](0) + \|\chi_{ps}\phi\|^2_{LE_{ps}} + \|\Box_g(\chi_{ps}\phi)\|^2_{LE^*_{ps}} \\ & + E[\chi_\infty\phi](0) + \|\chi_\infty\phi\|^2_{LE_M} + \|\Box_g(\chi_\infty\phi)\|^2_{LE^*_M}. \end{aligned}$$

Proving this requires commuting $\Box_g$ with the cutoffs. However this is straightforward since the LE and LE^* norms are equivalent to the H^1, respectively L^2, norm in the support of $\nabla\chi_{eh}$, $\nabla\chi_{ps}$ and $\nabla\chi_\infty$.

It remains to prove the $L^p_{\tilde v}\dot H^{-\rho,q}$ bound for each of the three terms in $\nabla\phi$. We consider the three cases separately:

I. The estimate near the event horizon. This is the easiest case. Given ϕ supported in $\{r < 11M/4\}$, we partition it on the unit scale with respect to $\tilde{v}$,

$$\phi = \sum_{j\in\mathbb{Z}} \chi(\tilde{v} - j)\phi,$$

where χ is a suitable smooth compactly supported bump function. Commuting the cut-offs with $\Box_g$ one easily obtains the square summability relation

$$\sum_{j\in\mathbb{N}} \|\chi(\tilde{v} - j)\phi\|^2_{H^1} + \|\Box_g(\chi(\tilde{v} - j)\phi)\|^2_{L^2} + E[\chi(\tilde{v} - j)\phi](0)$$
$$\lesssim \|\phi\|^2_{H^1} + \|\Box_g\phi\|^2_{L^2} + E[\phi](0),$$

where the energy term on the left is nonzero only for finitely many j. Since each of the functions $\chi(\tilde{v} - j)\phi$ have compact support, they satisfy the Strichartz estimates due to the local theory; see [32,39,47]. The above square summability with respect to j guarantees that the local estimates can be added up.

II. The estimate near the photon sphere. For ϕ supported in $\{5M/2 < r < 5M\}$ we need to show that

$$\|\nabla\phi\|^2_{L^p_{\tilde{v}} H^{-\rho,q}} \lesssim E[\phi](0) + \|\phi\|^2_{LE_{ps}} + \|\Box_g\phi\|^2_{LE^*_{ps}}.$$

We use again the Regge-Wheeler coordinates. Then the operator $\Box_g$ is replaced by L_{RW}. The potential V can be neglected due to the straightforward bound

$$\|V\phi\|_{LE^*_{ps}} \lesssim \|\phi\|_{LE_{ps}}.$$

Indeed, for ϕ at spherical frequency λ we have

$$\|V\phi\|_{LE^*_{ps}} \lesssim |\ln(2+\lambda)|^{\frac{1}{2}}\|\phi\|_{L^2} \lesssim \lambda|\ln(2+\lambda)|^{-\frac{1}{2}}\|\phi\|_{L^2} \lesssim \|\phi\|_{LE_{ps}}.$$

We introduce the auxiliary function

$$\psi = A^{-1}_{ps}\phi.$$

By the definition of the LE_{ps} norm we have

$$\|\psi\|_{H^1} \lesssim \|\phi\|_{LE_{ps}}. \tag{4.51}$$

We also claim that

$$\|L_{RW}\psi\|_{L^2} \lesssim \|\phi\|_{LE_{ps}} + \|L_{RW}\phi\|_{L^2}. \tag{4.52}$$

Since A^{-1}_{ps} is L^2 bounded, this is a consequence of the commutator bound

$$[A^{-1}_{ps}, L_{RW}] : LE_{ps} \to L^2,$$

or equivalently

$$[A^{-1}_{ps}, L_{RW}]A_{ps} : H^1 \to L^2. \tag{4.53}$$

It suffices to consider the first term in the symbol calculus, as the remainder belongs to $OPS^{\delta}_{1,\delta}$, mapping H^δ to L^2 for all $\delta > 0$. The symbol of the first term is

$$q(\xi, r^*, \lambda) = \{a_{ps}^{-1}(\lambda), \xi^2 + r^{-3}(r - 2M)\lambda^2\} a_{ps}(\lambda),$$

and a-priori we have $q \in S^{1+\delta}_{1,\delta}$. For a better estimate we compute the Poisson bracket

$$q(\xi, r^*, \lambda) = a_{ps}^{-1}(\lambda)\gamma_y(y, \ln \lambda) \frac{4\xi r^* - 2\xi \partial_{r^*}(r^{-3}(r - 2M))}{r^{*2} + \lambda^{-2}\xi^2},$$

where $y = r^{*2} + \lambda^{-2}\xi^2$. The first two factors on the right are bounded. The third is bounded by λ since $\partial_{r^*}(r^{-3}(r-2M))$ vanishes at $r^* = 0$. In addition, q is supported in $|\xi| \lesssim \lambda$. Hence we obtain $q \in \lambda S^0_{1-\delta,\delta}$. Then the commutator bound (4.53) follows.

Given (4.51) and (4.52), we argue as in the first case, namely we localize ψ to time intervals of unit length and then apply the local Strichartz estimates. By summing over these strips we obtain

$$\|\nabla\psi\|_{L^p H^{-\rho,q}} \lesssim \|\phi\|_{LE_{ps}} + \|L_{RW}\phi\|_{L^2}$$

for all sharp Strichartz pairs (ρ, p, q).

To return to ϕ we invert A_{ps}^{-1},

$$\phi = A_{ps}\psi + (1 - A_{ps}A_{ps}^{-1})\phi.$$

The second term is much more regular,

$$\|\nabla(1 - A_{ps}A_{ps}^{-1})\phi\|_{L^2 H^{1-\delta}} \lesssim \|\phi\|_{LE_{ps}}, \qquad \delta > 0;$$

therefore it satisfies all the Strichartz estimates simply by Sobolev embeddings.

For the main term $A_{ps}\psi$ we take advantage of the fact that we only seek to prove the nonsharp Strichartz estimates for ϕ. The nonsharp Strichartz estimates for ψ are obtained from the sharp ones via Sobolev embeddings,

$$\|\nabla\psi\|_{H^{-\rho_2,q_2}} \lesssim \|\nabla\psi\|_{H^{-\rho_1,q_1}}, \qquad \frac{3}{q_2} + \rho_2 = \frac{3}{q_1} + \rho_1, \quad \rho_1 < \rho_2.$$

To obtain the nonsharp estimates for ϕ instead, we need a slightly stronger form of the above bound, namely

Lemma 4.13. *Assume that* $1 < q_1 < q_2 < \infty$. *Then*

$$\|A_{ps}u\|_{H^{-\rho_2,q_2}} \lesssim \|u\|_{H^{-\rho_1,q_1}}, \qquad \frac{3}{q_2} + \rho_2 = \frac{3}{q_1} + \rho_1. \tag{4.54}$$

Proof. We need to prove that the operator

$$B = Op^w(\xi^2 + \lambda^2 + 1)^{-\frac{\rho_2}{2}} A_{ps} Op^w(\xi^2 + \lambda^2 + 1)^{\frac{\rho_1}{2}}$$

maps L^{q_1} into L^{q_2}. The principal symbol of B is

$$b_0(r^*, \xi, \lambda) = (\xi^2 + \lambda^2 + 1)^{\frac{\rho_1 - \rho_2}{2}} a_{ps}(r^*, \xi, \lambda),$$

and by the pdo calculus the remainder is easy to estimate,

$$B - b_0^w \in OPS_{1,0}^{\rho_1-\rho_2-1+\delta}, \qquad \delta > 0.$$

The conclusion of the lemma will follow from the Hardy-Littlewood-Sobolev inequality if we prove a suitable pointwise bound on the kernel K of b_0^w, namely

$$|K(r_1^*, \omega_1, r_2^*, \omega_2)| \lesssim (|r_1^* - r_2^*||\omega_1 - \omega_2|^2)^{-1+\frac{1}{q_1}-\frac{1}{q_2}}. \tag{4.55}$$

For fixed r^* we consider a smooth dyadic partition of unity in frequency as follows:

$$1 = \chi_{\{|\xi|>\lambda\}} + \sum_{\mu \text{ dyadic}} \chi_{\{\lambda\approx\mu\}} \left(\chi_{\{|\xi|\lesssim\nu_0\}} + \sum_{\nu=\nu_0}^{\mu} \chi_{\{|\xi|\approx\nu\}} \right),$$

where $\nu_0 = \nu_0(\lambda, r^*)$ is given by

$$\ln \nu_0(\lambda, r^*) = \ln \lambda + \max\{\ln r^*, -\sqrt{\ln \lambda}\}.$$

This leads to a similar decomposition for b_0, namely

$$b_0 = b_{00} + \sum_{\mu} \left(b_{\mu,<\nu_0} + \sum_{\nu=\nu_0}^{\mu} b_{\mu\nu} \right).$$

In the region $|\xi| \gtrsim \lambda$ the symbol b_0 is of class $S^{\rho_1-\rho_2}$, which yields a kernel bound for b_{00} of the form

$$|K_{00}(r_1^*, \omega_1, r_2^*, \omega_2)| \lesssim (|r_1^* - r_2^*| + |\omega_1 - \omega_2|)^{-3-\rho_1+\rho_2}.$$

The symbols of $b_{\mu\nu}$ are supported in $\{|\xi| \approx \nu,\ \lambda \approx \mu\}$, are smooth on the same scale and have size $\ln(\nu^{-1}\mu)\mu^{\rho_2-\rho_1}$. Hence their kernels satisfy bounds of the form

$$|K_{\mu,\nu}(r_1^*, \omega_1, r_2^*, \omega_2)| \lesssim \ln(\nu^{-1}\mu)\mu^{\rho_1-\rho_2}\nu(|r_1^* - r_2^*|\nu + 1)^{-N}\mu^2(|\omega_1 - \omega_2|\mu + 1)^{-N},$$

and similarly for $K_{\mu,<\nu_0}$. Then (4.55) follows after summation. □

III. The estimate near infinity. Let us first recall the setup from [29]. We fix a Littlewood-Paley dyadic decomposition of frequency space in $\mathbb{R}^3$,

$$1 = \sum_{k=-\infty}^{\infty} S_k(D), \quad \operatorname{supp} s_k \subset \{2^{k-1} < |\xi| < 2^{k+1}\}.$$

Functions u in $\mathbb{R} \times \mathbb{R}^3$ which are localized to frequency 2^k are measured in

$$\|u\|_{X_k} = 2^{k/2}\|u\|_{L^2(A_{<-k})} + \sup_{j\geq -k} \||x|^{-1/2}u\|_{L^2(A_j)}, \tag{4.56}$$

where

$$A_j = \mathbb{R} \times \{2^j \leq |x| \leq 2^{j+1}\}, \quad A_{<j} = \mathbb{R} \times \{|x| \leq 2^j\}.$$

As in [29], by X^0 we denote the space of functions in $\mathbb{R} \times \mathbb{R}^3$ with norm

$$\|u\|_{X^0}^2 = \sum_{k=-\infty}^{\infty} \|S_k u\|_{X_k}^2, \tag{4.57}$$

and by Y^0 the dual norm

$$\|u\|_{Y^0}^2 = \sum_{k=-\infty}^{\infty} \|S_k u\|_{X_k'}^2,$$

where $X_k^{'}$ is the dual norm of X_k.

One can establish the following (see [29, Lemma 1])

Lemma 4.14. *The following inequalities hold:*

$$\sup_j 2^{-j/2} \|\nabla u\|_{L^2(A_j)} \lesssim \|\nabla u\|_{X^0} \tag{4.58}$$

and its dual

$$\|u\|_{Y^0} \lesssim \|u\|_{LE_M^*}. \tag{4.59}$$

For small deviations from the Minkowski metric, one can also establish stronger local energy estimates involving the X^0 and Y^0 norms; more precisely, one can prove (see [29, Theorem 4]):

Lemma 4.15. *Let $\tilde{g}$ be a sufficiently small, long range perturbation of the Minkowski metric. Then, for all solutions u to the inhomogeneous problem $\Box_{\tilde{g}} u = f$, one has*

$$\|\nabla u\|_{L_t^\infty L_x^2}^2 + \|\nabla u\|_{X^0}^2 \lesssim E[u](0) + \|f\|_{Y^0 + L_t^1 L_x^2}^2.$$

We now return to proving our estimate. For ϕ supported in $\{r > 4M\}$ we need to show that

$$\|\nabla \phi\|_{L^p \dot{H}^{-\rho,q}}^2 \lesssim E[\phi](0) + \|\phi\|_{LE_M}^2 + \|\Box_g \phi\|_{LE_M^*}^2. \tag{4.60}$$

For large R we split ϕ into a near and a far part

$$\phi = \chi_{>R}\phi + \chi_{<R}\phi,$$

and estimate

$$\begin{aligned} E[\phi](0) + \|\phi\|_{LE_M}^2 + \|\Box_g \phi\|_{LE_M^*}^2 \gtrsim\ & E[\chi_{>R}\phi](0) + \|\chi_{>R}\phi\|_{LE_M}^2 + \|\Box_g(\chi_{>R}\phi)\|_{LE_M^*}^2 \\ & + E[\chi_{<R}\phi](0) + \|\chi_{<R}\phi\|_{H^1}^2 + \|\Box_g(\chi_{<R}\phi)\|_{L^2}^2. \end{aligned}$$

The term $\chi_{<R}\phi$ has compact support in r and can be treated as in the first case (i.e. near the event horizon). Hence without any restriction in generality we can restrict ourselves to the case when ϕ is supported in $\{r > R\}$. But in this region the operator $\Box_g$ is a small long range perturbation of $\Box$; therefore the results of [29] apply. More precisely, from [29, Theorem 7(a)] we obtain

$$\|\nabla \phi\|_{L^p \dot{H}^{-\rho,q}}^2 \lesssim E[\phi](0) + \|\nabla \phi\|_{X^0}^2 + \|\Box_g \phi\|_{LE^*}^2.$$

This does not directly imply (4.60), since the X^0 norm is stronger than LE_M. However, we can apply Lemma 4.15 and (4.59) to obtain the bound

$$\|\nabla\phi\|_{X^0}^2 \lesssim E[\phi](0) + \|\Box_g \phi\|_{LE_M^*}^2.$$

□

Proof of Proposition 4.12. We split f into

$$f = \chi_{eh} f + \chi_{ps} f + \chi_{\infty} f$$

and construct the parametrix separately in the three regions.
I. The parametrix near the event horizon. We further partition the term $\chi_{eh} f$ into unit intervals

$$\chi_{eh} f = \sum_j \chi(\tilde{v} - j)\chi_{eh} f$$

with χ supported in $[-1, 1]$, so that each component has compact support in the region

$$D_j = \{r_0 \le r < 11M/4,\ \ j-2 < \tilde{v} < j+2\}.$$

Let ψ_j be the forward solution to

$$\Box_g \psi_j = \chi(\tilde{v} - j)\chi_{eh} f.$$

Due to the local Strichartz estimates for variable coefficient wave equations, we obtain the uniform bounds

$$\|\nabla\psi_j\|_{L^{p_1}H^{-\rho_1,q_1}(D_j)} + \|\nabla\psi_j\|_{L^\infty L^2(D_j)} + \|\psi_j\|_{L^\infty L^2(D_j)} \lesssim \|\chi(\tilde{v}-j)\chi_{eh} f\|_{L^{p_2'}H^{\rho_2,q_2'}}.$$

Next we truncate ψ_j using a cutoff function $\tilde{\chi}(\tilde{v} - j, r)$ which is supported in D_j and equals 1 in the support of $\chi(\tilde{v}-j)\chi_{eh}$. Then the bound above also holds for the truncated functions $\phi_j = \tilde{\chi}(\tilde{v} - j, r)\psi_j$,

$$\|\nabla\phi_j\|_{L^{p_1}H^{-\rho_1,q_1}} + \|\nabla\phi_j\|_{L^\infty L^2} + \|\phi_j\|_{L^\infty L^2(D_j)} \lesssim \|\chi(\tilde{v} - j)\chi_{eh} f\|_{L^{p_2'}H^{\rho_2,q_2'}}. \tag{4.61}$$

In addition,

$$\Box_g \phi_j - \chi(\tilde{v} - j)\chi_{eh} f = [\Box_g, \tilde{\chi}(\tilde{v} - j, r)]\psi_j;$$

therefore

$$\|\Box_g \phi_j - \chi(\tilde{v} - j)\chi_{eh} f\|_{L^2} \lesssim \|\chi(\tilde{v} - j)\chi_{eh} f\|_{L^{p_2'}H^{\rho_2,q_2'}}. \tag{4.62}$$

Finally, by energy estimates we also obtain a bound for the energy of ϕ_j on the future space-like boundary of D_j at $r = r_0$,

$$\|\nabla\phi_j\|_{L^2(D_j\cap\{r=r_0\})} + \|\phi_j\|_{L^2(D_j\cap\{r=r_0\})} \lesssim \|\chi(\tilde{v} - j)\chi_{eh} f\|_{L^{p_2'}H^{\rho_2,q_2'}}. \tag{4.63}$$

To conclude we set

$$K_{eh} f = \sum_j \phi_j.$$

Summing up the bounds (4.61), (4.62) and (4.63) for ϕ_j we obtain the desired bounds for K_{eh}, namely

$$\sup_{\tilde{v}} E[K_{eh}f](\tilde{v}) + E[K_{eh}f](\Sigma_R^+) + \|K_{eh}f\|_{H^1}^2 + \|\nabla K_{eh}f\|^2_{L^{p_1}H^{-\rho_1,q_1}} \lesssim \|\chi_{eh}f\|^2_{L^{p_2'}H^{\rho_2,q_2'}},$$

respectively the error estimate

$$\|\Box_g K_{eh}f - \chi_{eh}f\|_{L^2} \lesssim \|\chi_{eh}f\|_{L^{p_2'}H^{\rho_2,q_2'}}.$$

II. The parametrix near the photon sphere. We work in the Regge-Wheeler coordinates. Arguing as in the previous case we produce a parametrix $\tilde{K}_{ps}$ with the property that, for each f supported in $\{5M/2+\epsilon < r < 5M-\epsilon\}$, the function $\tilde{K}_{ps}f$ is supported in $\{5M/2 < r < 5M\}$ and satisfies the bounds

$$\sup_t E[\tilde{K}_{ps}f](t) + \|\tilde{K}_{ps}f\|^2_{H^1_{x,t}} + \|\nabla \tilde{K}_{ps}f\|^2_{L^{p_1}H^{-\rho_1,q_1}} \lesssim \|f\|^2_{L^{p_2'}H^{\rho_2,q_2'}},$$

and the error estimate

$$\|L_{RW}\tilde{K}_{ps}f - f\|_{L^2} \lesssim \|f\|_{L^{p_2'}H^{\rho_2,q_2'}}.$$

Then we define the localized parametrix near the photon sphere K_{ps} as

$$K_{ps}f = A_{ps}^{-1}\tilde{K}_{ps}\tilde{\chi}_{ps}A_{ps}(\chi_{ps}f),$$

with $\tilde{\chi}_{ps} = 1$ in the support of χ_{ps} and slightly larger support. Then we show that K_{ps} satisfies the required bounds.

We recall that (ρ_2, p_2, q_2) is a nonsharp Strichartz pair. Then by (4.54) we can write

$$\|\tilde{\chi}_{ps}A_{ps}(\chi_{ps}f)\|_{L^{p_3'}H^{\rho_3,q_3'}} \lesssim \|\chi_{ps}f\|_{L^{p_2'}H^{\rho_2,q_2'}}$$

for some other Strichartz pair (ρ_3, p_3, q_3) with $p_3 = p_2$ and $q_3 < q_2$. Since A_{ps}^{-1} is L^2 bounded, from the above bounds for $\tilde{K}_{ps}$ we obtain

$$\sup_t E[K_{ps}f](t) + \|K_{ps}f\|^2_{H^1} \lesssim \|\chi_{ps}f\|^2_{L^{p_2'}H^{\rho_2,q_2'}}.$$

By using (4.54) with A_{ps} replaced by the weaker operator A_{ps}^{-1} we also obtain the $L^{p_1}H^{-\rho_1,q_1}$ bound for $K_{ps}f$:

$$\|\nabla K_{ps}f\|^2_{L^{p_1}H^{-\rho_1,q_1}} \lesssim \|\nabla \tilde{K}_{ps}\tilde{\chi}_{ps}A_{ps}(\chi_{ps}f)\|^2_{L^pH^{-\rho,q}} \lesssim \|\chi_{ps}f\|^2_{L^{p_2'}H^{\rho_2,q_2'}},$$

where (ρ, p, q) is another Strichartz pair with $p = p_1$ and $q < q_1$.

It remains to consider the error estimate,

$$\|L_{RW}K_{ps}f - \chi_{ps}f\|_{LE^*+L^1_{\tilde{v}}L^2} \lesssim \|\chi_{ps}f\|_{L^{p_2'}H^{\rho_2,q_2'}}, \tag{4.64}$$

for which we compute

$$L_{RW}K_{ps}f - \chi_{ps}f = [L_{RW}, A_{ps}^{-1}]\tilde{K}_{ps}\tilde{\chi}_{ps}A_{ps}(\chi_{ps}f) \\ +A_{ps}^{-1}(L_{RW}\tilde{K}_{ps} - I)\tilde{\chi}_{ps}A_{ps}(\chi_{ps}f) \\ +(A_{ps}^{-1}\tilde{\chi}_{ps}A_{ps} - \tilde{\chi}_{ps})(\chi_{ps}f).$$

We consider each term in the above decomposition. For the first term, due to the H^1 bound for $\tilde{K}$, we need the commutator bound

$$[L_{RW}, A_{ps}^{-1}] : H^1 \to LE^*$$

or equivalently

$$A_{ps}[L_{RW}, A_{ps}^{-1}] : H^1 \to L^2,$$

which is almost identical to (4.53) and is proved in the same manner.

The bound for the second term is a direct consequence of the L^2 error bound for $\tilde{K}$.

Finally, for the last term we know that $(A_{ps}^{-1}A_{ps} - I) \in OPS_{1,0}^{-1+\delta}$; therefore using Sobolev embeddings we estimate

$$\|(A_{ps}^{-1}\tilde{\chi}_{ps}A_{ps} - \tilde{\chi}_{ps})(\chi_{ps}f)\|_{L^{p_2'}H^{\frac{1}{2}}} \lesssim \|\chi_{ps}f\|_{L^{p_2'}H^{\rho_2,q_2'}}.$$

This concludes the proof of (4.64) since

$$L^{p_2'}H^{\frac{1}{2}} \subset L^2H^{\frac{1}{2}} + L^1H^{\frac{1}{2}} \subset LE_{ps}^* + L^1L^2.$$

III. The parametrix near infinity. We now consider the last component of f, namely $\chi_\infty f$. For some large R we separate it into two parts,

$$\chi_\infty f = (\chi_\infty - \chi_{>R})f + \chi_{>R}f.$$

The first part has compact support in r; therefore we can handle it as in the first case (i.e. near the event horizon), producing a parametrix $K_\infty^{<R}$. For the second part we modify the metric g for $r < R$ to a metric $\tilde{g}$ which is a small, long-range perturbation of $\Box$. We let ψ_∞ be the forward solution to

$$\Box_{\tilde{g}}\psi_\infty = \chi_{>R}f.$$

We consider a second cutoff function $\tilde{\chi}_{>R}$ which is supported in $r > R$ and equals 1 in the support of $\chi_{>R}$. Then we define

$$K_\infty^{>R}f = \tilde{\chi}_{>R}\psi_\infty.$$

It remains to show that $K_\infty^{>R}$ satisfies the appropriate bounds,

$$\sup_t E[K_\infty^{>R}f](t) + \|K_\infty^{>R}f\|_{LE_M}^2 + \|\nabla K_\infty^{>R}f\|_{L^{p_1}\dot{H}^{-\rho_1,q_1}}^2 \lesssim \|\chi_{>R}f\|_{L^{p_2'}\dot{H}^{\rho_2,q_2'}}^2,$$

respectively the error estimate

$$\|\Box_g K_\infty^{>R}f - \chi_{>R}f\|_{LE_M^*} \lesssim \|\chi_{>R}f\|_{L^{p_2'}\dot{H}^{\rho_2,q_2'}}.$$

These are easily obtained by applying the following lemma to ψ_∞:

Lemma 4.16. *Let $f \in L^{p_2'} \dot{H}^{\rho_2, q_2'}$. Then the forward solution ψ to $\Box_{\tilde{g}} \psi = f$ satisfies the bound*

$$\sup_t E[\psi](t) + \|\psi\|^2_{LE_M} + \|\nabla\psi\|^2_{L^{p_1} \dot{H}^{-\rho_1, q_1}} \lesssim \|f\|^2_{L^{p_2'} \dot{H}^{\rho_2, q_2'}}. \tag{4.65}$$

It remains to prove the lemma. This largely follows from [29, Theorem 6], but there is an interesting technical issue that needs clarification. Precisely, [29, Theorem 6] shows that we have the bound

$$\sup_t E[\psi](t) + \|\nabla\psi\|^2_{X^0} + \|\nabla\psi\|^2_{L^{p_1} \dot{H}^{-\rho_1, q_1}} \lesssim \|f\|^2_{L^{p_2'} \dot{H}^{\rho_2, q_2'}}. \tag{4.66}$$

By Lemma 4.14, we are left with proving that

$$\sup_{j \in \mathbb{Z}} 2^{-\frac{3j}{2}} \|\psi\|_{L^2(A_j)} \lesssim \|f\|_{L^{p_2'} \dot{H}^{\rho_2, q_2'}}. \tag{4.67}$$

We note that this does not follow from Lemma 4.14; this is a forbidden endpoint of the Hardy inequality in [29, Lemma 1(b)].

However, the bound (4.67) can still be obtained, although in a roundabout way. Precisely, from (4.66) we have

$$\sup_t E[\psi](t) \lesssim \|f\|^2_{L^{p_2'} \dot{H}^{\rho_2, q_2'}} \tag{4.68}$$

for the forward in time evolution, and similarly for the backward in time problem.

On the other hand, a straightforward modification of the classical Morawetz estimates (see e.g. [27]) for the wave equation shows that the solutions to the homogeneous wave equation $\Box_{\tilde{g}} \psi = 0$ satisfy

$$\sup_{j \in \mathbb{Z}} 2^{-3j} \|\psi\|^2_{L^2(A_j)} \lesssim E[\psi](0). \tag{4.69}$$

Denote by $1_{t>s} H(t,s)$ the forward fundamental solution for $\Box_{\tilde{g}}$ and by $H(t,s)$ its backward extension to a solution to the homogeneous equation, $\Box_{\tilde{g}} H(t,s) = 0$. Combining the bounds (4.68) and (4.69) shows that

$$\sup_j 2^{-3j} \left\| \int_{\mathbb{R}} H(t,s) f(s) ds \right\|^2_{L^2(A_j)} \lesssim \|f\|^2_{L^{p_2'} \dot{H}^{\rho_2, q_2'}}.$$

Since $p_2' < 2$, by the Christ-Kiselev lemma [11], it follows that

$$\sup_j 2^{-3j} \left\| \int_t^\infty H(t,s) f(s) ds \right\|^2_{L^2(A_j)} \lesssim \|f\|^2_{L^{p_2'} \dot{H}^{\rho_2, q_2'}},$$

which is exactly (4.67). □

5. The Critical NLW

In this section we prove Theorem 1.5. We first consider (1.18) in the compact region $\mathcal{M}_C$. We denote by ψ the solution to the homogeneous equation

$$\Box_g \psi = 0, \qquad \psi_{|\Sigma_0} = \phi_0, \ \tilde{K}\psi_{|\Sigma_0} = \phi_1,$$

and by Tf the solution to the inhomogeneous problem

$$\Box_g (Tf) = f, \qquad TF_{|\Sigma_0} = 0, \ \tilde{K}Tf_{|\Sigma_0} = 0.$$

Then we can rewrite the nonlinear equation (1.18) in the form

$$\phi = \psi \pm T(\phi^5). \tag{5.70}$$

We define Sobolev spaces in $\mathcal{M}_C$ by restricting to $\mathcal{M}_C$ functions in the same Sobolev space which are compactly supported in a larger open set. By the local Strichartz estimates we have

$$\|\psi\|_{H^{\frac{1}{2},4}(\mathcal{M}_C)} \lesssim E[\phi](\Sigma_0)$$

and

$$\|Tf\|_{H^{\frac{1}{2},4}(\mathcal{M}_C)} \lesssim \|f\|_{H^{\frac{1}{2},\frac{4}{3}}(\mathcal{M}_C)}.$$

At the same time we have the multiplicative estimate

$$\|\phi^5\|_{H^{\frac{1}{2},\frac{4}{3}}(\mathcal{M}_C)} \lesssim \|\phi\|^5_{H^{\frac{1}{2},4}(\mathcal{M}_C)}.$$

Then for small initial data we can use the contraction principle to solve (5.70) and obtain a solution $\phi \in H^{\frac{1}{2},4}(\mathcal{M}_C)$. In addition, still by local Strichartz estimates, the solution ϕ will have finite energy on any space-like surface, in particular on the forward and backward space-like boundary of $\mathcal{M}_C$. Thus we obtain

$$E[\phi](\Sigma_R^-) \lesssim E[\phi](\Sigma_0).$$

It remains to solve (1.18) in $\mathcal{M}_R$ (and its other three symmetrical copies). Using the $(\tilde{v}, r, \omega)$ coordinates in $\mathcal{M}_R$ we define ψ and T as above, but with Cauchy data on Σ_R^-.

By the global Strichartz estimates in Theorem 1.4, for (s, p) as in the theorem we have

$$\|\psi\|_{L^p\dot{H}^{s,p}(\mathcal{M}_R)} \lesssim E[\phi](\Sigma_R^-)$$

and

$$\|Tf\|_{L^p\dot{H}^{s,p}(\mathcal{M}_R)} \lesssim \|f\|_{L^1L^2}.$$

In particular we can take $p = 5$ which corresponds to $s = \frac{3}{10}$. By Sobolev embeddings we have

$$\|\phi\|_{L^5L^{10}} \lesssim \|\phi\|_{\dot{H}^{\frac{3}{10},5}};$$

therefore

$$\|\phi^5\|_{L^1L^2} \lesssim \|\phi\|^5_{\dot{H}^{\frac{3}{10},5}}.$$

Hence we can solve (5.70) using the contraction principle and obtain a solution $\phi \in \dot{H}^{\frac{3}{10},5}$. This implies that $\phi^5 \in L^1L^2$, which yields all of the other Strichartz estimates, as well as the energy bound on the forward boundary Σ_R^+ of $\mathcal{M}_R$. This concludes the proof of the theorem.

Acknowledgement. The authors are grateful to M. Dafermos and I. Rodnianski for pointing out their novel way of taking advantage of the red shift effect in [14], and to N. Burq and M. Zworski for useful conversations concerning the analysis near trapped null geodesics.

References

1. Alinhac, S.: On the Morawetz–Keel-Smith-Sogge inequality for the wave equation on a curved background. Publ. Res. Inst. Math. Sci. **42**, 705–720 (2006)
2. Blue, P., Soffer, A.: Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates. Adv. Differential Equations **8**, 595–614 (2003)
3. Blue, P., Soffer, A.: The wave equation on the Schwarzschild metric II: Local decay for the spin-2 Regge-Wheeler equation. J. Math. Phys. **46**(1), 012502 (2005)
4. Blue, P., Soffer, A.: *Errata for "Global existence and scatttering for the nonlinear Schrödinger equation on Schwarzschild manifolds", "Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates", and "The wave equation on the Schwarzschild metric II: Local decay for the spin 2 Regge Wheeler equation"*. http://arxiv.org/abs/gr-qc/0608073v3, 2006
5. Blue, P., Soffer, A.: Phase space analysis on some black hole manifolds. J. Funct. Anal. **256**(1), 1–90 (2009)
6. Blue, P., Soffer, A.: *Improved decay rates with small regularity loss for the wave equation about a Schwarzschild black hole*. http://arxiv.org/abs/math/0612168v1[math.Ap], 2006
7. Blue, P., Sterbenz, J.: Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. Commun. Math. Phys. **268**, 481–504 (2006)
8. Brenner, P.: On $L_p - L_{p'}$ estimates for the wave-equation. Math. Z. **145**, 251–254 (1975)
9. Burq, N.: Smoothing effect for Schrödinger boundary value problems. Duke Math. J. **123**, 403–427 (2004)
10. Burq, N., Gérard, P., Tzvetkov, N.: Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. Amer. J. Math. **126**, 569–605 (2004)
11. Christ, M., Kiselev, A.: Maximal functions associated to filtrations. J. Funct. Anal. **179**, 409–425 (2001)
12. Christianson, H.: Dispersive estimates for manifolds with one trapped orbit. Comm. Part. Diff. Eqs. **33**, 1147–1174 (2008)
13. Colinde Verdière, Y., Parisse, B.: Equilibre instable en Regime Semi-classique: I - concentration microlocale. Commun. PDE. **19**, 1535–1563 (1994)
14. Dafermos, M., Rodnianski, I.: *The red-shift effect and radiation decay on black hole spacetimes*. http://arXiv.org/abs/gr-qc/0512119v1, 2005
15. Dafermos, M., Rodnianski, I.: *A note on energy currents and decay for the wave equation on a Schwarzschild background*. http://arXiv.org/ab0710.0171v1[math.Ap], 2007
16. Dafermos, M., Rodnianski, I.: A proof of Price's law for the collapse of a self-gravitating scalar field. Invent. Math. **162**, 381–457 (2005)
17. Ginibre, J., Velo, G.: Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. **133**, 50–68 (1995)
18. Hawking, S.W., Ellis, G.F.R.: *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics, No. 1., London, New York: Cambridge University Press, 1973
19. Kapitanski, L.: Some generalizations of the Strichartz-Brenner inequality. Leningrad Math. J. **1**, 693–726 (1990)
20. Keel, M., Smith, H., Sogge, C.D.: Almost global existence for some semilinear wave equations. J. Anal. Math. **87**, 265–279 (2002)

21. Keel, M., Tao, T.: Endpoint Strichartz estimates. Amer. J. Math. **120**, 955–980 (1998)
22. Kenig, C.E., Ponce, G., Vega, L.: On the Zakharov and Zakharov-Schulman systems. J. Funct. Anal. **127**, 204–234 (1995)
23. Kronthaler, J.: *Decay Rates for Spherical Scalar Waves in the Schwarzschild Geometry*. http://arXiv.org/abs/0709.3703v1[gr-qc], 2007
24. Kay, B., Wald, R.: Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. Class. Quant. Grav. **4**, 893–898 (1987)
25. Laba, I., Soffer, A.: Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds. Helv. Phys. Acta **72**, 274–294 (1999)
26. Lindblad, H., Sogge, C.D.: On existence and scattering with minimal regularity for semilinear wave equations. J. Funct. Anal. **130**, 357–426 (1995)
27. Metcalfe, J., Sogge, C.D.: Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. SIAM J. Math. Anal. **38**, 391–420 (2006)
28. Metcalfe, J., Sogge, C.D.: Global existence of null-form wave equations in exterior domains. Math. Z. **256**, 521–549 (2007)
29. Metcalfe, J., Tataru, D.: *Global parametrices and dispersive estimates for variable coefficient wave equations*. http://arXiv.org/abs/0707.1191v2[math.Ap], 2007
30. Metcalfe, J., Tataru, D.: *Global parametrices and dispersive estimates for variable coefficient wave equations in exterior domains*. http://arXiv.org/abs/0806.3409v2[math.Ap], 2009
31. Misner, C.W., Thorne, K.S., Wheeler, J.A.: *Gravitation*. San Francisco, CA: W. H. Freeman and Co., 1973
32. Mockenhaupt, G., Seeger, A., Sogge, C.D.: Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. J. Amer. Math. Soc. **6**, 65–130 (1993)
33. Morawetz, C.: Time decay for the nonlinear Klein-Gordon equations. Proc. Roy. Soc. Ser. A. **306**, 291–296 (1968)
34. Nonnenmacher, S., Zworski, M.: *Quantum decay rates in chaotic scattering*. http://arXiv.org/abs/0706.3242v2[math-ph], 2007
35. Pecher, H.: Nonlinear small data scattering for the wave and Klein-Gordon equations. Math. Z. **185**, 261–270 (1984)
36. Price, R.: Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. Phys. Rev. D (3) **5**, 2419–2438 (1972)
37. Ralston, J.V.: Solutions of the wave equation with localized energy. Comm. Pure Appl. Math. **22**, 807–823 (1969)
38. Schlag, W., Soffer, A., Staubach, W.: *Decay for the wave and Schroedinger evolutions on manifolds with conical ends, Part II*. http://arXiv.org/abs/0801.2001v1[math.Ap], 2008
39. Smith, H.F.: A parametrix construction for wave equations with $C^{1,1}$ coefficients. Ann. Inst. Fourier (Grenoble) **48**, 797–835 (1998)
40. Smith, H.F., Sogge, C.D.: Global Strichartz estimates for nontrapping perturbations of the Laplacian. Comm. Part. Diff. Eqs. **25**, 2171–2183 (2000)
41. Sterbenz, J.: Angular regularity and Strichartz estimates for the wave equation. With an appendix by I. Rodnianski. Int. Math. Res. Not. **2005**, 187–231 (2005)
42. Strauss, W.A.: Dispersal of waves vanishing on the boundary of an exterior domain. Comm. Pure Appl. Math. **28**, 265–278 (1975)
43. Strichartz, R.S.: A priori estimates for the wave equation and some applications. J. Funct. Anal. **5**, 218–235 (1970)
44. Strichartz, R.S.: Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. Duke Math. J. **44**, 705–714 (1977)
45. Tataru, D.: Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. Amer. J. Math. **122**, 349–376 (2000)
46. Tataru, D.: Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients II. Amer. J. Math. **123**, 385–423 (2001)
47. Tataru, D.: Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III. J. Amer. Math. Soc. **15**, 419–442 (2002)
48. Tataru, D.: Parametrices and dispersive estimates for Schroedinger operators with variable coefficients. Amer. J. Math. **130**, 571–634 (2008)
49. Twainy, F.: *The Time Decay of Solutions to the Scalar Wave Equation in Schwarzschild Background*. Thesis, University of California San Diego, 1989
50. Wald, R.M.: Note on the stability of the Schwarzschild metric. J. Math. Phys. **20**, 1056–1058 (1979)
51. Wald, R.M.: *General Relativity*. Chicago, IL: University of Chicago Press, 1984

Communicated by S. Zelditch