A Sharpened Nuclearity Condition
and the Uniqueness of the Vacuum in QFT

Wojciech Dybalski

Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1,
D-37077 Göttingen, Germany. E-mail: dybalski@theorie.physik.uni-goettingen.de

Abstract: It is shown that only one vacuum state can be prepared with a finite amount of energy and it appears, in particular, as a limit of physical states under large timelike translations in any theory which satisfies a phase space condition proposed in this work. This new criterion, related to the concept of additivity of energy over isolated subsystems, is verified in massive free field theory. The analysis entails very detailed results about the momentum transfer of local operators in this model.

1. Introduction

Since the seminal work of Haag and Swieca [1] restrictions on the phase space structure of a theory formulated in terms of compactness and nuclearity conditions have proved very useful in the structural analysis of quantum field theories [2–6] and in the construction of interacting models [7,8]. However, the initial goal of Haag and Swieca, namely to characterize theories which have a reasonable particle interpretation, has not been accomplished to date. While substantial progress was made in our understanding of the timelike asymptotic behavior of physical states [9–15], several important convergence and existence questions remained unanswered. As a matter of fact, it turned out that the original compactness condition introduced in [1] is not sufficient to settle these issues.

Therefore, in the present article we propose a sharpened phase space condition, stated below, which seems to be more appropriate. We show that it is related to additivity of energy over isolated subregions and implies that there is only one vacuum state within the energy-connected component of the state space, as one expects in physical spacetime [16]. We stress that there may exist other vacua in a theory complying with our condition, but, loosely speaking, they are separated by an infinite energy barrier and thus not accessible to experiments. The convergence of physical states to the vacuum state under large timelike translations is a corollary of this discussion. A substantial part of this work is devoted to the proof that the new condition holds in massive scalar free field theory. As a matter of fact, it holds also in the massless case which will be treated

elsewhere. These last results demonstrate that the new criterion is consistent with the basic postulates of local relativistic quantum field theory [17] which we now briefly recall.

The theory is based on a local net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of von Neumann algebras, which are attached to open, bounded regions of spacetime $\mathcal{O} \subset \mathbb{R}^{s+1}$ and act on a Hilbert space $\mathcal{H}$. The global algebra of this net, denoted by $\mathfrak{A}$, is irreducibly represented on this space. Moreover, $\mathcal{H}$ carries a strongly continuous unitary representation of the Poincaré group $\mathbb{R}^{s+1} \rtimes L_+^\uparrow \ni (x, \Lambda) \to U(x, \Lambda)$ which acts geometrically on the net

$$\alpha_{(x,\Lambda)}\mathfrak{A}(\mathcal{O}) = U(x, \Lambda)\mathfrak{A}(\mathcal{O})U(x, \Lambda)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + x). \tag{1.1}$$

We adopt the usual notation for translated operators $\alpha_x A = A(x)$ and functionals $\alpha_x^* \varphi(A) = \varphi(A(x))$, where $A \in \mathfrak{A}$, $\varphi \in \mathfrak{A}^*$, and demand that the joint spectrum of the generators of translations $H, P_1, \ldots, P_s$ is contained in the closed forward lightcone $\overline{V}_+$. We denote by P_E the spectral projection of H (the Hamiltonian) on the subspace spanned by vectors of energy lower than E. Finally, we identify the predual of $B(\mathcal{H})$ with the space $\mathcal{T}$ of trace-class operators on $\mathcal{H}$ and denote by $\mathcal{T}_E = P_E \mathcal{T} P_E$ the space of normal functionals of energy bounded by E. We assume that there exists a vacuum state $\omega_0 \in \mathcal{T}_E$ and introduce the subspace $\mathring{\mathcal{T}}_E = \{\varphi - \varphi(I)\omega_0 \mid \varphi \in \mathcal{T}_E\}$ of functionals with the asymptotically dominant vacuum contribution subtracted.

The main object of our investigations is the family of maps $\Pi_E : \mathring{\mathcal{T}}_E \to \mathfrak{A}(\mathcal{O})^*$ given by

$$\Pi_E(\varphi) = \varphi|_{\mathfrak{A}(\mathcal{O})}, \quad \varphi \in \mathring{\mathcal{T}}_E. \tag{1.2}$$

Fredenhagen and Hertel argued in some unpublished work that in physically meaningful theories these maps should be subject to the following restriction:

Condition $C_\sharp$. *The maps Π_E are compact for any $E \geq 0$ and double cone $\mathcal{O} \subset \mathbb{R}^{s+1}$.*

This condition is expected to hold in theories exhibiting mild infrared behavior [19]. In order to restrict the number of local degrees of freedom also in the ultraviolet part of the energy scale, Buchholz and Porrmann proposed a stronger condition which makes use of the concept of nuclearity[1] [19]:

Condition $N_\sharp$. *The maps Π_E are p-nuclear for any $0 < p \leq 1$, $E \geq 0$ and double cone $\mathcal{O} \subset \mathbb{R}^{s+1}$.*

This condition is still somewhat conservative since it does not take into account the fact that for any $\varphi \in \mathring{\mathcal{T}}_E$ the restricted functionals $\alpha_x^* \varphi|_{\mathfrak{A}(\mathcal{O})}$ should be arbitrarily close to zero apart from translations varying in some compact subset of $\mathbb{R}^{s+1}$, depending on φ. It seems therefore desirable to introduce a family of norms on $\mathcal{L}(\mathring{\mathcal{T}}_E, X)$, where X is some Banach space, given for any $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in \mathbb{R}^{s+1}$ by

$$\|\Pi\|_{x_1,\ldots,x_N} = \sup_{\varphi \in \mathring{\mathcal{T}}_{E,1}} \left(\sum_{k=1}^N \|\Pi(\alpha_{x_k}^* \varphi)\|^2 \right)^{\frac{1}{2}}, \quad \Pi \in \mathcal{L}(\mathring{\mathcal{T}}_E, X), \tag{1.3}$$

---

[1] We recall that a map $\Pi : X \to Y$ is p-nuclear if there exists a decomposition $\Pi = \sum_n \Pi_n$ into rank-one maps s.t. $\nu^p := \sum_n \|\Pi_n\|^p < \infty$. The p-norm $\|\Pi\|_p$ of this map is the smallest such ν and it is equal to zero for $p > 1$ [18]. Note that for any norm on $\mathcal{L}(X, Y)$ one can introduce the corresponding class of p-nuclear maps. Similarly, we say that a map is compact w.r.t. a given norm on $\mathcal{L}(X, Y)$ if it can be approximated by finite rank mappings in this norm.

and the corresponding family of p-norms $\|\Pi\|_{p,x_1,\ldots,x_N}$, (see footnote 1). It is easily seen that if Π_E satisfies Condition $C_\sharp$, respectively $N_\sharp$, then Π_E is also compact, respectively p-nuclear, with respect to the above norms, and vice versa. Important additional information is contained in the dependence of the nuclear p-norms on N. In Sect. 2 we argue that the natural assumption is:

Condition $N_\natural$. *The maps Π_E are p-nuclear w.r.t. the norms $\|\cdot\|_{x_1,\ldots,x_N}$ for any $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \mathbb{R}^{s+1}$, $0 < p \leq 1$, $E \geq 0$ and double cone $\mathcal{O} \subset \mathbb{R}^{s+1}$. Moreover, there holds for their nuclear p-norms*

$$\limsup \|\Pi_E\|_{p,x_1,\ldots,x_N} \leq c_{p,E}, \tag{1.4}$$

where $c_{p,E}$ is independent of N and the limit is taken for configurations $x_1, \ldots, x_N$, where all $x_i - x_j$, $i \neq j$, tend to spacelike infinity.

Restricting attention to the case $N = 1$, it is easily seen that Condition $N_\natural$ implies Condition $N_\sharp$, but not vice versa.

Our paper is organized as follows: In Sect. 2 we show that Condition $N_\natural$ implies a certain form of additivity of energy over isolated subsystems and guarantees the physically meaningful vacuum structure of a theory. More technical part of this discussion is postponed to Appendix A. In Sect. 3 we recall some basic facts about massive scalar free field theory and its phase space structure. In Appendix B we provide a simple proof of the known fact that Condition $N_\sharp$ holds in this model. Section 4 contains our main technical result, namely the proof that Condition $N_\natural$ holds in this theory as well. The argument demonstrates, in this simple example, the interplay between locality and positivity of energy which allows to strengthen Condition $N_\sharp$. The paper concludes with a brief outlook where we apply our techniques to the harmonic analysis of translation automorphisms.

2. Physical Consequences of Condition $N_\natural$

In this section we show that theories satisfying Condition $N_\natural$ exhibit two physically desirable properties: a variant of additivity of energy over isolated subregions and the feature that only one vacuum state can be prepared given a finite amount of energy. Combining this latter property with covariance of a theory under Lorentz transformations we will conclude that physical states converge to the vacuum state under large timelike translations.

The concept of additivity of energy over isolated subsystems does not have an unambiguous meaning in the general framework of local relativistic quantum field theory and we rely here on the following formulation: We introduce the family of maps $\Theta_{E,x_1,\ldots,x_N} : \mathring{\mathcal{T}}_E \to \mathfrak{A}(\mathcal{O})^* \otimes \mathbb{C}^N_{\sup}$, given by

$$\Theta_{E,x_1,\ldots,x_N}(\varphi) = \left(\Pi_E(\alpha^*_{x_1}\varphi), \ldots, \Pi_E(\alpha^*_{x_N}\varphi)\right), \tag{2.1}$$

where $\mathbb{C}^N_{\sup}$ denotes the space $\mathbb{C}^N$ equipped with the norm $\|\underline{z}\| = \sup_{k\in\{1,\ldots,N\}} |z_k|$. We claim that a mild (polynomial) growth of the ε-contents[2] $\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N}$ of these maps

---

[2] The ε-content of a map $\Pi : X \to Y$ is the maximal natural number $\mathcal{N}(\varepsilon)$ for which there exist elements $\varphi_1, \ldots, \varphi_{\mathcal{N}(\varepsilon)} \in X_1$ s.t. $\|\Pi(\varphi_i) - \Pi(\varphi_j)\| > \varepsilon$ for $i \neq j$. Clearly, $\mathcal{N}(\varepsilon)$ is finite for any $\varepsilon > 0$ if and only if the map Π is compact.

with N, (when $x_i - x_j$, $i \neq j$, tend to spacelike infinity), is a signature of additivity of energy over isolated subregions. In order to justify this formulation we provide a heuristic argument: Given a functional $\varphi \in \mathring{T}_{E,1}$, we denote by E_k the 'local energy content' of the restricted functional $\varphi|_{\mathfrak{A}(\mathcal{O}+x_k)}$. Additivity of energy should then imply that $E_1 + \cdots + E_N \leq E$ for large spacelike distances between the regions $\mathcal{O}+x_1, \ldots, \mathcal{O}+x_N$. This suggests that to calculate $\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N}$ one should count all the families of functionals $(\varphi_1, \ldots, \varphi_N)$, $\varphi_k \in \mathring{T}_{E_k,1}$, $E_1 + \cdots + E_N \leq E$, which can be distinguished, up to accuracy ε, by measurements in $\mathcal{O}+x_1, \ldots, \mathcal{O}+x_N$. Relying on this heuristic reasoning we write

$$\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N} = \#\{ (n_1 \ldots n_N) \in \mathbb{N}^{* \times N} \,|\, n_1 \leq \mathcal{N}(\varepsilon)_{E_1}, \ldots, n_N \leq \mathcal{N}(\varepsilon)_{E_N},$$
$$\text{for some } E_1, \ldots, E_N \geq 0 \text{ s.t. } E_1 + \cdots + E_N \leq E \}, \quad (2.2)$$

where we made use of the fact that the number of functionals from $\mathring{T}_{E_k,1}$ which can be discriminated, up to ε, by observables localized in the region $\mathcal{O}+x_k$ is equal to the ε-content $\mathcal{N}(\varepsilon)_{E_k}$ of the map $\Pi_{E_k} : \mathring{T}_{E_k} \to \mathfrak{A}(\mathcal{O}+x_k)$ given by $\Pi_{E_k}(\varphi) = \varphi|_{\mathfrak{A}(\mathcal{O}+x_k)}$. Anticipating that $\mathcal{N}(\varepsilon)_{E_k}$ tends to one for small E_k we may assume that

$$\mathcal{N}(\varepsilon)_{E_k} \leq 1 + c(\varepsilon, E)E_k \quad (2.3)$$

for $E_k \leq E$. (This is valid e.g. in free field theory due to Sect. 7.2 of [20] and Proposition 2.5 (iii) of [21]). From the heuristic formula (2.2) and the bound (2.3) we obtain the estimate which grows only polynomially with N,

$$\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N} \leq \#\{ (n_1 \ldots n_N) \in \mathbb{N}^{* \times N} \,|\, n_1 + \cdots + n_N \leq N + c(\varepsilon, E)E \}$$
$$\leq (N+1)^{c(\varepsilon,E)E}, \quad (2.4)$$

where the last inequality can be verified by induction in N. Omitting the key condition $E_1 + \cdots + E_N \leq E$ in (2.2) and setting $E_k = E$ instead, one would arrive at an exponential growth of $\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N}$ as a function of N. Thus the moderate (polynomial) increase of this quantity with regard to N is in fact a clear-cut signature of additivity of energy over isolated subsystems. It is therefore of interest that this feature prevails in all theories complying with Condition $N_\natural$ as shown in the subsequent theorem whose proof is given in Appendix A.

Theorem 2.1. *Suppose that Condition $N_\natural$ holds. Then the ε-content $\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N}$ of the map $\Theta_{E,x_1,\ldots,x_N}$ satisfies*

$$\limsup \mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N} \leq (4eN)^{\frac{c(E)}{\varepsilon^2}}, \quad (2.5)$$

where the constant $c(E)$ is independent of N and the limit is taken for configurations $x_1, \ldots, x_N$, where all $x_i - x_j$, $i \neq j$, tend to spacelike infinity.

Now let us turn our attention to the vacuum structure of the theories under study. In physical spacetime one expects that there is a unique vacuum state which can be prepared with a finite amount of energy. This fact is related to additivity of energy and can be derived from Condition $N_\natural$.

Theorem 2.2. *Suppose that a state* $\omega \in \mathfrak{A}^*$ *belongs to the weak* closure of* $\mathcal{T}_{E,1}$ *for some* $E \geq 0$ *and is invariant under translations along some spacelike ray. Then the following assertions hold:*

(a) *If Condition* $C_\sharp$ *is satisfied,* ω *is a vacuum state.*
(b) *If Condition* $N_\natural$ *is satisfied,* ω *coincides with the vacuum state* ω_0.

Proof. (a) We pick any $A \in \mathfrak{A}(\mathcal{O})$, a test function $f \in S(\mathbb{R}^{s+1})$ s.t. supp $\tilde{f} \cap \overline{V}_+ = \emptyset$ and define the energy decreasing operator $A(f) = \int A(x)f(x)d^{s+1}x$. Next, we parametrize the ray from the statement of the theorem as $\{ \lambda\hat{e} \mid \lambda \in \mathbb{R} \}$, where $\hat{e} \in \mathbb{R}^{s+1}$ is some spacelike unit vector, choose a compact subset $K \subset \mathbb{R}$ and estimate

$$
\begin{aligned}
\omega(A(f)^*A(f))|K| &= \int_K d\lambda \, \omega\left((A(f)^*A(f))(\lambda\hat{e})\right) \\
&= \lim_{n\to\infty} \varphi_n\left(\int_K d\lambda \, (A(f)^*A(f))(\lambda\hat{e})\right) \\
&\leq \left\| P_E \int_K d\lambda \, (A(f)^*A(f))(\lambda\hat{e}) \, P_E \right\|.
\end{aligned}
\tag{2.6}
$$

In the first step we exploited invariance of the state ω under translations along the spacelike ray. In the second step we made use of local normality of this state, which follows from Condition $C_\sharp$, in order to exchange its action with integration. Approximating ω by a sequence of functionals $\varphi_n \in \mathcal{T}_{E,1}$, we arrived at the last expression. (Local normality of ω and existence of an approximating sequence can be shown as in [22] p. 49). Now we can apply a slight modification of Lemma 2.2 from [11], (see also Lemma 4.1 below), to conclude that the last expression on the r.h.s. of (2.6) is bounded uniformly in K. As $|K|$ can be made arbitrarily large, it follows that

$$
\omega(A(f)^*A(f)) = 0
\tag{2.7}
$$

for any $A \in \mathfrak{A}(\mathcal{O})$ and f as defined above. Since equality (2.7) extends to any $A \in \mathfrak{A}$, we conclude that ω is a vacuum state in the sense of Definition 4.3 from [23]. Invariance of ω under translations and validity of the relativistic spectrum condition in its GNS-representation follow from Theorem 4.5 of [23], provided that the functions $\mathbb{R}^{s+1} \ni x \to \omega(A^*B(x))$ are continuous for any $A, B \in \mathfrak{A}$. Since local operators form a norm-dense subspace of $\mathfrak{A}$, it is enough to prove continuity for $A, B \in \mathfrak{A}(\mathcal{O})$ for any open, bounded region $\mathcal{O}$. For this purpose we recall from [19] that Condition $C_\sharp$ has a dual formulation which says that the maps $\Xi_E : \mathfrak{A}(\mathcal{O}) \to B(\mathcal{H})$ given by $\Xi_E(A) = P_E A P_E$ are compact for any open, bounded region $\mathcal{O}$ and any $E \geq 0$. Given any sequence of spacetime points $x_n \to x$, there holds $A^*(B(x_n) - B(x)) \to 0$ in the strong topology and, by compactness of the maps Ξ_E, $P_E A^*(B(x_n) - B(x))P_E \to 0$ in the norm topology in $B(\mathcal{H})$. Now the required continuity follows from the bound

$$
|\omega\left(A^*(B(x_n) - B(x))\right)| \leq \|P_E A^*(B(x_n) - B(x))P_E\|
\tag{2.8}
$$

which can be established with the help of the approximating sequence $\varphi_n \in \mathcal{T}_{E,1}$.

(b) We note that for any open, bounded region $\mathcal{O}$, $E \geq 0$ and $\varepsilon > 0$, Condition $N_\natural$ allows for such N and $x_1, \ldots, x_N$, belonging to the spacelike ray, that $2N^{-\frac{1}{2}}\|\Pi_E\|_{x_1,\ldots,x_N} \leq \frac{\varepsilon}{3}$.

For arbitrary $A \in \mathfrak{A}(\mathcal{O})_1$ we can find $\varphi \in \mathcal{T}_{E,1}$ s.t. $\sup_{k \in \{1,...,N\}} |\omega(A(x_k)) - \varphi(A(x_k))| \le \frac{\varepsilon}{3}$ and $|1 - \varphi(I)| \le \frac{\varepsilon}{3}$. Next, we note that

$$|\omega(A) - \omega_0(A)| \le |\omega(A) - \varphi(I)\omega_0(A)| + \frac{\varepsilon}{3}$$

$$\le \frac{1}{N} \sum_{k=1}^{N} |\alpha_{x_k}^* \omega(A) - \alpha_{x_k}^* \varphi(A)| + \frac{1}{N} \sum_{k=1}^{N} |\alpha_{x_k}^* \varphi(A) - \varphi(I)\alpha_{x_k}^* \omega_0(A)| + \frac{\varepsilon}{3}$$

$$\le \sup_{k \in \{1,...,N\}} |\omega(A(x_k)) - \varphi(A(x_k))| + 2N^{-\frac{1}{2}} \|\Pi_E\|_{x_1,...,x_N} + \frac{\varepsilon}{3} \le \varepsilon, \quad (2.9)$$

where in the second step we made use of the fact that both ω and ω_0 are invariant under the translations $x_1, \ldots, x_N$ and in the third step we used the Hölder inequality and the fact that $\frac{1}{2}(\varphi - \varphi(I)\omega_0) \in \mathring{\mathcal{T}}_{E,1}$. We conclude that the states ω and ω_0 coincide on any local operator and therefore on the whole algebra $\mathfrak{A}$. $\square$

The above result is of relevance to the problem of convergence of physical states to the vacuum under large timelike translations. In fact, the following lemma asserts that the respective limit points are invariant under translations in some spacelike hyperplane.

Lemma 2.3 (D.Buchholz, private communication). *Suppose that Condition $C_\sharp$ holds. Let ω_0^+ be a weak* limit point as $t \to \infty$ of the net $\{\alpha_{t\hat{e}}^* \omega\}_{t \in \mathbb{R}_+}$ of states on $\mathfrak{A}$, where $\hat{e} \in \mathbb{R}^{s+1}$ is a timelike unit vector and ω is a state from $\mathcal{T}_E$ for some $E \ge 0$. Then ω_0^+ is invariant under translations in the spacelike hyperplane $\{\hat{e}^\perp\} = \{x \in \mathbb{R}^{s+1} \mid \hat{e} \cdot x = 0\}$, where dot denotes the Minkowski scalar product.*

Proof. Choose $x \in \{\hat{e}^\perp\}$, $x \ne 0$. Then there exists a Lorentz transformation Λ and $y^0, y^1 \in \mathbb{R} \backslash \{0\}$ s.t. $\Lambda \hat{e} = y^0 \hat{e}_0$, $\Lambda x = y^1 \hat{e}_1$, where $\hat{e}_\mu$, $\mu = 0, 1, \ldots, s$ form the canonical basis in $\mathbb{R}^{s+1}$. We set $v = \frac{y^1}{y^0}$ and introduce the family of Lorentz transformations $\Lambda_t = \Lambda^{-1} \tilde{\Lambda}_t \Lambda$, where $\tilde{\Lambda}_t$ denotes the boost in the direction of $\hat{e}_1$ with rapidity $\text{arsinh}(\frac{v}{t})$. By the composition law of the Poincaré group, the above transformations composed with translations in timelike direction give also rise to spacelike translations

$$(0, \Lambda_t)(t\hat{e}, I)(0, \Lambda_t^{-1}) = (t\Lambda_t \hat{e}, I), \quad t\Lambda_t \hat{e} = t\sqrt{1 + (v/t)^2} \hat{e} + x. \quad (2.10)$$

We make use of this fact in the following estimate:

$$|\alpha_{t\hat{e}}^* \omega(A) - \alpha_{t\hat{e}}^* \omega(A(x))| \le |\omega(\alpha_{t\hat{e}} A) - \omega(\alpha_{\Lambda_t} \alpha_{t\hat{e}} \alpha_{\Lambda_t^{-1}} A)|$$

$$+ |\alpha_{t\Lambda_t\hat{e}}^* \omega(A) - \alpha_{t\hat{e}}^* \omega(A(x))|, \quad (2.11)$$

where $A \in \mathfrak{A}(\mathcal{O})$. The first term on the r.h.s. of (2.11) satisfies the bound

$$\left| \omega(\alpha_{t\hat{e}} A) - \omega\left(\alpha_{\Lambda_t} \alpha_{t\hat{e}} \alpha_{\Lambda_t^{-1}} A\right) \right|$$

$$\le |\alpha_{t\hat{e}}^* \omega(A - \alpha_{\Lambda_t^{-1}} A)| + |(\omega - \alpha_{\Lambda_t}^* \omega)(\alpha_{t\hat{e}} \alpha_{\Lambda_t^{-1}} A)|$$

$$\le \|P_E(A - \alpha_{\Lambda_t^{-1}} A)P_E\| + \sup_{s \in \mathbb{R}_+} \|\omega - \alpha_{\Lambda_t}^* \omega\|_{\mathfrak{A}(\tilde{\mathcal{O}} + s\hat{e})} \|A\|, \quad (2.12)$$

where $\tilde{\mathcal{O}}$ is a slightly larger region than $\mathcal{O}$. Clearly, $\Lambda_t \to I$ for $t \to \infty$ and therefore $\alpha_{\Lambda_t} \to \text{id}$ in the point - weak open topology. Then the above expression tends to

zero in this limit by the dual form of Condition $C_\sharp$ and the assumption that Lorentz transformations are unitarily implemented. (The argument is very similar to the last step in the proof of Theorem 2.2 (a). We note that the restriction on Lorentz transformations can be relaxed to a suitable regularity condition). The second term on the r.h.s. of (2.11) converges to zero by the dual variant of Condition $C_\sharp$ and the following bound:

$$
\begin{aligned}
|\alpha^*_{t\Lambda_t \hat{e}}\omega(A) - \alpha^*_{t\hat{e}}\omega(A(x))| &\le \left|\omega\left(A\left(t\sqrt{1+(v/t)^2}\hat{e} + x\right) - A(t\hat{e} + x)\right)\right| \\
&\le \left\|P_E\left(A\left(\left\{\sqrt{1+(v/t)^2}+1\right\}^{-1}(v^2/t)\hat{e}\right) - A\right)P_E\right\|.
\end{aligned}
\tag{2.13}
$$

Thus we demonstrated that $\omega_0^+(A) = \omega_0^+(A(x))$ for any local operator A. This result extends by continuity to any $A \in \mathfrak{A}$. $\square$

It follows from Theorem 2.2 (a) that all the limit points ω_0^+ are vacuum states under the premises of the above lemma. On the other hand, adopting Condition $N_\natural$ we obtain a stronger result from Theorem 2.2 (b):

Corollary 2.4. *Let Condition $N_\natural$ be satisfied. Then, for any state $\omega \in \mathcal{T}_E$, $E \ge 0$, and timelike unit vector $\hat{e} \in \mathbb{R}^{s+1}$, there holds*

$$
\lim_{t\to\infty} \alpha^*_{t\hat{e}}\omega(A) = \omega_0(A), \text{ for } A \in \mathfrak{A}.
\tag{2.14}
$$

We note that in contrast to previous approaches to the problem of relaxation to the vacuum [9, 16] the present argument does not require the assumption of asymptotic completeness or asymptotic abelianess in time.

To conclude this survey of applications of Condition $N_\natural$ let us mention another physically meaningful procedure for preparation of vacuum states: It is to construct states with increasingly sharp values of energy and momentum and exploit the uncertainty principle. Let $P_{(p,r)}$ be the spectral projection corresponding to the ball of radius r centered around point p in the energy-momentum spectrum. Then, in a theory satisfying Condition $N_\natural$, any sequence of states $\omega_r \in P_{(p,r)}\mathcal{T}P_{(p,r)}$ converges, uniformly on local algebras, to the vacuum state ω_0 as $r \to 0$, since this is the only energetically accessible state which is completely dislocalized in spacetime. This fact is reflected in the following property of the map Π_E:

Proposition 2.5. *Suppose that Condition $N_\natural$ is satisfied. Then, for any $E \ge 0$ and $p \in \overline{V}_+$, there holds*

$$
\lim_{r\to 0} \|\Pi_E|_{\mathring{\mathcal{T}}_{(p,r)}}\| = 0,
\tag{2.15}
$$

where $\mathring{\mathcal{T}}_{(p,r)} = \{\varphi - \varphi(I)\omega_0 \mid \varphi \in P_{(p,r)}\mathcal{T}_E P_{(p,r)}\}$.

Proof. We pick $A \in B(\mathcal{H})$, $\varphi \in \mathring{\mathcal{T}}_{(p,r)}$ and estimate the deviation of this functional from translational invariance

$$
\begin{aligned}
|\varphi(A) - \alpha^*_x\varphi(A)| &= |\varphi(P_{(p,r)}AP_{(p,r)}) - \varphi(P_{(p,r)}e^{i(P-p)x}Ae^{-i(P-p)x}P_{(p,r)})| \\
&= |\varphi(P_{(p,r)}e^{i(P-p)x}A(1 - e^{-i(P-p)x})P_{(p,r)}) \\
&\quad + \varphi(P_{(p,r)}(1 - e^{i(P-p)x})AP_{(p,r)})| \le 2\|\varphi\|\,\|A\|\,|x|\,r,
\end{aligned}
\tag{2.16}
$$

where in the first step we used invariance of ω_0 under translations to insert the projections $P_{(p,r)}$ and in the last step we applied the spectral theorem. Consequently, for any $x_1, \ldots, x_N \in \mathbb{R}^{s+1}$ and open bounded region $\mathcal{O}$,

$$\|\varphi\|_{\mathfrak{A}(\mathcal{O})} \leq \frac{1}{N} \sum_{k=1}^{N} \|\alpha_{x_k}^* \varphi\|_{\mathfrak{A}(\mathcal{O})} + \sup_{k \in \{1,\ldots,N\}} \|\varphi - \alpha_{x_k}^* \varphi\|_{\mathfrak{A}(\mathcal{O})}$$

$$\leq \frac{1}{\sqrt{N}} \left(\sum_{k=1}^{N} \|\alpha_{x_k}^* \varphi\|_{\mathfrak{A}(\mathcal{O})}^2 \right)^{\frac{1}{2}} + 2\|\varphi\| \, r \sup_{k \in \{1,\ldots,N\}} |x_k|. \qquad (2.17)$$

To conclude the proof of the proposition we restate the above inequality as follows:

$$\|\Pi_E|_{\mathring{T}_{(p,r)}}\| \leq \frac{1}{\sqrt{N}} \|\Pi_E\|_{x_1,\ldots,x_N} + 2r \sup_{k \in \{1,\ldots,N\}} |x_k|, \qquad (2.18)$$

and make use of Condition $N_\natural$. $\square$

It is a consequence of the above proposition that $\lim_{E \searrow 0} \mathcal{N}(\varepsilon)_E = 1$ in any theory complying with Condition $N_\natural$, as anticipated in our heuristic discussion. Since $\mathcal{N}(\varepsilon)_E \geq 1$ and it decreases monotonically with decreasing E, the limit exists. If it was strictly larger than one, we could find nets of functionals $\varphi_{1,E}, \varphi_{2,E} \in \mathring{T}_{E,1}$ s.t. $\|\Pi_E(\varphi_{1,E} - \varphi_{2,E})\| > \varepsilon$ for any $E > 0$. But fixing some $E_0 > 0$ and restricting attention to $E \leq E_0/\sqrt{2}$ we obtain

$$\varepsilon < \|\Pi_E(\varphi_{1,E} - \varphi_{2,E})\| \leq 2\|\Pi_{E_0}|_{\mathring{T}_{(0,\sqrt{2}E)}}\|. \qquad (2.19)$$

The last expression on the r.h.s. tends to zero with $E \to 0$, by Proposition 2.5, leading to a contradiction.

Up to this point we discussed the physical interpretation and applications of the novel Condition $N_\natural$ from the general perspective of local relativistic quantum field theory. In order to shed more light on the mechanism which enforces this and related phase space criteria, we turn now to their verification in a model.

3. Condition $N_\natural$ in Massive Scalar Free Field Theory

In this section, which serves mostly to fix our notation, we recall some basic properties of scalar free field theory of mass $m > 0$ in s space dimensions. (See [24] Sect. X.7). The single particle space of this theory is $L^2(\mathbb{R}^s, d^s p)$. On this space there act the multiplication operators $\omega(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$ and $p_1, \ldots, p_s$ which are self-adjoint on a suitable dense domain and generate the unitary representation of translations

$$(U_1(x)f)(\vec{p}) = e^{i(\omega(\vec{p})x^0 - \vec{p}\vec{x})} f(\vec{p}), \quad f \in L^2(\mathbb{R}^s, d^s p). \qquad (3.1)$$

The full Hilbert space $\mathcal{H}$ of the theory is the symmetric Fock space over $L^2(\mathbb{R}^s, d^s p)$. By the method of second quantization we obtain the Hamiltonian $H = d\Gamma(\omega)$, and the momentum operators $P_i = d\Gamma(p_i)$, $i = 1, 2, \ldots, s$ defined on a suitable domain in $\mathcal{H}$.

The joint spectrum of this family of commuting, self adjoint operators is contained in the closed forward light cone. The unitary representation of translations in $\mathcal{H}$ given by

$$U(x) = \Gamma(U_1(x)) = e^{i(Hx^0 - \vec{P}\vec{x})} \tag{3.2}$$

implements the corresponding family of automorphisms of $B(\mathcal{H})$

$$\alpha_x(\cdot) = U(x) \cdot U(x)^*. \tag{3.3}$$

Next, we construct the local algebra $\mathfrak{A}(\mathcal{O})$ attached to the double cone $\mathcal{O}$, whose base is the s-dimensional ball $\mathcal{O}_r$ of radius r centered at the origin in configuration space. To this end we introduce the subspaces $\mathcal{L}^{\pm} = [\omega^{\mp\frac{1}{2}} \widetilde{D}(\mathcal{O}_r)]$, where tilde denotes the Fourier transform. (The respective projections are denoted by $\mathcal{L}^{\pm}$ as well.) Defining J to be the complex conjugation in configuration space we introduce the real linear subspace

$$\mathcal{L} = (1 + J)\mathcal{L}^+ + (1 - J)\mathcal{L}^- \tag{3.4}$$

and the corresponding von Neumann algebra

$$\mathfrak{A}(\mathcal{O}) = \{ W(f) \mid f \in \mathcal{L}\}'', \tag{3.5}$$

where $W(f) = e^{i(a^*(f) + a(f))}$ and $a^*(f)$, $a(f)$ are the creation and annihilation operators. With the help of the translation automorphisms α_x introduced above we define local algebras attached to double cones centered at any point x of spacetime

$$\mathfrak{A}(\mathcal{O} + x) = \alpha_x(\mathfrak{A}(\mathcal{O})). \tag{3.6}$$

The global algebra $\mathfrak{A}$ is the C^*-inductive limit of all such local algebras of different $r > 0$ and $x \in \mathbb{R}^{s+1}$. By construction, α_x leaves $\mathfrak{A}$ invariant.

Now we turn our attention to the phase space structure of the theory. Let Q_E be the projection on states of energy lower than E in the single particle space and $\beta \in \mathbb{R}$. We define operators $T_{E,\pm} = Q_E \mathcal{L}^{\pm}$, $T_{\beta,\pm} = e^{-\frac{1}{2}(\beta|\vec{p}|)^2} \mathcal{L}^{\pm}$. It follows immediately from [25], p. 137 that these operators satisfy $\||T_{E,\pm}|^p\|_1 < \infty$, $\||T_{\beta,\pm}|^p\|_1 < \infty$ for any $p > 0$, where $\|\cdot\|_1$ denotes the trace norm. We introduce their least upper bound T

$$T = \text{s-}\lim_{n\to\infty} \left(\frac{1}{4}(|T_{E,+}|^{2^n} + |T_{E,-}|^{2^n} + |T_{\beta,+}|^{2^n} + |T_{\beta,-}|^{2^n})\right)^{2^{-n}}. \tag{3.7}$$

Proceeding as in [26], pp. 316/317 one can show that this limit exists and that the operator T satisfies

$$T^n \geq |T_{E,\pm}|^n \text{ and } T^n \geq |T_{\beta,\pm}|^n \text{ for } n \in \mathbb{N}, \tag{3.8}$$

$$\|T\| \leq \max(\|T_{E,+}\|, \|T_{E,-}\|, \|T_{\beta,+}\|, \|T_{\beta,-}\|) \leq 1, \tag{3.9}$$

$$\|T^p\|_1 \leq \||T_{E,+}|^p\|_1 + \||T_{E,-}|^p\|_1 + \||T_{\beta,+}|^p\|_1 + \||T_{\beta,-}|^p\|_1 \text{ for } p > 0. \tag{3.10}$$

In particular T is a trace class operator. Since it commutes with the conjugation J, the orthonormal basis of its eigenvectors $\{e_j\}_1^\infty$ can be chosen so that $Je_j = e_j$. The corresponding eigenvalues will be denoted $\{t_j\}_1^\infty$. Given any pair of multiindices $\overline{\mu} = (\mu^+, \mu^-)$ we define the operator

$$B_{\overline{\mu}} = a(\mathcal{L}e)^{\overline{\mu}} = a(\mathcal{L}^+ e)^{\mu^+} a(\mathcal{L}^- e)^{\mu^-}. \tag{3.11}$$

We recall, that for any $f_1, \ldots, f_n \in L^2(\mathbb{R}^s, d^s p)$ there hold the so called energy bounds [19] which in the massive theory have the form

$$\|a(f_1) \ldots a(f_n) P_E\| = \|P_E a^*(f_n) \ldots a^*(f_1)\| \leq (M_E)^{\frac{n}{2}} \|f_1\| \ldots \|f_n\|, \quad (3.12)$$

where $M_E = \frac{E}{m}$. Consequently, the operators $B_{\overline{\mu}}$ are bounded on states of finite energy. We note the respective bound

$$\|B_{\overline{\mu}} P_E\| \leq \|a(Q_E \mathcal{L}e)^{\overline{\mu}} P_E\| \leq (M_E)^{\frac{|\overline{\mu}|}{2}} \|Q_E \mathcal{L}e\|^{\overline{\mu}}$$
$$\leq (M_E)^{\frac{|\overline{\mu}|}{2}} t^{\overline{\mu}}, \quad (3.13)$$

where $|\overline{\mu}| = |\mu^+| + |\mu^-|$, $t^{\overline{\mu}} = t^{\mu^+} t^{\mu^-}$, $\{t_j\}_1^\infty$ are the eigenvalues of T and in the last step we made use of the fact that $|Q_E \mathcal{L}^\pm|^2 \leq T^2$. We will construct the expansion of Π_E into rank-one maps with the help of the bounded linear functionals $S_{\overline{\mu}, \overline{\nu}} : \mathring{\mathcal{T}}_E \to \mathbb{C}$, given by

$$S_{\overline{\mu}, \overline{\nu}}(\varphi) = \varphi(B_{\overline{\mu}}^* B_{\overline{\nu}}). \quad (3.14)$$

In particular $S_{0,0} = 0$, since $\varphi(I) = 0$ for any $\varphi \in \mathring{\mathcal{T}}_E$. It follows from (3.13) that the norms of these maps satisfy the bound

$$\|S_{\overline{\mu}, \overline{\nu}}\| \leq M_E^{\frac{|\overline{\mu}| + |\overline{\nu}|}{2}} t^{\overline{\mu}} t^{\overline{\nu}}. \quad (3.15)$$

Clearly, we can assume that $M_E \geq 1$ as $\Pi_E \equiv 0$ otherwise. Since $S_{\overline{\mu}, \overline{\nu}} = 0$ for $|\overline{\mu}| > M_E$ or $|\overline{\nu}| > M_E$, the norms of the functionals $S_{\overline{\mu}, \overline{\nu}}$ are summable with any power $p > 0$. In fact

$$\sum_{\overline{\mu}, \overline{\nu}} \|S_{\overline{\mu}, \overline{\nu}}\|^p \leq M_E^{p M_E} \left(\sum_{\overline{\mu} : |\overline{\mu}| \leq M_E} t^{p\overline{\mu}} \right)^2 \leq M_E^{p M_E} \left(\sum_{\mu^+ : |\mu^+| \leq M_E} t^{p\mu^+} \right)^4$$
$$= M_E^{p M_E} \left(\sum_{k=0}^{[M_E]} \sum_{\mu^+ : |\mu^+| = k} t^{p\mu^+} \right)^4 \leq M_E^{p M_E} \left(\sum_{k=0}^{[M_E]} \|T^p\|_1^k \right)^4, \quad (3.16)$$

where in the last step we made use of the multinomial formula. With this information at hand it is easy to verify that Condition $N_\natural$ holds in massive scalar free field theory [19,20].

Theorem 3.1. *In massive scalar free field theory there exist functionals $\tau_{\overline{\mu}, \overline{\nu}} \in \mathfrak{A}(\mathcal{O})^*$ such that there holds in the sense of norm convergence in $\mathfrak{A}(\mathcal{O})^*$,*

$$\Pi_E(\varphi) = \sum_{\overline{\mu}, \overline{\nu}} \tau_{\overline{\mu}, \overline{\nu}} S_{\overline{\mu}, \overline{\nu}}(\varphi), \quad \varphi \in \mathring{\mathcal{T}}_E. \quad (3.17)$$

Moreover, $\|\tau_{\overline{\mu}, \overline{\nu}}\| \leq 2^{5 M_E}$ for all $\overline{\mu}, \overline{\nu}$ and $\sum_{\overline{\mu}, \overline{\nu}} \|S_{\overline{\mu}, \overline{\nu}}\|^p < \infty$ for any $p > 0$.

We give the proof of this theorem in Appendix B.

4. Condition $N_\natural$ in Massive Scalar Free Field Theory

At this point we turn to the main goal of this technical part of our investigations, namely to verification of Condition $N_\natural$ in the model at hand. By definition of the nuclear p-norms and Theorem 3.1 there holds the bound

$$\|\Pi_E\|_{p,x_1,\dots,x_N} \leq \left(\sum_{\overline{\mu},\overline{\nu}} \|\tau_{\overline{\mu},\overline{\nu}}\|^p \|S_{\overline{\mu},\overline{\nu}}\|^p_{x_1,\dots,x_N} \right)^{\frac{1}{p}} \leq 2^{5M_E} \left(\sum_{\overline{\mu},\overline{\nu}} \|S_{\overline{\mu},\overline{\nu}}\|^p_{x_1,\dots,x_N} \right)^{\frac{1}{p}}. \quad (4.1)$$

Consequently, we need estimates on the norms $\|S_{\overline{\mu},\overline{\nu}}\|_{x_1,\dots,x_N}$ whose growth with N can be compensated by large spacelike distances $x_i - x_j$ for $i \neq j$. This task will be accomplished in Proposition 4.4. The argument is based on the following lemma which is a variant of Lemma 2.2 from [11].

Lemma 4.1. *Let B be a (possibly unbounded) operator s.t. $\|B P_E\| < \infty$, $\|B^* P_E\| < \infty$ and $B P_E \mathcal{H} \subset P_{E-m}\mathcal{H}$ for any $E \geq 0$. Then, for any $x_1, \dots, x_N \in \mathbb{R}^{s+1}$, there hold the bounds*

(a) $\|P_E \sum_{k=1}^N (B^* B)(x_k) P_E\| \leq (M_E + 1) \{\|P_E[B, B^*]P_E\|$
$\quad + (N-1) \sup_{k_1 \neq k_2} \|P_E[B(x_{k_1}), B^*(x_{k_2})]P_E\|\}$,

(b) $\|P_E \int_K d^s x (B^* B)(\vec{x}) P_E\| \leq (M_E + 1) \int_{\Delta K} d^s x \|P_E[B(\vec{x}), B^*]P_E\|$,

where K is a compact subset of $\mathbb{R}^s$ and $\Delta K = \{\vec{x} - \vec{y} \mid \vec{x}, \vec{y} \in K\}$.

Proof. Part (b) coincides, up to minor modifications, with [11]. In the proof of part (a) the modifications are more substantial, so we provide some details. We will show, by induction in n, that there holds the following inequality:

$$\left\| P_{nm} \sum_{k=1}^N (B^* B)(x_k) P_{nm} \right\| \leq n \left\{ \|P_{(n-1)m}[B, B^*]P_{(n-1)m}\| \right.$$

$$\left. + (N-1) \sup_{k_1 \neq k_2} \|P_{(n-1)m}[B(x_{k_1}), B^*(x_{k_2})]P_{(n-1)m}\| \right\}, \quad (4.2)$$

where P_{nm} is the spectral projection of H on the subspace spanned by vectors of energy lower than nm. It clearly holds for $n = 0$. To make the inductive step we pick $\omega(\cdot) = (\Phi | \cdot | \Phi)$, $\Phi \in (P_{nm}\mathcal{H})_1$ and define $Q = \sum_{k=1}^N (B^* B)(x_k)$. Proceeding like in [11], with integrals replaced with sums, one arrives at

$$\omega(QQ) \leq \sum_{k=1}^N \omega((B^* B)(\vec{x}_k)) \left\{ \sum_{l=1}^N \|P_{(n-1)m}[B(\vec{x}_l), B^*(\vec{x}_k)]P_{(n-1)m}\| \right\}$$

$$+ \omega(Q)\|P_{(n-1)m} Q P_{(n-1)m}\|. \quad (4.3)$$

The sum w.r.t. l in the first term on the r.h.s. can be estimated by the expression in curly brackets in (4.2). To the second term on the r.h.s. of (4.3) we apply the induction hypothesis. Altogether

$$\omega(QQ) \leq n\omega(Q) \left\{ \|P_{(n-1)m}[B, B^*]P_{(n-1)m}\| \right.$$

$$\left. + (N-1) \sup_{k_1 \neq k_2} \|P_{(n-1)m}[B(x_{k_1}), B^*(x_{k_2})]P_{(n-1)m}\| \right\}. \quad (4.4)$$

Making use of the fact that $\omega(Q)^2 \leq \omega(QQ)$ and taking the supremum over states ω which are induced by vectors from $P_{nm}\mathcal{H}$ one concludes the proof of estimate (4.2). The statement of the lemma follows by choosing n s.t. $(n-1)m \leq E \leq nm$. $\quad\square$

In order to control the commutators appearing in the estimates in Lemma 4.1 we need a slight generalization of the result from [27] on the exponential decay of vacuum correlations between local observables.

Theorem 4.2. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ s.t. $Sp\,H = \{0\} \cup [m, \infty]$, $m > 0$ and there exists exactly one (up to a phase) eigenvector Ω of H with eigenvalue zero. Let A, B be operators such that Ω belongs to their domains and to the domains of their adjoints. If there holds*

$$(\Omega|\,[A, e^{itH}Be^{-itH}]\,\Omega) = 0 \text{ for } |t| < \delta, \tag{4.5}$$

then

$$|(\Omega|AB\Omega) - (\Omega|A\Omega)(\Omega|B\Omega)| \leq e^{-m\delta}\{\|A\Omega\|\,\|A^*\Omega\|\,\|B\Omega\|\,\|B^*\Omega\|\}^{\frac{1}{2}}. \tag{4.6}$$

With the help of the above theorem we prove the desired estimate.

Lemma 4.3. *Let $e \in L^2(\mathbb{R}^s, d^s p)$ be s.t. $\|e\| \leq 1$ and $Je = e$. Then there holds for any $x \in \mathbb{R}^{s+1}$, $0 < \varepsilon < 1$ and any combination of $\pm$ signs*

$$|\langle \mathcal{L}^{\pm}e|e^{-(\beta|\vec{p}|)^2}U(x)\mathcal{L}^{\pm}e\rangle| \leq c_{\varepsilon,\beta}e^{-m(1-\varepsilon)\delta(x)}, \tag{4.7}$$

where $c_{\varepsilon,\beta}$ does not depend on x and e. Here $\delta(x) = |\vec{x}| - |x^0| - 2r$ and r is the radius of the double cone entering into the definition of the projections $\mathcal{L}^{\pm}$.

Proof. We define the operators $\phi_+(e) = a^*(\mathcal{L}^+e) + a(\mathcal{L}^+e)$, $\phi_-(e) = a^*(i\mathcal{L}^-e) + a(i\mathcal{L}^-e)$ and their translates $\phi_{\pm}(e)(x) = U(x)\phi_{\pm}(e)U(x)^{-1}$. Since the projections $\mathcal{L}^{\pm}$ commute with J and $Je = e$, these operators are just the fields and canonical momenta of massive scalar free field theory. Assume that $\delta(x) > 0$. Then, by locality, $\phi_{\pm}(e)$ and $\phi_{\pm}(e)(x)$ satisfy the assumptions of Theorem 4.2. As they have vanishing vacuum expectation values, we obtain

$$|\langle \mathcal{L}^{\pm}e|U(x)\mathcal{L}^{\pm}e\rangle| = |(\Omega|\phi_{\pm}(e)\phi_{\pm}(e)(x)\Omega)| \leq e^{-m\delta(x)}. \tag{4.8}$$

Let us now consider the expectation value from the statement of the lemma. We fix some $0 < \varepsilon < 1$ and estimate

$$|\langle \mathcal{L}^{\pm}e|e^{-(\beta|\vec{p}|)^2}U(x)\mathcal{L}^{\pm}e\rangle|$$

$$\leq (2\sqrt{\pi}\beta)^{-s} \int_{\delta(\vec{y}+x) \geq (1-\varepsilon)\delta(x)} d^s y \, e^{-\frac{|\vec{y}|^2}{4\beta^2}} |\langle \mathcal{L}^{\pm}e|U(x+\vec{y})\mathcal{L}^{\pm}e\rangle|$$

$$+ (2\sqrt{\pi}\beta)^{-s} \int_{\delta(\vec{y}+x) \leq (1-\varepsilon)\delta(x)} d^s y \, e^{-\frac{|\vec{y}|^2}{4\beta^2}} |\langle \mathcal{L}^{\pm}e|U(x+\vec{y})\mathcal{L}^{\pm}e\rangle|$$

$$\leq e^{-m(1-\varepsilon)\delta(x)} + (2\sqrt{\pi}\beta)^{-s} \int_{|\vec{y}| \geq \varepsilon\delta(x)} d^s y \, e^{-\frac{|\vec{y}|^2}{4\beta^2}}$$

$$\leq e^{-m(1-\varepsilon)\delta(x)} \left(1 + (2\sqrt{\pi}\beta)^{-s} \int d^s y \, e^{-\frac{|\vec{y}|^2}{4\beta^2} + \frac{m(1-\varepsilon)|\vec{y}|}{\varepsilon}}\right). \tag{4.9}$$

In the first step we expressed the function $e^{-(\beta|\vec{p}|)^2}$ by its Fourier transform and divided the region of integration into two subregions. To the first integral we applied estimate (4.8). Making use of the fact that the second integral decays faster than exponentially with $\delta(x) \to \infty$ we arrived at the last expression which is of the form (4.7). Since $c_{\varepsilon,\beta} > 1$, the bound (4.9) holds also for $\delta(x) \leq 0$. $\quad\square$

It is a well known fact that any normal, self-adjoint functional on a von Neumann algebra can be expressed as a difference of two normal, positive functionals which are mutually orthogonal [28]. It follows that any $\varphi \in \mathcal{T}_{E,1}$ can be decomposed as

$$\varphi = \varphi_{\text{Re}}^+ - \varphi_{\text{Re}}^- + i(\varphi_{\text{Im}}^+ - \varphi_{\text{Im}}^-), \tag{4.10}$$

where $\varphi_{\text{Re}}^\pm$, $\varphi_{\text{Im}}^\pm$ are positive functionals from $\mathcal{T}_{E,1}$. This assertion completes the list of auxiliary results needed to establish the required estimate for $\|S_{\overline{\mu},\overline{\nu}}\|_{x_1,\dots,x_N}$.

Proposition 4.4. *The functionals $S_{\overline{\mu},\overline{\nu}}$ satisfy the bound*

$$\|S_{\overline{\mu},\overline{\nu}}\|_{x_1,\dots,x_N}^2 \leq 32t^{\overline{\mu}}t^{\overline{\nu}}(M_E)^{2M_E}e^{(\beta E)^2}\left\{1 + \sqrt{c_{\varepsilon,\beta}}(N-1)e^{-\frac{m}{2}(1-\varepsilon)\delta(\underline{x})}\right\}, \tag{4.11}$$

where $\{t_j\}_1^\infty$ are the eigenvalues of the operator T given by formula (3.7) and $\delta(\underline{x}) = \inf_{i\neq j}\delta(x_i - x_j)$. The function $\delta(x)$, the parameter ε and the constant $c_{\varepsilon,\beta}$ appeared in Lemma 4.3.

Proof. We denote by $\mathcal{T}_{E,1}^+$ the set of positive functionals from $\mathcal{T}_{E,1}$. Making use of the definition of $\|\cdot\|_{x_1,\dots,x_N}$, decomposition (4.10) and the Cauchy-Schwarz inequality we obtain

$$\|S_{\overline{\mu},\overline{\nu}}\|_{x_1,\dots,x_N}^2 = \sup_{\varphi\in\mathring{\mathcal{T}}_{E,1}}\sum_{k=1}^N |S_{\overline{\mu},\overline{\nu}}(\alpha_{x_k}^*\varphi)|^2 \leq 16 \sup_{\varphi\in\mathcal{T}_{E,1}^+}\sum_{k=1}^N |\alpha_{x_k}^*\varphi(B_{\overline{\mu}}^*B_{\overline{\nu}})|^2$$

$$\leq 16 \sup_{\varphi\in\mathcal{T}_{E,1}^+}\sum_{k=1}^N \alpha_{x_k}^*\varphi(B_{\overline{\mu}}^*B_{\overline{\mu}})\alpha_{x_k}^*\varphi(B_{\overline{\nu}}^*B_{\overline{\nu}})$$

$$\leq 16(M_E)^{|\overline{\mu}|}t^{2\overline{\mu}}\|P_E\sum_{k=1}^N (B_{\overline{\nu}}^*B_{\overline{\nu}})(x_k)P_E\|, \tag{4.12}$$

where in the last step we applied the bound (3.13). We can assume, without loss of generality, that $\overline{\nu} \neq 0$ and decompose it into two pairs of multiindices $\overline{\nu} = \overline{\nu}_a + \overline{\nu}_b$ in such a way that $|\overline{\nu}_b| = 1$. Since $B_{\overline{\nu}} = B_{\overline{\nu}_a}B_{\overline{\nu}_b}$, we get

$$P_E\sum_{k=1}^N (B_{\overline{\nu}}^*B_{\overline{\nu}})(x_k)P_E = P_E\sum_{k=1}^N (B_{\overline{\nu}_b}^*P_EB_{\overline{\nu}_a}^*B_{\overline{\nu}_a}P_EB_{\overline{\nu}_b})(x_k)P_E$$

$$\leq \|B_{\overline{\nu}_a}P_E\|^2 P_E\sum_{k=1}^N (B_{\overline{\nu}_b}^*B_{\overline{\nu}_b})(x_k)P_E$$

$$= M_E^{|\overline{\nu}_a|}t^{2\overline{\nu}_a}P_E\sum_{k=1}^N \left(a^*(\mathcal{L}e)^{\overline{\nu}_b}a(\mathcal{L}e)^{\overline{\nu}_b}\right)(x_k)P_E, \tag{4.13}$$

where in the last step we used again estimate (3.13). Next, let g be the operator of multiplication by $\frac{1}{2}(\beta|\vec{p}|)^2$ in $L^2(\mathbb{R}^s, d^s p)$ and let $G = d\Gamma(g) \geq 0$ be its second quantization. Since one knows explicitly the action of G and H on vectors of fixed particle number, it is easy to check that

$$e^G P_E = P_E e^G P_E \leq P_E e^{\frac{1}{2}(\beta H)^2} P_E \leq e^{\frac{1}{2}(\beta E)^2}. \tag{4.14}$$

Making use of this fact, Lemma 4.1 (a) and Lemma 4.3 we obtain from (4.13) the following string of inequalities:

$$\left\| P_E \sum_{k=1}^{N} (B_{\bar{v}}^* B_{\bar{v}})(x_k) P_E \right\|$$

$$\leq M_E^{|\bar{v}_a|} t^{2\bar{v}_a} \left\| P_E e^G \sum_{k=1}^{N} \left(a^*(e^{-\frac{1}{2}(\beta|\vec{p}|)^2} \mathcal{L}e)^{\bar{v}_b} e^{-2G} a(e^{-\frac{1}{2}(\beta|\vec{p}|)^2} \mathcal{L}e)^{\bar{v}_b} \right) (x_k) e^G P_E \right\|$$

$$\leq M_E^{|\bar{v}_a|} t^{2\bar{v}_a} e^{(\beta E)^2} \left\| P_E \sum_{k=1}^{N} \left(a^*(e^{-\frac{1}{2}(\beta|\vec{p}|)^2} \mathcal{L}e)^{\bar{v}_b} a(e^{-\frac{1}{2}(\beta|\vec{p}|)^2} \mathcal{L}e)^{\bar{v}_b} \right) (x_k) P_E \right\|$$

$$\leq M_E^{|\bar{v}_a|} t^{2\bar{v}_a} e^{(\beta E)^2} (M_E + 1) \left\{ \langle (\mathcal{L}e)^{\bar{v}_b} | e^{-(\beta|\vec{p}|)^2} (\mathcal{L}e)^{\bar{v}_b} \rangle \right.$$

$$\left. + (N-1) \sup_{i \neq j} |\langle (\mathcal{L}e)^{\bar{v}_b} | e^{-(\beta|\vec{p}|)^2} U(x_i - x_j)(\mathcal{L}e)^{\bar{v}_b} \rangle| \right\}$$

$$\leq 2 M_E^{|\bar{v}|} t^{\bar{v}} e^{(\beta E)^2} \left\{ 1 + (N-1)\sqrt{c_{\varepsilon,\beta}} \sup_{i \neq j} e^{-\frac{m}{2}(1-\varepsilon)\delta(x_i - x_j)} \right\}, \tag{4.15}$$

where in the last step we made use of the estimate $\quad |\langle \mathcal{L}^{\pm} e_j | e^{-(\beta|\vec{p}|)^2} U(x) \mathcal{L}^{\pm} e_j \rangle| \leq \langle e_j || T_{\beta,\pm}|^2 e_j \rangle \leq \langle e_j | T^2 e_j \rangle = t_j^2$ and the fact that $t_j \leq 1$ which follows from (3.9). Substituting inequality (4.15) to formula (4.12), estimating $t^{2\bar{\mu}} \leq t^{\bar{\mu}}$ and recalling that $S_{\bar{\mu},\bar{v}} = 0$ for $|\bar{\mu}| > M_E$ or $|\bar{v}| > M_E$ we obtain the bound from the statement of the proposition. $\square$

It is now straightforward to estimate the p-norms of the map Π_E. Substituting the bound from the above proposition to formula (4.1) and proceeding like in estimate (3.16) we obtain

$$\|\Pi_E\|_{p, x_1, \ldots, x_N}$$

$$\leq (4\sqrt{2})(2^5 M_E)^{M_E} e^{\frac{1}{2}(\beta E)^2} \left(\sum_{k=0}^{[M_E]} \|T^{\frac{p}{2}}\|_1^k \right)^{\frac{4}{p}} \left\{ 1 + \sqrt{c_{\varepsilon,\beta}}(N-1)e^{-\frac{m}{2}(1-\varepsilon)\delta(x)} \right\}^{\frac{1}{2}}. \tag{4.16}$$

It is clear from the above relation that $\limsup_{\delta(\underline{x}) \to \infty} \|\Pi_E\|_{p, x_1, \ldots, x_N}$ satisfies a bound which is independent of N. Consequently, we get

Theorem 4.5. *Condition $N_{\natural}$ holds in massive scalar free field theory for arbitrary dimension of space s.*

5. Conclusion and Outlook

In this work we proposed and verified in massive scalar free field theory the new Condition $N_\natural$. Since this phase space criterion encodes the firm physical principle that energy is additive over isolated subsystems, we expect that it holds in a large family of models. In fact, we will show in a future publication that massless scalar free field theory also satisfies this condition for $s \geq 3$. We recall that this model contains an infinite family of pure, regular vacuum states which are, however, mutually energy-disconnected [16]. In view of Theorem 2.2 (b), this decent vacuum structure is related to phase space properties of this model, as anticipated in [19].

Apart from more detailed information about the phase space structure of massive free field theory, our discussion offers also some new insights into the harmonic analysis of translation automorphisms. First, we recall from [11] that in all local, relativistic quantum field theories there holds the bound

$$\sup_{\varphi \in \mathcal{T}_{E,1}} \int d^s p |\vec{p}|^{s+1+\varepsilon} |\varphi(\widetilde{A}(\vec{p}))|^2 < \infty, \tag{5.1}$$

for any $\varepsilon > 0$, uniformly in $A \in \mathfrak{A}(\mathcal{O})_1$. It says that the distribution $\varphi(\widetilde{A}(\vec{p}))$, restricted to the domain $\{\vec{p} \mid |\vec{p}| \geq \delta\}$ for some $\delta > 0$, is represented by a square integrable function, but at $\vec{p} = 0$ it may have a power like singularity which is not square integrable. It turns out, however, that in massive scalar free field theory this distribution has a milder behavior at zero than one might expect from (5.1). Making use of Lemma 4.1 (b) and going through our argument once again one can easily establish that there holds, uniformly in $A \in \mathfrak{A}(\mathcal{O})_1$,

$$\sup_{\varphi \in \mathcal{T}_{E,1}} \int d^s x |\varphi(\mathring{A}(\vec{x}))|^2 < \infty, \tag{5.2}$$

where $\mathring{A} = A - \omega_0(A)I$. By the Plancherel theorem, we obtain

$$\sup_{\varphi \in \mathcal{T}_{E,1}} \int d^s p |\varphi(\widetilde{\mathring{A}}(\vec{p}))|^2 < \infty, \tag{5.3}$$

i.e. the distribution $\varphi(\widetilde{\mathring{A}}(\vec{p}))$ is represented by a square integrable function. Consequently, $\varphi(\widetilde{A}(\vec{p}))$ can deviate from square integrability only by a delta-like singularity at $\vec{p} = 0$. The above reasoning demonstrates the utility of phase space methods in harmonic analysis of automorphism groups [29]. One may therefore expect that they will be of further use in this interesting field.

A. Proof of Theorem 2.1

The argument is based on the following abstract lemma:

Lemma A.1. *Let X and Y be Banach spaces, $S_k \in X^*$ for $k \in \{1, \ldots, N\}$ and $\tau \in Y$ be s.t. $\|\tau\| = 1$. Then the ε-content of the map $\Theta : X \to Y \otimes \mathbb{C}_{\sup}^N$ given by*

$$\Theta(\varphi) = \tau \, (S_1(\varphi), \ldots, S_N(\varphi)), \quad \varphi \in X, \tag{A.1}$$

satisfies the bound

$$\mathcal{N}(\varepsilon)_\Theta \leq (4eN)^{\frac{2^7 \pi \|\Theta\|_2^2}{\varepsilon^2}}, \tag{A.2}$$

where $\|\Theta\|_2 = \sup_{\varphi \in X_1} (\sum_{k=1}^N |S_k(\varphi)|^2)^{\frac{1}{2}}$.

Proof. Fix $\varepsilon > 0$ and let $\mathbb{J}_0 = \{(n_1 + in_2)\varepsilon \mid n_1, n_2 \in \mathbb{Z}\}$. For each $k \in \{1, \ldots, N\}$ and $\varphi \in X_1$ we choose $J_k(\varphi) \in \mathbb{J}_0$ so that $|S_k(\varphi) - J_k(\varphi)| \leq \sqrt{2}\varepsilon$ and $|J_k(\varphi)| \leq |S_k(\varphi)|$. Define the set $\mathbb{J} = \{J_1(\varphi), \ldots, J_N(\varphi) \mid \varphi \in X_1\}$ of all N-tuples appearing in this way. We claim that $\#\mathbb{J} \geq \mathcal{N}(4\varepsilon)_\Theta$. In fact, assume that there are $\varphi_1, \ldots, \varphi_K \in X_1$, $K > \#\mathbb{J}$, s.t. for $i \neq j$ there holds

$$4\varepsilon < \|\Theta(\varphi_i) - \Theta(\varphi_j)\| = \sup_{k \in \{1, \ldots, N\}} |S_k(\varphi_i) - S_k(\varphi_j)|. \tag{A.3}$$

Then there exists such $\hat{k}$, depending on (i, j), that $4\varepsilon < |S_{\hat{k}}(\varphi_i) - S_{\hat{k}}(\varphi_j)|$. Consequently, by a 3ε-argument

$$|J_{\hat{k}}(\varphi_i) - J_{\hat{k}}(\varphi_j)| \geq |S_{\hat{k}}(\varphi_i) - S_{\hat{k}}(\varphi_j)| - 2\sqrt{2}\varepsilon > \varepsilon, \tag{A.4}$$

which shows that there are at least K different elements of $\mathbb{J}$ in contradiction to our assumption.

In order to estimate the cardinality of the set $\mathbb{J}$ we define $M = \left[\frac{\|\Theta\|_2^2}{\varepsilon^2}\right]$, assume for the moment that $0 < M \leq 2N$ and denote by $V_M(R) \leq e^{2\pi R^2}$ the volume of the M-dimensional ball of radius R. Then

$$\#\mathbb{J} \leq \sum_{\substack{n_1, \ldots, n_{2N} \in \mathbb{Z} \\ n_1^2 + \cdots + n_{2N}^2 \leq M}} 1 \leq \binom{2N}{M} 2^M V_M(2\sqrt{M}) \leq (4Ne)^{8\pi M}. \tag{A.5}$$

We note that each admissible combination of integers $n_1, \ldots, n_{2N}$ contains at most M non-zero entries. Thus to estimate the above sum we pick M out of $2N$ indices and consider the points $(n_{i_1}, \ldots, n_{i_M}) \in \mathbb{Z}^M$ which belong to the M-dimensional ball of radius $\sqrt{M}$. Each such point is a vertex of a unit cube which fits into a ball of radius $2\sqrt{M}$ (since $\sqrt{M}$ is the length of the diagonal of the cube). As in M dimensions a cube has 2^M vertices, there can be no more than $2^M V_M(2\sqrt{M})$ points $(n_{i_1}, \ldots, n_{i_M}) \in \mathbb{Z}^M$ satisfying the restriction $n_{i_1}^2 + \cdots + n_{i_M}^2 \leq M$. In the case $M \geq 2N$ a more stringent bound (uniform in N) can be established by a similar reasoning. For $M = 0$ there obviously holds $\#\mathbb{J} = 1$. $\square$

Proof of Theorem 2.1. Fix $0 < p < \frac{2}{3}$. Then Condition $N_\natural$ provides, for any $\delta > 0$, a decomposition of the map Π_E into rank-one mappings $\Pi_n(\cdot) = \tau_n S_n(\cdot)$, where $\tau_n \in \mathfrak{A}(\mathcal{O})^*$ and $S_n \in \mathring{\mathcal{T}}_E^*$, s.t.

$$\left(\sum_{n=1}^{\infty} \|\Pi_n\|_{x_1,\ldots,x_N}^p \right)^{\frac{1}{p}} \leq (1+\delta) \|\Pi_E\|_{p,x_1,\ldots,x_N}. \tag{A.6}$$

Assuming that the norms $\|\Pi_n\|_{x_1,\ldots,x_N}$ are given in descending order with n, we obtain the bound

$$\|\Pi_n\|_{x_1,\ldots,x_N} \leq \frac{(1+\delta)\|\Pi_E\|_{p,x_1,\ldots,x_N}}{n^{1/p}}. \tag{A.7}$$

Similarly, we can decompose the map $\Theta_{E,x_1,\ldots,x_N}$ into a sum of maps Θ_n of the form

$$\Theta_n(\varphi) = \left(\Pi_n(\alpha_{x_1}^* \varphi), \ldots, \Pi_n(\alpha_{x_N}^* \varphi) \right) = \tau_n \left(S_n(\alpha_{x_1}^* \varphi), \ldots, S_n(\alpha_{x_N}^* \varphi) \right). \tag{A.8}$$

Now we can apply Lemma A.1 with $\tau = \tau_n/\|\tau_n\|$ and $S_k(\cdot) = \|\tau_n\| S_n(\alpha_{x_k}^* \cdot)$. From estimate (A.7) we obtain

$$\|\Theta_n\|_2 = \sup_{\varphi \in \mathring{\mathcal{T}}_{E,1}} \left(\sum_{k=1}^{N} \|\tau_n\|^2 |S_n(\alpha_{x_k}^* \varphi)|^2 \right)^{\frac{1}{2}} = \|\Pi_n\|_{x_1,\ldots,x_N}$$

$$\leq \frac{(1+\delta)\|\Pi_E\|_{p,x_1,\ldots,x_N}}{n^{1/p}}. \tag{A.9}$$

Substituting this inequality to the bound (A.2) we get

$$\mathcal{N}(\varepsilon)_n \leq (4eN)^{\frac{2^7 \pi (1+\delta)^2 \|\Pi_E\|_{p,x_1,\ldots,x_N}^2}{\varepsilon^2 n^{2/p}}}. \tag{A.10}$$

We conclude with the help of Lemmas 2.3 and 2.4 from [21] that the ε-content of the map $\Theta_{E,x_1,\ldots,x_N}$ satisfies

$$\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N} \leq \prod_{n=1}^{\infty} \mathcal{N}(\varepsilon_n)_n \tag{A.11}$$

for any sequence $\{\varepsilon_n\}_1^{\infty}$ s.t. $\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{\varepsilon}{4}$. We choose $\varepsilon_n = \frac{\varepsilon}{4} \frac{n^{-\frac{2}{3p}}}{\sum_{n_1=1}^{\infty} n_1^{-\frac{2}{3p}}}$, make use of the bounds (A.10) and (A.11), and take the infinum w.r.t. $\delta > 0$. There follows

$$\mathcal{N}(\varepsilon)_{E,x_1,\ldots,x_N} \leq (4eN)^{\frac{2^{11} \pi \|\Pi_E\|_{p,x_1,\ldots,x_N}^2}{\varepsilon^2} (\sum_{n=1}^{\infty} n^{-\frac{2}{3p}})^3}. \tag{A.12}$$

With the help of Condition $N_\natural$ we obtain the bound in the statement of Theorem 2.1. $\quad\square$

B. Proof of Theorem 3.1

Since the expansion of Π_E into rank-one maps which appears in Theorem 3.1 differs slightly from those which are considered in the existing literature [19,20], we outline here the construction.

Proof of Theorem 3.1. First, we recall from [20], Sect. 7.2.B. that given any pair of multiindices $\overline{\mu} = (\mu^+, \mu^-)$ and an orthonormal sequence of J-invariant vectors (e.g. $\{e_j\}_1^\infty$), there exist weakly continuous linear functionals $\phi_{\overline{\mu}}$ on $\mathfrak{A}(\mathcal{O})$ s.t.

$$\phi_{\overline{\mu}}(W(f)) = e^{-\frac{1}{2}\|f\|^2} \langle e|f^+\rangle^{\mu^+} \langle e|f^-\rangle^{\mu^-}, \tag{B.1}$$

which satisfy the bound

$$\|\phi_{\overline{\mu}}\| \leq 4^{|\overline{\mu}|}(\overline{\mu}!)^{\frac{1}{2}}, \tag{B.2}$$

where $\overline{\mu}! = \mu^+!\mu^-!$. These functionals can be constructed making use of the equality

$$(\Omega|[a(e_1), [\ldots, [a(e_k), [a^*(e_{k+1}), [\ldots, [a^*(e_l), W(f)], \ldots]\Omega)$$
$$= e^{-\frac{1}{2}\|f\|^2} \prod_{n_1=1}^{k} \langle e_{n_1}|if\rangle \prod_{n_2=k+1}^{l} \langle if|e_{n_2}\rangle. \tag{B.3}$$

Next, we evaluate the Weyl operator on some $\varphi \in \mathring{\mathcal{T}}_E$, rewrite it in a normal ordered form and expand it into a power series

$$\varphi(W(f))$$
$$= e^{-\frac{1}{2}\|f\|^2} \sum_{m^\pm, n^\pm \in \mathbb{N}_0} \frac{i^{m^+ + n^+ + 2m^-}}{m^+!m^-!n^+!n^-!} \varphi(a^*(f^+)^{m^+} a^*(f^-)^{m^-} a(f^+)^{n^+} a(f^-)^{n^-}). \tag{B.4}$$

Subsequently, we expand each function $f^\pm$ in the orthonormal basis $\{e_j\}_1^\infty$ of J invariant eigenvectors of the operator T: $f^\pm = \sum_{j=1}^\infty e_j \langle e_j|f^\pm\rangle$. Then, making use of the multinomial formula, we obtain

$$a^*(f^+)^{m^+} = \sum_{\mu^+, |\mu^+|=m^+} \frac{m^+!}{\mu^+!} \langle e|f^+\rangle^{\mu^+} a^*(\mathcal{L}^+ e)^{\mu^+} \tag{B.5}$$

and similarly in the remaining cases. Altogether we get

$$\varphi(W(f)) = \sum_{\overline{\mu}, \overline{\nu}} \frac{i^{|\mu^+| + |\nu^+| + 2|\mu^-|}}{\overline{\mu}!\overline{\nu}!} \phi_{\overline{\mu}+\overline{\nu}}(W(f)) \varphi(a^*(\mathcal{L}e)^{\overline{\mu}} a(\mathcal{L}e)^{\overline{\nu}})$$
$$= \sum_{\overline{\mu}, \overline{\nu}} \tau_{\overline{\mu}, \overline{\nu}}(W(f)) S_{\overline{\mu}, \overline{\nu}}(\varphi), \tag{B.6}$$

where $\tau_{\overline{\mu}, \overline{\nu}}(\,\cdot\,) = \frac{i^{|\mu^+| + |\nu^+| + 2|\mu^-|}}{\overline{\mu}!\overline{\nu}!} \phi_{\overline{\mu}+\overline{\nu}}(\,\cdot\,)$. We recall that in the massive case $S_{\overline{\mu}, \overline{\nu}} = 0$ if $|\overline{\nu}| > M_E$ or $|\overline{\mu}| > M_E$. Consequently, for the relevant indices there holds

$$\|\tau_{\overline{\mu}, \overline{\nu}}\| \leq \frac{4^{|\overline{\mu}| + |\overline{\nu}|}}{(\overline{\mu}!\overline{\nu}!)^{\frac{1}{2}}} \left(\frac{(\overline{\mu}+\overline{\nu})!}{\overline{\mu}!\overline{\nu}!}\right)^{\frac{1}{2}} \leq 2^{\frac{5}{2}(|\overline{\mu}|+|\overline{\nu}|)} \leq 2^{5M_E}, \tag{B.7}$$

where we made use of the bound (B.2) and properties of the binomial coefficients. Now it follows from estimate (3.16) that for any $p > 0$,

$$\sum_{\overline{\mu},\overline{\nu}} \|\tau_{\overline{\mu},\overline{\nu}}\|^p \|S_{\overline{\mu},\overline{\nu}}\|^p \leq 2^{5pM_E} M_E^{pM_E} \left(\sum_{k=0}^{[M_E]} \|T^p\|_1^k \right)^4. \tag{B.8}$$

In view of this fact and of weak continuity of the functionals $\tau_{\overline{\mu},\overline{\nu}}$, equality (B.6) can be extended to any $A \in \mathfrak{A}(\mathcal{O})$. In other words

$$\Pi_E(\varphi)(A) = \varphi(A) = \sum_{\overline{\mu},\overline{\nu}} \tau_{\overline{\mu},\overline{\nu}}(A) S_{\overline{\mu},\overline{\nu}}(\varphi), \tag{B.9}$$

what concludes the proof of the theorem. $\square$

References

1. Haag, R., Swieca, J.A.: When Does a Quantum Field Theory Describe Particles?. Commun. Math. Phys. **1**, 308–320 (1965)
2. Buchholz, D., Wichmann, E.H.: Causal Independence and the Energy-Level Density of States in Local Quantum Field Theory. Commun. Math. Phys. **106**, 321–344 (1986)
3. Buchholz, D., Junglas, P.: On the Existence of Equilibrium States in Local Quantum Field Theory. Commun. Math. Phys. **121**, 255–270 (1989)
4. Buchholz, D.: Phase space properties of local observables and structure of scaling limits. Ann. Inst. H. Poincaré **64**, 433–459 (1996)
5. Bostelmann, H.: Phase space properties and the short distance structure in quantum field theory. J. Math. Phys. **46**, 052301–052317 (2005)
6. Bostelmann, H.: Operator product expansions as a consequence of phase space properties. J. Math. Phys. **46**, 082304–082316 (2005)
7. Lechner, G.: On the existence of local observables in theories with a factorizing S-matrix. J. Phys. A: Math. Gen. **38**, 3045–3056 (2005)
8. Lechner, G.: Construction of Quantum Field Theories with Factorizing S-Matrices. Commun. Math. Phys. **277**(3), 821–860 (2008)
9. Araki, H., Haag, R.: Collision Cross Sections in Terms of Local Observables. Commun. Math. Phys. **4**, 77–91 (1967)
10. Enss, V.: Characterization of Particles by Means of Local Observables. Commun. Math. Phys. **45**, 35–52 (1975)
11. Buchholz, D.: Harmonic Analysis of Local Operators. Commun. Math. Phys. **129**, 631–641 (1990)
12. Buchholz, D., Porrmann, M., Stein, U.: Dirac versus Wigner: Towards a universal particle concept in quantum field theory. Phys. Lett. B **267**, 377–381 (1991)
13. Porrmann, M.: Particle Weights and their Disintegration I. Commun. Math. Phys. **248**, 269–304 (2004)
14. Porrmann, M.: Particle Weights and their Disintegration II. Commun. Math. Phys. **248**, 305–333 (2004)
15. Buchholz, D., Summers, S.J.: Scattering in Relativistic Quantum Field Theory: Fundamental Concepts and Tools. In: Francoise, J.-P., Naber, G., Tsun, T.S. (eds.) Encyclopedia of Mathematical Physics. Elsevier, New York, 2006
16. Buchholz, D., Wanzenberg, R.: The Realm of the Vacuum. Commun. Math. Phys. **143**, 577–589 (1992)
17. Haag, R.: *Local Quantum Physics*, Berlin-Heidelberg-New York: Springer, 1992
18. Fewster, C.J., Ojima, I., Porrmann, M.: *p*-Nuclearity in a New Perspective. Lett. Math. Phys. **73**, 1–15 (2005)
19. Buchholz, D., Porrmann, M.: How small is the phase space in quantum field theory?. Ann. Inst. H. Poincaré **52**, 237–257 (1990)
20. Bostelmann, H.: *Lokale Algebren und Operatorprodukte am Punkt*. PhD Thesis, Universität Göttingen, 2000; http://webdoc.sub.gwdg.de/diss/2000/bostelmann/, 2000
21. Buchholz, D., D'Antoni, C.: Phase space properties of charged fields in theories of local observables. Rev. Math. Phys. **7**, 527–557 (1995)
22. Glimm, J., Jaffe, A.: Quantum Field Theory Models. In: DeWitt, C., Stora, R. (eds.), *Statistical Mechanics and Quantum Field Theory*, (Université de Grenoble - Summer School of Theoretical Physics, Les Houches 1970), New York: Gorden and Breach Science Pub., 1971

23. Araki, H.: *Mathematical Theory of Quantum Fields*. Oxford: Oxford Science Publications, 1999
24. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. Part II: Fourier Analysis, Self-Adjointness.* New York-San Francisco-London: Academic Press, 1975
25. Buchholz, D., D'Antoni, C., Longo, R.: Nuclear Maps and Modular Structures II: Applications to Quantum Field Theory. Commun. Math. Phys. **129**, 115–138 (1990)
26. Buchholz, D., Jacobi, P.: On the Nuclearity Condition for Massless Fields. Lett. Math. Phys. **13**, 313–323 (1987)
27. Fredenhagen, K.: A Remark on the Cluster Theorem. Commun. Math. Phys. **97**, 461–463 (1985)
28. Sakai, S.: *C^*-Algebras and W^*-Algebras.* Berlin-Heidelberg-New York: Springer-Verlag, 1971
29. Arveson, W.: The harmonic analysis of automorphism groups. In: Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., **38**, Providence, R.I.: Amer. Math. Soc., 1982, pp. 199–269