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Erratum

Propagation Effects on the Breakdown of a Linear Amplifier Model: Complex-Mass Schrödinger Equation Driven by the Square of a Gaussian Field

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The proof of the inequality $\lambda_q(x,t) \leq (q\mu_{x,t} - 0^+)^{-1}$ [p. 750, below Eq. (29)] is based on the statement that $\mathcal{E}(x,t;s)$ is an entire function of $s \in \mathbb{C}^M$ [see below Eq. (30)]. But according to Eq. (9) and Lemma 1, all we know is that $\mathcal{E}(x,t;s)$ is an entire function of $k(s) \in \mathbb{R}^N$. Nevertheless, the above inequality holds, hence Proposition 1. To prove it we replace (31) with the following lemma.

Lemma. *If* $\langle|\mathcal{E}(x,t)|^q\rangle < +\infty$*, then*

$$\limsup_{||s||\to+\infty} \mathrm{e}^{-||s||^2}|\mathcal{E}(x,t;s)|^q < +\infty, \tag{31}$$

along almost every direction $\hat{s}$ *in* $\mathbb{C}^M$.

Proof. For given x and t, write $\mathcal{E}(x,t;s) = E(\hat{s},\kappa)$, where $\kappa = ||k(s)|| = ||s||^2$. Making this change of notation in (29) it is easily seen that if $\langle|\mathcal{E}(x,t)|^q\rangle < +\infty$, then

$$\int_0^{+\infty} |E(\hat{s},\kappa)|^q \mathrm{e}^{-\kappa}\kappa^{M-1}d\kappa < +\infty, \tag{32}$$

for almost every direction $\hat{s}$ in $\mathbb{C}^M$. Fix $\hat{s}$ such that (32) is fulfilled. By Eq. (9), Lemma 1, and Lemma 2, $E(\hat{s},z)$ is an entire function of $z \in \mathbb{C}$ of finite exponential type [1]. Since $q \geq 1$ and $M \geq 1$ are integers, the function $f(z) = E(\hat{s},z)^q \mathrm{e}^{-z} z^{M-1}$ is also an entire

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function of $z \in \mathbb{C}$ of finite exponential type, say γ_f. Let R be a fixed positive number, let $\gamma > \max(0, \gamma_f)$, and define

$$\varphi_{\pm}(z) = \int_0^R |\mathrm{e}^{\pm i\gamma z} f(z+u)| du. \tag{33}$$

The functions $\varphi_{\pm}(z)$ are logarithmically subharmonic and bounded by (32) on the positive real axis. Furthermore, $\varphi_{+}(z)$ and $\varphi_{-}(z)$ are bounded respectively on the positive and negative imaginary axis. Following then the same argument as in the proof of the Plancherel-Pólya theorem (see [1], p 51), we apply the Phragmén-Lindelöf theorem [1] to the subharmonic functions $\ln \varphi_{+}(z)$ in the sector $0 < \arg z < \pi/2$ and $\ln \varphi_{-}(z)$ in the sector $-\pi/2 < \arg z < 0$. One finds that $\exists A > 0$ such that, $\forall z$ with $\mathrm{Re}(z) > 0$,

$$\int_0^R |f(z+u)| du \leq A \mathrm{e}^{\gamma |\mathrm{Im}(z)|}. \tag{34}$$

Now, for all $\kappa > R/2$ one has, by the subharmonicity of $|f(z)|$ and (34),

$$\begin{aligned} |f(\kappa)| &\leq \frac{4}{\pi R^2} \int\int_{|z|<R/2} |f(\kappa + z)|\, d^2 z \\ &\leq \frac{4}{\pi R^2} \int_{-R/2}^{+R/2} dy \int_0^R |f(\kappa - R/2 + iy + u)|\, du \\ &\leq \frac{8A}{\gamma \pi R^2} \left(\mathrm{e}^{\gamma R/2} - 1\right) < +\infty, \end{aligned} \tag{35}$$

and since $|f(\kappa)| = |E(\hat{s}, \kappa)|^q \mathrm{e}^{-\kappa} \kappa^{M-1}$, it follows that, for all $\kappa > R/2$,

$$|E(\hat{s}, \kappa)|^q \mathrm{e}^{-\kappa} \leq \frac{8A}{\gamma \pi R^2} \left(\mathrm{e}^{\gamma R/2} - 1\right) \frac{1}{\kappa^{M-1}} < +\infty, \tag{36}$$

(hence $\lim_{\kappa \to +\infty} |E(\hat{s}, \kappa)|^q \mathrm{e}^{-\kappa} = 0$ for $M > 1$). Getting back to the original notation yields the new Eq. (31), which completes the proof of the lemma.

We can now proceed with the proof of $\lambda_q(x,t) \leq (q\mu_{x,t} - 0^+)^{-1}$. Since every element of the matrix $\int_0^t \gamma(x(\tau), \tau)\, d\tau$ is a continuous functional of $x(\cdot) \in B(x,t)$ with the uniform norm on $[0,t]$ (see Appendix B), its largest eigenvalue, $\mu_1[x(\cdot)]$, is also a continuous functional of $x(\cdot)$. Accordingly, $\forall \varepsilon > 0\ \exists x_\varepsilon(\cdot) \in B(x,t)$ such that $\mu_{x,t} - \varepsilon/2 \leq \mu_1[x_\varepsilon(\cdot)] \leq \mu_{x,t}$. Let $\sigma_\varepsilon \in \mathbb{C}^M$ (with $||\sigma_\varepsilon|| = 1$) be an eigenvector of $\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau$ associated with the eigenvalue $\mu_1[x_\varepsilon(\cdot)]$. Fix $x_\varepsilon(\cdot)$ and σ_ε. The quadratic form $\hat{s}^\dagger \left[\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau\right] \hat{s}$ is a continuous function of the direction $\hat{s}$. Thus, $\exists \delta > 0$ such that $\forall \hat{s}$ with $||\hat{s} - \sigma_\varepsilon|| \leq \delta$,

$$\left| \hat{s}^\dagger \left[\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau\right] \hat{s} - \sigma_\varepsilon^\dagger \left[\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau\right] \sigma_\varepsilon \right| \leq \frac{\varepsilon}{2}, \tag{37}$$

and

$$\begin{aligned} H_{x,t}(\hat{s}) &= \sup_{x(\cdot) \in B(x,t)} \hat{s}^\dagger \left[\int_0^t \gamma(x(\tau), \tau)\, d\tau\right] \hat{s} \\ &\geq \hat{s}^\dagger \left[\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau\right] \hat{s} \\ &\geq \sigma_\varepsilon^\dagger \left[\int_0^t \gamma(x_\varepsilon(\tau), \tau)\, d\tau\right] \sigma_\varepsilon - \varepsilon/2 \geq \mu_{x,t} - \varepsilon. \end{aligned} \tag{38}$$

From the latter inequality and Lemma 2 it follows that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\forall\hat{s}$ with $||\hat{s} - \sigma_\varepsilon|| \leq \delta$,

$$\limsup_{||s||\to+\infty} \frac{\ln|\mathcal{E}(x,t;s)|^q}{||s||^2} \geq q\lambda(\mu_{x,t} - \varepsilon),$$

and for every $\lambda > (q\mu_{x,t} - q\varepsilon)^{-1}$,

$$\limsup_{||s||\to+\infty} \mathrm{e}^{-||s||^2}|\mathcal{E}(x,t;s)|^q = +\infty. \tag{39}$$

Since $\delta > 0$, the set of all the directions $\hat{s}$ fulfilling (39) is of strictly positive measure and, according to the lemma above, $\langle|\mathcal{E}(x,t)|^q\rangle = +\infty$. Therefore, $\lambda_q(x,t) \leq (q\mu_{x,t} - q\varepsilon)^{-1}$ and taking ε arbitrarily small one obtains $\lambda_q(x,t) \leq (q\mu_{x,t} - 0^+)^{-1}$. (We use the notation $q\mu_{x,t} - 0^+$ to emphasize the fact that $\langle|\mathcal{E}(x,t)|^q\rangle$ may be finite at $\lambda = 1/q\mu_{x,t}$. Our approach does not yield the behavior of $\langle|\mathcal{E}(x,t)|^q\rangle$ at $\lambda = 1/q\mu_{x,t}$ sharp.)

Reference

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