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Erratum

The Hamiltonian Operator Associated with Some Quantum Stochastic Evolutions

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It was kindly pointed out to us by W. von Waldenfels that Section 3.2 of [1] contains an error when the trace operator is introduced for functions in the Sobolev space $H^\Sigma(\mathbb{R}^n_*;\mathfrak{H})$: we claimed that there exists a *bounded* operator

$$\cdot|_{\{r_\ell=s\}} : H^\Sigma(\mathbb{R}^n_*;\mathfrak{H}) \to L^2(\mathbb{R}^{n-1};\mathfrak{H})$$

which naturally defines the trace of each v in $H^\Sigma(\mathbb{R}^n_*;\mathfrak{H})$ as a function $v|_{\{r_\ell=s\}}$ in $L^2(\mathbb{R}^{n-1};\mathfrak{H})$, but actually such trace $v|_{\{r_\ell=s\}}$ is naturally defined only as a function in $L^2_{\text{loc}}(\mathbb{R}^{n-1}_*;\mathfrak{H})$ and a trace operator from $H^\Sigma(\mathbb{R}^n_*;\mathfrak{H})$ to $L^2(\mathbb{R}^{n-1};\mathfrak{H})$ can only be *closed*, with a domain to be specified.

Nevertheless the main result of [1], Theorem 3, is correct and provable through an adjustment of the argument.

We refer to [2] for a detailed introduction of the traces $\cdot|_{\{r_\ell=s\}}$ and we list below the points which require an adjustment, that is the points involving $\cdot|_{\{r_\ell=s\}}$ which are to be handled taking into account domain constraints.

1. The integration by parts formula (22) needs to be generalized [2] because $\langle u|_{\partial Q_m} \big| v|_{\partial Q_m}\rangle_\mathfrak{H}$ is not necessarily in $L^1(\partial Q_m)$ for every u and v in $H^\Sigma(\mathbb{R}^n_*;\mathfrak{H})$. Therefore, for $\epsilon > 0$, we introduce on $\mathbb{R}^n$ the totally symmetric indicator function $I_\epsilon(r) = \prod_{\ell<\ell'}\big\{1 - I_{(-\infty,0)}(r_\ell r_{\ell'})\ I_{[0,\epsilon]}(|r_\ell| + |r_{\ell'}|)\big\}$, which vanishes when r has two small coordinates of opposite sign. Then $I_\epsilon(r) \uparrow 1$ as $\epsilon \downarrow 0$ and for every u and v in $H^\Sigma(\mathbb{R}^n_*;\mathfrak{H})$ the following generalized integration by parts formula holds:

$$\int_{Q_m} \langle u|\sum_{\ell=1}^n \partial_\ell v\rangle_\mathfrak{H} = -\int_{Q_m} \langle \sum_{\ell=1}^n \partial_\ell u|v\rangle_\mathfrak{H} + \lim_{\epsilon\downarrow 0}\int_{\partial Q_m} \Big(\sum_{\ell=1}^n \eta_m \cdot e_\ell\Big)\langle (I_\epsilon u)|_{\partial Q_m}\big|(I_\epsilon v)|_{\partial Q_m}\rangle_\mathfrak{H}, \qquad \text{(22b)}$$

which reduces to (22), by dominated convergence, every time $\langle u|_{\partial Q_m} \big| v|_{\partial Q_m}\rangle_{\mathfrak{H}}$ is in $L^1(\partial Q_m)$. This happens if u and v have traces $u|_{\partial Q_m}$ and $v|_{\partial Q_m}$ in $L^2(\partial Q_m; \mathfrak{H})$, or also if, independently of v, $u|_{\partial Q_m} = (I_\epsilon u)|_{\partial Q_m}$ for some ϵ.

Analogously, for every u and v in $H^{\Sigma}_{\mathrm{symm}}\big((\mathbb{R}_* \times J)^n; \mathcal{H}\big)$, the correct version of (23) is the following generalized integration by parts formula [2]:

$$\begin{aligned}\Big\langle u \Big| \sum_{\ell=1}^{n} \partial_\ell v \Big\rangle_{L^2((\mathbb{R}\times J)^n;\mathcal{H})} &= -\Big\langle \sum_{\ell=1}^{n} \partial_\ell u \Big| v \Big\rangle_{L^2((\mathbb{R}\times J)^n;\mathcal{H})} \\ &\quad + n \lim_{\epsilon\downarrow 0} \Big\{ \langle (I_\epsilon\, u)|_{\{r_n=0^-\}} \big| (I_\epsilon\, v)|_{\{r_n=0^-\}} \rangle_{\mathfrak{Z}\otimes L^2((\mathbb{R}\times J)^{n-1};\mathcal{H})} \\ &\quad - \langle (I_\epsilon\, u)|_{\{r_n=0^+\}} \big| (I_\epsilon\, v)|_{\{r_n=0^+\}} \rangle_{\mathfrak{Z}\otimes L^2((\mathbb{R}\times J)^{n-1};\mathcal{H})} \Big\}.\end{aligned} \tag{23b}$$

2. The unbounded operators $a(s)$ and their domains $\mathcal{V}_s$ are to be defined just by Eqs. (32) and (25) of [1], which therefore imply that a vector Φ in $\mathcal{V}_s$ needs to have every single component Φ_n with square integrable trace ($\big\| \Phi_n|_{\{r_n=s\}} \big\|_{\mathfrak{Z}\otimes L^2((\mathbb{R}\times J)^{n-1};\mathcal{H})} < \infty \forall n$).

3. Proposition 3 can still be proved as in [1], but domain constraints for $a(0^-)$ and $a(0^+)$ are to be dealt with more carefully. Clearly Eq. (36) can always be extended by linearity and it can also be extended by continuity (bounded convergence) to a vector Φ in $\mathcal{V}_{0^\pm}$ every time there is a sequence of vectors Φ_N in $\mathcal{V}_{0^\pm}$ satisfying (36) such that $\Phi_N \to \Phi$ in $\mathcal{K}$, $E\,\Phi_N \to E\,\Phi$ in $\mathcal{K}$ and $a(s)\,\Phi_N \to a(s)\,\Phi$ in $\mathfrak{Z}\otimes\mathcal{K}$ for $s = 0^-, 0^+$. So the validity of (36) can be extended from $\mathcal{E}\big(H^1(\mathbb{R}_*; \mathfrak{Z})\big)$ to n-particle vectors in span $\Big\{ v^{\otimes n} \otimes h \Big| v \in H^1(\mathbb{R}_*; \mathfrak{Z}), h \in \mathcal{H} \Big\}$ and then to n-particle vectors in $H^1(\mathbb{R}_*; \mathfrak{Z})^{\circledS n} \otimes \mathcal{H}$; thanks to Theorem 4 in [2], since the latter space includes $\mathfrak{D}(\mathbb{R}^n_*; \mathfrak{Z}^{\otimes n} \otimes \mathcal{H}) \cap L^2_{\mathrm{symm}}\big((\mathbb{R}\times J)^{n-1}; \mathcal{H}\big)$, Eq. (36) can be extended also to all n-particle vectors belonging to $\mathcal{V}_{0^\pm}$ and finally to all vectors in $\mathcal{V}_{0^\pm}$.

4. Proposition 6 can still be proved as in [1], even if only the generalized integration by parts formula (23b) is available. The integration by parts formula is applied to prove that $U_t\,\Phi$ belongs to $\mathcal{V}_{0^-}$ and with (23b) there is a limit w.r.t. $\epsilon \downarrow 0$ which has to be commuted with the integrations in the scalar products. Such operations can be commuted if the vector Υ in $\mathcal{V}_0$ is assumed to have components Υ_n vanishing in a neighborhood of all the coordinate hyperedges $\{r_j = r_\ell = 0\}$, $j \neq \ell$. Then, thanks to Lemma 8 in [2], this class of vectors is large enough to get the thesis.

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