Theoretical Chemistry Accounts (2020) 139:178
https://doi.org/10.1007/s00214-020-02685-x

REGULAR ARTICLE

The eigenvalue problem of one-dimensional Dirac operator

Jacek Karwowski[1] · **Artur Ishkhanyan**[2,3] · **Andrzej Poszwa**[4]

Received: 23 June 2020 / Accepted: 6 October 2020

Abstract
The properties of the eigenvalue problem of the one-dimensional Dirac operator are discussed in terms of the mutual relations between vector, scalar and pseudo-scalar contributions to the potential. Relations to the exact solubility are analyzed.

Keywords One-dimensional Dirac equation · Vector potential · Scalar potential · Pseudo-scalar potential · Supersymmetry · Effective mass · Schrödinger equation · Lévy-Leblond equation · Bound states · Non-relativistic limit · Pauli approximation

1 Introduction

We are concerned with the simplest quantum system: a single particle in a one-dimensional space. Its time evolution in the Schrödinger picture and in the coordinate representation is governed by the evolution equation:

$$\left[i\hbar\frac{\partial}{\partial t} - \mathsf{H}(x)\right]\Psi(x,t) = 0. \tag{1}$$

Solutions of the eigenvalue problem of the Hamiltonian

$$\mathsf{H}\,\psi_E(x) = E\,\psi_E(x) \tag{2}$$

give the stationary state energies and wave functions. From here solution of (1) may be obtained as

$$\Psi(x,t) = \int\!\!\!\!\!\!\sum_E \mathcal{C}(E)\,\mathrm{e}^{-iEt/\hbar}\,\psi_E(x)\,dE. \tag{3}$$

We have chosen to describe a single fermion using the Lorentz-covariant quantum theory. The space-time is two-dimensional, and physical quantities may be described by scalars, pseudo-scalars and two vectors composed of time and space components. The relativistic dispersion relation for a free particle

$$\mathsf{E}^2 = c^2\mathsf{p}^2 + m^2c^4, \quad \mathsf{E} = \mathsf{H}_0, \quad \mathsf{p} = \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{dx}}, \tag{4}$$

implies that the free-particle Hamiltonian H_0 is a Hermitian square root of the second-order differential operator in the rhs of this relation, i.e. it is the one-dimensional free Dirac operator [1]. By applying the classical procedure of Dirac, reduced to one dimension, we get

$$\mathsf{H}_0 = c\boldsymbol{\alpha}\,\mathsf{p} + \boldsymbol{\beta}\,mc^2, \tag{5}$$

with $\boldsymbol{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta}\boldsymbol{\alpha} = 0$, $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = 1$ and $x \in \langle -\infty, \infty\rangle$[1]. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have to be Hermitian, their eigenvalues can only be ± 1. It is easy to see that $\mathrm{Tr}(\boldsymbol{\alpha}) = \mathrm{Tr}(\boldsymbol{\beta}) = 0$ and, consequently, the dimension of these matrices has to be even. The Pauli matrices

Published as part of the special collection of articles "Festschrift in honour of Prof. Ramon Carbó-Dorca".

✉ Jacek Karwowski
jka@fizyka.umk.pl

Artur Ishkhanyan
aishkhanyan@gmail.com

Andrzej Poszwa
poszwa@matman.uwm.edu.pl

[1] Institute of Physics, Nicolaus Copernicus University, 87-100 Toruń, Poland

[2] Russian-Armenian University, 0051 Yerevan, Armenia

[3] Institute for Physical Research, NAS of Armenia, 0203 Ashtarak, Armenia

[4] Faculty of Mathematics and Computer Science, University of Warmia and Mazury, 10-710 Olsztyn, Poland

[1] The momentum operator is Hermitian if $x \in \langle -\infty, \infty\rangle$, but it is non-Hermitian on a semi-axis. By properly choosing boundary conditions one can construct a Hermitian extensions of p for x in a finite interval $x \in \langle -L, L\rangle$ [2, 3].

Published online: 21 November 2020

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

form a complete set of 2×2 matrices. One can readily check that any two of three anti-commuting matrices fulfill the above conditions. Therefore two of them, say $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$, can be set for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

In realistic physical and chemical models, complicated multi-particle interactions are replaced by *external fields*—effective one-particle potentials which reflect the main features of these interactions in a simplified way. Models of a single particle in external fields may be applied to describe nearly all systems of chemical or physical interest. For example, a many-electron system may be reduced to a one-electron problem using the mean-field approach which results in Hartree–Fock or Dirac–Fock equations. The external potentials may have different transformation properties. In the "complete", four-dimensional world the external potentials may transform as vectors, scalars, pseudo-scalars, pseudo-vectors, tensors. In our case we have a two-component vector, scalar and pseudo-scalar only. In the most general form, the one-dimensional Dirac operator is a linear combination of four Pauli matrices:

$$\begin{aligned} \mathsf{H} = {} & \boldsymbol{\sigma}_0\, \mathsf{V}(x) + c\,\boldsymbol{\sigma}_1 \left[\mathsf{p} + \mathsf{U}(x)\right] + c\,\boldsymbol{\sigma}_2\, \mathsf{P}(x) \\ & + \boldsymbol{\sigma}_3 \left[mc^2 + \mathsf{S}(x)\right] \end{aligned} \tag{6}$$

where V and $c\mathsf{U}$ stand, respectively, for time and space components of the two-vector potential, $c\mathsf{P}$ for pseudo-scalar and S—for scalar potential. Note that factor c in the definitions of the space component of the two-vector and of the pseudo-scalar potentials has been introduced for convenience and is related to the choice of units in which U and P are expressed.

The eigenvalue problem of the Dirac operator (6) reads

$$\mathsf{H}\,\psi(x) = E\,\psi(x), \tag{7}$$

where ψ is a two-component wave function. Using the gauge transformation

$$\psi(x) \;\rightarrow\; \exp\left(-\frac{i}{\hbar}\int \mathsf{U}(x)\mathrm{d}x\right)\psi(x),$$

one can eliminate $\mathsf{U}(x)$. Then, without any loss of generality, we can assume that $\mathsf{U}(x) = 0$.

Discussions of the eigenvalue problem of the one-dimensional Dirac operator (6) have appeared in the literature about half a century after the four-dimensional Dirac equation was formulated, mostly for didactic purposes. The discussion of the one-dimensional step potential has commonly been used as an illustration of the Klein paradox. A detailed early study has been given by Greiner [4]. General properties of the one-dimensional Dirac equation have been presented in a series of important works by Sukumar [5], Cooper et al. [6], Nogami, Toyama and coworkers [7–10]. According to a general observation by Wolfram [11]: *"Even when the underlying rules for a system are extremely simple, the behavior of the system as a whole can be essentially arbitrarily rich"*, many non-trivial aspects of the one-dimensional Dirac operator eigenvalue problem have been discovered with applications ranging from solid-state theory to nuclear and elementary particle physics. From the quantum-chemical perspective the most important are applications of the one-dimensional Dirac equation to the description of quantum systems whose spectrum has an energy gap and whose properties may be conveniently simulated by a properly taylored Dirac equation. To this category belong works on the edge states, on the Landau levels, on the theoretical modeling of graphene [12–15], also including external electromagnetic fields [16], and advanced studies on radially twisted carbon nanotubes [17, 18], specific calculations relevant to the theoretical simulation of one-dimensional graphene structures performed, among others, in [19, 20]. One-dimensional Dirac equation has also been used to the description of the π-electron systems in conjugated molecules [21].

During the last two decades, the number of papers concerned with one-dimensional Dirac eigenvalue problem has grown up exponentially. The subject covers many areas of modern physics, as, for example, statistical properties of spectra [22, 23], properties of nonlinear equations [24] and studies on the Cornell potential describing quark interactions [25–27]. A great deal of work has been done on the one-dimensional hydrogen atom—an issue apparently simple, but in fact, containing many mathematical subtleties and still, to a certain extent controversial. Already the choice of the potential is not obvious. The solution of the Poisson equation in one dimension for the point charge is proportional to $|x|$. Thus, some authors describe the 1D hydrogen atom using this potential [28–30] or its screened form [31]. In some other works, the singular 1D limit $a/|x|$ of the 3D Coulomb potential a/r or its regularized forms is used instead [32, 33]. Different kinds of potentials lead to different forms of equations and, consequently, to different, sometimes unexpected, spectral properties. For example, as shown in [34, 35] the pseudo-scalar Coulomb problem has no bound states, also in 2D and 3D cases [34]. On the other hand de Castro, in a very interesting paper [36], demonstrated that there exists an infinite sequence of bound states if the problem is formulated in a slightly different way. Spin and pseudo-spin symmetries of the 1D Dirac equation with Coulomb potential were studied in [37]. A unified treatment of Coulomb and harmonic potentials in D dimensions, including $D = 1$, have been discussed by Lévai et al. [38].

The one-dimensional Dirac equation is a nice playground for designing analytical solutions in some non-trivial environments. Here are some examples: Bound states in double-well potentials [39], pseudo-scalar potential barrier [40], construction of transparent potentials [41, 42], quadratic plus inversely quadratic potential [43], Kratzer potential [44], Wood-Saxon potential and effective mass problem [45], hyperbolic tangent potential [46] and Dirac–Moshinsky oscillator [47], just to mention several examples. Among innovative, non-standard, applications one should mention a work by Correa and Jakubsky on the description of optical systems, in the coupled mode theory of the Bragg gratings using 1D Dirac equation with a non-Hermitian Hamiltonian [48]. For some potentials the 1D Dirac equation is also quasi-exactly solvable. For example, in Ref. [49] quasi-exact solutions for the hyperbolic secant potential have been found.

The one-dimensional Dirac equation is reducible to a Schrödinger form in which the original Dirac potential generates a two- or three-term effective potential. The integrability of this Schrödinger equation implies that the Dirac equation is also solvable and its solutions are directly expressible in terms of the solutions of the Schrödinger equation. During the last decade, a large family of Schrödinger equations, reducible to different forms of the Heun equation, have been solved analytically (see, for example, [50–52]). Consequently, also the set of solvable Dirac equations is extended. An example, the analytical solution for the inverse square root potential, has recently been presented [53].

Due to its simplicity, the one-dimensional Dirac equation is a practical and convenient tool to study the interplay between different kinds of potentials. Among many works devoted to this subject, let us mention some representative examples. Thus, the dependence of the asymptotic behavior of the effective confining potentials on the relative strength of vector and scalar contributions was studied in [54], a joint treatment of mixed vector–scalar potentials in [55, 56] and mixed vector–scalar–pseudo-scalar potentials in [57].

In this paper we analyze some general properties of the one-dimensional Dirac operator. In the consecutive sections, we discuss the structure of the energy spectrum, the non-relativistic limit of the equation and, formally related to it, spin and pseudo-spin symmetries. Then, the large-component equation and the Pauli approximation, important in analyzing the asymptotic behavior of the eigen-problem at the non-relativistic limit, are presented. Next, the position dependent mass is discussed in a general context of the one-dimensional relativistic model. Finally, in the last two sections we discuss the reduction of the Dirac equation to the Schrödinger form and the equations describing massless fermions.

The paper is dedicated to our old friend, Professor Ramon Carbó-Dorca on the occasion of his 80th birthday.

2 General properties of the spectrum

The energy spectrum of the one-dimensional free Dirac operator, consists of two continua. The positive one spreads from the rest energy of the particle, mc^2, to $+\infty$. It corresponds to the non-relativistic energies of a free particle. The negative continuum, spreading from $-mc^2$ to $-\infty$, has no physical meaning. But in the charge-conjugate equation the negative energy states correspond to positrons with positive energies. The two continua in the spectrum of the Dirac operator are referred to as its *essential spectrum*. The essential spectrum of a Hermitian operator is a subset of its complete spectrum; its complement is the discrete spectrum, i.e. the set of isolated eigenvalues of finite multiplicity. For most of potentials of physical significance the discrete energies of the bound states are located in the energy gap between $-mc^2$ and mc^2. For some specific potentials resonances embedded in the continuum, described by the wave functions containing both localized and continuum-type contributions, and corresponding to auto-ionizing states, are also present [58, 59].

Many symmetries, present in three-dimensional space, are absent in one dimension. One of the consequences is that there is no degeneracy of energy levels in one dimension [7]. Since angular momentum does not exist in one dimension, spin effects, in particular spin-orbit interaction, are absent in one-dimensional Dirac equation. This simplifies the mathematical structure of the Dirac operator eigenvalue equation, compared to the three-dimensional case. Therefore, in one dimension some detailed studies on the interplay between different kinds of interactions are easier and more transparent than in the three-dimensional case. Let us note that the vector potentials couple to the electric charges. Then, the coupling is different for particles and for antiparticles and spectrum is not symmetric with respect to change of the energy sign. The scalar interactions couple potentials to the mass, the same for particles and for antiparticles. As a consequence, the energy spectrum is symmetric: to each bound state of a particle with energy E, there exists an antiparticle counterpart with energy $-E$ [7]. For the same reasons the effect of confinement due to vector and scalar potentials is different. In the case of vector coupling if $\mathsf{V}(x) \to \infty$ for $|x| \to \infty$, bound states do not exist—all states are metastable. For scalar coupling, similarly as in the non-relativistic models, we have an infinite series of bound states in such cases [9].

3 Non-relativistic limit

The eigenvalue equation (7), with the gauge leading to $\mathsf{U}(x) = 0$, may be written as

$$\begin{pmatrix} \mathsf{V}^+ - E^-, & c\,\pi^\dagger \\ c\,\pi, & \mathsf{V}^- - E^+ \end{pmatrix} \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{S} \end{pmatrix} = 0, \tag{8}$$

where

$$E^\pm = E \pm mc^2 \quad \mathsf{V}^\pm = \mathsf{V} \pm \mathsf{S}, \quad \pi = \mathsf{p} + i\mathsf{P}$$

and

$$\psi(x) = \begin{pmatrix} \psi_\mathrm{L}(x) \\ \psi_\mathrm{S}(x) \end{pmatrix}.$$

Two components of the wave function, ψ_L and ψ_S, are referred to, respectively, as large and small components—the names restricted to the present representation of the Dirac operator in the two-dimensional spinor space, known as the Pauli representation.

In order to get the non-relativistic limit of Eq. (8) it is convenient to introduce the "non-relativistic energy scale", i.e. to shift the energy by the rest energy of the particle. Then, Eq. (8) becomes

$$\begin{pmatrix} \mathsf{V}^+ - \mathcal{E}, & c\,\pi^\dagger \\ c\,\pi, & \mathsf{V}^- - 2mc^2 - \mathcal{E} \end{pmatrix} \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{S} \end{pmatrix} = 0, \tag{9}$$

where $\mathcal{E} = E^-$. After dividing the second of the two equations by c, introducing $c\,\psi_\mathrm{S}$ instead of ψ_S and setting $(\mathsf{V}^- - \mathcal{E})/(2mc^2) = 0$, Eq. (9) transforms to

$$\begin{pmatrix} \mathsf{V}^+ - \mathcal{E}, & \pi^\dagger \\ \pi, & -2m \end{pmatrix} \begin{pmatrix} \psi_\mathrm{L} \\ c\,\psi_\mathrm{S} \end{pmatrix} = 0, \tag{10}$$

i.e. to the non-relativistic limit defined in the same Hilbert space as the original Dirac-operator eigenvalue problem. This equation, known as the *one-dimensional Lévy-Leblond equation*, transforms according to a spinor representation of the Galilei group. The elimination of $c\,\psi_\mathrm{S}$ leads to the Schrödinger form of the non-relativistic limit:

$$\left[\frac{\pi^\dagger \pi}{2m} + \mathsf{V}^+\right]\psi_\mathrm{L} = \mathcal{E}\psi_\mathrm{L}. \tag{11}$$

As one can see, in the non-relativistic limit V and S are combined to form the non-relativistic "scalar" potential V^+. The relativistic pseudo-scalar potential P enters the kinetic energy operator in a similar way as a single component of the non-relativistic "vector potential":

$$\pi^\dagger \pi = (\mathsf{p} - i\mathsf{P})(\mathsf{p} + i\mathsf{P}) = -\hbar^2 \frac{d^2}{dx^2} + \hbar \frac{d\mathsf{P}}{dx} + \mathsf{P}^2.$$

4 Spin and pseudo-spin symmetries

A straightforward reduction of the eigenvalue equation (8) to a Schrödinger-like equation is obtained in the case of *spin symmetry*, or *pseudo-spin symmetry*, i.e. when, respectively, $\mathsf{V}^- = v^- = \text{const}$ or $\mathsf{V}^+ = v^+ = \text{const}$. In both cases one of the components of $\psi(x)$ in Eq. (8) may be eliminated and the equations for the second component read, respectively:

$$\left[\frac{\pi^\dagger \pi}{2\mathfrak{M}_v^-} + \mathsf{V}^+\right]\psi_\mathrm{L} = E^-\,\psi_\mathrm{L}, \quad \text{and} \quad \left[\frac{\pi\,\pi^\dagger}{2\mathfrak{M}_v^+} + \mathsf{V}^-\right]\psi_\mathrm{S} = E^+\,\psi_\mathrm{S}, \tag{12}$$

where

$$\mathfrak{M}_v^\mp = m\left(\frac{E^\mp - v^\mp}{2mc^2} \pm 1\right).$$

Note that in the limit of small external potentials, the dominant component is ψ_L for spin symmetry and ψ_S for pseudo-spin symmetry. In both cases, if for a given potential analytic solutions of the Schrödinger equation are known, then the corresponding solutions of the one-dimensional Dirac equation may be easily obtained by a direct substitution of the appropriate parameters. A highly instructive note on this subject, with wealth of references, has been presented in a short comment by Castro and de Castro [26].

It has hardly been noticed that the reduction of the Dirac eigenvalue problem to the Schrödinger-like one by suppressing the coupling between components of the Dirac wave function has originally been done by Moore nearly half a century ago. The resulting equation

$$\mathsf{H}_\mathrm{d}\,\psi_\mathrm{L} = \mathcal{E}\psi_\mathrm{L}, \tag{13}$$

where

$$\mathsf{H}_\mathrm{d} = \frac{\pi^\dagger \pi}{2\,\mathfrak{M}} + \mathsf{V}, \quad \mathfrak{M} = m\left(1 + \frac{\mathcal{E}}{2mc^2}\right)$$

is known as the *direct Dirac equation* [60].[2]

5 Large-component equation

After some simple algebra, the separation of Eq. (8) yields

$$\left[\frac{\pi^\dagger\,\mathsf{R}\,\pi}{2\mathfrak{M}} + \mathsf{V}^+\right]\psi_\mathrm{L} = \mathcal{E}\psi_\mathrm{L}, \tag{14}$$

[2] An intuitive meaning of $\mathfrak{M}$ may be derived from the observation that the relativistic free-particle energy–momentum relation (4), with $E = \mathcal{E} + mc^2$, may be rewritten as $\mathcal{E} = \mathsf{p}^2/(2\mathfrak{M})$.

where

$$\mathsf{R} = \frac{1}{1-\mathsf{w}}, \qquad \mathsf{w} = \frac{\mathsf{V}^-}{2\mathfrak{M}\,c^2} \tag{15}$$

and

$$\psi_{\mathrm{S}} = \frac{\mathsf{R}\,\pi}{2\mathfrak{M}\,c}\,\psi_{\mathrm{L}}. \tag{16}$$

The pair of Eqs. (14) and (16) is equivalent to the original Dirac equation. But, Eq. (14) itself is not an eigenvalues problem—parameter $\mathfrak{M}$ depends on $\mathcal{E}$ and, consequently, functions corresponding to different $\mathcal{E}$ are not orthogonal. The orthonormality condition involves both ψ_{L} and ψ_{S} and reads

$$\int_{-\infty}^{\infty} \psi^{(m)\dagger}(x)\,\psi^{(n)}(x)\,\mathrm{d}x = \delta_{mn},$$

where $\psi^{(m)}$ and $\psi^{(n)}$ are two-component wave functions (8) corresponding, respectively, to energies $\mathcal{E}_m$ and $\mathcal{E}_n$.

Equation (14) may be rewritten as

$$\left[\mathsf{H}_0 + \mathsf{H}_{\mathrm{D}} + \mathsf{H}_{\mathrm{mv}}\right]\psi_{\mathrm{L}} = \mathcal{E}\psi_{\mathrm{L}}, \tag{17}$$

where

$$\mathsf{H}_0 = \frac{\pi^\dagger\,\pi}{2\,m} + \mathsf{V}^+,$$

and

$$\mathsf{H}_{\mathrm{D}} = \frac{\mathsf{R}}{2\mathfrak{M}}\,\frac{\mathsf{p}(\mathsf{V}^-)\pi}{2m\,c^2}, \qquad \mathsf{H}_{\mathrm{mv}} = -\frac{(\mathsf{V}^+ - \mathcal{E})(\mathsf{V}^- - \mathcal{E})}{2mc^2};$$

H_{D} and H_{mv} may be recognized as the exact form of the one-dimensional Darwin correction and the mass-velocity term. The standard Pauli approximation results from Eq. (17) by the replacement of R and $\mathfrak{M}$ by, respectively, 1 and m.

5.1 Position-dependent effective mass

Equation (14) may also be interpreted as the Schrödinger equation with a position-dependent effective mass, commonly used in solid-state physics [61]. If we define the effective mass as

$$m^*(x) = \frac{\mathfrak{M}}{\mathsf{R}(x)} = \mathfrak{M}\left(1 - \frac{\mathsf{V}^-}{\mathfrak{M}\,c^2}\right) \tag{18}$$

then, Eq. (14) yields

$$\left[\pi^\dagger\,\frac{1}{2\,m^*(x)}\,\pi + \mathsf{V}^+\right]\Phi(x) = \mathcal{E}\Phi(x), \tag{19}$$

where $\Phi(x)$ is the wave function which, effectively, contains relativistic effects carried implicitly by the effective mass m^*. The model is formally valid if

$$F(x) \equiv \frac{1}{m^*(x)}\,\frac{d\,\Phi(x)}{dx}$$

is continuous [61].

6 Reduction to a Schrödinger form

In some cases the reduction of the one-dimensional Dirac equation to the Schrödinger form is associated with the factorization of the Hamiltonian to two mutually Hermite conjugate operators which, in turn, is the first step in an analysis of super-symmetry (SUSY) properties of the equation. Studies on super-symmetry of the one-dimensional Dirac equation were initiated by fundamental works by Sukumar [5], Cooper et al. [6], Nogami and Toyama [8]. Basic ideas of SUSY quantum mechanics, its relations to exact solubility of quantum problems and a list of exactly solvable potentials have been given by Dutt at al. [62]. In the next subsections two special cases are considered.

6.1 Constant P and $\mathsf{V}(x) \sim \mathsf{S}(x)$

Let $\mathsf{P}(x) = \mathsf{P}_0$, where P_0 is x-independent, and

$$\mathsf{V}^+(x) = q\,\mathsf{W}(x), \qquad \mathsf{V}^-(x) = -r\,\mathsf{W}(x), \tag{20}$$

where $q, r \neq 0$ are independent of x. If $q = 0$ or $r = 0$, then we have special cases of spin or pseudo-spin symmetries, considered in Sect. 4. Equation (8) may be rewritten as

$$\begin{pmatrix} c\,(\mathsf{p} + i\,\mathsf{P}_0), & -r\,\mathsf{W}(x) - E^+ \\ [5pt] q\,\mathsf{W}(x) - E^-, & c\,(\mathsf{p} - i\,\mathsf{P}_0) \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{L}}(x) \\ [5pt] \psi_{\mathrm{S}}(x) \end{pmatrix} = 0. \tag{21}$$

By a similarity transformation:

$$\begin{aligned} \mathsf{T} &= \frac{1}{\sqrt{2qr}}\begin{pmatrix} \sqrt{qr}, & i\,r \\ [3pt] i\,q, & \sqrt{qr} \end{pmatrix}, \\ \mathsf{T}^{-1} &= \frac{1}{\sqrt{2qr}}\begin{pmatrix} \sqrt{qr}, & -i\,r \\ [3pt] -i\,q, & \sqrt{qr} \end{pmatrix}, \end{aligned} \tag{22}$$

Eq. (21) becomes

$$\begin{pmatrix} c(\mathsf{p} + i\,\mathsf{Q}(x)), & r\,g^- \\ [5pt] q\,g^+, & c(\mathsf{p} - i\,\mathsf{Q}(x)) \end{pmatrix}\begin{pmatrix} \varphi_1(x) \\ [5pt] \varphi_2(x) \end{pmatrix} = 0, \tag{23}$$

where

$$\mathsf{Q}(x) = \mathsf{Y}(x) + \mathfrak{p}_{\mathsf{V}}, \tag{24}$$

with

$$\mathsf{Y}(x) = \frac{\sqrt{qr}}{c}\,\mathsf{W}(x), \qquad \mathfrak{p}_{\mathsf{V}} = \frac{q\,E^+ - r\,E^-}{2c\,\sqrt{qr}} \tag{25}$$

and

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \mathsf{T}\,\psi(x),$$

$$g^{\pm} = -\frac{1}{\sqrt{qr}}\left(\frac{q\,E^{+} + r\,E^{-}}{2\sqrt{qr}} \pm c\,\mathsf{P}_0\right).$$

By introducing

$$A^{\pm}(x) = i\left(-\hbar\frac{d}{dx} \pm \mathsf{Q}(x)\right), \tag{26}$$

Eq. (23) implies

$$\begin{aligned} \left(\mathsf{A}^{-}\,\mathsf{A}^{+} + \mathfrak{p}_{\mathsf{V}}^2\right)\varphi_1(x) &= \mathfrak{E}_{\mathsf{V}}\,\varphi_1(x), \\ \left(\mathsf{A}^{+}\,\mathsf{A}^{-} + \mathfrak{p}_{\mathsf{V}}^2\right)\varphi_2(x) &= \mathfrak{E}_{\mathsf{V}}\,\varphi_2(x), \end{aligned} \tag{27}$$

$\mathfrak{p}_{\mathsf{V}}^2/2m$ is the *factorization energy* [6] and

$$\mathfrak{E}_{\mathsf{V}} = \frac{E^{+}\,E^{-}}{c^2} - \mathsf{P}_0^2 = 2\,\mathfrak{M}\,\mathcal{E} - P_0^2. \tag{28}$$

Eq. (27) may be rewritten in the Schrödinger form:

$$\left(-\hbar^2\frac{d^2}{dx^2} + \mathsf{Y}^2 + 2\mathfrak{p}_{\mathsf{V}}\mathsf{Y} \mp \hbar\frac{d\,\mathsf{Y}}{dx}\right)\varphi_{1,2} = \mathfrak{E}_{\mathsf{V}}\varphi_{1,2}. \tag{29}$$

Hamiltonian in Eq. (29) depends on parameter E. For a fixed value of E, the Hamiltonian has a complete spectrum of eigenvalues. If an eigenvalue is equal to the parameter E, then it is also equal to the corresponding eigenvalue of the Dirac Hamiltonian. The tuning of an eigenvalue to the value of the parameter is straightforward if the equation is solved analytically. In case of numerical solutions, an iterative loop in which the two numbers are adjusted to each other is necessary. In any case, at most one eigenvalue may be equal to a fixed parameter.

According to (20)

$$\frac{\mathsf{S}^2 - \mathsf{V}^2}{\mathsf{W}^2} = qr.$$

If $qr > 0$, i.e. if the scalar part dominates in the total potential, then the eigenvalue problem (29) is Hermitian. If $qr < 0$, then the Hamiltonian is complex, i.e. it is non-Hermitian. But, if $\mathsf{W}(x) = \mathsf{W}^*(-x)$ then Eq. (29) is $\mathcal{PT}$–symmetric.[3]

Two cases deserve some particular attention:

1. $q = r = 1$. In this case $\mathsf{V} = 0$, i.e. $\mathsf{V}^{+} = -\mathsf{V}^{-} = \mathsf{S}$, $\mathsf{Y} = \mathsf{W}/c$ and $\mathfrak{p}_{\mathsf{V}} = mc$. Equation (29) simplifies to a Hermitian eigenvalue problem with E-independent Hamiltonian: It is interesting to note that the Hamiltonian does not depend on E only if $q = r$.
2. $q = -r = 1$. In this case $\mathsf{S} = 0$, i.e. $\mathsf{V}^{+} = \mathsf{V}^{-} = \mathsf{V}$, $\mathsf{Y} = i\,\mathsf{W}/c$ and $\mathfrak{p}_{\mathsf{V}} = E/c$. The Hamiltonian depends on E and in general is non-Hermitian, but if $\mathsf{W}(x) = \mathsf{W}^*(-x)$ then Eq. (29) is $\mathcal{PT}$-symmetric:

$$\left[-\hbar^2\frac{d^2}{dx^2} - \left(\frac{\mathsf{W}}{c}\right)^2 + \frac{2\,E}{c}\left(\frac{\mathsf{W}}{c}\right) \mp i\hbar\frac{d}{dx}\left(\frac{\mathsf{W}}{c}\right)\right]\varphi_{1,2} = \mathfrak{E}_{\mathsf{V}}\varphi_{1,2}.$$

6.2 Constant V^{+} and V^{-}

If $\mathsf{V}^{+}(x) = v^{+}$, $\mathsf{V}^{-}(x) = v^{-}$, where v^{+}, v^{-} are x independent, then Eq. (8) may be rewritten as

$$\begin{pmatrix} v^{+} - E^{-}, & c\left(\mathsf{p} - i\widetilde{\mathsf{P}}(x)\right) \\ [5pt] c\left(\mathsf{p} + i\widetilde{\mathsf{P}}(x)\right), & v^{-} - E^{+} \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{L}}(x) \\ [5pt] \psi_{\mathrm{S}}(x) \end{pmatrix} = 0, \tag{30}$$

with $\widetilde{\mathsf{P}}(x) = \mathsf{P}(x) + \mathsf{P}_0$, where P_0 is a constant *background potential* [36, 64]. Equation (30) may be easily decoupled:

$$\psi_{\mathrm{s,l}}(x) = \frac{c\left(\mathsf{p} \pm i\widetilde{\mathsf{P}}(x)\right)}{E^{\pm} - v^{\mp}}\,\psi_{\mathrm{l,s}}(x) \tag{31}$$

and

$$\begin{aligned} &\left(-\hbar^2\frac{d^2}{dx^2} + \mathsf{P}^2(x) + 2\mathsf{P}_0\,\mathsf{P}(x) \pm \hbar\frac{d\mathsf{P}}{dx}\right) \\ &\qquad \psi_{\mathrm{l,s}}(x) = \mathfrak{E}_{\mathsf{P}}\,\psi_{\mathrm{l,s}}(x) \end{aligned} \tag{32}$$

where

$$\mathfrak{E}_{\mathsf{P}} = \left(\mathcal{E} - v^{+}\right)\left(\frac{\mathcal{E} - v^{-}}{c^2} + 2m\right) - \mathsf{P}_0^2. \tag{33}$$

Note, that if we set $\mathsf{P} = \mathsf{W}/c$ and $\mathsf{P}_0 = mc$, then the Hamiltonians in Eq. (32) are the same as in Eq. (29), in the case of the scalar potential. By introducing

$$A^{\pm}(x) = i\left(-\hbar\frac{d}{dx} \pm \widetilde{\mathsf{P}}(x)\right)$$

we can rewrite Eq. (32) as a SUSY pair of equations:

$$\mathsf{A}^{+}\,\mathsf{A}^{-}\,\psi_{\mathrm{L}}(x) = \mathfrak{E}_{\mathsf{P}}\,\psi_{\mathrm{S}}(x), \quad \text{and} \quad \mathsf{A}^{-}\,\mathsf{A}^{+}\,\psi_{\mathrm{S}}(x) = \mathfrak{E}_{\mathsf{P}}\,\psi_{\mathrm{S}}(x)$$

[3] Spectrum of a non-Hermitian Hamiltonian may be real if the Hamiltonian is $\mathcal{PT}$-symmetric, i.e. if $\mathsf{H}\,\mathcal{PT} = \mathcal{PT}\mathsf{H}$. In the present case H is $\mathcal{PT}$-symmetric if $\mathsf{W}(x) = \mathsf{W}^*(-x)$ [63].

7 Massless fermions and shift of potentials

The time component of the vector potential, $\mathsf{V}(x)$, according to the minimal coupling principle, enters the Dirac equation in the same way as energy. Therefore, a shift by a constant, $\mathsf{V} \to \mathsf{V} + e$, is equivalent to shifting the energy scale by e. Consequently, under this transformation the energy spectrum is shifted by e and the form of the eigenfunctions does not change.

The scalar potential S, under the Lorentz transformation, behaves like a scalar and is included to the Dirac equation in the same way as the particle mass. In the non-relativistic limit S and V are non-distinguishable (the time component of the vector potential, under the Galilei transformation also behaves like a scalar) and combine to form a single potential. But, in the Dirac equation the behavior of S and V is different. In particular, neither energies nor eigenfunctions are invariant with respect to a shift of S. In general, S is like a position-dependent mass. By setting $\mathsf{S} = mc^2$ we get Dirac equation describing massless fermions, quasi-particles playing an important role in the modeling of graphene [14].

Finally, the pseudo-scalar potential P adds to the momentum operator. Its influence on the spectrum of the Dirac operator is complicated. In particular, by adding a constant to a pseudo-scalar potential one may change the spectrum of an operator which does not support bound states to another one, with infinite number of bound states, as it was demonstrated for an inversely linear [64] and for a screened Coulomb [65] potentials.

The 1D Dirac equation for massless fermions has a simple, symmetric, form

$$\begin{pmatrix} \mathsf{V} - E, & c\,\pi^\dagger \\ c\,\pi, & \mathsf{V} - E \end{pmatrix} \begin{pmatrix} \psi_\mathrm{a} \\ \psi_\mathrm{b} \end{pmatrix} = 0. \tag{34}$$

This means that particle and antiparticle spectra are the same. If $\mathsf{P} = 0$, i.e. we have pure vector potential, the equation can be decoupled by transforming to the Weyl representation. By introducing $\phi_{1,2} = \left(\psi_\mathrm{a} \pm \psi_\mathrm{b}\right)/\sqrt{2}$ we get

$$\frac{i}{\phi_1}\frac{d\phi_1}{dx} = \frac{\mathsf{V} - E}{\hbar\, c}$$
$$-\frac{i}{\phi_2}\frac{d\phi_2}{dx} = \frac{\mathsf{V} - E}{\hbar\, c}.$$

After the integration

$$\phi_{1,2} \sim \exp\left[\pm i \int \frac{(\mathsf{V} - E)}{\hbar\, c} dx\right],$$

where E is arbitrary. Then, the wave functions are not square integrable and the spectrum is continuous.

The introduction of a constant pseudo-scalar contribution, P_0, changes this picture. The 1D Dirac equation expressed in the Weyl representation

$$\begin{pmatrix} c\,\mathsf{p} + \mathsf{V} - E, & i c\,\mathsf{P}_0 \\ -i c\,\mathsf{P}_0, & -c\,\mathsf{p} + \mathsf{V} - E \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0, \tag{35}$$

may be written in the second-order form

$$\left[c^2\,\mathsf{p}^2 - (\mathsf{V} - E)^2 \pm (\mathsf{pV}) + c^2\,\mathsf{P}_0^2\right]\phi_{1,2} = 0, \tag{36}$$

with

$$\phi_{2,1} = \frac{\pm c\,\mathsf{p} + \mathsf{V} - E}{\mp i c\,\mathsf{P}_0}\,\phi_{1,2}.$$

Eq. (36) may easily be transformed to an eigenvalue problem with a complex and energy-dependent effective potential.

In the case of a pure pseudo-scalar potential, i.e. $\mathsf{V} = 0$, Eq. (34) yields

$$\left[c^2(\mathsf{p} \mp i\mathsf{P})(\mathsf{p} \pm i\mathsf{P}) - E^2\right]\phi_{\mathrm{a,b}} = 0. \tag{37}$$

The Hamiltonian in the corresponding eigenvalue problem is Hermitian. The symmetry of particle–antiparticle spectrum is reflected in a symmetry of the wave functions: the replacement $(+E \leftrightarrow -E)$ corresponds to the replacement of the components $(\phi_1 \leftrightarrow \phi_2)$ in the two-component spinors.

8 Final remarks

In all cases considered in Sects. 6 and 7 the Schrödinger forms of the Dirac equation, i.e. Eqs. (29) and (30), after the substitution $x/\hbar \to x$, may be written as

$$-\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} + \mathsf{V}_\mathrm{eff}(x)\,\chi(x) = \mathfrak{E}\,\chi(x), \tag{38}$$

where

$$\mathsf{V}_\mathrm{eff}(x) = \mathsf{Y}^2(x) + 2b\,\mathsf{Y}(x) \mp \frac{\mathrm{d}\mathsf{Y}}{\mathrm{d}x},$$

and, depending on the case, $\mathsf{Y}(x) = \sqrt{qr}\,\mathsf{W}/c$ with $b = \mathfrak{p}_\mathsf{V}$ or $\mathsf{Y}(x) = \mathsf{P}(x)$ with $b = \mathsf{P}_0$ and $\mathfrak{E}$ stands for $\mathfrak{E}_\mathsf{V}$ or $\mathfrak{E}_\mathsf{P}$. The resulting Hamiltonian is Hermitian, unless $qr < 0$—then, for $\mathsf{W}(x) = \mathsf{W}^*(-x)$, it is $\mathcal{PT}$-symmetric. Let us note that if we allow for imaginary potentials, then Eq. (38) corresponds to the one-dimensional Dirac equation with scalar potential

$$\mathsf{S}(x) = c\,\mathsf{Y}(x) + b.$$

The effective potential may be factorized and represented in a SUSY form. Many solvable representations of this potential have already been discussed in detail by authors mentioned in the Introduction. For a large class of potentials the

ground-state energies and wave functions can be obtained by a direct integration of the equation.

Many new potentials, particularly the ones linked with different members of the Heun family of equations remain to be explored. Also the problem of metastable states has scarcely been discussed [66] and very little was done on Hamiltonians which are neither Hermitian nor $\mathcal{PT}$-symmetric. We conclude that the subject "One-dimensional Dirac equation" is far from being exhausted.

Acknowledgements AMI thanks to the Armenian State Committee of Science, Armenian National Science and Education Fund, and the Russian-Armenian (Slavonic) University at the expense of the Ministry of Education and Science of the Russian Federation for support

Funding

Compliance with ethical standards

Conflict of interest We (Jacek Karwowski, Artur Ishkhanyan and Andrzej Poszwa) herewith declare that we have no conflict of interest

References

1. Karwowski J (2017) Dirac operator and its properties. In: Liu W (ed) Handbook of relativistic quantum chemistry. Springer, Berlin, pp 3–49
2. Bonneau G, Faraut J, Valent G (2001) Self-adjoint extensions of operators and the teaching of quantum mechanics. Am J Phys 69:322–331
3. Gitman DM, Tyutin IV, Voronov BL (2012) Self-adjoint extensions in quantum mechanics general theory and applications to Schrdinger and Dirac equations with singular potentials. Springer, New York, pp 207–212
4. Greiner W (1981) Theoretische Physik, Vol. 6: Relativistische Quantenmechanik, Chapter 9. Harri Deutsch, Frankfurt am Main
5. Sukumar CV (1985) Supersymmetric quantum mechanics in one-dimensional systems. J Phys A Math Gen 18:2917–2936
6. Cooper F, Khare A, Musto R, Wipf A (1988) Supersymmetry and the Dirac equation. Ann. Phys. (N. Y.) 187:1–28
7. Coutinho FAB, Y, Nogami and F. M. Toyama. (1988) General aspects of the bound-state solutions of the one-dimensional Dirac equation. Am J Phys 56:904–907
8. Nogami Y, Toyama FM (1993) Supersymmetry aspects of the Dirac equation in one dimension with a Lorentz scalar potential. Phys Rev A 47:1708–1714
9. Nogami Y, Toyama FM, van Dijk W (2003) The Dirac equation with a confining potential. Am J Phys 71:950–951
10. Coutinho FAB, Nogami Y (1987) Conditions for the existence of bound states of a Dirac particle in one dimension. Phys Lett A 124:211–214
11. Wolfram S (2020) *Stephen Wolfram writings, April 14*https://www.wolframphysics.org/
12. Castro Neto AH, Guinea F, Peres F, Novoselov KS, Geim AK (2009) The electronic properties of graphene. Rev Mod Phys 81:109
13. Geim A (2009) Graphene: status and prospects. Science 324:1530
14. Katsnelson MI (2012) Graphene: carbon in two dimensions. Cambridge University Press, Cambridge
15. Wehling TO, Black-Schaffer AM, Balatsky AV (2014) Dirac materials. Adv Phys 63:1–76
16. Jakubsky V, Plyushchay MS (2012) Supersymmetric twisting of carbon nanotubes. Phys Rev D 85:045035
17. Correa F, Jakubsky V (2013) Finite-gap twists of carbon nanotubes and an emergent hidden supersymmetry. Phys Rev D 87:085019
18. Jakubsky V (2015) Spectrally isomorphic Dirac systems: graphene in electromagnetic field. Phys Rev D 91:045039
19. Alhaidari AD, El Aaoud E (2011) Solution of the Dirac equation in a one-dimensional box. AIP Conf Proc 1370:21–25
20. Assi IA, Bahlouli H (2017) Analytical solutions of the 1D Dirac equation using the tridiagonal representation approach. J Appl Math Phys 5:2072–2092
21. Ernzerhof M, Goyer F (2010) Conjugated molecules described by a one-dimensional Dirac equation. J Chem Theory Comput 6:1818–1824
22. Bocquet M (1999) Some spectral properties of the one-dimensional disordered Dirac equation. Nucl Phys B 546:621–646
23. Texier CB, Hagendorf C (2010) The effect of boundaries on the spectrum of a one-dimensional random mass Dirac Hamiltonian. J Phys A: Math Theor 43:025002
24. Machihara S (2005) One dimensional Dirac equation with quadratic nonlinearities. Discret Cont Dyn Syst 13:277–290
25. de Castro AS (2002) Comment on "Fun and frustration with quarkonium in 1 + 1 dimension" by R. S. Bhalerao and B. Ram. Am J Phys 70:450–451
26. Castro LB, de Castro AS (2014) Unsuitable use of spin and pseudospin symmetries with a pseudoscalar Cornell potential. Chin Phys B 23:090301
27. Castro LB, De Castro AS (2013) Missing solution in a Cornell potential. Ann Phys 338:278–282
28. Capri AZ, Ferrari R (1985) Hydrogenic atoms in one plus one dimensions. Can J Phys 63:1029–1031
29. Galić H (1988) Fun and frustration with hydrogen in 1 + 1 dimension. Am J Phys 56:312–317
30. Hiller JR (2002) Solution of the one-dimensional Dirac equation with a linear scalar potential. Am J Phys 70:522–524
31. Domínguez-Adame F, Rodríguez A (1995) A one-dimensional relativistic screened Coulomb potential. Phys Lett A 198:275–278
32. Sveshnikov KA, Khomovskii DI (2012) The Dirac particle in a one-dimensional hydrogen atom. Moscow Univ Phys Bull 67:358–363
33. Downing CA, Portnoi ME (2014) One-dimensional Coulomb problem in Dirac materials. Phys Rev A 90:052116
34. McKeon DGC, van Leeuwen G (2002) The Dirac equation in a pseudoscalar Coulomb potential. Mod Phy Lett A 17:1961–1963
35. Villalba VM (1997) Exact solution of the Dirac equation in the presence of pseudoscalar potentials. Nuevo Cim 112B:109–116
36. de Castro AS (2003) Bound states by a pseudoscalar Coulomb potential in one-plus-one dimensions. Phys Lett A 318:40–47

37. Castro LB, de Castro AS, Alberto P (2015) Pseudospin and spin symmetries in 1 + 1 dimensions: the case of the Coulomb potential. Ann Phys 356:83–94
38. Lévai G, Kónya B, Papp Z (1998) Unified treatment of the Coulomb and harmonic oscillator potentials in *D* dimensions. J Math Phys 39:5811–5823
39. Hasegawa H (2014) Bound states of the one-dimensional Dirac equation for scalar and vector double-well potentials. Physica E 59:192–201
40. Haouat S, Benzekka M (2013) On the quantum behavior of a neutral fermion in a pseudoscalar potential barrier. Phys Lett A 377:2298–2306
41. Nogami Y, Toyama FM (1992) Transparent potential for the one-dimensional Dirac equation. Phys Rev A 45:5258–5261
42. Correa F, Jakubsky V (2014) Twisted kinks, Dirac transparent systems and Darboux transformations. Phys Rev A 90:125003
43. de Castro AS (2004) Bound states of the Dirac equation for a class of effective quadratic plus inversely quadratic potentials. Ann Phys (N.Y.) 311:170–181
44. Arda A, Sever R (2019) Bound-state solutions of Dirac equation for Kratzer potential with pseudoscalar-Coulomb term. Eur Phys J Plus 134:29–37
45. Chargui Y (2016) Effective mass and pseudoscalar interaction in the Dirac equation with Woods–Saxon potential. Few-Body Syst 57:289–306
46. Eshghi M, Mehraban H, Ikhdair SM (2015) Bound states of (1 + 1)-dimensional Dirac equation with kink-like vector potential and delta interaction. Acta Math Appl Sin 31:1131–1140
47. Contreras-Astorga A (2015) One dimensional Dirac-Moshinsky oscillator-like system and isospectral partners. J Phys C 624:012013
48. Correa F, Jakubsky V (2017) Confluent Crum-Darboux transformations in Dirac Hamiltonians with PT−symmetric Bragg gratings. Phys Rev A 95:033807
49. Hartmann RR, Portnoi ME (2014) Quasi-exact solutions to the Dirac equation for the hyperbolic secant potential. Phys Rev A 89:012101
50. Ishkhanyan AM (2016) Schrödinger potentials solvable in terms of the confluent Heun functions. Theor Math Phys 188:980–993
51. Ishkhanyan TA, Ishkhanyan AM (2017) Solutions of the biconfluent Heun equation in terms of the Hermite functions. Ann Phys 383:79–91
52. Ishkhanyan AM (2018) Schrödinger potentials solvable in terms of the general Heun functions. Ann Phys 388:456–471
53. Ishkhanyan AM (2020) Exact solution of the 1D Dirac equation for the inverse-square-root potential $1/\sqrt{x}$. Z. Naturforsch A 75: 771–779
54. de Castro AS (2002) Confinement of fermions by mixed vector-scalar linear potentials in two-dimensional space-time. Phys. Lett. A 305:100–104
55. Castro LB, de Castro AS (2007) Unified treatment of mixed vector-scalar screened Coulomb potentials for fermions. Int J Mod Phys E 16:2998–3001
56. Castro LB, de Castro AS (2008) Bound states for bosons and fermions in a mixed vector-scalar coupling with unequal shapes for the potentials. Phys Scr 77:045007
57. Castro LB, A. S, De Castro and M. Hott. (2007) Relativistic effects of mixed vector-scalar-pseudoscalar potentials for fermions in 1 + 1 dimensions. Int J Mod Phys E 16:3002–3005
58. Thaller B (1992) The Dirac equation. Springer, Berlin
59. Moiseyew N (2011) Non-Hermitian quantum mechanics. Cambridge University Press, New York
60. Moore RA (1975) An alternative method of obtaining approximate solutions to the Dirac equation. Can J Phys 53:1240–1246
61. Lévy-Leblond J-M (1995) Position-dependent effective mass and Galilean invariance. Phys Rev A 52:1845–1849
62. Dutt R, Khare A, Sukhatme UP (1988) Supersymmery, shape invariance and exactly solvable potentials. Am J Phys 56:163–168
63. Bender CM, Boettcher S (1998) Real spectra in non-Hermitian Hamiltonians having PT symmetry. Phys Rev Lett 80:5243–5246
64. de Castro AS (2006) The peremptory influence of a uniform background for trapping neutral fermions with an inversely linear potential. Int J Mod Phys A 21:2321–2329
65. de Castro AS (2005) Bounded solutions of neutral fermions with screened Coulomb potential. Ann Phys 320:56–70
66. Giachetti R, Grecchi V (2011) PT−symmetric operators and metastable states of the 1D relativistic oscillators. J Phys A: Math Theor 44:095308