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Adaptive guaranteed lower eigenvalue bounds with optimal convergence rates

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Abstract

Guaranteed lower Dirichlet eigenvalue bounds (GLB) can be computed for the m-th Laplace operator with a recently introduced extra-stabilized nonconforming Crouzeix–Raviart ($m = 1$) or Morley ($m = 2$) finite element eigensolver. Striking numerical evidence for the superiority of a new adaptive eigensolver motivates the convergence analysis in this paper with a proof of optimal convergence rates of the GLB towards a simple eigenvalue. The proof is based on (a generalization of) known abstract arguments entitled as the axioms of adaptivity. Beyond the known a priori convergence rates, a medius analysis is enfolded in this paper for the proof of best-approximation results. This and subordinated L^2 error estimates for locally refined triangulations appear of independent interest. The analysis of optimal convergence rates of an adaptive mesh-refining algorithm is performed in 3D and highlights a new version of discrete reliability.

Mathematics Subject Classification 65N12 · 65N15 · 65N25 · 65N30 · 65N50 · 65Y20

1 Introduction

Motivation. Guaranteed lower Dirichlet eigenvalue bounds (GLB) can be computed for the m-th Laplace operator from a global postprocessing of respective nonconforming finite element eigensolvers like the Crouzeix–Raviart resp. Morley finite element method (FEM) for $m = 1$ resp. $m = 2$ [15, 16]. The maximal mesh-size $h_{\max}$ enters

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as an explicit parameter and this can be non-effective for an imperative adaptive mesh-refinement. This has recently motivated the design of extra-stabilized nonconforming finite element eigensolvers for $m = 1, 2$ that directly compute GLB under moderate mesh-size restrictions and allow an efficacious adaptive mesh-refinement [11, 24, 27]. The striking superiority of those adaptive schemes has been displayed in numerical experiments in [11, 24] and motivates the mathematical analysis of optimal convergence rates in this paper. This appears to be the first method that combines the localization of eigenvalues as GLB with their efficient approximation.

Model problem. The continuous eigenvalue problem (EVP) seeks eigenpairs $(\lambda, u) \in \mathbb{R}^+ \times (V\backslash\{0\})$ with

$$a(u, v) = \lambda\, b(u, v) \quad \text{for all } v \in V \tag{1.1}$$

in the Hilbert space $V := H_0^m(\Omega)$ with its energy scalar product $a(\bullet, \bullet) := (D^m \bullet, D^m \bullet)_{L^2(\Omega)}$ with the gradient $D^1 := \nabla$ or the Hessian D^2 and the L^2 scalar product $b(\bullet, \bullet) := (\bullet, \bullet)_{L^2(\Omega)}$ on a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^3$. The infinite but countably many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lim_{j\to\infty} \lambda_j = \infty$ in (1.1) are enumerated in ascending order counting multiplicities [6, 7].

Discretization. The discrete space $\boldsymbol{V_h} = P_m(\mathcal{T}) \times V(\mathcal{T}) \subset P_m(\mathcal{T}) \times P_m(\mathcal{T})$ consists of piecewise polynomials of degree at most m on the shape-regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^3$ into closed tetrahedra. Throughout this paper, $V(\mathcal{T})$ abbreviates the Crouzeix–Raviart finite element space $CR_0^1(\mathcal{T})$ [25] for $m = 1$ and the Morley finite element space $M(\mathcal{T})$ [40, 41] for $m = 2$. The algebraic eigenvalue problem seeks eigenpairs $(\lambda_h, \boldsymbol{u_h}) \in \mathbb{R}^+ \times (\boldsymbol{V_h}\backslash\{0\})$ with

$$\boldsymbol{a_h}(\boldsymbol{u_h}, \boldsymbol{v_h}) = \lambda_h \boldsymbol{b_h}(\boldsymbol{u_h}, \boldsymbol{v_h}) \quad \text{for all } \boldsymbol{v_h} \in \boldsymbol{V_h}. \tag{1.2}$$

The discrete scalar product $\boldsymbol{a_h}$ contains the scalar product $a_{\mathrm{pw}}(\bullet, \bullet) := (D^m_{\mathrm{pw}} \bullet, D^m_{\mathrm{pw}} \bullet)_{L^2(\Omega)}$ of the piecewise derivatives of order m and some stabilization with explicit (known) constant $\kappa_m > 0$ from [24], while the bilinear form $\boldsymbol{b_h}$ is the L^2 scalar product $b(\bullet, \bullet)$ of the piecewise polynomial components,

$$\begin{aligned}
\boldsymbol{a_h}(\boldsymbol{v_h}, \boldsymbol{w_h}) &= a_{\mathrm{pw}}(v_{\mathrm{nc}}, w_{\mathrm{nc}}) + \kappa_m^{-2} (h_{\mathcal{T}}^{-2m} (v_{\mathrm{pw}} - v_{\mathrm{nc}}), w_{\mathrm{pw}} - w_{\mathrm{nc}})_{L^2(\Omega)},\\
\boldsymbol{b_h}(\boldsymbol{v_h}, \boldsymbol{w_h}) &= b(v_{\mathrm{pw}}, w_{\mathrm{pw}}) \qquad \text{for all } \boldsymbol{v_h} = (v_{\mathrm{pw}}, v_{\mathrm{nc}}),\ \boldsymbol{w_h} = (w_{\mathrm{pw}}, w_{\mathrm{nc}}) \in \boldsymbol{V_h}.
\end{aligned}$$

The piecewise constant mesh-size function $h_{\mathcal{T}} \in P_0(\mathcal{T})$ has the value $h_{\mathcal{T}}|_T = h_T := \mathrm{diam}(T)$ in each tetrahedron $T \in \mathcal{T}$ and $h_{\max} := \max_{T\in\mathcal{T}} h_T$ denotes the maximal mesh-size. The $M := \dim(P_m(\mathcal{T}))$ finite discrete eigenvalues of (1.2) are enumerated in ascending order $0 < \lambda_h(1) \leq \lambda_h(2) \leq \cdots \leq \lambda_h(M) < \infty$ counting multiplicity.

GLB. For the biharmonic operator ($m = 2$) the discrete eigenvalue problem (1.2) is analysed in [24]. For the Laplace operator ($m = 1$) in 2D, (1.2) describes the lowest-order skeleton method in [27]; for 3D it is different and suggested in [24]. The discrete eigenvalue problem (1.2) directly computes guaranteed lower bounds [24, Thm. 1.1] in that

$$\min\{\lambda_h(k), \lambda_k\}\kappa_m^2 h_{\max}^{2m} \le 1 \quad \text{implies} \quad \lambda_h(k) \le \lambda_k \quad \text{for all } k = 1, \dots, M. \tag{1.3}$$

AFEM. The adaptive algorithm [12, 26, 30, 39] is based on the refinement indicator $\eta(T)$ defined in (1.4) below for any triangulation $\mathcal{T}$ and any tetrahedron $T \in \mathcal{T}$. Let $(\lambda_h, \boldsymbol{u_h}) \in \mathbb{R}^+ \times \boldsymbol{V_h}$ denote the k-th eigenpair of (1.2) with $\lambda_h := \lambda_h(k)$ and $\boldsymbol{u_h} = (u_{\text{pw}}, u_{\text{nc}}) \in \boldsymbol{V_h}$. For any tetrahedron $T \in \mathcal{T}$ with volume $|T|$ and set of faces $\mathcal{F}(T)$, the local estimator contribution $\eta^2(T) = (\eta(T))^2$ reads

$$\eta^2(T) = |T|^{2m/3}\|\lambda_h u_{\text{nc}}\|_{L^2(T)}^2 + |T|^{1/3} \sum_{F\in\mathcal{F}(T)} \|[D_{\text{pw}}^m u_{\text{nc}}]_F \times \nu_F\|_{L^2(F)}^2 \tag{1.4}$$

with the tangential components $[D_{\text{pw}}^m u_{\text{nc}}]_F \times \nu_F$ of the jump $[D_{\text{pw}}^m u_{\text{nc}}]_F$ along any face $F \in \mathcal{F}(T)$ and the (piecewise) gradient $D_{\text{pw}}^1 = \nabla_{\text{pw}}$ ($m = 1$) or Hessian D_{pw}^2 ($m = 2$). Let $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ denote the set of all admissible regular triangulations computed by successive newest-vertex bisection (NVB) [35, 48] of a regular initial triangulation $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^3$. The AFEM algorithm with Dörfler marking and newest-vertex bisection abbreviates $\eta_\ell(T)$ for any $T \in \mathcal{T} := \mathcal{T}_\ell \in \mathbb{T}$ and $\eta_\ell^2 := \eta^2(\mathcal{T}_\ell) := \sum_{T\in\mathcal{T}_\ell} \eta_\ell^2(T)$. The selection of the set $\mathcal{M}_\ell$ in the step Mark of AFEM4EVP with *minimal cardinality* is possible at linear cost [44].

AFEM 4EVP

Input: regular triangulation $\mathcal{T}_0$ and parameters $0 < \theta \le 1$ and $k \in \mathbb{N}$
for $\ell = 0, 1, 2, \dots$ **do**
Solve the discrete problem (1.2) exactly and compute the k-th algebraic eigenpair $(\lambda_\ell(k), \boldsymbol{u_\ell}(k))$ with $\boldsymbol{u_\ell}(k) = (u_{\text{pw}}, u_{\text{nc}}) \in P_m(\mathcal{T}_\ell) \times V(\mathcal{T}_\ell)$ and $\mathcal{T}$ replaced by $\mathcal{T}_\ell$
Compute $\eta_\ell(T)$ for any $T \in \mathcal{T}_\ell$ from (1.4) with $(\lambda_h, u_{\text{nc}}, \mathcal{T})$ replaced by $(\lambda_\ell(k), u_{\text{nc}}, \mathcal{T}_\ell)$
Mark minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with $\theta\eta_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_\ell^2(T)$
Refine $\mathcal{T}_\ell$ with newest-vertex bisection to compute $\mathcal{T}_{\ell+1}$ with $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ **od**
Output: sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ with $(\lambda_\ell(k), \boldsymbol{u_\ell}(k))_{\ell\in\mathbb{N}_0}$ and $(\eta_\ell)_{\ell\in\mathbb{N}_0}$

Optimal convergence rates. The optimal convergence rates of AFEM4EVP in the error estimator means that the outputs $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ and $(\eta_\ell)_{\ell\in\mathbb{N}_0}$ of AFEM4EVP satisfy

$$\sup_{\ell\in\mathbb{N}_0} (1 + |\mathcal{T}_\ell| - |\mathcal{T}_0|)^s \eta_\ell \approx \sup_{N\in\mathbb{N}_0} (1 + N)^s \min\{\eta(\mathcal{T}) : \mathcal{T} \in \mathbb{T} \text{ with } |\mathcal{T}| \le |\mathcal{T}_0| + N\} \tag{1.5}$$

for any $s > 0$ and the counting measure $|\bullet| = \text{card}(\bullet)$. In other words, if the estimator $\eta(\mathcal{T})$ converges with rate $s > 0$ for some optimal selection of triangulations $\mathcal{T} \in \mathbb{T}$, then the output η_ℓ of AFEM4EVP converges with the same rate.

Theorem 1.1 (rate optimality of AFEM4EVP) *Suppose that $\lambda_k = \lambda$ is a simple eigenvalue of* (1.1)*, then there exist $\varepsilon > 0$ and $0 < \theta_0 < 1$ such that $\mathcal{T}_0 \in \mathbb{T}(\varepsilon) := \{\mathcal{T} \in \mathbb{T} : h_{\max} := \max_{T\in\mathcal{T}} h_T \le \varepsilon\}$ and θ with $0 < \theta \le \theta_0$ imply* (1.5) *for any $s > 0$.*

At first glance the discrete problem (1.2) involves a stabilization that is expected to generate the additional term $\kappa_m^{-2}|T|^{-2m/3}\|u_{\mathrm{pw}}-u_{\mathrm{nc}}\|_{L^2(T)}^2$ in the error estimator (1.4). The negative power of the mesh-size in the latter term prevents a reduction property [12, 26, 39] and has to be circumvented. The only other known affirmative result for optimal convergence rates of an adaptive algorithm with stabilization (and negative powers of the mesh-size in the discrete problem) is [5] on discontinuous Galerkin (dG) schemes. An over-penalization therein diminishes the influence of the stabilization and eventually shows the dominance of the remaining a posteriori error terms. In the present case, the stabilization parameter κ_m is fixed to maintain the GLB property and this requires a different argument: Since (1.2) is equivalent to a rational eigenvalue problem for a nonconforming scheme, a careful perturbation analysis eventually shows efficiency and reliability of the nonconforming error estimator (1.4) for sufficiently small mesh-sizes. The verification requires a medius analysis [37], which applies arguments from a posteriori error analysis (e.g., efficiency in (3.10) below) in an a priori error analysis.

Outline. The remaining parts of this paper are devoted to the proof of Theorem 1.1 and are organized as follows. A general interpolation operator I and a right-inverse J in Sect. 2 allow for a simultaneous analysis for $m = 1$ and $m = 2$ in the Crouzeix–Raviart and Morley FEM. The medius analysis in Sect. 3 provides new best-approximation results and thereby prepares the proof of Theorem 1.1 in Sect. 4–5. The proof of the optimal convergence rates requires a framework extended from [12, 26] in Appendix A.

While more general boundary conditions appear feasible as in [15, 31], non-constant coefficients in a general elliptic differential operator of order $2m$ appear a less straightforward extension from the m-harmonic operator $(-1)^m\Delta^m$. An expected extension revisits [24] for the question of lower eigenvalue bounds, while the convergence analysis of an adaptive algorithm expects extra terms for the perturbations of the piecewise polynomial approximation of inhomogeneous coefficients as in [22]; this is therefore left for future research. This first paper on optimal convergence rates of an adaptive algorithm for the direct guaranteed lower eigenvalue bounds focuses on a model problem. The results hold in 2D and 3D and are presented in 3D for brevity.

2 Preliminaries

This section summarizes abstract conditions (I1)–(I4) on an interpolation operator $I : V \to V(\mathcal{T})$ and (J1)–(J4) on a right inverse $J : V(\mathcal{T}) \to V$. The conditions hold for the Crouzeix–Raviart and the Morley finite element space in the two model examples for the Laplacian $m = 1$ and the bi-Laplacian $m = 2$.

2.1 Notation

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper; $(\bullet,\bullet)_{L^2(\Omega)}$ abbreviates the L^2 scalar product and $H^m(T)$ abbreviates $H^m(\mathrm{int}(T))$ for a tetrahedron $T \in \mathcal{T}$. The vector space $H^m(\mathcal{T}) := \{v \in L^2(\Omega) : v|_T \in$

$H^m(T)\}$ consists of piecewise H^m functions and is equipped with the semi-norm $|||\bullet|||_{\mathrm{pw}}^2 := (D_{\mathrm{pw}}^m\bullet, D_{\mathrm{pw}}^m\bullet)_{L^2(\Omega)}$. The piecewise gradient D_{pw}^1 or piecewise Hessian D_{pw}^2 is understood with respect to the (non-displayed) regular triangulation $\mathcal{T} \in \mathbb{T}$ of the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^3$ into tetrahedra. The triangulation $\mathcal{T}$ is computed by successive newest-vertex bisection (NVB) [35, 48] of a regular initial triangulation $\mathcal{T}_0$ (plus some initialization of tagged tetrahedra) of $\Omega \subset \mathbb{R}^3$. The set $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ of all admissible triangulations is (uniformly) shape-regular. For any $\mathcal{T} \in \mathbb{T}$, let $\mathbb{T}(\mathcal{T})$ abbreviate the set of all admissible refinements of $\mathcal{T}$. For any $0 < \varepsilon < 1$ let $\mathbb{T}(\varepsilon) := \{\mathcal{T} \in \mathbb{T} : h_{\max} := \max_{T\in\mathcal{T}} h_T \le \varepsilon\}$ denote the set of all admissible triangulations with maximal mesh-size $h_{\max} \le \varepsilon$. The context-depending notation $|\bullet|$ denotes the Euclidean length of a vector, the cardinality of a finite set, as well as the non-trivial three-, two-, or one-dimensional Lebesgue measure of a subset of $\mathbb{R}^3$. For any positive, piecewise polynomial $\varrho \in P_k(\mathcal{T})$ with $\varrho \ge 0$, $k \in \mathbb{N}_0$, $(\bullet,\bullet)_\varrho := (\varrho\bullet,\bullet)_{L^2(\Omega)}$ abbreviates the weighted L^2 scalar product with induced ϱ-weighted L^2 norm $\|\bullet\|_\varrho := \|\varrho^{1/2}\bullet\|_{L^2(\Omega)}$. The discrete space $P_m(\mathcal{T}) := \{p_m \in L^2(\Omega) : p_m|_T \in P_m(T)$ is a polynomial of degree at most m for any $T \in \mathcal{T}\}$ consists of piecewise polynomials, the spaces $CR_0^1(\mathcal{T})$ resp. $M(\mathcal{T})$ will be defined in Sect. 2.4.1 resp. 2.4.2 below. Given a function $v \in L^2(\omega)$, define the integral mean $⨍_\omega v\,\mathrm{d}x := 1/|\omega| \int_\omega v\,\mathrm{d}x$. The L^2 projection Π_0 onto the piecewise constant functions $P_0(\mathcal{T})$ reads $(\Pi_0 f)|_T := ⨍_T f\,\mathrm{d}x$ for all $f \in L^2(\Omega)$ and $T \in \mathcal{T}$. Let $\sigma := \min\{1, \sigma_{\mathrm{reg}}\}$ denote the minimum of one and the index of elliptic regularity $\sigma_{\mathrm{reg}} > 0$ for the source problem of the m-Laplacian $(-1)^m\Delta^m$ in $H_0^m(\Omega)$: Given any right-hand side $f \in L^2(\Omega)$, the weak solution $u \in V$ to $(-1)^m\Delta^m u = f$ satisfies

$$u \in H^{m+\sigma}(\Omega) \text{ and } \|u\|_{H^{m+\sigma}(\Omega)} \le C(\sigma)\|f\|_{L^2(\Omega)}. \tag{2.1}$$

(This is well-established for $m = 1$ [1, 28, 34, 36, 42] and $m = 2$ in 2D [8] with $\sigma_{\mathrm{reg}} > 1/2$ and otherwise a hypothesis throughout this paper.) The Sobolev space $H^{m+s}(\Omega)$ is defined for $0 < s < 1$ by complex interpolation of $H^m(\Omega)$ and $H^{m+1}(\Omega)$, $m \in \mathbb{N}_0$. Throughout this paper, $a \lesssim b$ abbreviates $a \le Cb$ with a generic constant C depending on σ in (2.1) and the shape-regularity of $\mathcal{T} \in \mathbb{T}$ only; $a \approx b$ stands for $a \lesssim b \lesssim a$.

2.2 Interpolation

The operators I and J concern the (nonconforming) discrete space $V(\mathcal{T}) \subset P_m(\mathcal{T})$ and $V := H_0^m(\Omega)$ for an admissible triangulation $\mathcal{T} \in \mathbb{T}$. An advantage of separate interest is that the analysis with I and J is performed simultaneously for $m \ge 1$, while the examples in Sect. 2.4 below concern $m = 1, 2$.

Suppose that, for each admissible triangulation $\mathcal{T} \in \mathbb{T}$, there exists a linear interpolation operator I onto $V(\mathcal{T})$ that is defined on $V + V(\widehat{\mathcal{T}})$ for any refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ and that satisfies the following properties with universal positive constants κ_m and κ_d; in all examples below κ_m is known and the existence of κ_d is clarified.

(I1) Any $T \in \mathcal{T}$ and $v \in H^m(T)$ satisfy $\|v - Iv\|_{L^2(T)} \le \kappa_m h_T^m |v - Iv|_{H^m(T)}$.

(I2) The piecewise derivative D^m_{pw} of any $v \in V + V(\widehat{\mathcal{T}})$ satisfies $D^m_{\mathrm{pw}} I v = \Pi_0 D^m_{\mathrm{pw}} v$.
(I3) The operator I acts as identity in non-refined tetrahedra in that $(1 - I)\widehat{v}_{\mathrm{nc}}|_T = 0$ in $T \in \mathcal{T} \cap \widehat{\mathcal{T}}$ for all $\widehat{v}_{\mathrm{nc}} \in V(\widehat{\mathcal{T}})$. The interpolation operator $\widehat{I}$ associated with $V(\widehat{\mathcal{T}})$ satisfies $I \circ \widehat{I} = I$ in $V + V(\widehat{\mathcal{T}})$.
(I4) Any $T \in \mathcal{T}$ and $\widehat{v}_{\mathrm{nc}} \in V(\widehat{\mathcal{T}})$ satisfy $\|\widehat{v}_{\mathrm{nc}} - I\widehat{v}_{\mathrm{nc}}\|_{L^2(T)} \le \kappa_d h_T^m |\widehat{v}_{\mathrm{nc}} - I\widehat{v}_{\mathrm{nc}}|_{H^m(T)}$.

Corollary 2.1 (properties of I)

(a) Given $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, *any* $v \in V + V(\widehat{\mathcal{T}})$ *and* $w_{\mathrm{nc}} \in V(\mathcal{T})$ *satisfy* $a_{\mathrm{pw}}(v - Iv, w_{\mathrm{nc}}) = 0$ *and* $|||v - Iv|||_{\mathrm{pw}} = \min_{v_{\mathrm{nc}} \in V(\mathcal{T})} |||v - v_{\mathrm{nc}}|||_{\mathrm{pw}}$.
(b) Any $v \in H^{m+s}(\Omega)$ *with* $1/2 < s \le 1$ *satisfies* $|||(1 - I)v|||_{\mathrm{pw}} \le (h_{\max}/\pi)^s \|v\|_{H^{m+s}(\Omega)}$.
(c) Any $v, w \in V$ *and* $v_{\mathrm{nc}} \in V(\mathcal{T})$ *satisfy* $a_{\mathrm{pw}}(v, v_{\mathrm{nc}}) = a_{\mathrm{pw}}(Iv, v_{\mathrm{nc}})$ *and* $a_{\mathrm{pw}}(v, (1 - I)w) = a_{\mathrm{pw}}((1 - I)v, (1 - I)w) \le \min_{v_{\mathrm{nc}} \in V(\mathcal{T})} |||v - v_{\mathrm{nc}}|||_{\mathrm{pw}} \min_{w_{\mathrm{nc}} \in V(\mathcal{T})} |||w - w_{\mathrm{nc}}|||_{\mathrm{pw}}$.
(d) Any $w \in V$ *and* $v \in V + V(\mathcal{T})$ *satisfy* $b(v, (1 - I)w) \le \|h_{\mathcal{T}}^m v\|_{L^2(\Omega)} \|h_{\mathcal{T}}^{-m}(1 - I)w\|_{L^2(\Omega)} \le \kappa_m \|h_{\mathcal{T}}^m v\|_{L^2(\Omega)} \min_{w_{\mathrm{nc}} \in V(\mathcal{T})} |||w - w_{\mathrm{nc}}|||_{\mathrm{pw}}$.

Proof Since $D^m_{\mathrm{pw}} w_{\mathrm{nc}} \in P_0(\mathcal{T}; \mathbb{R}^{3^m})$, (I2) implies (a). In combination with a piecewise Poincaré inequality, (I2) implies (b) (see [24, Cor. 2.2.a] for details). The first claim in (c) follows from (a). The combination of (a) with the Cauchy–Schwarz inequality proves (c). The Cauchy–Schwarz inequality, the approximation property (I1), and (c) conclude the proof of (d). □

2.3 Conforming companion

Given any tetrahedron $T \in \mathcal{T}$ in a triangulation $\mathcal{T} \in \mathbb{T}$, let $\mathcal{V}(T)$ denote the set of its vertices (0-subsimplices) and let $\mathcal{F}(T)$ denote the set of its faces (2-subsimplices). A linear operator $J : V(\mathcal{T}) \to V$ is called *conforming companion* if (J1)–(J4) hold with universal constants M_1, M_2, M_4 (that exclusively depend on $\mathbb{T}$).

(J1) J is a right inverse to the interpolation I in the sense that $I \circ J$ acts as identity in $V(\mathcal{T})$.
(J2) $\|h_{\mathcal{T}}^{-m}(1-J)v_{\mathrm{nc}}\|_{L^2(\Omega)} + |||(1-J)v_{\mathrm{nc}}|||_{\mathrm{pw}} \le \Big(M_1 \sum_{T \in \mathcal{T}} |T|^{1/3} \sum_{F \in \mathcal{F}(T)} \|[D^m_{\mathrm{pw}} v_{\mathrm{nc}}]_F \times \nu_F\|^2_{L^2(F)}\Big)^{1/2} \le M_2 \min_{v \in V} |||v_{\mathrm{nc}} - v|||_{\mathrm{pw}}$ for any $v_{\mathrm{nc}} \in V(\mathcal{T})$.
(J3) $(1 - J)(V(\mathcal{T})) \perp P_m(\mathcal{T})$ holds in $L^2(\Omega)$.
(J4) $|v_{\mathrm{nc}} - Jv_{\mathrm{nc}}|^2_{H^m(K)} \le M_4 \sum_{T \in \mathcal{T}(\Omega(K))} |T|^{1/3} \sum_{F \in \mathcal{F}(T)} \|[D^m_{\mathrm{pw}} v_{\mathrm{nc}}]_F \times \nu_F\|^2_{L^2(F)}$ holds for any $v_{\mathrm{nc}} \in V(\mathcal{T})$ and $K \in \mathcal{T}$ with the set $\mathcal{T}(\Omega(K)) := \{T \in \mathcal{T} : \mathrm{dist}(T, K) = 0\}$ of adjacent tetrahedra.

The properties (J1)–(J4) [18, 24, 32] are stated for convenient quotation throughout this paper. The localized version (J4) applies at the very end (in Theorem 4.6) and implies parts of (J2). The second inequality in (J2) is the efficiency of a posteriori error estimators.

Remark 2.2 (on (J4)) For any refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of a triangulation $\mathcal{T} \in \mathbb{T}$, let $\mathcal{R}_1 := \{K \in \mathcal{T} : \exists T \in \mathcal{T}\setminus\widehat{\mathcal{T}} \text{ with } \mathrm{dist}(K,T) = 0\} \subset \mathcal{T}$ denote the set of coarse but not fine tetrahedra plus one layer of coarse tetrahedra around. Then (J4) and a finite overlap argument imply the existence of $M_5 > 0$ such that any $v_{\mathrm{nc}} \in V(\mathcal{T})$ satisfies

$$\|D^m_{\mathrm{pw}}(v_{\mathrm{nc}} - Jv_{\mathrm{nc}})\|^2_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} \le M_5 \sum_{T\in\mathcal{R}_1} |T|^{1/3} \sum_{F\in\mathcal{F}(T)} \|[D^m_{\mathrm{pw}} v_{\mathrm{nc}}]_F \times \nu_F\|^2_{L^2(F)}.$$

The superset $\mathcal{R}_1$ of $\mathcal{T}\setminus\widehat{\mathcal{T}}$ serves as a simple example and could indeed be replaced by $\mathcal{T}\setminus\widehat{\mathcal{T}}$ provided J may depend on $\widehat{\mathcal{T}}$; cf. [23, §6] for details in the two model problems below. □

Corollary 2.3 (properties of J) *Any $w \in V$ and $v_{\mathrm{nc}} \in V(\mathcal{T})$ satisfy*

(a) $\|v_{\mathrm{nc}} - Jv_{\mathrm{nc}}\|_{L^2(\Omega)} = \|(1-I)Jv_{\mathrm{nc}}\|_{L^2(\Omega)} \le \kappa_m |||h^m_{\mathcal{T}}(v_{\mathrm{nc}} - Jv_{\mathrm{nc}})|||_{\mathrm{pw}}$
$\le h^m_{\max}\kappa_m M_2 \min_{v\in V} |||v_{\mathrm{nc}} - v|||_{\mathrm{pw}}$;

(b) $b(w, v_{\mathrm{nc}} - Jv_{\mathrm{nc}}) = b(w - Iw, v_{\mathrm{nc}} - Jv_{\mathrm{nc}}) \le \|w - Iw\|_{L^2(\Omega)}\|v_{\mathrm{nc}} - Jv_{\mathrm{nc}}\|_{L^2(\Omega)}$
$\le h^{2m}_{\max}\kappa^2_m M_2 \min_{w_{\mathrm{nc}}\in V(\mathcal{T})} |||w - w_{\mathrm{nc}}|||_{\mathrm{pw}} \min_{v\in V} |||v_{\mathrm{nc}} - v|||_{\mathrm{pw}}$;

(c) $a_{\mathrm{pw}}(w, v_{\mathrm{nc}} - Jv_{\mathrm{nc}}) = a_{\mathrm{pw}}(w - Iw, v_{\mathrm{nc}} - Jv_{\mathrm{nc}}) \le |||w - Iw|||_{\mathrm{pw}} |||v_{\mathrm{nc}} - Jv_{\mathrm{nc}}|||_{\mathrm{pw}}$
$\le M_2 \min_{w_{\mathrm{nc}}\in V(\mathcal{T})} |||w - w_{\mathrm{nc}}|||_{\mathrm{pw}} \min_{v\in V} |||v - v_{\mathrm{nc}}|||_{\mathrm{pw}}$.

Proof The combination of (J1), (I1), and (J2) proves (a). The claim (b) follows from (J3), the Cauchy–Schwarz inequality, (I1), and (a). Corollary 2.1.c and (J1)–(J2) lead to (c). □

2.4 Examples

Two examples for $V(\mathcal{T}) \subset P_m(\mathcal{T})$ are analysed simultaneously in this paper for $m = 1, 2$. It is appealing to follow our methodology for $m \ge 3$ [52] in future research.

2.4.1 Crouzeix–Raviart finite elements for the Laplacian ($m = 1$)

Given the shape-regular triangulation $\mathcal{T} \in \mathbb{T}$, let $\mathcal{F}$ (resp. $\mathcal{F}(\Omega)$ or $\mathcal{F}(\partial\Omega)$) denote the set of all (resp. interior or boundary) faces. Throughout this paper, the model problem with $m = 1$ approximates the Dirichlet eigenvectors $u \in H^1_0(\Omega)$ of the Laplacian $-\Delta u = \lambda u$ in the Crouzeix–Raviart finite element space [25]

$$V(\mathcal{T}) := CR^1_0(\mathcal{T}) := \{v \in P_1(\mathcal{T}) : v \text{ is continuous at } \mathrm{mid}(F) \text{ for all } F \in \mathcal{F}(\Omega) \text{ and } v(\mathrm{mid}(F)) = 0 \text{ for all } F \in \mathcal{F}(\partial\Omega)\}.$$

Given the face-oriented basis functions $\psi_F \in CR^1(\mathcal{T})$ with $\psi_F(\mathrm{mid}(E)) = \delta_{EF}$ for all faces $E, F \in \mathcal{F}$ (δ_{EF} is Kronecker's delta), the standard interpolation operator reads

$$I_{\mathrm{CR}}(v) := \sum_{F\in\mathcal{F}(\Omega)} \left(⨍_F v\,\mathrm{d}\sigma\right)\psi_F \quad \text{for any } v \in H^1_0(\Omega) + CR^1_0(\widehat{\mathcal{T}}).$$

The interpolation operator I_{CR} satisfies (I1)–(I4) with $\kappa_1 := \sqrt{1/\pi^2 + 1/120}$, see [23, Sec. 4.2–4.4] and the references therein. The constant κ_1 is provided in [15, 16, 27].

The design of the conforming companion $J : CR_0^1(\mathcal{T}) \to S_0^5(\mathcal{T}) := P_5(\mathcal{T}) \cap C_0(\Omega)$ with (J1)–(J4) is a straightforward generalization of [18, Prop. 2.3] to 3D. The arguments in [18, Prop. 2.3] can be localized [10, Thm. 5.1] and lead with [9, Thm. 3.2], [17, Thm. 4.9] to (J2) and (J4).

2.4.2 Morley finite elements for the bi-Laplacian ($m = 2$)

Given the shape-regular triangulation $\mathcal{T} \in \mathbb{T}$, let $\mathcal{E}$ (resp. $\mathcal{E}(\Omega)$ or $\mathcal{E}(\partial\Omega)$) denote the set of all (resp. interior or boundary) edges. Let $\mathcal{F}(E) := \{F \in \mathcal{F} : E \subset \overline{F}\}$ denote the set of all faces containing the edge $E \in \mathcal{E}$. For any face $F \in \mathcal{F}$, let ν_F denote the unit normal with fixed orientation and $[\bullet]_F$ the jump across F. The model problem with $m = 2$ approximates the Dirichlet eigenvectors $u \in H_0^2(\Omega)$ of the bi-Laplacian $\Delta^2 u = \lambda u$ in the discrete Morley finite element space [40, 41]

$$V(\mathcal{T}) := M(\mathcal{T}) := \Big\{ v \in P_2(\mathcal{T}) : \fint_E [v]_F \,\mathrm{d}s = 0 \text{ for all } E \in \mathcal{E} \text{ and } F \in \mathcal{F}(E), \\ \text{and } \fint_F [\nabla v]_F \cdot \nu_F \,\mathrm{d}\sigma = 0 \text{ for all } F \in \mathcal{F} \Big\}.$$

Given the nodal basis functions Φ_E, Φ_F for any $E \in \mathcal{E}$ and $F \in \mathcal{F}$ (see [24, Eq. (2.1)–(2.2)] for details), the standard interpolation operator [15, 23, 24, 32] reads

$$I_M(v) := \sum_{E \in \mathcal{E}(\Omega)} \Big(\fint_E v \,\mathrm{d}s \Big) \phi_E + \sum_{F \in \mathcal{F}(\Omega)} \Big(\fint_F \nabla v \cdot \nu_F \,\mathrm{d}\sigma \Big) \phi_F \\ \text{for any } v \in H_0^2(\Omega) + M(\widehat{\mathcal{T}}).$$

The operator I_M satisfies (I1)–(I4) with $\kappa_2 := \kappa_1/\pi + \sqrt{(3\kappa_1^2 + 2\kappa_1)/80}$ as discussed in [23, 24]; κ_2 is provided in [15, 24]; cf. also [38] for GLB in $2m$-th order eigenvalue problems in n-dimension.

There exists a conforming companion $J : M(\mathcal{T}) \to V$ based on the Hsieh–Clough–Tocher FEM [21, Chap. 6] with (J1)–(J4) in [23, 32, 50] in 2D and on the Worsey–Farin FEM [46, 51] with (J1)–(J3) in [24] in 3D. Since the arguments in the proof of (J2) in [24, Thm. 3.1.b] are local, (J4) follows in 3D as well.

3 Medius analysis

This section shows that (I1)–(I2) and (J1)–(J3) lead to best-approximation and error estimates in weaker Sobolev norms.

3.1 Main result and layout of the proof

Throughout this paper, $k \in \mathbb{N}$ is the number of a *simple* exact eigenvalue $\lambda \equiv \lambda_k$. The aim of this section is the proof of Theorem 3.1 with $\| \bullet \|_\delta$ defined in (3.1) below.

Theorem 3.1 (best-approximation) *Let* $(\lambda, u) \in \mathbb{R}^+ \times V$ *denote the k-th continuous eigenpair of* (1.1) *with a* simple *eigenvalue* $\lambda \equiv \lambda_k$ *and* $\|u\|_{L^2(\Omega)} = 1$. *There exist* $\varepsilon_5 > 0$ *and* $C_0 > 0$ *such that, for all* $\mathcal{T} \in \mathbb{T}(\varepsilon_5) := \{\mathcal{T} \in \mathbb{T} : h_{\max} \le \varepsilon_5\}$, *there exists a discrete eigenpair* $(\lambda_h, \boldsymbol{u_h}) \in \mathbb{R}^+ \times \boldsymbol{V_h}$ *of number k of* (1.2) *with* $\lambda_h \equiv \lambda_h(k)$, $\boldsymbol{u_h} = (u_{\mathrm{pw}}, u_{\mathrm{nc}})$, $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$, *and* $b(u, u_{\mathrm{nc}}) > 0$ *such that*

(a) $\lambda_h(k)$ *is a simple algebraic eigenvalue of* (1.2) *with* $\lambda_k/2 \le \lambda_h(k)$,
(b) $\lambda_h(j) \le \lambda_j$ *for all* $j = 1, \dots, k+1$,
(c) $|\lambda - \lambda_h| + |||u - u_{\mathrm{nc}}|||^2_{\mathrm{pw}} + h_{\max}^{-2\sigma} \|u - u_{\mathrm{nc}}\|^2_{L^2(\Omega)} + \|u_{\mathrm{nc}}\|^2_\delta \le C_0 |||u - Iu|||^2_{\mathrm{pw}}$.

Some comments on related results and an outline of the proof of Theorem 3.1 are in order before Sects. 3.2–3.5 provide details.

Remark 3.2 (known convergence results) The analysis in [24] (§ 2.3.3 for $m = 1$ and Thm. 1.2 for $m = 2$) guarantees the convergence of the eigenvalues λ_h to λ and the component $u_{\mathrm{pw}} \in P_m(\mathcal{T})$ to $u \in V$. The assumption that $\lambda = \lambda_k$ is a simple eigenvalue of (1.1) and the convergence $\lambda_h(k) \equiv \lambda_h \to \lambda$ as $h_{\max} \to 0$ lead to the existence of $\varepsilon_0 > 0$ such that the number $M := \dim(P_m(\mathcal{T}))$ of discrete eigenvalues of (1.2) is larger than $k + 1$ and $\lambda_h(k-1) < \lambda_h(k) \equiv \lambda_h < \lambda_h(k+1)$ as well as $\lambda_k/2 \le \lambda_h(k)$ for all $\mathcal{T} \in \mathbb{T}(\varepsilon_0)$. Then the eigenfunction $\boldsymbol{u_h} = (u_{\mathrm{pw}}, u_{\mathrm{nc}}) \in \boldsymbol{V_h} \setminus \{0\}$ is unique.

The convergence analysis in [24] displays convergence of the eigenvector $u_{\mathrm{pw}} \in P_m(\mathcal{T})$ but not for the nonconforming component $u_{\mathrm{nc}} \in V(\mathcal{T})$. This section focusses on the convergence analysis for $u_{\mathrm{nc}} \in V(\mathcal{T})$. Recall that $k \in \mathbb{N}$ is fixed and (λ, u) denotes the k-th eigenpair of (1.1) with a simple eigenvalue $\lambda \equiv \lambda_k > 0$ and $\|u\|_{L^2(\Omega)} = 1$. Set $\varepsilon_1 := \min\{\varepsilon_0, (2\lambda_{k+1}\kappa_m^2)^{-1/(2m)}\}$ and suppose $\mathcal{T} \in \mathbb{T}(\varepsilon_1)$. Let $(\lambda_h, \boldsymbol{u_h})$ denote the k-th discrete eigenpair in (1.2) with $\lambda_h \equiv \lambda_h(k) > 0$, $\boldsymbol{u_h} = (u_{\mathrm{pw}}, u_{\mathrm{nc}}) \in \boldsymbol{V_h}$, $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$, and $b(u, u_{\mathrm{nc}}) \ge 0$.

Proof of Theorem 3.1.a. This follows from Remark 3.2 for $\varepsilon_1 := \min\{\varepsilon_0, (2\lambda_{k+1}\kappa_m^2)^{-1/(2m)}\}$. □

Proof of Theorem 3.1.b. The choice $\varepsilon_1 := \min\{\varepsilon_0, (2\lambda_{k+1}\kappa_m^2)^{-1/(2m)}\}$ implies for all $j = 1, \dots, k$ that $\lambda_j \kappa_m^2 \, h_{\max}^{2m} \le \lambda_{k+1}\kappa_m^2 \varepsilon_1^{2m} \le 1/2$. Hence (1.3) proves Theorem 3.1.b. □

Remark 3.3 (weight δ) The piecewise constant weight $\delta \in P_0(\mathcal{T})$ in the weighted L^2 norm $\| \bullet \|_\delta := \|\sqrt{\delta} \bullet \|_{L^2(\Omega)}$ on the left-hand side of Theorem 3.1.c reads

$$\delta := \frac{1}{1 - \lambda_h \kappa_m^2 h_{\mathcal{T}}^{2m}} - 1 = \frac{\lambda_h \kappa_m^2 h_{\mathcal{T}}^{2m}}{1 - \lambda_h \kappa_m^2 h_{\mathcal{T}}^{2m}} = \lambda_h \kappa_m^2 h_{\mathcal{T}}^{2m} (1 + \delta) \in P_0(\mathcal{T}). \tag{3.1}$$

Notice that $h_{\max} \le \varepsilon_1$ implies $\delta \le \delta_{\max} := (1 - \lambda_h \kappa_m^2 h_{\max}^{2m})^{-1} - 1 \le 1$. The constant $C_\delta := 2\lambda\kappa_m^2$ satisfies $\delta \le C_\delta h_{\mathcal{T}}^{2m} \le C_\delta h_{\max}^{2m}$ (because $\lambda_h \le \lambda$ from Theorem 3.1.b) and δ converges to zero as the maximal mesh-size $h_{\max} \to 0$ approaches zero.

Remark 3.4 (related work) This section extends the analysis in [18, Section 2–3] to a simultaneous analysis of the Crouzeix–Raviart and Morley FEM and to the extra-stabilized discrete eigenvalue problem (EVP) (1.2) and to 3D.

Remark 3.5 (equivalent problem) Since $\lambda_h\kappa_m^2\, h_{\max}^{2m} \le \lambda_{k+1}\kappa_m^2\varepsilon_1^{2m} \le 1/2$, (1.2) is equivalent to a reduced rational eigenvalue problem that seeks $(\lambda_h, u_{\rm nc}) \in \mathbb{R}^+ \times (V(\mathcal{T})\setminus\{0\})$ with

$$a_{\rm pw}(u_{\rm nc}, v_{\rm nc}) = \lambda_h \Big(\frac{u_{\rm nc}}{1-\lambda_h\kappa_m^2 h_{\mathcal{T}}^{2m}}, v_{\rm nc}\Big)_{L^2(\Omega)} \qquad \text{for all } v_{\rm nc} \in V(\mathcal{T}) \tag{3.2}$$

and $u_{\rm pw} = (1-\lambda_h\kappa_m^2 h_{\mathcal{T}}^{2m})^{-1} u_{\rm nc}$ [24, Prop. 2.5, § 2.3.3].

Outline of the proof of Theorem 3.1.c. The outline of the proof of Theorem 3.1.c provides an overview and clarifies the various steps for a reduction of ε_1 to ε_5, before the technical details follow in the subsequent subsections. The coefficient $(1-\lambda_h\kappa_m^2 h_{\mathcal{T}}^{2m})^{-1} = 1+\delta \in P_0(\mathcal{T})$ with $\lambda_h \equiv \lambda_h(k)$ on the right-hand side of (3.2) is frozen in the intermediate EVP.

Definition 3.6 (intermediate EVP) Recall $(\bullet,\bullet)_{1+\delta} := ((1+\delta)\bullet,\bullet)_{L^2(\Omega)}$. Let $(\mu,\phi) \in \mathbb{R}^+ \times (V(\mathcal{T})\setminus\{0\})$ solve the (algebraic) eigenvalue problem

$$a_{\rm pw}(\phi, v_{\rm nc}) = \mu(\phi, v_{\rm nc})_{1+\delta} \quad \text{for all } v_{\rm nc} \in V(\mathcal{T}). \tag{3.3}$$

The two coefficient matrices in (3.3) are SPD and there exist $N := \dim V(\mathcal{T})$ (algebraic) eigenpairs $(\mu_1,\phi_1),\dots,(\mu_N,\phi_N)$ of (3.3). The eigenvectors $\phi_1,\dots,\phi_N$ are $(\bullet,\bullet)_{1+\delta}$-orthonormal and the eigenvalues $\mu_1 \le \cdots \le \mu_N$ are enumerated in ascending order counting multiplicities.

Since λ_h is an eigenvalue of the rational problem (3.2), $\lambda_h \in \{\mu_1,\dots,\mu_N\}$ belongs to the eigenvalues of (3.3). Lemma 3.9 below guarantees the convergence $|\mu_j-\lambda_h(j)| \to 0$ as $h_{\max} \to 0$ for $j = 1,\dots,k+1$. Hence there exist positive $\varepsilon_2 \le \min\{1/2,\varepsilon_1\}$ and M_6 such that $\mathcal{T} \in \mathbb{T}(\varepsilon_2)$ implies

(H1) $\mu_k = \lambda_h(k)$ is a simple algebraic eigenvalue of (3.3),

(H2) $\displaystyle\max_{\substack{j=1,\dots,N\\ j\neq k}} \frac{\lambda_k}{|\lambda_k-\mu_j|} \le M_6.$

The intermediate EVP and the following associated source problem allow for the control of the extra-stabilization.

Definition 3.7 (auxiliary source problem) Let $z_{\rm nc} \in V(\mathcal{T})$ denote the solution to

$$a_{\rm pw}(z_{\rm nc}, v_{\rm nc}) = (\lambda u, v_{\rm nc})_{1+\delta} \quad \text{for all } v_{\rm nc} \in V(\mathcal{T}). \tag{3.4}$$

For any $\mathcal{T} \in \mathbb{T}(\varepsilon_2)$, Sect. 3.3 below provides $C_1, C_2 > 0$ that satisfy

$$\|u-u_{\rm nc}\|_{L^2(\Omega)} \le C_1\|u-z_{\rm nc}\|_{L^2(\Omega)}, \tag{3.5}$$

$$C_2^{-1}\|u-z_{\rm nc}\|_{L^2(\Omega)} \le h_{\max}^{\sigma}|||u-z_{\rm nc}|||_{\rm pw} + \|\delta\lambda u\|_{L^2(\Omega)}. \tag{3.6}$$

The proof of (3.5) in Sect. 3.3 extends [18, Lem. 2.4]. The proof of (3.6) utilizes another continuous source problem with the right-hand side $u - Jz_{\mathrm{nc}}$. For all $\mathcal{T} \in \mathbb{T}(\varepsilon_2)$, Sect. 3.4 below provides a constant $C_3 > 0$ such that

$$C_3^{-1}|||u - z_{\mathrm{nc}}|||_{\mathrm{pw}} \le |||u - Iu|||_{\mathrm{pw}} + \|\delta\lambda u\|_{L^2(\Omega)}. \tag{3.7}$$

The proof of (3.7) below rests upon a decomposition of $|||u - z_{\mathrm{nc}}|||^2_{\mathrm{pw}}$ into terms controlled by the conditions (I1)–(I2) and (J1)–(J3). Since $h_{\max} \le 1$, the combination of (3.5)–(3.7) reads

$$\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \le C_1C_2\big(C_3h^{\sigma}_{\max}|||u - Iu|||_{\mathrm{pw}} + (1 + C_3)\|\delta\lambda u\|_{L^2(\Omega)}\big). \tag{3.8}$$

The control of $\|\delta\lambda u\|_{L^2(\Omega)}$ on the right-hand side of (3.8) consists of two steps and leads to $c_1 := 2\lambda^2\kappa_m^2C_1C_2(1 + C_3)$ and $\varepsilon_3 := \min\{\varepsilon_2, (2c_1)^{-1/2m}\}$. A triangle inequality $\|\delta\lambda u\|_{L^2(\Omega)} \le \|\delta\lambda(u - u_{\mathrm{nc}})\|_{L^2(\Omega)} + \|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}$, the estimate $\delta \le 2\lambda\kappa_m^2h^{2m}_{\max}$ in Remark 3.3, and (3.8) imply

$$\|\delta\lambda u\|_{L^2(\Omega)} \le \frac{c_1C_3h^{2m}_{\max}}{1 + C_3}h^{\sigma}_{\max}|||u - Iu|||_{\mathrm{pw}} + c_1h^{2m}_{\max}\|\delta\lambda u\|_{L^2(\Omega)} + \|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}.$$

The choice of ε_3 shows $c_1h^{2m}_{\max}\|\delta\lambda u\|_{L^2(\Omega)} \le \|\delta\lambda u\|_{L^2(\Omega)}/2$ for any $\mathcal{T} \in \mathbb{T}(\varepsilon_3)$. Therefore

$$\|\delta\lambda u\|_{L^2(\Omega)} \le C_3/(1 + C_3)h^{\sigma}_{\max}|||u - Iu|||_{\mathrm{pw}} + 2\|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}. \tag{3.9}$$

Notice that $\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)} \le 2\lambda\kappa_m^2h^m_{\max}\|h^m_{\mathcal{T}}u_{\mathrm{nc}}\|_{L^2(\Omega)}$ (from Remark 3.3) allows for the application of an efficiency estimate

$$C_4^{-1}\|h^m_{\mathcal{T}}u_{\mathrm{nc}}\|_{L^2(\Omega)} \le h^m_{\max}\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} + \lambda^{-1}|||u - Iu|||_{\mathrm{pw}} \tag{3.10}$$

based on Verfürth's bubble-function methodology [49]; see Sect. 3.4 for the proof of (3.10). Abbreviate $c_2 := 4\lambda^2\kappa_m^2C_1C_2(1+C_3)C_4$ and $C_5 := 2C_1C_2\big(2C_3 + 4\lambda\kappa_m^2(1 + C_3)C_4\big)$. The combination of (3.9)–(3.10) controls $\|\delta\lambda u\|_{L^2(\Omega)}$ in (3.8) and shows

$$\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \le \frac{C_5}{2}h^{\sigma}_{\max}|||u - Iu|||_{\mathrm{pw}} + c_2h^{2m}_{\max}\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}. \tag{3.11}$$

The choice $\varepsilon_4 := \min\{\varepsilon_3, (2c_2)^{-1/2m}\} < 1$ shows $c_2h^{2m}_{\max}\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \le \|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}/2$ for $\mathcal{T} \in \mathbb{T}(\varepsilon_4)$. This and (3.11) show the central estimate in Theorem 3.1.c

$$\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \le C_5h^{\sigma}_{\max}|||u - Iu|||_{\mathrm{pw}}. \tag{3.12}$$

Note that (3.12) and Corollary 2.1.b from (I2) imply the convergence $\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \to 0$ as $h_{\max} \to 0$. This and some $\varepsilon_5 \le \varepsilon_4$ ensures $b(u, u_{\mathrm{nc}}) > 0$ for all $\mathcal{T} \in \mathbb{T}(\varepsilon_5)$. Based on this outline, it remains to prove (3.5)–(3.7), (3.10), and hence (3.12) and to identify $C_0, \dots, C_4$ below. The remaining estimates in Theorem 3.1.c follow in Sect. 3.5.

3.2 Intermediate EVP

Recall $\varepsilon_1 := \min\{\varepsilon_0, (2\lambda_{k+1}\kappa_m^2)^{-1/(2m)}\}$ and that $(\lambda_h, \boldsymbol{u_h})$ denotes the k-th eigenpair of (1.2) with $\lambda_h \equiv \lambda_h(k) > 0$, $\boldsymbol{u_h} = (u_{\mathrm{pw}}, u_{\mathrm{nc}}) \in \boldsymbol{V_h}$, $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$, and $b(u, u_{\mathrm{nc}}) \geq 0$. Recall the intermediate EVP (3.3) and that $(\lambda_h, u_{\mathrm{nc}}) \in \mathbb{R}^+ \times V(\mathcal{T})$ solves the rational EVP (3.2).

Remark 3.8 ($\| \bullet \|_{1+\delta} \approx \| \bullet \|_{L^2(\Omega)}$) The weighted norm $\| \bullet \|_{1+\delta}$ is equivalent to the L^2-norm. Since $\lambda_h\kappa_m^2\varepsilon_1^{2m} < \lambda_{k+1}\kappa_m^2\varepsilon_1^{2m} \leq 1/2$ and $1 \leq (1+\delta)|_T \leq 2$ for all $T \in \mathcal{T} \in \mathbb{T}(\varepsilon_1)$, $\|v_{\mathrm{nc}}\|_{L^2(\Omega)} \leq \|v_{\mathrm{nc}}\|_{1+\delta} \leq \sqrt{2}\|v_{\mathrm{nc}}\|_{L^2(\Omega)}$ holds for any $v_{\mathrm{nc}} \in V(\mathcal{T})$. □

Lemma 3.9 (comparison of (1.2) with (3.3)) *Given $\mathcal{T} \in \mathbb{T}(\varepsilon_1)$, let $\lambda_h(j)$ denote the j-th eigenvalue of (1.2), and μ_j the j-th eigenvalue of (3.3) for any $j = 1, \dots, k+1$. Then*

$$(1 - \lambda_{k+1}\kappa_m^2 h_{\max}^{2m})\mu_j \leq (1 - \lambda_h(j)\kappa_m^2 h_{\max}^{2m})\mu_j \leq \lambda_h(j) \leq \mu_j + 2\lambda_h^2\kappa_m^2 h_{\max}^{2m}. \tag{3.13}$$

The upper bound $\lambda_h(j) \leq \mu_j + 2\lambda_h^2\kappa_m^2 h_{\max}^{2m}$ holds for all $j = 1, \dots, N$; $N :=$ $\dim V(\mathcal{T})$.

Proof of the upper bound Since the eigenfunctions $\phi_1, \dots, \phi_N$ of (3.3) are $(\bullet, \bullet)_{1+\delta}$-orthonormal, $a_{\mathrm{pw}}(\phi_j, \phi_\ell) = \mu_j\delta_{j\ell}$ and $(\phi_j, \phi_\ell)_{1+\delta} = \delta_{j\ell}$ for all $j, \ell = 1, \dots, N$. Set $\psi_j := (1+\delta)\phi_j$ and $\boldsymbol{U_j} := \mathrm{span}\{(\psi_1, \phi_1), \dots, (\psi_j, \phi_j)\} \subset \boldsymbol{V_h}$. Since $b(\psi_j, \phi_\ell) = (\phi_j, \phi_\ell)_{1+\delta} = \delta_{j\ell}$, the functions $\phi_1, \dots, \phi_N$ are linear independent and so $\dim(\boldsymbol{U_j}) = j$ for any $j = 1, \dots, N$. The discrete min-max principle [7, 45] for the algebraic eigenvalue problem (1.2) shows

$$\lambda_h(j) \leq \max_{\boldsymbol{v_h} \in \boldsymbol{U_j} \setminus \{0\}} \boldsymbol{a_h}(\boldsymbol{v_h}, \boldsymbol{v_h}) / \boldsymbol{b_h}(\boldsymbol{v_h}, \boldsymbol{v_h}). \tag{3.14}$$

The maximum in (3.14) is attained for some $\boldsymbol{v_h} = (\psi, \phi) \in \boldsymbol{U_j} \setminus \{0\}$ with $\phi = \sum_{\ell=1}^j \alpha_\ell\phi_\ell \in V(\mathcal{T})$, $\psi = \sum_{\ell=1}^j \alpha_\ell\psi_\ell = (1+\delta)\phi \in P_m(\mathcal{T})$, and $1 = \|\phi\|_{1+\delta}^2 = \sum_{\ell=1}^j \alpha_\ell^2$. Then $\boldsymbol{b_h}(\boldsymbol{v_h}, \boldsymbol{v_h}) = \|(1+\delta)\phi\|_{L^2(\Omega)}^2 \geq 1$ and $\boldsymbol{a_h}(\boldsymbol{v_h}, \boldsymbol{v_h}) = |||\phi|||_{\mathrm{pw}}^2 + \|\kappa_m^{-1}h_{\mathcal{T}}^{-m}(\psi - \phi)\|_{L^2(\Omega)}^2$. Since $a_{\mathrm{pw}}(\phi_j, \phi_\ell) = \mu_j\delta_{j\ell}$ for $\ell, j = 1, \dots, N$, $\sum_{\ell=1}^j \alpha_\ell^2 = 1$ implies $|||\phi|||_{\mathrm{pw}}^2 = \sum_{\ell=1}^j \alpha_\ell^2\mu_\ell \leq \mu_j$. Since $\delta = \lambda_h\kappa_m^2 h_{\mathcal{T}}^{2m}(1+\delta)$ a.e. in Ω, the stabilization term in $\boldsymbol{a_h}$ reads

$$\|\kappa_m^{-1}h_{\mathcal{T}}^{-m}(\psi - \phi)\|_{L^2(\Omega)}^2 = \|\kappa_m^{-1}h_{\mathcal{T}}^{-m}\delta\phi\|_{L^2(\Omega)}^2 = \lambda_h^2\kappa_m^2\|h_{\mathcal{T}}^m(1+\delta)\phi\|_{L^2(\Omega)}^2.$$

The bound $1+\delta \leq 2$ from Remark 3.3 and $\|\phi\|_{1+\delta} = 1$ imply $\|h_{\mathcal{T}}^m(1+\delta)\phi\|_{L^2(\Omega)}^2 \leq 2h_{\max}^{2m}$. Consequently, $\|\kappa_m^{-1}h_{\mathcal{T}}^{-m}(\psi - \phi)\|_{L^2(\Omega)}^2 \leq 2\lambda_h^2\kappa_m^2 h_{\max}^{2m}$. The substitution of the resulting estimates $\boldsymbol{b_h}(\boldsymbol{v_h}, \boldsymbol{v_h}) \geq 1$ and $\boldsymbol{a_h}(\boldsymbol{v_h}, \boldsymbol{v_h}) \leq \mu_j + 2\lambda_h^2\kappa_m^2 h_{\max}^{2m}$ in (3.14) concludes the proof of $\lambda_h(j) \leq \mu_j + 2\lambda_h^2\kappa_m^2 h_{\max}^{2m}$ in (3.13) for $j = 1, \dots, N$. □

Proof of the lower bound This situation is similar to [27, Thm. 6.4] and adapted below for completeness. For $j = 1, \dots, k+1$, let $(\lambda_h(j), \boldsymbol{\phi_h}(j)) \in \mathbb{R}^+ \times \boldsymbol{V_h}$ denote the first $\boldsymbol{b_h}$-orthonormal eigenpairs of (1.2) with $\boldsymbol{\phi_h}(j) = (\phi_{\mathrm{pw}}(j), \phi_{\mathrm{nc}}(j))$. The test functions $(v_{\mathrm{nc}}, v_{\mathrm{nc}}) \in V(\mathcal{T}) \times V(\mathcal{T}) \subset \boldsymbol{V_h}$ and $(v_{\mathrm{pw}}, 0) \in \boldsymbol{V_h}$ in (1.2) show

$$\begin{aligned} a_{\mathrm{pw}}(\phi_{\mathrm{nc}}(j), v_{\mathrm{nc}}) &= \lambda_h(j) b(\phi_{\mathrm{pw}}(j), v_{\mathrm{nc}}) \quad \text{and} \\ \phi_{\mathrm{pw}}(j) - \phi_{\mathrm{nc}}(j) &= \lambda_h(j) \kappa_m^2 h_{\mathcal{T}}^{2m} \phi_{\mathrm{pw}}(j). \end{aligned} \tag{3.15}$$

For $\xi = (\xi_1, \dots, \xi_j) \in \mathbb{R}^j$ with $\sum_{\ell=1}^{j} \xi_\ell^2 = 1$, set

$$v_{\mathrm{nc}} := \sum_{\ell=1}^{j} \xi_\ell \phi_{\mathrm{nc}}(\ell), \quad v_{\mathrm{pw}} := \sum_{\ell=1}^{j} \xi_\ell \phi_{\mathrm{pw}}(\ell), \quad \text{and} \quad w_{\mathrm{pw}} := \sum_{\ell=1}^{j} \xi_\ell \lambda_h(\ell) \phi_{\mathrm{pw}}(\ell).$$

Since $(\phi_{\mathrm{pw}}(\alpha), \phi_{\mathrm{pw}}(\beta))_{L^2(\Omega)} = \delta_{\alpha\beta}$ for $\alpha, \beta = 1, \dots, k+1$, $\|v_{\mathrm{pw}}\|_{L^2(\Omega)} = 1$ and $\|w_{\mathrm{pw}}\|_{L^2(\Omega)} = \sqrt{\sum_{\ell=1}^{j} \xi_\ell^2 \lambda_h(\ell)^2} \le \lambda_h(j)$. The combination of this with (3.15) and a Cauchy–Schwarz inequality leads to $|||v_{\mathrm{nc}}|||_{\mathrm{pw}}^2 = b(w_{\mathrm{pw}}, v_{\mathrm{nc}}) \le \lambda_h(j) \|v_{\mathrm{nc}}\|_{L^2(\Omega)}$ and $v_{\mathrm{pw}} - v_{\mathrm{nc}} = \kappa_m^2 h_{\mathcal{T}}^{2m} w_{\mathrm{pw}}$. This and a reverse triangle inequality result in

$$\begin{aligned} 0 < 1 - \lambda_h(j) \kappa_m^2 h_{\max}^{2m} &\le 1 - \kappa_m^2 h_{\max}^{2m} \|w_{\mathrm{pw}}\|_{L^2(\Omega)} \\ &\le \|v_{\mathrm{pw}} - \kappa_m^2 h_{\mathcal{T}}^{2m} w_{\mathrm{pw}}\|_{L^2(\Omega)} = \|v_{\mathrm{nc}}\|_{L^2(\Omega)}. \end{aligned} \tag{3.16}$$

This holds for all $v_{\mathrm{nc}} \in U_j := \mathrm{span}\{\phi_{\mathrm{nc}}(1), \dots, \phi_{\mathrm{nc}}(j)\} \subset V(\mathcal{T})$ with coefficients $(\xi_1, \dots, \xi_j) \in \mathbb{R}^j$ of Euclidean norm one. Hence $\dim(U_j) = j$ and the discrete min-max principle [7, 45] for (3.3) show

$$\mu_j \le \max_{v_{\mathrm{nc}} \in U_j \setminus \{0\}} |||v_{\mathrm{nc}}|||_{\mathrm{pw}}^2 / \|v_{\mathrm{nc}}\|_{1+\delta}^2. \tag{3.17}$$

Let $v_{\mathrm{nc}} = \sum_{\ell=1}^{j} \alpha_\ell \phi_{\mathrm{nc}}(\ell) \in U_j$ denote a maximizer in (3.17) with $\sum_{\ell=1}^{j} \alpha_\ell^2 = 1$. The combination of $|||v_{\mathrm{nc}}|||_{\mathrm{pw}}^2 \le \lambda_h(j) \|v_{\mathrm{nc}}\|_{L^2(\Omega)}$, (3.16)–(3.17), and $\|v_{\mathrm{nc}}\|_{L^2(\Omega)} \le \|v_{\mathrm{nc}}\|_{1+\delta}$ from Remark 3.8 provides

$$\mu_j \le \frac{|||v_{\mathrm{nc}}|||_{\mathrm{pw}}^2}{\|v_{\mathrm{nc}}\|_{1+\delta}^2} \le \frac{|||v_{\mathrm{nc}}|||_{\mathrm{pw}}^2}{\|v_{\mathrm{nc}}\|_{L^2(\Omega)}^2} \le \frac{\lambda_h(j)}{1 - \lambda_h(j) \kappa_m^2 h_{\max}^{2m}}.$$

Recall $\lambda_h(j) \le \lambda_h(k+1) \le \lambda_{k+1}$ from the lower bound property in Theorem 3.1.b to conclude the proof of the associated lower bound for all $j = 1, \dots, k$. □

The subsequent corollaries adapt the notation μ_j, $\lambda_h(j)$, λ_j from Lemma 3.9.

Corollary 3.10 *For any $j = 1, \dots, k+1$, it holds $|\mu_j - \lambda_h(j)| + |\mu_j - \lambda_j| \to 0$ as $h_{\max} \to 0$.*

Proof The a priori convergence analysis [24, Thm. 1.2] implies $\lim_{h_{\max}\to 0}\lambda_h(j)\to\lambda_j$. Lemma 3.9 shows $|\lambda_h(j)-\mu_j|\le h_{\max}^{2m}\kappa_m^2\max\{2\lambda_h^2,\lambda_h(j)\mu_j\}\to 0$ as $h_{\max}\to 0$. □

Corollary 3.11 *There exists* $0<\varepsilon_2\le\min\{1/2,\varepsilon_1\}$ *such that (H1)–(H2) hold for* $\mathcal{T}\in\mathbb{T}(\varepsilon_2)$.

Proof Corollary 3.10 and $\lambda_h=\lambda_h(k)\in\{\mu_1,\dots,\mu_N\}$ lead to $\varepsilon_a>0$ such that $\lambda_h=\lambda_h(k)=\mu_k$ has the correct index k for all $\mathcal{T}\in\mathbb{T}(\varepsilon_a)$. It also leads to some $\varepsilon_b>0$ such that $\mu_{k-1}<\mu_k<\mu_{k+1}$ for all $\mathcal{T}\in\mathbb{T}(\varepsilon_b)$. Then $\varepsilon_2:=\min\{1/2,\varepsilon_1,\varepsilon_a,\varepsilon_b\}$ and $\mathcal{T}\in\mathbb{T}(\varepsilon_2)$ imply (H1)–(H2). □

3.3 Proof of (3.5)–(3.6) for the L^2 error control

Recall M_6 from (H2), δ from Remark 3.3, the norm equivalence from Remark 3.8, and the auxiliary source problem (3.4).

Proof of (3.5) Recall the following straightforward result from [18, Eq. (2.8)]: Any $u,v\in L^2(\Omega)$ with $\|u\|_{L^2(\Omega)}=\|v\|_{L^2(\Omega)}=1$ satisfy

$$\big(1+b(u,v)\big)\|u-v\|_{L^2(\Omega)}^2=2\min_{t\in\mathbb{R}}\|u-tv\|_{L^2(\Omega)}^2.$$

This, a triangle inequality, $t:=(z_{\mathrm{nc}},u_{\mathrm{nc}})_{1+\delta}\|\phi_k\|_{L^2(\Omega)}^2$, and $v_{\mathrm{nc}}:=z_{\mathrm{nc}}-tu_{\mathrm{nc}}$ lead to

$$2^{-1/2}\|u-u_{\mathrm{nc}}\|_{L^2(\Omega)}\le\|u-tu_{\mathrm{nc}}\|_{L^2(\Omega)}\le\|u-z_{\mathrm{nc}}\|_{L^2(\Omega)}+\|v_{\mathrm{nc}}\|_{L^2(\Omega)}. \tag{3.18}$$

Since the eigenvectors $\phi_1,\dots,\phi_N$ of (3.3) are $(\bullet,\bullet)_{1+\delta}$-orthonormal and form a basis of $V(\mathcal{T})$, there exist Fourier coefficients $\alpha_1,\dots,\alpha_N\in\mathbb{R}$ with $v_{\mathrm{nc}}=\sum_{j=1}^N\alpha_j\phi_j$ and $\|v_{\mathrm{nc}}\|_{1+\delta}^2=\sum_{j=1}^N\alpha_j^2$. Since $(\lambda_h,u_{\mathrm{nc}})$ solves (3.2), (H1) implies $u_{\mathrm{nc}}\in\operatorname{span}\{\phi_k\}$ with $\|u_{\mathrm{nc}}\|_{L^2(\Omega)}=1$. Hence $u_{\mathrm{nc}}=\pm\phi_k/\|\phi_k\|_{L^2(\Omega)}$, $t=\pm(z_{\mathrm{nc}},\phi_k)_{1+\delta}\|\phi_k\|_{L^2(\Omega)}$, and $(u_{\mathrm{nc}},\phi_k)_{1+\delta}=\pm\|\phi_k\|_{L^2(\Omega)}^{-1}$. Consequently,

$$\alpha_k=(v_{\mathrm{nc}},\phi_k)_{1+\delta}=(z_{\mathrm{nc}},\phi_k)_{1+\delta}-t(u_{\mathrm{nc}},\phi_k)_{1+\delta}=0.$$

Since $(u_{\mathrm{nc}},\phi_j)_{1+\delta}=0$ for all $j=1,\dots,N$ with $j\neq k$, $\alpha_j=(v_{\mathrm{nc}},\phi_j)_{1+\delta}=(z_{\mathrm{nc}},\phi_j)_{1+\delta}$. Since ϕ_j is an eigenvector in (3.3) and z_{nc} solves (3.4), it follows

$$\alpha_j=(z_{\mathrm{nc}},\phi_j)_{1+\delta}=\frac{1}{\mu_j}a_{\mathrm{pw}}(z_{\mathrm{nc}},\phi_j)=\frac{\lambda}{\mu_j}(u,\phi_j)_{1+\delta}.$$

Hence $(u - z_{\rm nc}, \phi_j)_{1+\delta} = (\mu_j/\lambda - 1)\alpha_j$. These values for the coefficients α_j and the separation condition (H2) imply

$$\begin{aligned}\|v_{\rm nc}\|_{1+\delta}^2 &= \sum_{j\neq k}\alpha_j^2 = \sum_{j\neq k}\left|\frac{\lambda}{\mu_j - \lambda}\right||\alpha_j||(u - z_{\rm nc}, \phi_j)_{1+\delta}| \\ &\le M_6 \sum_{j\neq k}(u - z_{\rm nc}, \alpha_j'\phi_j)_{1+\delta}\end{aligned}$$

for a sign in $\alpha_j' \in \{\pm\alpha_j\}$ such that $|(u - z_{\rm nc}, \alpha_j\phi_j)_{1+\delta}| = (u - z_{\rm nc}, \alpha_j'\phi_j)_{1+\delta}$ and with the abbreviation $\sum_{j\neq k} = \sum_{j=1, j\neq k}^N$. This and a Cauchy–Schwarz inequality show

$$M_6^{-1}\|v_{\rm nc}\|_{1+\delta}^2 \le \Big(u - z_{\rm nc}, \sum_{j\neq k}\alpha_j'\phi_j\Big)_{1+\delta} \le \|u - z_{\rm nc}\|_{1+\delta}\|v_{\rm nc}\|_{1+\delta}.$$

The norm equivalence in Remark 3.8 proves $\|v_{\rm nc}\|_{L^2(\Omega)} \le \|v_{\rm nc}\|_{1+\delta} \le \sqrt{2}M_6\|u - z_{\rm nc}\|_{L^2(\Omega)}$. This and (3.18) conclude the proof of (3.5) with $C_1 := \sqrt{2}(1 + \sqrt{2}M_6)$. □

Proof of (3.6) Given the solution $z_{\rm nc} \in V(\mathcal{T})$ to (3.4), let $w \in V := H_0^m(\Omega)$ solve

$$a(w, \varphi) = b(u - Jz_{\rm nc}, \varphi) \quad \text{for all } \varphi \in V. \tag{3.19}$$

Since $u - Jz_{\rm nc} \in V \subset L^2(\Omega)$, the elliptic regularity (2.1) guarantees $w \in H^{m+\sigma}(\Omega)$ and

$$\|w\|_{H^{m+\sigma}(\Omega)} \le C(\sigma)\|u - Jz_{\rm nc}\|_{L^2(\Omega)}. \tag{3.20}$$

The combination of (3.20) with Corollary 2.1.b shows

$$|||w - Iw|||_{\rm pw} \le \left(\frac{h_{\max}}{\pi}\right)^{\sigma}\|w\|_{H^{m+\sigma}(\Omega)} \le C(\sigma)\left(\frac{h_{\max}}{\pi}\right)^{\sigma}\|u - Jz_{\rm nc}\|_{L^2(\Omega)}. \tag{3.21}$$

The test function $\varphi = u - Jz_{\rm nc}$ in the auxiliary problem (3.19) leads to

$$\begin{aligned}\|u - Jz_{\rm nc}\|_{L^2(\Omega)}^2 &= a(u, w - JIw) + a_{\rm pw}(w, z_{\rm nc} - Jz_{\rm nc}) \\ &\quad + a(u, JIw) - a_{\rm pw}(w, z_{\rm nc}).\end{aligned} \tag{3.22}$$

Since (J1) asserts $I(w - JIw) = 0$, Corollary 2.1.c and a triangle inequality show

$$\begin{aligned}a(u, w - JIw) &= a_{\rm pw}(u, (1 - I)(w - JIw)) \\ &\le |||u - z_{\rm nc}|||_{\rm pw}(|||w - Iw|||_{\rm pw} + |||Iw - JIw|||_{\rm pw}).\end{aligned}$$

Then (J2) implies that $a(u, w - JIw) \le (1 + M_2)|||w - Iw|||_{pw}|||u - z_{nc}|||_{pw}$. Corollary 2.3.c proves for the second term in the right-hand side of (3.22) that

$$a_{pw}(w, z_{nc} - Jz_{nc}) \le M_2|||w - Iw|||_{pw}|||u - z_{nc}|||_{pw}.$$

Corollary 2.1.c ensures $a_{pw}(w, z_{nc}) = a_{pw}(Iw, z_{nc})$. Since (λ, u) is an eigenpair of (1.1) and z_{nc} satisfies (3.4), this implies

$$a(u, JIw) - a_{pw}(w, z_{nc}) = b(\lambda u, JIw) - a_{pw}(Iw, z_{nc}) = \lambda b(u, JIw - Iw - \delta Iw).$$

Corollary 2.3.b shows $b(u, JIw - Iw) \le M_2\kappa_m^2 h_{max}^{2m}|||u - z_{nc}|||_{pw}|||w - Iw|||_{pw}$. The discrete Friedrichs inequality

$$\|v_{nc}\|_{L^2(\Omega)} \le C_{dF}|||v_{nc}|||_{pw} \text{ for all } v_{nc} \in V(\mathcal{T}) \text{ with } C_{dF} := C_F(1 + M_2) + M_2 h_{max}^m \tag{3.23}$$

is a direct consequence of the Friedrichs inequality $\|v\|_{L^2(\Omega)} \le C_F|||v|||$ for any $v \in V$ and (J2); cf. [19, Cor. 4.11] for details in case $m = 1$; the proof for $m = 2$ is analogous. This, (I2), and the boundedness of Π_0 imply $C_{dF}^{-1}\|Iw\|_{L^2(\Omega)} \le |||Iw|||_{pw} = \|\Pi_0 D^m w\|_{L^2(\Omega)} \le \|w\|_{H^m(\Omega)}$. The Cauchy–Schwarz inequality leads to

$$-b(\lambda u, \delta Iw) \le \|\delta\lambda u\|_{L^2(\Omega)}\|Iw\|_{L^2(\Omega)} \le C_{dF}\|\delta\lambda u\|_{L^2(\Omega)}\|w\|_{H^{m+\sigma}(\Omega)}.$$

This bounds the last term on the right-hand side of (3.22). The substitution in (3.22) and $\lambda\kappa_m^2 h_{max}^{2m} \le 1/2$ result in

$$\begin{aligned}\|u - Jz_{nc}\|_{L^2(\Omega)}^2 &\le (1 + 5M_2/2)|||w - Iw|||_{pw}|||u - z_{nc}|||_{pw}\\ &\quad + C_{dF}\|\delta\lambda u\|_{L^2(\Omega)}\|w\|_{H^{m+\sigma}(\Omega)}.\end{aligned}$$

This and (3.20)–(3.21) imply

$$C(\sigma)^{-1}\|u - Jz_{nc}\|_{L^2(\Omega)} \le (h_{max}/\pi)^\sigma(1 + 5M_2/2)|||u - z_{nc}|||_{pw} + C_{dF}\|\delta\lambda u\|_{L^2(\Omega)}.$$

Corollary 2.3.a implies $\|z_{nc} - Jz_{nc}\|_{L^2(\Omega)} \le M_2\kappa_m h_{max}^m|||u - z_{nc}|||_{pw}$. This, $0 < \sigma \le 1 \le m$, $h_{max} < 1$, and a triangle inequality show

$$\begin{aligned}\|u - z_{nc}\|_{L^2(\Omega)} &\le \|Jz_{nc} - z_{nc}\|_{L^2(\Omega)} + \|u - Jz_{nc}\|_{L^2(\Omega)}\\ &\le C_2\big(h_{max}^\sigma|||u - z_{nc}|||_{pw} + \|\delta\lambda u\|_{L^2(\Omega)}\big)\end{aligned}$$

with the constant $C_2 := \max\big\{C(\sigma)(1 + 5M_2/2)/\pi^\sigma + M_2\kappa_m, C(\sigma)C_{dF}\big\}$. □

3.4 Proof of (3.7) and (3.10) for the energy error control

Recall δ from Remark 3.3 and that $z_{\mathrm{nc}} \in V(\mathcal{T})$ solves (3.4).

Proof of (3.7) Elementary algebra with $a_{\mathrm{pw}}(z_{\mathrm{nc}}, u) = a_{\mathrm{pw}}(z_{\mathrm{nc}}, Iu)$ from Corollary 2.1.c shows

$$\begin{aligned} |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}^2 &= a(u, u - JIu) + a_{\mathrm{pw}}(u, Jz_{\mathrm{nc}} - z_{\mathrm{nc}}) \\ &\quad + a(u, JIu - Jz_{\mathrm{nc}}) + a_{\mathrm{pw}}(z_{\mathrm{nc}}, z_{\mathrm{nc}} - Iu). \end{aligned} \tag{3.24}$$

Corollary 2.1.c and Corollary 2.3.c control the first two terms in the decomposition

$$\begin{aligned} &a(u, u - JIu) + a_{\mathrm{pw}}(u, Jz_{\mathrm{nc}} - z_{\mathrm{nc}}) \\ &\quad = a_{\mathrm{pw}}(u, u - Iu) + a_{\mathrm{pw}}(u, Iu - JIu) + a_{\mathrm{pw}}(u, Jz_{\mathrm{nc}} - z_{\mathrm{nc}}) \\ &\quad \le (1 + M_2)|||u - Iu|||_{\mathrm{pw}}^2 + M_2|||u - Iu|||_{\mathrm{pw}}|||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}. \end{aligned}$$

Recall that (λ, u) is an eigenpair of (1.1) and z_{nc} satisfies (3.4). Consequently,

$$\begin{aligned} a(u, JIu - Jz_{\mathrm{nc}}) + a_{\mathrm{pw}}(z_{\mathrm{nc}}, z_{\mathrm{nc}} - Iu) &= b(\lambda u, JIu - Jz_{\mathrm{nc}} + (1 + \delta)(z_{\mathrm{nc}} - Iu)) \\ &= \lambda b(u, (J - 1)(Iu - z_{\mathrm{nc}})) \\ &\quad + \lambda b(\delta u, z_{\mathrm{nc}} - Iu). \end{aligned}$$

Corollary 2.3.b, $\kappa_m^2 \lambda h_{\max}^{2m} \le 1/2$, and a triangle inequality show

$$\lambda b(u, (J - 1)(Iu - z_{\mathrm{nc}})) \le M_2/2\, |||u - Iu|||_{\mathrm{pw}}(|||u - Iu|||_{\mathrm{pw}} + |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}).$$

Since Cauchy–Schwarz and triangle inequalities show $b(\delta\lambda u, z_{\mathrm{nc}} - Iu) \le \|\delta\lambda u\|_{L^2(\Omega)} (\|u - z_{\mathrm{nc}}\|_{L^2(\Omega)} + \|u - Iu\|_{L^2(\Omega)})$, (I1) provides the first and (3.6) the second estimate in

$$\begin{aligned} b(\delta\lambda u, z_{\mathrm{nc}} - Iu) &\le \|\delta\lambda u\|_{L^2(\Omega)}(\|u - z_{\mathrm{nc}}\|_{L^2(\Omega)} + \kappa_m h_{\max}^m |||u - Iu|||_{\mathrm{pw}}) \\ &\le \|\delta\lambda u\|_{L^2(\Omega)}(C_2 h_{\max}^\sigma |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}} + C_2\|\delta\lambda u\|_{L^2(\Omega)} \\ &\quad + \kappa_m h_{\max}^m |||u - Iu|||_{\mathrm{pw}}). \end{aligned}$$

Since $h_{\max}^m |||u - Iu|||_{\mathrm{pw}} \le h_{\max}^\sigma |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}$ from Corollary 2.1.a, a weighted Young inequality shows $b(\delta\lambda u, z_{\mathrm{nc}} - Iu) \le ((C_2 + \kappa_m)^2 h_{\max}^{2\sigma} + C_2)\|\delta\lambda u\|_{L^2(\Omega)}^2 + |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}^2/4$. The substitution of the displayed estimates in (3.24) shows

$$\begin{aligned} |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}^2 \le &(1 + 3M_2/2)|||u - Iu|||_{\mathrm{pw}}^2 + 3M_2/2\, |||u - Iu|||_{\mathrm{pw}}|||u - z_{\mathrm{nc}}|||_{\mathrm{pw}} \\ &+ ((C_2 + \kappa_m)^2 h_{\max}^{2\sigma} + C_2)\|\delta\lambda u\|_{L^2(\Omega)}^2 + |||u - z_{\mathrm{nc}}|||_{\mathrm{pw}}^2/4. \end{aligned}$$

This and $3M_2/2|||u-Iu|||_{\mathrm{pw}}|||u-z_{\mathrm{nc}}|||_{\mathrm{pw}} \le 9M_2^2/4|||u-Iu|||_{\mathrm{pw}}^2+|||u-z_{\mathrm{nc}}|||_{\mathrm{pw}}^2/4$ conclude the proof of (3.7) with $C_3^2 := 2\max\{1+3M_2/2+9{M_2}^2/4, (C_2+\kappa_m)^2 h_{\max}^{2\sigma}+C_2\}$. □

Proof of (3.10) The proof of the efficiency estimate of the volume residual is based on Verfürth's bubble-function methodology [49], comparable to [3, Thm. 2], [33, Prop. 3.1], and given here for completeness. Let $\varphi_z \in S^1(\mathcal{T}) := P_1(\mathcal{T})\cap C(\Omega)$ denote the nodal basis function associated with the vertex $z \in \mathcal{V}$. For any $T \in \mathcal{T}$, let $b_T := 4^{4m}\prod_{z\in\mathcal{V}(T)}\varphi_z^m \in P_{4m}(T)\cap W_0^{m,\infty}(T) \subset V$ denote the volume-bubble-function with $\mathrm{supp}(b_T) = T$ and $\|b_T\|_\infty = 1$. An inverse estimate $\|p\|_{L^2(T)} \le c_b\|p\|_{b_T}$ for any polynomial $p \in P_m(T)$ leads to

$$c_b^{-2}\|u_{\mathrm{nc}}\|_{L^2(T)}^2 \le \|u_{\mathrm{nc}}\|_{b_T}^2 = (u_{\mathrm{nc}}, u)_{b_T} - (u_{\mathrm{nc}}, u-u_{\mathrm{nc}})_{b_T}. \tag{3.25}$$

The Cauchy–Schwarz inequality and $\|b_T\|_\infty = 1$ show $(u_{\mathrm{nc}}, u-u_{\mathrm{nc}})_{b_T} \le \|u_{\mathrm{nc}}\|_{L^2(T)}\|u-u_{\mathrm{nc}}\|_{L^2(T)}$. An integration by parts proves $\int_T D^m(b_T u_{\mathrm{nc}})\,\mathrm{d}x = 0$ since $b_T u_{\mathrm{nc}} \in H_0^m(T)$, i.e., $D^m(b_T u_{\mathrm{nc}})$ is L^2-orthogonal to $P_0(T)$. Recall that (λ, u) is an eigenpair of (1.1) and the support of $b_T u_{\mathrm{nc}}$ is T. This, (I2), and the Cauchy–Schwarz inequality result in

$$\begin{aligned}\lambda b(u, b_T u_{\mathrm{nc}}) &= a_{\mathrm{pw}}(u, b_T u_{\mathrm{nc}}) = (D^m u, D^m(b_T u_{\mathrm{nc}}))_{L^2(T)}\\ &\le |u-Iu|_{H^m(T)}|b_T u_{\mathrm{nc}}|_{H^m(T)}.\end{aligned}$$

An inverse estimate for polynomials in $P_{5m}(T)$ with the constant c_{inv} and the boundedness of b_T show $\lambda b(u, b_T u_{\mathrm{nc}}) \le c_{\mathrm{inv}} h_T^{-m}|u-Iu|_{H^m(T)}\|u_{\mathrm{nc}}\|_{L^2(T)}$. This provides $c_b^{-2}h_T^m\|u_{\mathrm{nc}}\|_{L^2(T)} \le h_T^m\|u-u_{\mathrm{nc}}\|_{L^2(T)}+c_{\mathrm{inv}}\lambda^{-1}|u-Iu|_{H^m(T)}$ for all $T \in \mathcal{T}$ in (3.25). The sum over all $T \in \mathcal{T}$ concludes the proof of (3.10) with $C_4 = c_b^2\max\{1, c_{\mathrm{inv}}\}$. □

3.5 Proof of Theorem 3.1.c

Proof of (3.12) for $\varepsilon_4 > 0$ Recall $c_1 := 2\lambda^2\kappa_m^2 C_1 C_2(1+C_3)$ and (3.8) as a result of (3.5)–(3.7). A triangle inequality, Remark 3.3, and (3.8) show

$$\begin{aligned}\|\delta\lambda u\|_{L^2(\Omega)} &\le 2\lambda^2\kappa_m^2 h_{\max}^{2m}\|u-u_{\mathrm{nc}}\|_{L^2(\Omega)} + \|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}\\ &\le \frac{c_1 C_3 h_{\max}^{2m}}{1+C_3}h_{\max}^\sigma|||u-Iu|||_{\mathrm{pw}} + c_1 h_{\max}^{2m}\|\delta\lambda u\|_{L^2(\Omega)} + \|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}.\end{aligned}$$

Since $0 < \varepsilon_3 := \min\{\varepsilon_2, (2c_1)^{-1/2m}\}$ ensures $c_1 h_{\max}^{2m} \le 1/2$ for all $\mathcal{T} \in \mathbb{T}(\varepsilon_3)$, the previously displayed estimate reads $\|\delta\lambda u\|_{L^2(\Omega)} \le \frac{c_1 C_3 h_{\max}^{2m}}{1+C_3}h_{\max}^\sigma|||u-Iu|||_{\mathrm{pw}} + \|\delta\lambda u\|_{L^2(\Omega)}/2 + \|\delta\lambda u_{\mathrm{nc}}\|_{L^2(\Omega)}$. This implies (3.9). The bound (3.9) for $\|\delta\lambda u\|_{L^2(\Omega)}$ recasts (3.8) as

$$C_1^{-1}C_2^{-1}\|u-u_{\mathrm{nc}}\|_{L^2(\Omega)} \le 2C_3 h_{\max}^\sigma|||u-Iu|||_{\mathrm{pw}} + 2(1+C_3)\lambda\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}.$$

Remark 3.3 and (3.10) control the last term in

$$(2\kappa_m^2 C_4)^{-1}\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)} \leq C_4^{-1}\lambda h_{\max}^m \|h_{\mathcal{T}}^m u_{\mathrm{nc}}\|_{L^2(\Omega)}$$
$$\leq \lambda h_{\max}^{2m}\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} + h_{\max}^m |||u - Iu|||_{\mathrm{pw}}.$$

Recall that $c_2 := 4\lambda^2\kappa_m^2 C_1 C_2(1 + C_3)C_4$ and $\varepsilon_4 := \min\{\varepsilon_3, (2c_2)^{-1/2m}\} < 1$ ensure $c_2 h_{\max}^{2m} \leq 1/2$. Hence the last term in (3.11) is $\leq \|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}/2$ and can be absorbed. This concludes the proof of (3.12) with $C_5 := 2C_1C_2\big(2C_3 + 4\kappa_m^2\lambda(1 + C_3)C_4\big)$. □

Recall $0 < \varepsilon_5 \leq \varepsilon_4$ such that $b(u, u_{\mathrm{nc}}) > 0$ for any $\mathcal{T} \in \mathbb{T}(\varepsilon_5)$.

Proof of Theorem 3.1.c for $\varepsilon_5 > 0$ Recall $\lambda_h \leq \lambda$ and $\|u\|_{L^2(\Omega)} = \|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$. The continuous eigenpair (λ, u) in (1.1) satisfies $\lambda = |||u|||^2$. The discrete eigenpair $(\lambda_h, u_{\mathrm{nc}})$ solves (3.2) and so $\lambda_h = |||u_{\mathrm{nc}}|||_{\mathrm{pw}}^2/\|u_{\mathrm{nc}}\|_{1+\delta}^2$ with $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$. Then

$$|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2 = \lambda - 2a_{\mathrm{pw}}(u, u_{\mathrm{nc}}) + \lambda_h\|u_{\mathrm{nc}}\|_{1+\delta}^2 \quad \text{and}$$
$$\|u_{\mathrm{nc}}\|_{1+\delta}^2 - 1 = b(\delta u_{\mathrm{nc}}, u_{\mathrm{nc}}) = \|u_{\mathrm{nc}}\|_\delta^2.$$

This and elementary algebra show for the left-hand side of Theorem 3.1.c that

$$\mathrm{LHS} := \lambda - \lambda_h + |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2 + \|u_{\mathrm{nc}}\|_\delta^2 = 2\lambda - 2a_{\mathrm{pw}}(u, u_{\mathrm{nc}}) + (1 + \lambda_h)\|u_{\mathrm{nc}}\|_\delta^2.$$

Since u is the eigenfunction in (1.1) and $2b(u, u - u_{\mathrm{nc}}) = \|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}^2$ from $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1 = \|u\|_{L^2(\Omega)}$, it follows

$$\lambda = \lambda b(u, u_{\mathrm{nc}}) + \lambda b(u, u - u_{\mathrm{nc}})$$
$$= \lambda b(u, u_{\mathrm{nc}} - Ju_{\mathrm{nc}}) + a_{\mathrm{pw}}(u, Ju_{\mathrm{nc}}) + \lambda/2\ \|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}^2.$$

The combination of the last two displayed identities eventually leads to

$$\mathrm{LHS} = (1 + \lambda_h)\|u_{\mathrm{nc}}\|_\delta^2 + \lambda\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 + 2\lambda b(u, u_{\mathrm{nc}} - Ju_{\mathrm{nc}})$$
$$+ 2a_{\mathrm{pw}}(u, Ju_{\mathrm{nc}} - u_{\mathrm{nc}}). \tag{3.26}$$

Recall $2\lambda\kappa_m^2 h_{\max}^{2m} \leq 1$. The combination of Remark 3.3 and (3.10) implies that

$$\|u_{\mathrm{nc}}\|_\delta \leq \sqrt{2}\kappa_m\lambda^{1/2}\|h_{\mathcal{T}}^m u_{\mathrm{nc}}\|_{L^2(\Omega)} \leq C_4\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} + \sqrt{2/\lambda}\kappa_m C_4|||u - Iu|||_{\mathrm{pw}}$$

and (3.12) controls $\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \leq C_5 h_{\max}^\sigma |||u - Iu|||_{\mathrm{pw}}$. Corollary 2.3.b asserts $2\lambda b(u, u_{\mathrm{nc}} - Ju_{\mathrm{nc}}) \leq M_2|||u - Iu|||_{\mathrm{pw}}|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}$. Corollary 2.3.c shows $a_{\mathrm{pw}}(u, Ju_{\mathrm{nc}} - u_{\mathrm{nc}}) \leq M_2|||u - Iu|||_{\mathrm{pw}}|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}$. Since $\lambda_h \leq \lambda$, these estimates lead in (3.26) to

$$\mathrm{LHS} \leq \big((1 + \lambda)C_4^2(C_5 h_{\max}^\sigma + \sqrt{2/\lambda}\kappa_m)^2 + \lambda C_5^2 h_{\max}^{2\sigma}\big)|||u - Iu|||^2$$
$$+ 3M_2|||u - Iu|||_{\mathrm{pw}}|||u_{\mathrm{nc}} - u|||_{\mathrm{pw}}.$$

A weighted Young inequality and the absorption of $|||u_{\mathrm{nc}} - u|||^2_{\mathrm{pw}}/2$ conclude the proof of Theorem 3.1.c with $C_0 := \max\{C_5^2, 2((1+\lambda)C_4^2(C_5 h^\sigma_{\max} + \sqrt{2/\lambda}\kappa_m)^2 + \lambda C_5^2 h^{2\sigma}_{\max}) + 9M_2^2\}$. □

4 Optimal convergence rates

This section verifies some general axioms of adaptivity [12, 26] sufficient for optimal rates for AFEM4EVP and prepares the conclusion of the proof of Theorem 1.1 in Sect. 5.

4.1 Stability and reduction

The 2-level notation of Table 1 concerns one coarse triangulation $\mathcal{T} \in \mathbb{T}$ and one fine triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$. Let $(\lambda, u) \in \mathbb{R}^+ \times V$ denote the k-th continuous eigenpair of (1.1) with a *simple* eigenvalue $\lambda \equiv \lambda_k$ and the normalization $\|u\|_{L^2(\Omega)} = 1$. Choose $\varepsilon_5 > 0$ as in Theorem 3.1, suppose $\mathcal{T} \in \mathbb{T}(\varepsilon_5)$, and let $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ be any admissible refinement of $\mathcal{T}$.

Definition 4.1 (2-level notation) Let $(\lambda_h, \boldsymbol{u_h}) \in \mathbb{R}^+ \times \boldsymbol{V_h}$ (resp. $(\widehat{\lambda}_h, \widehat{\boldsymbol{u}}_{\boldsymbol{h}}) \in \mathbb{R}^+ \times \widehat{\boldsymbol{V}}_{\boldsymbol{h}}$) with $\boldsymbol{u_h} = (u_{\mathrm{pw}}, u_{\mathrm{nc}}) \in \boldsymbol{V_h} := P_m(\mathcal{T}) \times V(\mathcal{T})$ (resp. $\widehat{\boldsymbol{u}}_{\boldsymbol{h}} = (\widehat{u}_{\mathrm{pw}}, \widehat{u}_{\mathrm{nc}}) \in \widehat{\boldsymbol{V}}_{\boldsymbol{h}} := P_m(\widehat{\mathcal{T}}) \times V(\widehat{\mathcal{T}})$) denote the k-th discrete eigenpair of (1.2) with the simple algebraic eigenvalue $\lambda_h \equiv \lambda_h(k)$ (resp. $\widehat{\lambda}_h \equiv \widehat{\lambda}_h(k)$), the normalization $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$ (resp. $\|\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$), and the sign convention $b(u, u_{\mathrm{nc}}) > 0$ (resp. $b(u, \widehat{u}_{\mathrm{nc}}) > 0$). Recall $\widehat{h}_{\max} := \max_{T\in\widehat{\mathcal{T}}} h_T \le h_{\max} := \max_{T\in\mathcal{T}} h_T \le \varepsilon_5$, λ_h, $\widehat{\lambda}_h \le \lambda$ from Theorem 3.1.b, and δ from Remark 3.3 with its analogue $\widehat{\delta} := (1 - \widehat{\lambda}_h \kappa_m^2 h^{2m}_{\widehat{\mathcal{T}}})^{-1} - 1 \in P_0(\widehat{\mathcal{T}})$ on the fine level. The constant $C_\delta := 2\lambda\kappa_m^2$ satisfies $\delta \le C_\delta h^{2m}_{\mathcal{T}}$ and $\widehat{\delta} \le C_\delta h^{2m}_{\widehat{\mathcal{T}}}$. Recall the estimator $\eta^2(T)$ for any $T \in \mathcal{T}$ from (1.4) and define $\widehat{\eta}^2(T)$, for any $T \in \widehat{\mathcal{T}}$ with volume $|T|$ and the set of faces $\widehat{\mathcal{F}}(T)$, by

$$\widehat{\eta}^2(T) := |T|^{2m/3}\|\widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}\|^2_{L^2(T)} + |T|^{1/3} \sum_{F\in\widehat{\mathcal{F}}(T)} \|[D^m_{\mathrm{pw}}\widehat{u}_{\mathrm{nc}}]_F \times \nu_F\|^2_{L^2(F)}. \tag{4.1}$$

The sum conventions $\eta^2(\mathcal{M}) := \sum_{T\in\mathcal{M}} \eta^2(T)$ for $\mathcal{M} \subset \mathcal{T}$ and $\widehat{\eta}^2(\widehat{\mathcal{M}}) := \sum_{T\in\widehat{\mathcal{M}}} \widehat{\eta}^2(T)$ for $\widehat{\mathcal{M}} \subset \widehat{\mathcal{T}}$ from Table 1 apply throughout this section. Abbreviate the distance function

$$\delta^2(\mathcal{T}, \widehat{\mathcal{T}}) := \|\lambda_h u_{\mathrm{nc}} - \widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}\|^2_{L^2(\Omega)} + |||u_{\mathrm{nc}} - \widehat{u}_{\mathrm{nc}}|||^2_{\mathrm{pw}}. \tag{4.2}$$

Theorem 4.2 (stability and reduction) *There exist $\Lambda_1, \Lambda_2 > 0$, such that, for any $\mathcal{T}$ and $\widehat{\mathcal{T}}$ from Definition 4.1, the following holds*

(A1) Stability. $\left|\eta(\mathcal{T} \cap \widehat{\mathcal{T}}) - \widehat{\eta}(\mathcal{T} \cap \widehat{\mathcal{T}})\right| \le \Lambda_1 \delta(\mathcal{T}, \widehat{\mathcal{T}})$,
(A2) Reduction. $\widehat{\eta}(\widehat{\mathcal{T}} \backslash \mathcal{T}) \le 2^{-1/12}\eta(\mathcal{T} \backslash \widehat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \widehat{\mathcal{T}})$.

Table 1 : 2-level notation with respect to $\mathcal{T} \in \mathbb{T}(\varepsilon)$ (left) and an admissible refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ (right)

$(\lambda_h, \boldsymbol{u_h}) \in \mathbb{R}^+ \times \boldsymbol{V_h}$ k-th eigenpair in (1.2)	$(\widehat{\lambda}_h, \widehat{\boldsymbol{u}}_{\boldsymbol{h}}) \in \mathbb{R}^+ \times \widehat{\boldsymbol{V}}_{\boldsymbol{h}}$ k-th eigenpair in (1.2)
with $\boldsymbol{u_h} = (u_{\text{pw}}, u_{\text{nc}}) \in P_m(\mathcal{T}) \times V(\mathcal{T})$	with $\widehat{\boldsymbol{u}}_{\boldsymbol{h}} = (\widehat{u}_{\text{pw}}, \widehat{u}_{\text{nc}}) \in P_m(\widehat{\mathcal{T}}) \times V(\widehat{\mathcal{T}})$
$\|u_{\text{nc}}\|_{L^2(\Omega)} = 1, b(u, u_{\text{nc}}) > 0, \lambda_h \le \lambda$	$\|\widehat{u}_{\text{nc}}\|_{L^2(\Omega)} = 1, b(u, \widehat{u}_{\text{nc}}) > 0, \widehat{\lambda}_h \le \lambda$
$h_{\max} := \max_{T\in\mathcal{T}} h_T$	$\widehat{h}_{\max} := \max_{T\in\widehat{\mathcal{T}}} h_T$
$\delta := (1 - \lambda_h\kappa_m^2 h_{\mathcal{T}}^{2m})^{-1} - 1 \le C_\delta h_{\mathcal{T}}^{2m} \le 1$	$\widehat{\delta} := (1 - \widehat{\lambda}_h\kappa_m^2 h_{\widehat{\mathcal{T}}}^{2m})^{-1} - 1 \le C_\delta h_{\widehat{\mathcal{T}}}^{2m} \le 1$
$\eta^2(T)$ from (1.4) for $T \in \mathcal{T}$	$\widehat{\eta}^2(T)$ from (4.1) for $T \in \widehat{\mathcal{T}}$
$\eta^2(\mathcal{M}) := \sum_{T\in\mathcal{M}} \eta^2(T)$ for $\mathcal{M} \subseteq \mathcal{T}$	$\widehat{\eta}^2(\widehat{\mathcal{M}}) := \sum_{T\in\widehat{\mathcal{M}}} \widehat{\eta}^2(T)$ for $\widehat{\mathcal{M}} \subseteq \widehat{\mathcal{T}}$

Proof A reverse triangle inequality in $\mathbb{R}^L$ for the number $L := |\mathcal{T} \cap \widehat{\mathcal{T}}|$ of tetrahedra in $\mathcal{T} \cap \widehat{\mathcal{T}}$ and one for each common tetrahedra $T \in \mathcal{T} \cap \widehat{\mathcal{T}}$ and each of its faces $F \in \mathcal{F}(T)$ lead to

$$\big|\eta(\mathcal{T} \cap \widehat{\mathcal{T}}) - \widehat{\eta}(\mathcal{T} \cap \widehat{\mathcal{T}})\big|^2 \le \sum_{T\in\mathcal{T}\cap\widehat{\mathcal{T}}} \Big(|T|^{2m/3}\|\lambda_h u_{\text{nc}} - \widehat{\lambda}_h\widehat{u}_{\text{nc}}\|^2_{L^2(T)} + |T|^{1/3} \sum_{F\in\mathcal{F}(T)} \|[D^m_{\text{pw}}(u_{\text{nc}} - \widehat{u}_{\text{nc}})]_F \times \nu_F\|^2_{L^2(F)}\Big).$$

The discrete jump control from [26, Lem. 5.2] with constant $C_{\text{jc}}(\ell)$ (that only depends on the shape-regularity of $\mathbb{T}$ and the polynomial degree $\ell \in \mathbb{N}_0$) reads

$$\sum_{T\in\mathcal{T}} |T|^{1/3} \sum_{F\in\mathcal{F}(T)} \|[g]_F\|^2_{L^2(F)} \le C_{\text{jc}}(\ell)^2\|g\|^2_{L^2(\Omega)} \quad \text{for any } g \in P_\ell(\mathcal{T}).$$

The combination of the two displayed estimates concludes the proof of (A1) with $\Lambda_1^2 = \max\big\{\max_{T\in\mathcal{T}_0} |T|^{2m/3}, C_{\text{jc}}(0)^2\big\}$. For any tetrahedron $K \in \mathcal{T}\setminus\widehat{\mathcal{T}}$, let $\widehat{\mathcal{T}}(K) := \{T \in \widehat{\mathcal{T}} : T \subset K\}$ denote its fine triangulation. The newest-vertex bisection guarantees $|T| \le |K|/2$ for the volume $|T|$ of any $T \in \widehat{\mathcal{T}}(K)$. This, a triangle inequality, and $(a + b)^2 \le (1 + \beta)a^2 + (1 + 1/\beta)b^2$ for $a,\ b \ge 0,\ \beta = 2^{1/6} - 1 > 0$ show

$$\widehat{\eta}^2(\widehat{\mathcal{T}}(K)) \le 2^{-1/6}\eta^2(K) + (1 + 1/\beta) \sum_{T\in\widehat{\mathcal{T}}(K)} \Big(|T|^{2m/3}\|\lambda_h u_{\text{nc}} - \widehat{\lambda}_h\widehat{u}_{\text{nc}}\|^2_{L^2(K)} + |T|^{1/3} \sum_{F\in\widehat{\mathcal{F}}(T)} \|[D^m_{\text{pw}}(\widehat{u}_{\text{nc}} - u_{\text{nc}})]_F \times \nu_F\|^2_{L^2(F)}\Big).$$

The summation over all $K \in \mathcal{T} \setminus \widehat{\mathcal{T}}$ and the above jump control conclude the proof of (A2) with $\Lambda_2^2 = 2^{1/6}/(2^{1/6} - 1)\ \max\big\{\max_{T\in\mathcal{T}_0} |T|^{2m/3}, C_{\text{jc}}(0)^2\big\}$. The arguments for (A1)–(A2) are similar for other problems; cf., e.g., [12, 20, 22, 26] for more details. □

4.2 Towards discrete reliability

Given the 2-level notation of Definition 4.1 with respect to $\mathcal{T}$ and $\widehat{\mathcal{T}}$, let $\mathcal{R}_1 := \{K \in \mathcal{T} : \exists T \in \mathcal{T}\setminus\widehat{\mathcal{T}} \text{ with } \operatorname{dist}(K,T)=0\} \subset \mathcal{T}$ denote the set of coarse but not fine tetrahedra plus one layer of coarse tetrahedra around. Lemma 4.3–4.5 prepare the proof of the discrete reliability in Theorem 4.6 below. Let $\widehat{I} : V + V(\widehat{\mathcal{T}}) \to V(\widehat{\mathcal{T}})$ denote the interpolation operator on the fine level of $\widehat{\mathcal{T}}$ so that (I3) and a Cauchy–Schwarz inequality show, for any $v \in V + V(\widehat{\mathcal{T}})$ and any $w \in V + V(\mathcal{T}) + V(\widehat{\mathcal{T}})$, that

$$\begin{aligned} |b((I-\widehat{I})v, w)| &\le \|(I-\widehat{I})v\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\|w\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})},\\ |a_{\mathrm{pw}}((I-\widehat{I})v, w)| &\le \|D^m_{\mathrm{pw}}(I-\widehat{I})v\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\|D^m_{\mathrm{pw}}w\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}. \end{aligned} \tag{4.3}$$

Lemma 4.3 (distance control I) *There exists $C_6 > 0$ such that any $\mathcal{T} \in \mathbb{T}(\varepsilon_5)$ and the difference $e := \widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}}$ satisfy*

$$\begin{aligned} C_6^{-1}|||e|||^2_{\mathrm{pw}} \le \|D^m_{\mathrm{pw}}(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|^2_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} + \|h^m_{\mathcal{T}}\lambda_h u_{\mathrm{nc}}\|^2_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} + \|e\|^2_{L^2(\Omega)}\\ + \|\delta u_{\mathrm{nc}}\|^2_{L^2(\Omega)} + \|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|^2_{L^2(\Omega)}. \end{aligned}$$

Proof Corollary 2.1.c shows $a_{\mathrm{pw}}(e, \widehat{u}_{\mathrm{nc}} - Ju_{\mathrm{nc}}) = a_{\mathrm{pw}}\big(\widehat{u}_{\mathrm{nc}}, \widehat{u}_{\mathrm{nc}} - \widehat{I}Ju_{\mathrm{nc}}\big) - a_{\mathrm{pw}}\big(u_{\mathrm{nc}}, I(\widehat{u}_{\mathrm{nc}} - Ju_{\mathrm{nc}})\big)$. Since $(\lambda_h, u_{\mathrm{nc}})$ and $(\widehat{\lambda}_h, \widehat{u}_{\mathrm{nc}})$ solve (3.2), this and (J1) lead to

$$\begin{aligned} a_{\mathrm{pw}}(e, \widehat{u}_{\mathrm{nc}} - Ju_{\mathrm{nc}}) &= b\big(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}}, (1+\widehat{\delta})(\widehat{u}_{\mathrm{nc}} - \widehat{I}Ju_{\mathrm{nc}})\big)\\ &\quad - b\big(\lambda_h u_{\mathrm{nc}}, (1+\delta)(I\widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}})\big)\\ &= b(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}} - \lambda_h u_{\mathrm{nc}}, e) + b(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}}, \widehat{\delta}e) - b(\lambda_h u_{\mathrm{nc}}, \delta e)\\ &\quad + b\big(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}}, (1+\widehat{\delta})(u_{\mathrm{nc}} - \widehat{I}Ju_{\mathrm{nc}})\big)\\ &\quad + b\big(\lambda_h u_{\mathrm{nc}}, (1+\delta)(\widehat{u}_{\mathrm{nc}} - I\widehat{u}_{\mathrm{nc}})\big). \end{aligned} \tag{4.4}$$

Elementary algebra with $\|u_{\mathrm{nc}}\|_{L^2(\Omega)} = \|\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} = 1$ shows (as, e.g., in [13, Lem. 3.1])

$$\begin{aligned} b(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}} - \lambda_h u_{\mathrm{nc}}, e) &= \frac{\widehat{\lambda}_h + \lambda_h}{2}\|e\|^2_{L^2(\Omega)} + \frac{\widehat{\lambda}_h - \lambda_h}{2} b(\widehat{u}_{\mathrm{nc}} + u_{\mathrm{nc}}, \widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}})\\ &= \frac{\widehat{\lambda}_h + \lambda_h}{2}\|e\|^2_{L^2(\Omega)}. \end{aligned}$$

Cauchy–Schwarz inequalities verify

$$b(\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}}, \widehat{\delta}e) - b(\lambda_h u_{\mathrm{nc}}, \delta e) \le \|e\|_{L^2(\Omega)}\big(\widehat{\lambda}_h\|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} + \lambda_h\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}\big).$$

Since $1+\widehat{\delta}\le 2$ and $\widehat{\lambda}_h\le\lambda$ from Table 1, the right inverse property (J1) and (4.3) result in

$$\begin{aligned}b\big((1+\widehat{\delta})\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}},u_{\mathrm{nc}}-\widehat{I}Ju_{\mathrm{nc}}\big)&=b\big((1+\widehat{\delta})\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}},(I-\widehat{I})Ju_{\mathrm{nc}}\big)\\&\le 2\|h_{\mathcal{T}}^m\lambda\widehat{u}_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\|h_{\mathcal{T}}^{-m}(I-\widehat{I})Ju_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}.\end{aligned}$$

The triangle inequality $\|h_{\mathcal{T}}^m\lambda\widehat{u}_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\le h_{\max}^m\lambda\|e\|_{L^2(\Omega)}+\|h_{\mathcal{T}}^m\lambda u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}$ and $\lambda/\lambda_h\le 2$ from Theorem 3.1.a imply $\|h_{\mathcal{T}}^m\lambda u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\le 2\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}$. Since the interpolation operators I and $\widehat{I}$ satisfy (I3)–(I4), it follows that

$$\begin{aligned}\|h_{\mathcal{T}}^{-m}(I-\widehat{I})Ju_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}&=\|h_{\mathcal{T}}^{-m}(1-I)\widehat{I}Ju_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\\&\le\kappa_d\|D_{\mathrm{pw}}^m(1-I)\widehat{I}Ju_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}.\end{aligned}$$

Recall $D_{\mathrm{pw}}^m u_{\mathrm{nc}}\in P_0(\mathcal{T};\mathbb{R}^{3^m})$. The condition (I2) and the L^2-orthogonal projections Π_0 (resp. $\widehat{\Pi}_0$) onto $P_0(\mathcal{T})$ (resp. $P_0(\widehat{\mathcal{T}})$) lead to the estimate

$$\begin{aligned}&\kappa_d^{-1}\|h_{\mathcal{T}}^{-m}(I-\widehat{I})Ju_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\le\|(\Pi_0-\widehat{\Pi}_0)D^mJu_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\\&\quad=\|(\Pi_0-\widehat{\Pi}_0)D_{\mathrm{pw}}^m(Ju_{\mathrm{nc}}-u_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\le\|D_{\mathrm{pw}}^m(Ju_{\mathrm{nc}}-u_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}.\end{aligned}$$

The estimate (4.3) and $\delta\le 1$ from Table 1 imply the first inequality and (I4) and Corollary 2.1.a the second in

$$\begin{aligned}b\big(\lambda_h u_{\mathrm{nc}},(1+\delta)(\widehat{u}_{\mathrm{nc}}-I\widehat{u}_{\mathrm{nc}})\big)&=b\big(\lambda_h u_{\mathrm{nc}},(1+\delta)(\widehat{I}-I)\widehat{u}_{\mathrm{nc}}\big)\\&\le 2\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\|h_{\mathcal{T}}^{-m}(\widehat{u}_{\mathrm{nc}}-I\widehat{u}_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\\&\le 2\kappa_d\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}|||e|||_{\mathrm{pw}}.\end{aligned}$$

The combination of the six previously displayed estimates and $\lambda_h,\widehat{\lambda}_h\le\lambda$ lead in (4.4) to

$$\begin{aligned}a_{\mathrm{pw}}(e,\widehat{u}_{\mathrm{nc}}-Ju_{\mathrm{nc}})&\le 2\kappa_d\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}}-Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\big(\lambda h_{\max}^m\|e\|_{L^2(\Omega)}\\&\quad+2\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\big)+\lambda\|e\|_{L^2(\Omega)}\big(\|e\|_{L^2(\Omega)}+\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}\\&\quad+\|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}\big)+2\kappa_d|||e|||_{\mathrm{pw}}\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}.\end{aligned}$$

Additionally, Corollary 2.3.c and (4.3) show

$$\begin{aligned}a_{\mathrm{pw}}(e,Ju_{\mathrm{nc}}-u_{\mathrm{nc}})&=a_{\mathrm{pw}}((1-I)e,Ju_{\mathrm{nc}}-u_{\mathrm{nc}})=a_{\mathrm{pw}}\big((\widehat{I}-I)\widehat{u}_{\mathrm{nc}},Ju_{\mathrm{nc}}-u_{\mathrm{nc}}\big)\\&\le\|D_{\mathrm{pw}}^m(1-I)e\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}}-Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}.\end{aligned}$$

Condition (I2) and the boundedness of Π_0 show $\|D_{\mathrm{pw}}^m(1-I)e\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\le|||e|||_{\mathrm{pw}}$. This and the combination of the two previously displayed estimates with a triangle inequality prove

$$
\begin{aligned}
|||e|||_{\mathrm{pw}}^2 &= a_{\mathrm{pw}}(e, Ju_{\mathrm{nc}} - u_{\mathrm{nc}}) + a_{\mathrm{pw}}(e, \widehat{u}_{\mathrm{nc}} - Ju_{\mathrm{nc}}) \\
&\le \|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\big(|||e|||_{\mathrm{pw}} + 2\kappa_d\lambda h_{\max}^m\|e\|_{L^2(\Omega)} \\
&\quad + 4\kappa_d\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\big) + \lambda\|e\|_{L^2(\Omega)}\big(\|e\|_{L^2(\Omega)} \\
&\quad + \|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)} + \|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}\big) + 2\kappa_d|||e|||_{\mathrm{pw}}\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} \\
&\le (1 + 4\kappa_d^2 + \kappa_d^2\lambda^2 h_{\max}^{2m})\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + \|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 \\
&\quad + \|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}^2 + (1 + \lambda + \lambda^2/2)\|e\|_{L^2(\Omega)}^2 \\
&\quad + (1 + 4\kappa_d^2)\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + |||e|||_{\mathrm{pw}}^2/2
\end{aligned}
$$

with weighted Young inequalities in the last step. This concludes the proof with $C_6 := 2\max\{1 + 4\kappa_d^2 + \kappa_d^2\lambda^2 h_{\max}^{2m}, 1 + \lambda + \lambda^2/2\}$. □

4.2.1 Reliability and efficiency

A first consequence of Lemma 4.3 is the reliability of the error estimator $\eta(\mathcal{T})$ from (1.4).

Theorem 4.4 (reliability and efficiency) *There exist C_{rel}, C_{eff}, and $\varepsilon_6 > 0$ such that $C_{\mathrm{eff}}^{-1}\eta(\mathcal{T}) \le |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}} \le C_{\mathrm{rel}}\,\eta(\mathcal{T})$ holds for $\mathcal{T} \in \mathbb{T}(\varepsilon_6)$.*

Proof of reliability Lemma 4.3 holds for any refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of $\mathcal{T} \in \mathbb{T}(\varepsilon_5)$ and we may consider a sequence $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ of uniform mesh-refinements of $\mathcal{T}$. The reliability follows in the limit as $\widehat{h}_{\max} \to 0$ for $\ell \to \infty$ and $|||u - \widehat{u}_{\mathrm{nc}}|||_{\mathrm{pw}} \to 0$ from Theorem 3.1.c. The left-hand side of Lemma 4.3 converges to $C_6^{-1}|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}$. On the right-hand side, $\|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} \le C_\delta\widehat{h}_{\max}^{2m}$ converges to zero and $\|e\|_{L^2(\Omega)} \to \|u - u_{\mathrm{nc}}\|_{L^2(\Omega)}$ as $\widehat{h}_{\max} \to 0$. Moreover the shape-regularity $h_T \le C_{\mathrm{sr}}|T|^{1/3}$ for $T \in \mathcal{T} \in \mathbb{T}$, (J2), and $\|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)} \le 2\kappa_m^2 h_{\max}^m\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\Omega)}$ show

$$
\begin{aligned}
&\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\Omega)}^2 + \|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 + \|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 \\
&\quad \le \max\{M_1, C_{\mathrm{sr}}^{2m}(1 + 4\kappa_m^4 h_{\max}^{2m})\}\eta^2(\mathcal{T}).
\end{aligned}
$$

For the remaining term on the right-hand side, (3.12) and Corollary 2.1.a show

$$
C_5^{-1}\|u - u_{\mathrm{nc}}\|_{L^2(\Omega)} \le h_{\max}^\sigma|||u - Iu|||_{\mathrm{pw}} \le h_{\max}^\sigma|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}.
$$

A reduction to $\varepsilon_6 := \min\{\varepsilon_5, (2C_5^2C_6)^{-1/(2\sigma)}\}$ such that $C_5^2C_6h_{\max}^{2\sigma} \le 1/2$ allows for the absorption of $C_5^2C_6h_{\max}^{2\sigma}|||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2 \le |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2/2$ and concludes the proof with $C_{\mathrm{rel}}^2 := 2C_6\max\{M_1, C_{\mathrm{sr}}^{2m}(1 + 4\kappa_m^4 h_{\max}^{2m})\}$. □

Proof of efficiency The condition (J2) guarantees

$$M_1/M_2^2 \sum_{T\in\mathcal{T}} |T|^{1/3} \sum_{F\in\mathcal{F}(T)} \|[D^m_{\rm pw} u_{\rm nc}]_F \times \nu_F\|^2_{L^2(F)} \le \min_{v\in V} |||v - u_{\rm nc}|||^2_{\rm pw}$$
$$\le |||u - u_{\rm nc}|||^2_{\rm pw}.$$

The combination of $|T|^{1/3} \le h_T$, $\lambda_h \le \lambda$, and the efficiency (3.10) with $|||u - Iu|||_{\rm pw} \le |||u - u_{\rm nc}|||_{\rm pw}$ from Corollary 2.1.a implies that

$$\sum_{T\in\mathcal{T}} |T|^{2m/3} \|\lambda_h u_{\rm nc}\|^2_{L^2(T)} \le \|h^m_{\mathcal{T}} \lambda_h u_{\rm nc}\|^2_{L^2(\Omega)}$$
$$\le 2C_4^2\big(\lambda^2 h^{2m}_{\max} \|u - u_{\rm nc}\|^2_{L^2(\Omega)} + |||u - u_{\rm nc}|||^2_{\rm pw}\big).$$

Theorem 3.1.c concludes the proof with $C^2_{\rm eff} := M_2^2/M_1 + 2C_4^2 + 2C_4^2 C_0 \lambda^2 h^{2m+2\sigma}_{\max}$. □

4.2.2 Discrete reliability

Lemma 4.5 (distance control II) *There exists a constant $C_7 > 0$ such that $\|\widehat{\lambda}_h \widehat{u}_{\rm nc} - \lambda_h u_{\rm nc}\|_{L^2(\Omega)} + \|\widehat{u}_{\rm nc} - u_{\rm nc}\|_{L^2(\Omega)} + \|\widehat{\delta} \widehat{u}_{\rm nc}\|_{L^2(\Omega)} + \|\delta u_{\rm nc}\|_{L^2(\Omega)} \le C_7 h^\sigma_{\max} |||u - u_{\rm nc}|||_{\rm pw} \le C_7 C_{\rm rel} h^\sigma_{\max} \eta(\mathcal{T})$ holds for any $\mathcal{T} \in \mathbb{T}(\varepsilon_6)$.*

Proof Triangle inequalities and the normalization $\|u\|_{L^2(\Omega)} = 1$ show

$$\|\widehat{\lambda}_h \widehat{u}_{\rm nc} - \lambda_h u_{\rm nc}\|_{L^2(\Omega)} \le \lambda_h \|u - u_{\rm nc}\|_{L^2(\Omega)} + \widehat{\lambda}_h \|u - \widehat{u}_{\rm nc}\|_{L^2(\Omega)} + |\widehat{\lambda}_h - \lambda_h|.$$

Theorem 3.1.c and Corollary 2.1.b imply $|\lambda - \lambda_h| \le C_0 |||u - Iu|||^2_{\rm pw} \le C_0 (h_{\max}/\pi)^{2\sigma} \|u\|^2_{H^{m+\sigma}(\Omega)}$. Since the eigenfunction $u \in V$ in (1.1) solves the source problem with right-hand side $\lambda u \in L^2(\Omega)$, (2.1) implies $\|u\|_{H^{m+\sigma}(\Omega)} \le C(\sigma)\|\lambda u\|_{L^2(\Omega)} = C(\sigma)\lambda$. The same arguments apply to $|\lambda - \widehat{\lambda}_h|$. This and $\widehat{h}^\sigma_{\max} |||u - \widehat{I} u|||_{\rm pw} \le h^\sigma_{\max} |||u - Iu|||_{\rm pw}$ result in

$$|\widehat{\lambda}_h - \lambda_h| \le |\lambda - \lambda_h| + |\lambda - \widehat{\lambda}_h| \le 2C_0 C(\sigma)\lambda/\pi^\sigma h^\sigma_{\max} |||u - Iu|||_{\rm pw}.$$

Recall $\lambda_h, \widehat{\lambda}_h \le \lambda$, $\|\delta u_{\rm nc}\|_{L^2(\Omega)} \le C_\delta^{1/2} h^m_{\max} \|u_{\rm nc}\|_\delta$, and $\|\widehat{\delta} \widehat{u}_{\rm nc}\|_{L^2(\Omega)} \le C_\delta^{1/2} \widehat{h}^m_{\max} \|\widehat{u}_{\rm nc}\|_{\widehat{\delta}}$ from Table 1. The last two displayed estimates, a triangle inequality, and Theorem 3.1.c show

$$\|\widehat{\lambda}_h \widehat{u}_{\rm nc} - \lambda_h u_{\rm nc}\|_{L^2(\Omega)} + \|\widehat{u}_{\rm nc} - u_{\rm nc}\|_{L^2(\Omega)} + \|\widehat{\delta} \widehat{u}_{\rm nc}\|_{L^2(\Omega)} + \|\delta u_{\rm nc}\|_{L^2(\Omega)}$$
$$\le 2\big((C_0 C(\sigma)\lambda/\pi^\sigma + C_0^{1/2}(1+\lambda))h^\sigma_{\max} + (C_\delta C_0)^{1/2} h^m_{\max}\big)|||u - Iu|||_{\rm pw}$$

with $|||u - \widehat{I}u|||_{\rm pw} \le |||u - Iu|||_{\rm pw}$ and $\widehat{h}_{\max} \le h_{\max}$. Since $h_{\max} \le \varepsilon_6 < 1$ and $1/2 < \sigma \le 1 \le m$, Corollary 2.1.a concludes the proof of the first bound in Lemma 4.5

with $C_7 := 2C_0C(\sigma)\lambda/\pi^\sigma + 2C_0^{1/2}(1+\lambda+C_\delta^{1/2})$. The second claim follows from Theorem 4.4. □

Theorem 4.6 (discrete reliability) *There exist constants $\Lambda_3,\ M_3 > 0$ such that $\mathcal{T} \in \mathbb{T}(\varepsilon_6)$ with maximal mesh-size $h_{\max} \le \varepsilon_6$ (ε_6 from Theorem 4.4) and $\epsilon_3 := M_3 h_{\max}^{2\sigma}$ imply*
(A3$_\varepsilon$) Discrete reliability. $\delta^2(\mathcal{T},\widehat{\mathcal{T}}) \le \Lambda_3\eta^2(\mathcal{R}_1) + \epsilon_3\eta^2(\mathcal{T})$.

Proof Recall that Lemma 4.3 shows

$$C_6^{-1}|||\widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2 \le \|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + \|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + \|\widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 + \|\widehat{\delta u}_h\|_{L^2(\Omega)}^2 + \|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)}^2.$$

This and Lemma 4.5 lead with $M_3 := C_7^2C_{\mathrm{rel}}^2\max\{1, C_6\}$ to

$$\begin{aligned}\delta^2(\mathcal{T},\widehat{\mathcal{T}}) &= \|\widehat{\lambda}_h\widehat{u}_{\mathrm{nc}} - \lambda_h u_{\mathrm{nc}}\|_{L^2(\Omega)}^2 + |||\widehat{u}_{\mathrm{nc}} - u_{\mathrm{nc}}|||_{\mathrm{pw}}^2\\ &\le C_6\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + C_6\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 + M_3h_{\max}^{2\sigma}\eta^2(\mathcal{T}).\end{aligned}$$

The shape regularity $h_T \le C_{\mathrm{sr}}|T|^{1/3}$ for any $T \in \mathcal{T} \in \mathbb{T}$ guarantees

$$\|h_{\mathcal{T}}^m\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} \le C_{\mathrm{sr}}^m|T|^{m/3}\|\lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} \le C_{\mathrm{sr}}^m\eta(\mathcal{T}\setminus\widehat{\mathcal{T}}) \le C_{\mathrm{sr}}^m\eta(\mathcal{R}_1).$$

with $\mathcal{T}\setminus\widehat{\mathcal{T}} \subset \mathcal{R}_1$ in the last step. Remark 2.2 asserts

$$\begin{aligned}M_5^{-1}\|D_{\mathrm{pw}}^m(u_{\mathrm{nc}} - Ju_{\mathrm{nc}})\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}^2 &\le \sum_{T\in\mathcal{R}_1}|T|^{1/3}\sum_{F\in\mathcal{F}(T)}\|[D_{\mathrm{pw}}^m u_{\mathrm{nc}}]_F \times \nu_F\|_{L^2(F)}^2\\ &\le \eta^2(\mathcal{R}_1).\end{aligned}$$

The combination of the last three displayed inequalities concludes the proof of (A3$_\varepsilon$) with $\Lambda_3 := C_6(C_{\mathrm{sr}}^{2m} + M_5)$. □

4.3 Quasiorthogonality

The quasiorthogonality in Theorem 4.7 below concerns the outcome $(\mathcal{T}_j)_{j\in\mathbb{N}_0}$ of AFEM4EVP. Let $u_j \in V(\mathcal{T}_j)$ abbreviate the nonconforming component of the discrete solution $\boldsymbol{u}_j = (u_{\mathrm{pw}}, u_{\mathrm{nc}}) =: (u_{\mathrm{pw}}, u_j) \in P_m(\mathcal{T}_j) \times V(\mathcal{T}_j)$ with $b(u, u_j) > 0$, $\|u_j\|_{L^2(\Omega)} = 1$, and $\lambda_j(k) \le \lambda$ the associated eigenvalue from AFEM4EVP on the level $j \in \mathbb{N}_0$. Recall the distance

$$\delta^2(\mathcal{T}_j, \mathcal{T}_{j+1}) = \|\lambda_j(k)u_j - \lambda_{j+1}(k)u_{j+1}\|_{L^2(\Omega)}^2 + |||u_j - u_{j+1}|||_{\mathrm{pw}}^2$$

for the triangulations $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$. Set $h_0 := \max_{T\in\mathcal{T}_0} h_T$ and recall $\varepsilon_6 > 0$ from Theorem 4.4.

Theorem 4.7 (quasiorthogonality) *For any* $0 < \beta \le C_{\mathrm{eff}}^2/C_{\mathrm{rel}}^2$, *there exist* Λ_4, $\widetilde{\Lambda}_4$, *and* $\epsilon_4 := \widetilde{\Lambda}_4(\beta + h_0^{2\sigma}(1+\beta^{-1})) > 0$, *such that* $\mathcal{T}_0 \in \mathbb{T}(\varepsilon_6)$ *implies that the output* $(\eta_j)_{j\in\mathbb{N}_0}$ *and* $(\mathcal{T}_j)_{j\in\mathbb{N}_0}$ *of AFEM4EVP satisfies*

(A4$_\varepsilon$) Quasiorthogonality. $\sum_{j=\ell}^{\ell+L} \delta^2(\mathcal{T}_j, \mathcal{T}_{j+1}) \le \Lambda_4(1+\beta^{-1})\eta_\ell^2 + \epsilon_4 \sum_{j=\ell}^{\ell+L} \eta_j^2$ *for any* $\ell, L \in \mathbb{N}_0$.

The following Lemma 4.8 in the 2-level notation of Definition 4.1 prepares the proof of Theorem 4.7 below.

Lemma 4.8 (2-level quasiorthogonality) *There exists* $C_{\mathrm{qo}} > 0$ *such that, for* $\mathcal{T} \in \mathbb{T}(\varepsilon_6)$, $a_{\mathrm{pw}}(u - \widehat{u}_{\mathrm{nc}}, u_{\mathrm{nc}} - \widehat{u}_{\mathrm{nc}}) \le C_{\mathrm{qo}}\big(h_{\max}^\sigma |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}} + \|h_{\mathcal{T}}^m \lambda u\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})}\big)$ $|||u - \widehat{u}_{\mathrm{nc}}|||_{\mathrm{pw}}$ *holds.*

Proof Since $(\lambda_h, u_{\mathrm{nc}})$ (resp. $(\widehat{\lambda}_h, \widehat{u}_{\mathrm{nc}})$) solves (3.2) with respect to $\mathcal{T}$ and $\delta \in P_0(\mathcal{T})$ (resp. $\widehat{\mathcal{T}}$ and $\widehat{\delta} \in P_0(\widehat{\mathcal{T}})$ from Table 1), Corollary 2.1.c and elementary algebra show that

$$
\begin{aligned}
a_{\mathrm{pw}}(u_{\mathrm{nc}} - \widehat{u}_{\mathrm{nc}}, u - \widehat{u}_{\mathrm{nc}}) &= a_{\mathrm{pw}}\big(u_{\mathrm{nc}}, I(u - \widehat{u}_{\mathrm{nc}})\big) - a_{\mathrm{pw}}\big(\widehat{u}_{\mathrm{nc}}, \widehat{I}u - \widehat{u}_{\mathrm{nc}}\big)\\
&= b\big(\lambda_h u_{\mathrm{nc}}(1+\delta), I(u - \widehat{u}_{\mathrm{nc}})\big) - b\big(\widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}(1+\widehat{\delta}), \widehat{I}u - \widehat{u}_{\mathrm{nc}}\big)\\
&= b\big(\lambda_h u_{\mathrm{nc}}(1+\delta) - \widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}(1+\widehat{\delta}), \widehat{I}u - \widehat{u}_{\mathrm{nc}}\big) + \big(\lambda_h u_{\mathrm{nc}}, (I - \widehat{I})(u - \widehat{u}_{\mathrm{nc}})\big)_{1+\delta}.
\end{aligned} \tag{4.5}
$$

The Cauchy–Schwarz inequality, $\lambda_h, \widehat{\lambda}_h \le \lambda$, and Lemma 4.5 in the last step prove

$$
\begin{aligned}
t_1 &:= b\big(\lambda_h u_{\mathrm{nc}}(1+\delta) - \widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}(1+\widehat{\delta}), \widehat{I}u - \widehat{u}_{\mathrm{nc}}\big)\\
&\le \Big(\|\lambda_h u_{\mathrm{nc}} - \widehat{\lambda}_h \widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} + \lambda_h \|\delta u_{\mathrm{nc}}\|_{L^2(\Omega)} + \widehat{\lambda}_h \|\widehat{\delta}\widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}\Big) \|\widehat{I}u - \widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}\\
&\le \max\{1, \lambda\} C_7 h_{\max}^\sigma |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}} \|\widehat{I}u - \widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)}.
\end{aligned}
$$

The discrete Friedrichs inequality (3.23) with respect to $V(\widehat{\mathcal{T}})$, (I2), and the L^2-projection $\widehat{\Pi}_0$ onto $P_0(\widehat{\mathcal{T}})$ lead to

$$
C_{\mathrm{dF}}^{-1}\|\widehat{I}u - \widehat{u}_{\mathrm{nc}}\|_{L^2(\Omega)} \le |||\widehat{I}u - \widehat{u}_{\mathrm{nc}}|||_{\mathrm{pw}} = \|\widehat{\Pi}_0 D_{\mathrm{pw}}^m (u - \widehat{u}_{\mathrm{nc}})\|_{L^2(\Omega)} \le |||u - \widehat{u}_{\mathrm{nc}}|||_{\mathrm{pw}}.
$$

Consequently, $t_1 \le \max\{1, \lambda\} C_7 C_{\mathrm{dF}} h_{\max}^\sigma |||u - u_{\mathrm{nc}}|||_{\mathrm{pw}} |||u - \widehat{u}_{\mathrm{nc}}|||_{\mathrm{pw}}$. Since $1+\delta \le 2$ from Table 1, the arguments behind (4.3) also show

$$
t_2 := \big(\lambda_h u_{\mathrm{nc}}, (I - \widehat{I})(u - \widehat{u}_{\mathrm{nc}})\big)_{1+\delta} \le 2\|h_{\mathcal{T}}^m \lambda_h u_{\mathrm{nc}}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} \|h_{\mathcal{T}}^{-m}(I - \widehat{I})(u - \widehat{u}_{\mathrm{nc}})\|_{L^2(\Omega)}.
$$

Since (I3) implies $I(\widehat{I}u) = Iu$, (I2) and (I4) for I and (I2) for $\widehat{I}$ show $\|h_{\mathcal{T}}^{-m}(I - \widehat{I})(u - \widehat{u}_{\mathrm{nc}})\|_{L^2(\Omega)} = \|h_{\mathcal{T}}^{-m}(1 - I)(\widehat{I}u - \widehat{u}_{\mathrm{nc}})\|_{L^2(\Omega)} \le \kappa_d |||(1 - I)\widehat{I}(u - \widehat{u}_{\mathrm{nc}})|||_{\mathrm{pw}} \le \kappa_d |||u -$

$\widehat{u}_{\rm nc}|||_{\rm pw}$. On the other hand, $\lambda_h \le \lambda$, a triangle inequality, (3.12), and Corollary 2.1.a imply

$$\begin{aligned}\|h_{\mathcal{T}}^m \lambda_h u_{\rm nc}\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} &\le \|h_{\mathcal{T}}^m \lambda u\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} + \lambda h_{\max}^m \|u-u_{\rm nc}\|_{L^2(\Omega)}\\ &\le \|h_{\mathcal{T}}^m \lambda u\|_{L^2(\mathcal{T}\setminus\widehat{\mathcal{T}})} + C_5 \lambda h_{\max}^{m+\sigma} |||u-u_{\rm nc}|||_{\rm pw}.\end{aligned}$$

Hence the upper bound t_1+t_2 in (4.5) is controlled and the above estimates lead to the assertion with $C_{\rm qo} := \max\{2\kappa_d, \max\{1,\lambda\}C_7 C_{\rm dF} + 2C_5\lambda h_{\max}^m \kappa_d\}$. □

Proof of Theorem 4.7 Recall that $u_j \in V(\mathcal{T}_j)$ is the nonconforming component of the discrete solution $\boldsymbol{u}_j = (u_{\rm pw}, u_{\rm nc}) =: (u_{\rm pw}, u_j) \in P_m(\mathcal{T}_j)\times V(\mathcal{T}_j)$ and that $\lambda_j(k) \le \lambda$ is the associated eigenvalue from AFEM4EVP on the j-th level for $\ell \le j \le \ell + L$. Since $\mathcal{T}_j, \mathcal{T}_{j+1} \in \mathbb{T}(\mathcal{T}_0)$ for $\ell \le j \le \ell+L$, Lemma 4.5 shows

$$\delta^2(\mathcal{T}_j, \mathcal{T}_{j+1}) \le |||u_j - u_{j+1}|||_{\rm pw}^2 + C_7^2 C_{\rm rel}^2 h_0^{2\sigma}\eta_j^2.$$

Elementary algebra, Lemma 4.8, and two weighted Young inequalities show

$$\begin{aligned}|||u_j-u_{j+1}|||_{\rm pw}^2 - |||u-u_j|||_{\rm pw}^2 + |||u-u_{j+1}|||_{\rm pw}^2 &= 2a_{\rm pw}(u-u_{j+1}, u_j-u_{j+1})\\ &\le 2C_{\rm qo}\Big(h_0^\sigma |||u-u_j|||_{\rm pw} + \|h_{\mathcal{T}_j}^m \lambda u\|_{L^2(\mathcal{T}_j\setminus\mathcal{T}_{j+1})}\Big)|||u-u_{j+1}|||_{\rm pw}\\ &\le \frac{2C_{\rm qo}^2}{\beta} h_0^{2\sigma} C_{\rm rel}^2\eta_j^2 + \beta C_{\rm rel}^2 \eta_{j+1}^2 + \frac{2C_{\rm qo}^2}{\beta}\|h_{\mathcal{T}_j}^m\lambda u\|_{L^2(\mathcal{T}_j\setminus\mathcal{T}_{j+1})}^2\end{aligned}$$

with Theorem 4.4 in the last step. Theorem 4.4 controls the telescoping sum

$$\begin{aligned}\sum_{j=\ell}^{\ell+L}\left(|||u-u_j|||_{\rm pw}^2 - |||u-u_{j+1}|||_{\rm pw}^2\right) &= |||u-u_\ell|||_{\rm pw}^2 - |||u-u_{\ell+L+1}|||_{\rm pw}^2\\ &\le C_{\rm rel}^2\eta_\ell^2 - C_{\rm eff}^2\eta_{\ell+L+1}^2.\end{aligned}$$

Since $\beta \le C_{\rm eff}^2/C_{\rm rel}^2$ implies $(\beta C_{\rm rel}^2 - C_{\rm eff}^2)\,\eta_{\ell+L+1}^2 \le 0$, the last three displayed estimates show

$$\begin{aligned}\sum_{j=\ell}^{\ell+L}\delta^2(\mathcal{T}_j,\mathcal{T}_{j+1}) \le &\sum_{j=\ell}^{\ell+L}\left(|||u-u_j|||_{\rm pw}^2 - |||u-u_{j+1}|||_{\rm pw}^2\right)\\ &+ \left(\left(\frac{2C_{\rm qo}^2}{\beta} + C_7^2\right)h_0^{2\sigma} + \beta\right) C_{\rm rel}^2 \sum_{k=\ell}^{\ell+L}\eta_j^2\\ &+ \beta C_{\rm rel}^2\eta_{\ell+L+1}^2 + \frac{2C_{\rm qo}^2}{\beta}\sum_{j=\ell}^{\ell+L}\|h_{\mathcal{T}_j}^m \lambda u\|_{L^2(\mathcal{T}_j\setminus\mathcal{T}_{j+1})}^2\end{aligned}$$

$$\leq C_{\mathrm{rel}}^2 \eta_\ell^2 + \left(\left(\frac{2C_{\mathrm{qo}}^2}{\beta} + C_7^2 \right) h_0^{2\sigma} + \beta \right) C_{\mathrm{rel}}^2 \sum_{k=\ell}^{\ell+L} \eta_j^2 + \frac{2C_{\mathrm{qo}}^2}{\beta} \sum_{j=\ell}^{\ell+L} \| h_{\mathcal{T}_j}^m \lambda u \|_{L^2(\mathcal{T}_j \setminus \mathcal{T}_{j+1})}^2. \tag{4.6}$$

Recall that $h_{\mathcal{T}_j}|_T := \mathrm{diam}(T)$ for any $T \in \mathcal{T}_j$ and compare it with the piecewise constant function $\tilde{h}_j \in P_0(\mathcal{T}_j)$ defined by $\tilde{h}_j|_T := |T|^{1/3} \leq h_T \leq C_{\mathrm{sr}} |T|^{1/3}$ (from shape-regularity) for any $T \in \mathcal{T}_j$ and $j \in \mathbb{N}_0$. Then $\tilde{h}_j \approx h_{\mathcal{T}_j} \in P_0(\mathcal{T}_j)$ and $\tilde{h}_j \in P_0(\mathcal{T}_j)$ satisfies the reduction $\tilde{h}_{j+1} \leq \tilde{h}_j / 2^{1/3}$ a.e. in the set of refined tetrahedra $\bigcup (\mathcal{T}_j \setminus \mathcal{T}_{j+1})$. Hence $\tilde{h}_j^m \leq \frac{2^{m/3}}{\sqrt{4^{m/3}-1}} \sqrt{\tilde{h}_j^{2m} - \tilde{h}_{j+1}^{2m}}$ a.e. in $\bigcup (\mathcal{T}_j \setminus \mathcal{T}_{j+1})$ and

$$C_{\mathrm{sr}}^{-2m} \frac{4^{m/3}-1}{4^{m/3}} \sum_{j=\ell}^{\ell+L} \| h_{\mathcal{T}_j}^m \lambda u \|_{L^2(\mathcal{T}_j \setminus \mathcal{T}_{j+1})}^2 \leq \frac{4^{m/3}-1}{4^{m/3}} \sum_{j=\ell}^{\ell+L} \| \tilde{h}_j^m \lambda u \|_{L^2(\mathcal{T}_j \setminus \mathcal{T}_{j+1})}^2$$
$$\leq \sum_{j=\ell}^{\ell+L} \left\| \sqrt{\tilde{h}_j^{2m} - \tilde{h}_{j+1}^{2m}} \lambda u \right\|_{L^2(\Omega)}^2 = \int_\Omega (\tilde{h}_\ell^{2m} - \tilde{h}_{\ell+L+1}^{2m}) (\lambda u)^2 \mathrm{d}x \leq \| \tilde{h}_\ell^m \lambda u \|_{L^2(\Omega)}^2.$$

Since $\tilde{h}_\ell \leq h_{\mathcal{T}_\ell} \leq h_0 := \max_{T \in \mathcal{T}_0} h_T \leq \varepsilon_6$, a triangle inequality implies

$$\| \tilde{h}_\ell^m \lambda u \|_{L^2(\Omega)}^2 \leq 2 (\lambda / \lambda_\ell(k))^2 \| \tilde{h}_\ell^m \lambda_\ell(k) u_\ell \|_{L^2(\Omega)}^2 + 2 \lambda^2 h_0^{2m} \| u - u_\ell \|_{L^2(\Omega)}^2.$$

Theorem 3.1.a and (1.4) show $(\lambda/\lambda_\ell(k))^2 \| \tilde{h}_\ell^m \lambda_\ell(k) u_\ell \|_{L^2(\Omega)}^2 \leq 4 \eta_\ell^2$. Corollary 2.1.a, Theorem 4.4, and (3.12) imply $\| u - u_\ell \|_{L^2(\Omega)}^2 \leq h_0^{2\sigma} C_5^2 C_{\mathrm{rel}}^2 \eta_\ell^2$. The substitution in (4.6) concludes the proof with $\Lambda_4 := \max\{C_{\mathrm{rel}}^2, C_{\mathrm{qo}}^2 C_{\mathrm{sr}}^{2m} \frac{4^{m/3+1}}{4^{m/3}-1} (4 + h_0^{2m+2\sigma} C_5^2 C_{\mathrm{rel}}^2 \lambda^2)\}$ and $\widetilde{\Lambda}_4 := C_{\mathrm{rel}}^2 \max\{1, 2C_{\mathrm{qo}}^2, C_7^2\}$. □

5 Conclusion and comments

5.1 Proof of Theorem 1.1

The proven properties (A1)–(A4$_\varepsilon$) are the axioms of adaptivity in [12, 26] and known to imply (1.5). Compared to [12, 26] the discrete reliability in Theorem 4.6 is extended in that (A3$_\varepsilon$) includes the additional term $M_3 h_{\max}^{2\sigma} \eta^2(\mathcal{T})$. Minor modifications of the arguments in [12, 26] prove that (A1)–(A4$_\varepsilon$) imply (1.5). This is stated and proven as Theorem A.1 in Appendix A for some $\varepsilon := \varepsilon_8 \leq \varepsilon_6$. □

5.2 Optimal convergence rates of the error

The reliability and efficiency in Theorem 4.4 provide the equivalence $|||u - u_\ell|||_{\mathrm{pw}} \approx \eta_\ell(\mathcal{T}_\ell)$. This and Theorem 1.1 lead to optimal convergence rates for the error as well.

5.3 Global convergence

This paper on the asymptotic convergence rates justifies that a small initial mesh-size guarantees the asymptotic convergence from the beginning. Although the reasons are presented in several steps for $\varepsilon_0, \ldots, \varepsilon_8$, the computation of ε_8 may be cumbersome and a huge overestimation in practice. To guarantee global convergence without a priori knowledge of ε_8, we may modify the marking step in AFEM4EVP as follows: Enlarge the set $\mathcal{M}_\ell$ in AFEM4EVP by one tetrahedron of maximal mesh-size in $\mathcal{T}_\ell$. This guarantees that the maximal mesh-size tends to zero as the level $\ell \to \infty$. Consequently there exists some $L \in \mathbb{N}$ such that $\mathcal{T}_\ell \in \mathbb{T}(\varepsilon_8)$ for all $\ell = L, L+1, L+2, \ldots$ Relabel $\mathcal{T}_L$ by $\mathcal{T}_0$ so that Theorem 1.1 leads to optimal convergence rates for $\eta_L, \eta_{L+1}, \eta_{L+2}, \ldots$, whence for the entire outcome of the adaptive algorithm. However, the constant in the overhead control [48, Thm. 6.1] depends on $\mathcal{T}_L$ and this possibly enlarges the equivalence constants in (1.5).

5.4 Numerical experiments

Numerical experiments in [11, 24] show an asymptotic convergences of AFEM4EVP with $\theta = 0.5$ even for coarse initial triangulation and confirm the optimal convergence rates of Theorem 1.1 even for one example with a multiple eigenvalue. The extension to eigenvalue clusters requires an algorithm from [4, 29, 33]. This paper assumes *exact solve* of the algebraic eigenvalue problem (1.2), but perturbation results in numerical linear algebra [43] can be included as in [14].

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A. Appendix – A review and extension of the axioms of adaptivity

The framework (A1)–(A4$_\varepsilon$) in Sect. 4 is a modification of [12, 26] with a more general discrete reliability (A3$_\varepsilon$). Theorem A.1 below proves that the modified axioms are sufficient for optimal convergence rates of the AFEM algorithm with Dörfler marking and newest-vertex bisection [12, Algorithm 2.2]. On level $\ell \in \mathbb{N}_0$ of the general purpose adaptive algorithm AFEM there is given a regular triangulation $\mathcal{T}_\ell$ of $\Omega \subset \mathbb{R}^n$

into closed simplices and an undisplayed discrete problem with a discrete solution u_ℓ. These allow for the computation of $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$ in the step compute. The step mark uses the sum convention $\eta_\ell^2(\mathcal{M}) := \sum_{T\in\mathcal{M}} \eta_\ell^2(T)$ for any $\mathcal{M} \subseteq \mathcal{T}_\ell$ and $\eta_\ell^2 := \eta_\ell^2(\mathcal{T}_\ell)$. The selection of a set $\mathcal{M}_\ell$ with *almost minimal cardinality* in this step means that there exists a constant $\Lambda_{\mathrm{opt}} \geq 1$ such that the cardinality satisfies $|\mathcal{M}_\ell| \leq \Lambda_{\mathrm{opt}}|\mathcal{M}_\ell^\star|$, where $\mathcal{M}_\ell^\star \subset \mathcal{T}_\ell$ denotes some set of minimal cardinality $|\mathcal{M}_\ell^\star|$ with $\theta\eta_\ell^2 \leq \sum_{T\in\mathcal{M}_\ell^\star} \eta_\ell^2(T)$; cf. [12, 26, 47] for details; this is more general than in AFEM4EVP, which utilizes a minimal set M_ℓ with $\Lambda_{\mathrm{opt}} = 1$ constructed at linear cost in [44].

AFEM

Input: regular initial triangulation $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^n$ and bulk parameter $0 < \theta \leq 1$
for $\ell = 0, 1, 2, \ldots$ **do**
Solve the discrete problem for the discrete solution u_ℓ based on $\mathcal{T}_\ell$
Compute $\eta_\ell(T)$ for any $T \in \mathcal{T}_\ell$ with respect to the discrete solution
Mark almost minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with $\theta\eta_\ell^2 \leq \eta_\ell^2(\mathcal{M}_\ell)$
Refine $\mathcal{T}_\ell$ with newest vertex bisection to compute $\mathcal{T}_{\ell+1}$ with $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ **od**
Output: sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ with $(u_\ell)_{\ell\in\mathbb{N}_0}$ and $(\eta_\ell)_{\ell\in\mathbb{N}_0}$

This appendix is written in a self-contained way based on the set $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ of all admissible triangulation computed by successive newest-vertex bisection [35, 48] of a regular initial triangulation $\mathcal{T}_0$ (plus some initialization of tagged n-simplices) of the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$ into closed simplices and the subset $\mathbb{T}(\mathcal{T})$ of admissible refinements of $\mathcal{T} \in \mathbb{T}$. For $N \in \mathbb{N}_0$, set $\mathbb{T}(N) := \{\mathcal{T} \in \mathbb{T} : |\mathcal{T}| \leq |\mathcal{T}_0| + N\}$. To analyse the error estimates $\eta_\ell(\mathcal{T}_\ell)$ and their rates and in particular to compare with error estimators $\eta(\mathcal{T}, \bullet)$ for *any* admissible triangulation $\mathcal{T} \in \mathbb{T}$, we need to assume that the error estimators are computable for any $\mathcal{T} \in \mathbb{T}$. This leads to a family $\eta(\mathcal{T}, \bullet) \in \mathbb{R}^{\mathcal{T}}$ of error estimators parametrized by $\mathcal{T} \in \mathbb{T}$ with $\eta(\mathcal{T}, K) \geq 0$ for all $K \in \mathcal{T}$. For any subset $\mathcal{M} \subseteq \mathcal{T} \in \mathbb{T}$, the sum convention reads

$$\eta^2(\mathcal{T}, \mathcal{M}) := \big(\eta(\mathcal{T}, \mathcal{M})\big)^2 := \sum_{T\in\mathcal{M}} \eta^2(\mathcal{T}, T) \quad \text{and} \quad \eta^2(\mathcal{T}) := \eta(\mathcal{T}, \mathcal{T}). \tag{1}$$

For any triangulation $\mathcal{T}_\ell$ in the AFEM algorithm, we abbreviate $\eta_\ell(\bullet) := \eta(\mathcal{T}_\ell, \bullet)$ and $\eta_\ell := \eta_\ell(\mathcal{T}_\ell) \equiv \eta(\mathcal{T}_\ell, \mathcal{T}_\ell)$. Recall the Axioms (A1)–(A4$_\varepsilon$) with constants Λ_1, Λ_2, Λ_3, Λ_4, $\Lambda_{\mathrm{ref}} > 0$, $\widehat{\Lambda}_3$, ϵ_3, $\epsilon_4 \geq 0$, and $0 < \rho_2 < 1$ for convenient reading. For any $\mathcal{T} \in \mathbb{T}$ and admissible refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, there exists a set $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$ with $\mathcal{T}\backslash\widehat{\mathcal{T}} \subset \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})$ and $|\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})| \leq \Lambda_{\mathrm{ref}}|\mathcal{T}\backslash\widehat{\mathcal{T}}|$, such that $\mathcal{T} \in \mathbb{T}$, $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})$, and the output $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ and $(\eta_k)_{k\in\mathbb{N}_0}$ of AFEM satisfy (A1)–(A4$_\varepsilon$).

(A1) Stability. $\big|\eta(\mathcal{T}, \mathcal{T} \cap \widehat{\mathcal{T}}) - \eta(\widehat{\mathcal{T}}, \mathcal{T} \cap \widehat{\mathcal{T}})\big| \leq \Lambda_1\delta(\mathcal{T}, \widehat{\mathcal{T}})$.

(A2) Reduction. $\eta(\widehat{\mathcal{T}}, \widehat{\mathcal{T}}\backslash\mathcal{T}) \leq \rho_2\eta(\mathcal{T}, \mathcal{T}\backslash\widehat{\mathcal{T}}) + \Lambda_2\delta(\mathcal{T}, \widehat{\mathcal{T}})$.

(A3$_\varepsilon$) Discrete reliability. $\delta^2(\mathcal{T}, \widehat{\mathcal{T}}) \leq \Lambda_3\eta^2(\mathcal{T}, \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})) + \widehat{\Lambda}_3\eta^2(\widehat{\mathcal{T}}) + \epsilon_3\eta^2(\mathcal{T})$.

(A4$_\varepsilon$) Quasiorthogonality. $\displaystyle\sum_{j=\ell}^{\ell+m} \delta^2(\mathcal{T}_j, \mathcal{T}_{j+1}) \leq \Lambda_4\eta_\ell^2 + \epsilon_4 \sum_{j=\ell}^{\ell+m} \eta_j^2$ for any $\ell, m \in \mathbb{N}_0$.

Theorem A.1 below contains smallness assumptions for the constants $\widehat{\Lambda}_3$, ϵ_3, and ϵ_4. In a typical application such as Theorem 1.1 the quantities $\widehat{\Lambda}_3$, ϵ_3, ϵ_4 contain a power of the initial mesh-size $h_0 := \max_{T\in\mathcal{T}_0} h_T$ such that the assumptions are satisfied for a sufficiently fine initial triangulation $\mathcal{T}_0$. Given $\epsilon_3 < \Lambda_1^{-2}$, set $\Theta := (1-\Lambda_1^2\epsilon_3)/(1+\Lambda_1^2\Lambda_3)$. Any choice of μ and ξ with $0<\mu<\rho_2^{-2}-1$ and $0<\xi<(1-(1+\mu)\rho_2^2)\Theta/(1-\Theta)$ implies

$$\rho_{12} := \Theta\rho_2^2(1+\mu)+(1-\Theta)(1+\xi) < 1 \quad \text{and}$$
$$\Lambda_{12} := (1+1/\xi)\Lambda_1^2+(1+1/\mu)\Lambda_2^2 < \infty.$$

Theorem A.1 (rate optimality of the adaptive algorithm) *Suppose (A1)–(A4$_\varepsilon$) with*

$$\Lambda_1^2\epsilon_3 < 1, \quad \widehat{\Lambda}_3(\Lambda_1^2+\Lambda_2^2) < 1, \quad \epsilon_4 < (1-\rho_{12})/\Lambda_{12}, \quad \textit{and} \quad 0<\theta<\Theta.$$

The output $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ and $(\eta_\ell)_{\ell\in\mathbb{N}_0}$ of AFEM satisfy, for any $s>0$, the equivalence

$$\sup_{\ell\in\mathbb{N}_0}(1+|\mathcal{T}_\ell|-|\mathcal{T}_0|)^s\eta_\ell \approx \sup_{N\in\mathbb{N}_0}(1+N)^s \min_{\mathcal{T}\in\mathbb{T}(N)}\eta(\mathcal{T}).$$

The proof of Theorem A.1 reviews parts of the analysis in [12, 26] and focusses on the relevant extensions in Theorem A.2 and Theorem A.3 below. The following results (A12), (A4), and (2) follow verbatim as in [12, 26]: (A1)–(A2) and the Dörfler marking strategy with bulk parameter $\theta<\Theta<1$ provide the estimator reduction [26, Thm. 4.1]

$$\eta^2(\widehat{\mathcal{T}}) \le \varrho_{12}\eta^2(\mathcal{T})+\Lambda_{12}\delta^2(\mathcal{T},\widehat{\mathcal{T}}) \tag{A12}$$

for any $\mathcal{T}\in\mathbb{T}$ and any admissible refinement $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$. The estimator reduction (A12), (A4$_\varepsilon$), and $\Lambda_{\text{qo}} := \Lambda_4+\epsilon_4(1+\Lambda_{12}\Lambda_4)/(1-\rho_{12}-\epsilon_4\Lambda_{12}) > 0$ guarantee the stricter quasi-orthogonality [26, Thm. 3.1]

$$\sum_{k=\ell}^{\ell+m}\delta^2(\mathcal{T}_k,\mathcal{T}_{k+1}) \le \Lambda_{\text{qo}}\eta_\ell^2 \quad \text{for any } \ell,m\in\mathbb{N}_0. \tag{A4}$$

This and (A12) imply plain and R-linear convergence on each level for the output $(\eta_\ell)_{\ell\in\mathbb{N}_0}$ of AFEM in [26, Thm. 4.2]: The constants $\Lambda_c := (1+\Lambda_{12}\Lambda_{\text{qo}})/(1-\rho_{12}) > 0$ and $q_c := \Lambda_c/(1+\Lambda_c) < 1$ satisfy

$$\sum_{k=\ell}^{\ell+m}\eta_k^2 \le \Lambda_c\eta_\ell^2 \quad \text{and} \quad \eta_{\ell+m}^2 \le \frac{q_c^m}{1-q_c}\eta_\ell^2 \quad \text{for any } \ell,m\in\mathbb{N}_0. \tag{2}$$

On the other hand, (A1)–(A3) are sufficient for the quasimonotonicity (QM) and the comparison lemma. But the discrete reliability is relaxed in (A3$_\varepsilon$) in this paper, so the proofs of (QM) and the comparison lemma are revisited below.

Theorem A.2 (QM) *The axioms (A1), (A2), (A3$_\varepsilon$), and $\widehat{\Lambda}_3(\Lambda_1^2+\Lambda_2^2)<1$ imply the existence of $\Lambda_{\mathrm{mon}}>0$ such that $\eta(\widehat{\mathcal{T}})\le\Lambda_{\mathrm{mon}}\eta(\mathcal{T})$ holds for any $\mathcal{T}\in\mathbb{T}$ and $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$.*

Proof This proof extends [12, Lem. 3.5] and [26, Thm. 3.2]. The axioms (A1)–(A2) apply to the decomposition $\eta^2(\widehat{\mathcal{T}})=\eta^2(\widehat{\mathcal{T}},\mathcal{T}\cap\widehat{\mathcal{T}})+\eta^2(\widehat{\mathcal{T}},\widehat{\mathcal{T}}\backslash\mathcal{T})$ of the estimator of the fine triangulation $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$ and show

$$\begin{aligned}\eta^2(\widehat{\mathcal{T}})&\le\big(\eta(\mathcal{T},\mathcal{T}\cap\widehat{\mathcal{T}})+\Lambda_1\delta(\mathcal{T},\widehat{\mathcal{T}})\big)^2+\big(\rho_2\eta(\mathcal{T},\mathcal{T}\setminus\widehat{\mathcal{T}})+\Lambda_2\delta(\mathcal{T},\widehat{\mathcal{T}})\big)^2\\&\le(1+1/\alpha)\eta^2(\mathcal{T})+(1+\alpha)(\Lambda_1^2+\Lambda_2^2)\delta^2(\mathcal{T},\widehat{\mathcal{T}})\end{aligned}$$

with $(a+b)^2\le(1+\alpha)a^2+(1+1/\alpha)b^2$ for any positive a, b and $0<\alpha<\big((\Lambda_1^2+\Lambda_2^2)\widehat{\Lambda}_3\big)^{-1}-1$ in the second step. (For $\widehat{\Lambda}_3=0$, the upper bound for $0<\alpha<\infty$ is understood as infinity.) The Axiom (A3$_\varepsilon$) controls the distance $\delta^2(\mathcal{T},\widehat{\mathcal{T}})$ and leads to

$$\eta^2(\widehat{\mathcal{T}})\le\big(1+1/\alpha+(1+\alpha)(\Lambda_1^2+\Lambda_2^2)(\Lambda_3+\epsilon_3)\big)\eta^2(\mathcal{T})+(1+\alpha)(\Lambda_1^2+\Lambda_2^2)\widehat{\Lambda}_3\eta^2(\widehat{\mathcal{T}}).$$

Since $(1+\alpha)(\Lambda_1^2+\Lambda_2^2)\widehat{\Lambda}_3<1$, this proves $\eta^2(\widehat{\mathcal{T}})\le\Lambda_{\mathrm{mon}}^2\eta^2(\mathcal{T})$ for

$$\Lambda_{\mathrm{mon}}^2:=\frac{1+1/\alpha+(1+\alpha)(\Lambda_1^2+\Lambda_2^2)(\Lambda_3+\epsilon_3)}{1-(1+\alpha)(\Lambda_1^2+\Lambda_2^2)\widehat{\Lambda}_3}.$$

□

The convergence is guaranteed with (2) and the optimality requires the sufficient smallness of the bulk parameter $\theta<\Theta$ in the adaptive algorithm. This enters with the help of the comparison lemma, where some $\theta_0(\varkappa,\alpha)$ depends on parameter $\varkappa,\alpha$ that allow for $\theta\le\theta_0(\varkappa,\alpha)<\Theta$. The lemma dates back to the seminal contribution [47].

Lemma A.3 (comparison) *Suppose (QM), i.e., the axioms (A1), (A2), (A3$_\varepsilon$), and $\widehat{\Lambda}_3(\Lambda_1^2+\Lambda_2^2)<1$. Let $0<\varkappa<1$, $0<\alpha<\infty$, and let $s>0$ satisfy*

$$M:=\sup_{N\in\mathbb{N}_0}(N+1)^s\min_{\mathcal{T}\in\mathbb{T}(N)}\eta(\mathcal{T})<\infty.$$

Then for any level $\ell\in\mathbb{N}_0$, there exist $\widehat{\mathcal{T}}_\ell\in\mathbb{T}(\mathcal{T}_\ell)$ and

$$\theta_0(\alpha,\varkappa):=\frac{1-\varkappa^2\big((1+\alpha)+(1+1/\alpha)\Lambda_1^2\widehat{\Lambda}_3\big)-(1+1/\alpha)\Lambda_1^2\epsilon_3}{1+(1+1/\alpha)\Lambda_1^2\Lambda_3}<1$$

such that

(a) $\eta(\widehat{\mathcal{T}}_\ell)\le\varkappa\eta(\mathcal{T}_\ell)\le\Lambda_{\mathrm{mon}}M|\mathcal{T}_\ell\backslash\widehat{\mathcal{T}}_\ell|^{-s}$ and
(b) $\theta_0(\alpha,\varkappa)\eta^2(\mathcal{T}_\ell)\le\eta^2(\mathcal{T}_\ell,\mathcal{R}_\ell)$ with $\mathcal{T}_\ell\backslash\widehat{\mathcal{T}}_\ell\subset\mathcal{R}_\ell:=\mathcal{R}(\mathcal{T}_\ell,\widehat{\mathcal{T}}_\ell)$ and $|\mathcal{R}_\ell|\le\Lambda_{\mathrm{ref}}|\mathcal{T}_\ell\backslash\widehat{\mathcal{T}}_\ell|$.

Proof The proof of (a) is verbatim that of [12, Prop. 4.12] or that of [26, Lem. 4.3] based on the overlay control (i.e., (6) below) and Theorem A.2. It remains to modify the proofs in [12, Prop. 4.12] or [26, Lem. 4.3] for the verification of (b). Axiom (A1) and (a) imply that

$$\eta(\mathcal{T}_\ell, \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) \le \eta(\widehat{\mathcal{T}}_\ell, \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + \Lambda_1 \delta(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell) \le \varkappa \eta(\mathcal{T}_\ell) + \Lambda_1 \delta(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell). \tag{3}$$

Recall $\eta_\ell^2(\mathcal{M}_\ell) := \eta^2(\mathcal{T}_\ell, \mathcal{M}_\ell) := \sum_{T\in\mathcal{M}_\ell} \eta^2(\mathcal{T}_\ell, T)$ for any $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ and $\eta_\ell := \eta(\mathcal{T}_\ell) \equiv \eta(\mathcal{T}_\ell, \mathcal{T}_\ell)$ and abbreviate $\widehat{\eta}_\ell := \eta(\widehat{\mathcal{T}}_\ell) \equiv \eta(\widehat{\mathcal{T}}_\ell, \widehat{\mathcal{T}}_\ell)$. A weighted Young inequality with $\alpha > 0$, the Axiom (A3$_\varepsilon$) with $\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ replaced by $\mathcal{R}_\ell$ defined in (b), and (a) show that

$$\begin{aligned}\big(\varkappa\eta_\ell + \Lambda_1\delta(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)\big)^2 &\le (1+\alpha)\varkappa^2\eta_\ell^2 + (1+1/\alpha)\Lambda_1^2\big(\Lambda_3\eta_\ell^2(\mathcal{R}_\ell) + \widehat{\Lambda}_3\widehat{\eta}_\ell^2 + \epsilon_3\eta_\ell^2\big)\\ &\le (1+\alpha)\varkappa^2\eta_\ell^2 + (1+1/\alpha)\Lambda_1^2\big(\Lambda_3\eta_\ell^2(\mathcal{R}_\ell) + \widehat{\Lambda}_3\varkappa^2\eta_\ell^2 + \epsilon_3\eta_\ell^2\big).\end{aligned} \tag{4}$$

Recall $\varkappa < 1$, $\alpha > 0$, and set

$$C_a := (1+\alpha)\varkappa^2 + (1+1/\alpha)\Lambda_1^2(\epsilon_3 + \widehat{\Lambda}_3\varkappa^2) \quad \text{and} \quad C_b := (1+1/\alpha)\Lambda_1^2\Lambda_3.$$

Then the combination of (3)–(4) reads

$$\eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) \le C_a\eta_\ell^2 + C_b\eta_\ell^2(\mathcal{R}_\ell). \tag{5}$$

Since $\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell \subseteq \mathcal{R}_\ell$, the estimate (5) implies

$$\eta_\ell^2 \le \eta_\ell^2(\mathcal{R}_\ell) + \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) \le C_a\eta_\ell^2 + (1+C_b)\eta_\ell^2(\mathcal{R}_\ell).$$

This proves (b) with

$$\frac{1-C_a}{1+C_b} = \frac{1-\big((1+\alpha)\varkappa^2 + (1+1/\alpha)\Lambda_1^2(\epsilon_3+\widehat{\Lambda}_3\varkappa^2)\big)}{1+(1+1/\alpha)\Lambda_1^2\Lambda_3} = \theta_0(\varkappa,\alpha) < 1.$$

□

The proof of Theorem A.1 can be concluded as in [12, Proof of Theorem 4.1 (ii)] or [26, Section 4.3]. The function $\theta_0(\alpha, \varkappa)$ in Theorem A.3.b is bounded from above by $\lim_{\alpha\to\infty}\theta_0(0,\alpha) = (1-\Lambda_1^2\epsilon_3)/(1+\Lambda_1^2\Lambda_3)$ and there exist a choice of $0 < \varkappa < 1$ and $0 < \alpha < \infty$ such that $0 < \theta < \theta_0(\alpha, \varkappa) < \Theta$. This is the first formula on page 2655 in [26] and the remaining parts of the proof are summarized below for convenient reading and almost verbatim to Case A in [26]. The choice of θ and Theorem A.3.b show

$$\theta\eta^2(\mathcal{T}_\ell) \le \theta_0(\alpha,\varkappa)\eta^2(\mathcal{T}_\ell) \le \eta^2(\mathcal{T}_\ell, \mathcal{R}_\ell),$$

i.e., $\mathcal{R}_\ell$ satisfies the Dörfler marking condition. Recall that $\mathcal{M}_\ell$ denotes the set of marked elements on level ℓ in AFEM, while $\mathcal{M}_\ell^\star$ with $|\mathcal{M}_\ell^\star| = M_\ell$ is a minimal set of marked elements. Then there exists $\Lambda_{\text{opt}} \geq 1$ with $|\mathcal{M}_\ell| \leq \Lambda_{\text{opt}} M_\ell \leq \Lambda_{\text{opt}}|\mathcal{R}_\ell|$. The control over $\mathcal{R}_\ell := \mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ and Theorem A.3.a ensure

$$|\mathcal{R}_\ell| \leq \Lambda_{\text{ref}}|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \leq \Lambda_{\text{ref}}\big(\Lambda_{\text{mon}} M/(\varkappa\eta_\ell)\big)^{1/s}.$$

Hence $|\mathcal{M}_\ell| \leq C_c M^{1/s}\eta_\ell^{-1/s}$ with $C_c := \Lambda_{\text{opt}}\Lambda_{\text{ref}}\Lambda_{\text{mon}}^{1/s}\varkappa^{-1/s}$. One important ingredient of NVB is the overhead control [2, 48]

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq \Lambda_{\text{BDdV}} \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \tag{6}$$

with a universal constant Λ_{BDdV} that exclusively depends on $\mathcal{T}_0$. The combination of the above with the overhead control leads to

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq \Lambda_{\text{BDdV}} C_c M^{1/s} \sum_{k=0}^{\ell-1} \eta_k^{-1/s}. \tag{7}$$

The R-linear convergence (2) bounds the sum $\sum_{k=0}^{\ell-1}\eta_k^{-1/s}$ as in [26, Thm. 4.2.c]. For all $0 \leq k < \ell$, the second identity in (2) implies $\eta_k^{-1/s} \leq \eta_\ell^{-1/s} q_c^{(\ell-k)/(2s)}(1-q_c)^{-1/(2s)}$. Hence the formula for the partial sum of the geometric series shows

$$\sum_{k=0}^{\ell-1} \eta_k^{-1/s} \leq C_d \eta_\ell^{-1/s} \quad \text{with} \quad C_d := \frac{q_c^{1/(2s)}}{\big(1 - q_c^{1/(2s)}\big)(1-q_c)^{1/(2s)}}. \tag{8}$$

The combination of (7)–(8) reads $|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq \Lambda_{\text{BDdV}} C_c C_d M^{1/s}\eta_\ell^{-1/s}$. Hence $1 \leq |\mathcal{T}_\ell| - |\mathcal{T}_0|$ implies $(1 + |\mathcal{T}_\ell| - |\mathcal{T}_0|) \leq 2(|\mathcal{T}_\ell| - |\mathcal{T}_0|) \leq 2\Lambda_{\text{BDdV}} C_c C_d M^{1/s}\eta_\ell^{-1/s}$, while $|\mathcal{T}_\ell| = |\mathcal{T}_0|$ implies $1 \leq M^{1/s}\eta_\ell^{-1/s}$. This concludes the proof of

$$\eta_\ell(1 + |\mathcal{T}_\ell| - |\mathcal{T}_0|)^s \leq \max\{1, (2\Lambda_{\text{BDdV}} C_c C_d)^s\} M \quad \text{with } M := \sup_{N\in\mathbb{N}_0} (N+1)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T})$$

and so of "$\lesssim$" in Theorem A.1.

For the proof of the converse implication, assume, without loss of generality, that $0 < \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T})$ and so $0 < \eta_\ell$ for any $\ell \in \mathbb{N}_0$ with $N_\ell := |\mathcal{T}_\ell| - |\mathcal{T}_0| \leq N$. AFEM leads to $N_\ell < N_{\ell+1}$ (since no refinement only occurs for $\eta_\ell = 0$). Hence there exists a level ℓ with $N_\ell < N \leq N_{\ell+1}$ and $(N+1)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T}) \leq (N_{\ell+1} + 1)^s\eta_\ell$. On each refinement level ℓ each simplex creates at most a finite number $K(n)$ (depending only on the spatial dimension n) of children in the next level $\ell + 1$ [35]. In other words $|\mathcal{T}_{\ell+1}| \leq K(n)|\mathcal{T}_\ell|$ and $(N_{\ell+1} + 1)/(N_\ell + 1) \leq K(n) + (K(n) - 1)(|\mathcal{T}_0| - 1) \lesssim 1$.

This concludes the proof of rate optimality for AFEM in Theorem A.1. □

Proof of Theorem 1.1. The AFEM4EVP in Theorem 1.1 is a particular case with $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) := \mathcal{R}_1 := \{K \in \mathcal{T} : \exists\, T \in \mathcal{T}\backslash\widehat{\mathcal{T}} \text{ with } \operatorname{dist}(K,T) = 0\}$. Theorem 4.2, 4.6, and 4.7 guarantee (A1)–(A4$_\varepsilon$) with $\widehat{\Lambda}_3 := 0, \epsilon_3 := M_3 h_{\max}^{2\sigma}$, and $\epsilon_4 := \widetilde{\Lambda}_4(\beta + h_0^{2\sigma}(1 + 1/\beta)) > 0$. Let $\varepsilon_7 := \min\left\{\varepsilon_6, (2\Lambda_1^2 M_3)^{-1/(2\sigma)}\right\}$ such that $\epsilon_3 < \Lambda_1^{-2}$ and select ρ_{12} and Λ_{12}, then abbreviate $c_3 := (1-\rho_{12})/(2\Lambda_{12}\widetilde{\Lambda}_4)$, $\beta := \min\{C_{\mathrm{eff}}^2/C_{\mathrm{rel}}^2, c_3/2\}$, and define

$$\varepsilon := \varepsilon_8 := \min\left\{\varepsilon_7, ((c_3-\beta)/(1+1/\beta))^{1/(2\sigma)}\right\}. \tag{9}$$

Then $\widehat{\Lambda}_3(\Lambda_1^2 + \Lambda_2^2) = 0$, $\epsilon_3\Lambda_1^2 \le 1/2$, and $\epsilon_4 \le (1-\rho_{12})/(2\Lambda_{12})$ in Theorem A.1.

Remark A.4 (smallness assumptions on $\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8$) The reduction to ε_5 guarantees the best approximation result in Theorem 3.1, while $\varepsilon_6 := \min\{\varepsilon_5, (2C_5^2)^{-1/(2\sigma)}\}$ is sufficient for reliability in Theorem 4.4. Optimal rates follow with $\varepsilon := \varepsilon_8$ from (9). Since C_5 from (3.12), $c_3 := (1-\rho_{12})/(2\Lambda_{12}\widetilde{\Lambda}_4)$, and M_3 are bounded $\mathcal{O}(1)$, independent of the mesh-size, $\varepsilon_6 = \min\{\varepsilon_5, \mathcal{O}(1)\}$, $\varepsilon_7 = \min\{\varepsilon_6, \mathcal{O}(1)\}$, and $\varepsilon_8 = \min\{\varepsilon_7, \mathcal{O}(1)\}$ are *not* expected to be dramatically smaller than ε_5.

Remark A.5 (modification with global convergence) The modified algorithm of Sect. 5.3, with $\mathcal{T}_L, \mathcal{T}_{L+1}, \dots$ has no influence on the constants $1/2 \le \Theta(1 + \Lambda_1^2\Lambda_3) \le 1$, $\Lambda_4 \le \Lambda_{\mathrm{qo}} \le 2\Lambda_4 + 1/\Lambda_{12}$, $1 + (\Lambda_1^2 + \Lambda_2^2)\Lambda_3 \le \Lambda_{\mathrm{mon}}^2 \le \left(1 + \sqrt{(\Lambda_1^2 + \Lambda_2^2)(\Lambda_3 + \Lambda_1^{-2}/2)}\right)^2$. But Λ_{BDdV} in the overhead control (6) (e.g. [48, Thm. 6.1]) depends on $\mathcal{T}_L$ and could become larger (when replacing $\mathcal{T}_0$ by $\mathcal{T}_L$) and leads to larger equivalence constants in Theorem A.1. Fortunately, the asymptotic convergence rate remains optimal and the choice of θ is not affected.

Remark A.6 (parameter choice in practice) In a practical computation, we suggest uniform mesh-refinement until the eigenvalue λ_k of interest is resolved in that $5h_{\max}$ is smaller or equal the estimated wavelength of λ_k. This triangulation serves as initial triangulation in $\mathcal{T}_0$ in the modified algorithm of Sect. 5.3 with some bulk parameter θ smaller than $(1-\Lambda_1^2\Lambda_3)^{-1}$. In this way, the pre-asymptotic range is (hopefully) kept small while the asymptotic convergence rate remains optimal.

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