Galerkin approximation of holomorphic eigenvalue problems: weak T-coercivity and T-compatibility

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Abstract

We consider Galerkin approximations of eigenvalue problems for holomorphic Fredholm operator functions for which the operators do not have the structure “coercive+compact”. In this case the regularity (in the vocabulary of *discrete approximation schemes*) of Galerkin approximations is not unconditionally satisfied and the question of convergence is delicate. We report a technique to prove regularity of approximations which is applicable to a wide range of eigenvalue problems. The technique is based on the knowledge of a suitable *Test function operator*. In particular, we introduce the concepts of weak T-coercivity and T-compatibility and prove that for weakly T-coercive operators, T-compatibility of Galerkin approximations implies their regularity. Our framework can be successfully applied to analyze e.g. complex scaling/perfectly matched layer methods, problems involving sign-changing coefficients due to meta-materials and also (boundary element) approximations of Maxwell-type equations. We demonstrate the application of our framework to the Maxwell eigenvalue problem for a conductive material.

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1 Introduction

In this article we consider the approximation of eigenvalue problems for holomorphic Fredholm operator functions of the following form: find $\lambda \in \Lambda \subset \mathbb{C}$ and non-homogeneous eigenelements u in a Hilbert space X such that $A(\lambda)u = 0$, where it is assumed that $A(\cdot): \Lambda \rightarrow L(X)$ is an operator function which depends holomor-

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phically on λ and that for all $\lambda \in \Lambda$ the operator $A(\lambda): X \rightarrow X$ is linear and bounded, and satisfies a so-called weak T-coercivity condition. The analysis of approximations of eigenvalue problems for holomorphic Fredholm operator has a long history and was contributed by e.g. Anselone, Grigorieff, Jeggli, Karma, Stummel, Treuden, Vainikko and Wendland (in alphabetic order). It is usually performed in the framework of discrete approximation schemes [30] and regular approximations of operator functions [1,18]. This tradition roots in the seventies [19,25,33,34]. In this framework a complete convergence analysis and asymptotic error estimates for eigenvalues are given by Karma in [26,27]. If the discrete approximation scheme is chosen as a Galerkin scheme, then the assumptions of [26,27] reduce to a single non-trivial assumption: the regular approximation property (see Definition 5 for the meaning of regularity). If the operators are of the form “coercive+compact”, the regularity of Galerkin approximations is unconditionally satisfied [19, (32)]. However, if the operator values are not of this kind the question of reliable eigenvalue approximations is very delicate. This can already be observed for linear eigenvalue problems, see e.g. [2,4]. Thus it is little known how to prove regularity of approximations for non-injective Fredholm operators being not weakly coercive. In Theorem 1 we report a new condition on the Galerkin spaces to ensure the regularity of Galerkin approximations such that [26,27] can be applied. This condition is stronger than the classical regularity condition. However, it is satisfied for a wide variety of applications. We combine our approach with the results of [26,27] in Theorem 3. On the side, we report in Theorem 3 vii) new asymptotic error estimates on eigenspaces (which are not provided by [26,27]). Further, we demonstrate how to apply our framework to the Maxwell eigenvalue problem for a conductive material.

As preparation for the forthcoming concept of weakly T-coercive operators (operator functions) we remind the reader how Fredholmness of operators is usually established. In the case of coercive operators Fredholmness is trivial. The same holds for weakly coercive operators A , i.e. A is a compact perturbation of a coercive operator. Alternatively we may construct an isomorphism T such that T^*A is weakly coercive (T^* denotes the adjoint operator of T), which yields the Fredholmness of A . The name “T-coercivity” originates from Bonnet-Ben Dhia, Ciarlet, Zwölf [6]. The notion was introduced to analyze differential operators with sign-changing coefficients in the principal part which occur e.g. in the modeling of meta materials. The technique is also applied in the analysis of interior transmission eigenvalue problems, see e.g. [11,12]. However, as far as we know the concept goes back at least to an article [8] of Buffa, Costabel, Schwab on Maxwell’s equations (wherein $T = \theta$). For an operator A to be (weakly) T -coercive means that T^*A is already (weakly) coercive. However, in eigenvalue problems the operators will be in general not bijective (precisely at the eigenvalues). Thus the nomenclature of T-coercivity is not meaningful for our purposes and we will rely on the term *weak* T-coercivity. In general the Galerkin spaces will not be T -invariant and hence one cannot reproduce the above analysis on the approximation level. An invariance condition is indeed not necessary, but can be relaxed. We will make precise in which sense the Galerkin spaces have to interact with the operator T to ensure regularity. It will turn out that the existence of bounded linear operators T_n from the Galerkin spaces X_n to themselves such that

$$\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0, \quad \text{with} \quad \|T - T_n\|_n := \sup_{u_n \in X_n \setminus \{0\}} \frac{\|(T - T_n)u_n\|_X}{\|u_n\|_X}$$

is sufficient. We call this property “ T -compatibility”. The previous “norm” $\|\cdot\|_n$ was termed “discrete norm” by Descloux, Nassif and Rappaz [14,15] wherein it was used in a different but familiar context. In our context it was already employed by Hohage and Nannen [24] for the analysis of complex scaling/perfectly matched layer and Hardy space infinite element methods in cylindrical waveguides; and also by Bonnet-Ben Dhia, Ciarlet and Carvalho [5,10] for the analysis of finite element methods for equations which involve meta materials. Both works [5,24] prove weak T -coercivity and T -compatibility. Thus our results can directly be applied to improve the results of [24] and to establish convergence results for approximations of the eigenvalue problems related to [5]. Note that the negative material parameters in meta materials are e.g. of the kind $1 - \frac{1}{\omega^2}$ with ω^2 being the eigenvalue parameter. Hence such eigenvalue problems are indeed non-linear. Recently we applied our framework to study the approximation of such problems in [22], wherein we analyze the approximation of plasmonic electromagnetic eigenvalue problems in 2D.

However, the original motivation for this article was to provide a framework for the convergence analysis of boundary element discretizations of boundary integral formulations of Maxwell eigenvalue problems. Indeed the results of Unger [32] are based on an earlier version of this article. Although in the non-dispersive case the Maxwell eigenvalue problem is of linear nature, its formulation as boundary integral equation becomes non-linear due to the dependency of the fundamental solution on the frequency.

This article is an extract of the thesis [20]. Therein (see also [23]) the framework is applied to establish convergence results for radial complex scaling/perfectly matched layer methods for scalar resonance problems in homogeneous open systems. Although these eigenvalue problems are linear, classical theory [3,4] can’t be applied.

Indeed our presented framework is of interest even for linear eigenvalue problems for which [3,4] could be applied. It serves as an alternative and has the advantage that no transformation of the problem to a standard stencil $A - \lambda I$ by means of a solution operator is necessary. We want to mention that we apply our framework in this sense also in [21] for electromagnetic Steklov eigenvalue problems.

The remainder of this article is structured as follows. In Sect. 2 we introduce the notion of weak T -coercivity and T -compatibility. In Theorem 1 we prove that T -compatibility implies regularity. In Sect. 3 we report in Theorem 3 convergence results for $T(\cdot)$ -compatible Galerkin approximations of eigenvalue problems for weakly $T(\cdot)$ -coercive holomorphic operator functions. In Sect. 4 we apply our framework to the Maxwell eigenvalue problem for a conductive material.

2 Weak T -coercivity and T -compatibility

Let X be a Hilbert space over $\mathbb{C}$. We denote its scalar product as $\langle \cdot, \cdot \rangle_X$ and its associated norm as $\|\cdot\|_X$. Further, let $L(X)$ be the space of bounded linear operators from X to X with operator norm $\|A\|_{L(X)} := \sup_{u \in X \setminus \{0\}} \|Au\|_X / \|u\|_X$

for $A \in L(X)$. For an operator $A \in L(X)$ we denote its adjoint operator by $A^* \in L(X)$, i.e. $\langle u, A^*v \rangle_X = \langle Au, v \rangle_X$ for all $u, v \in X$. For a closed subspace $X_n \subset X$ let $L(X_n)$ be the space of bounded linear operators from X_n to X_n with norm $\|A_n\|_{L(X_n)} := \sup_{u_n \in X_n \setminus \{0\}} \|A_n u_n\|_X / \|u_n\|_X$ for $A_n \in L(X_n)$ and denote P_n the orthogonal projection from X to X_n . Henceforth we assume that $(X_n)_{n \in \mathbb{N}}$ is a sequence of closed subspaces of X such that P_n converges point-wise to the identity, i.e. $\lim_{n \rightarrow \infty} \|u - P_n u\|_X = 0$ for each $u \in X$. Further we denote $(A_n := P_n A|_{X_n} \in L(X_n))_{n \in \mathbb{N}}$ the Galerkin approximation of A .

Definition 1 Let $A, T \in L(X)$ and T be bijective. The operator A is called

1. *Coercive*, if $\inf_{u \in X \setminus \{0\}} |\langle Au, u \rangle_X| / \|u\|_X^2 > 0$,
2. *Weakly coercive*, if there exists a compact operator $K \in L(X)$ such that $A + K$ is coercive,
3. *T -Coercive* if T^*A is coercive,
4. *Weakly T -coercive* if T^*A is weakly coercive.

Due to the Lemma of Lax-Milgram every coercive operator is invertible. Every weakly T -coercive operator is Fredholm with index zero. For a (weakly) coercive operator A it is true that the Galerkin approximations $A_n = P_n A|_{X_n} \in L(X_n)$ inherit the (weak) coercivity, while for (weakly) T -coercive operators it is in general wrong.

We note that if T^*A is weakly coercive, then AT^{-1} is so too. Vice-versa, if AT is weakly coercive, then so is $T^{-*}A$. Hence we could alternatively define A to be (weakly) *right T -coercive*, if AT is (weakly) coercive. However, we stick to the former variant because it is more convenient.

For an operator $T \in L(X)$ or $T \in L(X_n)$, or a sum of such we define the “discrete norm”

$$\|T\|_n := \sup_{u_n \in X_n \setminus \{0\}} \frac{\|T u_n\|_X}{\|u_n\|_X} = \|T\|_{L(X_n, X)} = \|T P_n\|_{L(X)}.$$

Note that we use $\|\cdot\|_n$ simultaneously for operators defined on different spaces X, X_n , and hence $\|\cdot\|_n$ is only a norm on $L(X_n)$, but not on $L(X)$ (if $X_n \neq X$).

Definition 2 Consider $T \in L(X)$ and $(T_n \in L(X_n))_{n \in \mathbb{N}}$. We say that $(T_n)_{n \in \mathbb{N}}$ *converges to T in discrete norm*, if $\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0$.

We define in the following what we mean by T -compatible approximations of weakly T -coercive operators.

Definition 3 Let $A \in L(X)$ be weakly T -coercive. Then we call the sequence of Galerkin approximations $(A_n)_{n \in \mathbb{N}}$ to be *T -compatible*, if $(A_n)_{n \in \mathbb{N}}$ is a sequence of index-zero Fredholm operators and there exists a sequence of index-zero Fredholm operators $(T_n \in L(X_n))_{n \in \mathbb{N}}$ such that T_n converges to T in discrete norm: $\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0$.

Definition 4 A sequence $(u_n \in X)_{n \in \mathbb{N}}$ is said to be *compact*, if for every subsequence exists a converging subsubsequence.

Definition 5 A sequence $(A_n \in L(X_n))_{n \in \mathbb{N}}$ is called *regular*, if for every bounded sequence $(u_n \in X_n)_{n \in \mathbb{N}}$ the compactness of $(A_n u_n)_{n \in \mathbb{N}}$ already implies the compactness of $(u_n)_{n \in \mathbb{N}}$.

Next we briefly elaborate on the notion of regularity. To this end we recall that for a bijective operator $A \in L(X)$ its Galerkin approximation $(A_n)_{n \in \mathbb{N}}$ is called *stable*, if there exists $n_0 > 0$ such that A_n is invertible for each $n > n_0$ and $\sup_{n > n_0} \|A_n^{-1}\|_{L(X_n)} < \infty$. It is well known that for stable Galerkin approximations the solution $u_n \in X_n$ to $A_n u_n = P_n f$ converges to the solution $u \in X$ of $Au = f$. However for the approximation of eigenvalue problems it is necessary to approximate also non-bijective operators. In this case the notion of stability is not meaningful. E.g. if $u \in \ker A \setminus \{0\}$, then

$$\lim_{n \rightarrow \infty} \inf_{u_n \in X_n, \|u_n\|_X = 1} \|A_n u_n\|_X \leq \lim_{n \rightarrow \infty} \|A P_n u\|_X / \|P_n u\|_X = 0$$

and hence $(A_n)_{n \in \mathbb{N}}$ cannot be stable. Thus it is necessary to introduce a generalized notion of stability, i.e. regularity. Indeed, the regularity of approximations of bijective operators implies their stability: Assume that $(A_n)_{n \in \mathbb{N}}$ is not stable. Thus there exists a sequence $(u_n \in X_n)_{n \in \mathbb{N}}$ with $\|u_n\|_X = 1$ for each $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \|A_n u_n\|_X = 0$. If $(A_n)_{n \in \mathbb{N}}$ is regular, there exists a subsequence $(n(m))_{m \in \mathbb{N}}$ and $u \in X$ such that $\lim_{m \rightarrow \infty} u_{n(m)} = u$. It follows $Au = \lim_{m \rightarrow \infty} A_{n(m)} u_{n(m)} = 0$. Since A is bijective, it follows $u = 0$ which is a contradiction to $\|u\|_X = \lim_{m \rightarrow \infty} \|u_{n(m)}\|_X = 1$.

Our next goal is to prove in Theorem 1 that T -compatible Galerkin approximations of weakly T -coercive operators are regular. In preparation we formulate the next two lemmata.

Lemma 1 Let $T \in L(X) \setminus \{0\}$ and $(T_n \in L(X_n))_{n \in \mathbb{N}}$ be a sequence of operators with $\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0$. Then there exist a constant $c > 0$ and an index $n_0 \in \mathbb{N}$ such that

$$\|T_n\|_{L(X_n)}, \|T_n\|_{L(X_n)}^{-1} \leq c$$

for all $n > n_0$. If T is bijective and T_n is Fredholm with index zero for each $n \in \mathbb{N}$, then there exist a constant $c > 0$ and an index $n_0 \in \mathbb{N}$ such that T_n is also bijective for all $n > n_0$ and

$$\|(T_n)^{-1}\|_{L(X_n)} \leq c.$$

Proof Let $u_n \in X_n$. With the triangle inequality we deduce

$$\|T_n u_n\|_X \leq \|T u_n\|_X + \|(T - T_n) u_n\|_X$$

and hence

$$\|T_n\|_{L(X_n)} \leq \|T\|_{L(X)} + \|T - T_n\|_n.$$

Since $\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0$ the right hand side of the previous inequality is bounded. Similar, with the inverse triangle inequality we deduce

$$\|T_n u_n\|_X \geq \|T u_n\|_X - \|(T - T_n)u_n\|_X$$

and hence

$$\|T_n\|_{L(X_n)} \geq \|T\|_n - \|T - T_n\|_n.$$

Since $\lim_{n \rightarrow \infty} \|T\|_n = \|T\|_{L(X)} > 0$ and $\lim_{n \rightarrow \infty} \|T - T_n\|_n = 0$ it follows that there exist $n_0 > 0$ and $c > 0$ such that $\|T_n\|_n > c$ for all $n > n_0$. For the last claim let $n_0 > 0$ be such that $\|T - T_n\|_n < \frac{1}{2\|T^{-1}\|_{L(X)}}$ for all $n > n_0$. With the inverse triangle inequality and

$$\inf_{u \in X, \|u\|_X=1} \|Tu\|_X = 1/\|T^{-1}\|_{L(X)} > 0$$

it follows

$$\inf_{u_n \in X_n, \|u_n\|_X=1} \|T_n u_n\|_X \geq \inf_{u \in X, \|u\|_X=1} \|Tu\|_X - \|T - T_n\|_n \geq \frac{1}{2\|T^{-1}\|_{L(X)}}$$

for all $n > n_0$. We deduce that T_n is injective. Since T_n is a Fredholm operator with index zero its bijectivity follows. The norm estimate holds due to $\inf_{u_n \in X_n, \|u_n\|_X=1} \|T_n u_n\|_X = 1/\|T_n^{-1}\|_{L(X_n)}$. $\square$

Lemma 2 *Let $A \in L(X)$ be weakly T -coercive and $K \in L(X)$ be compact such that $T^*A + K$ is coercive. Let $(A_n)_{n \in \mathbb{N}}$ be a T -compatible Galerkin approximation of A . Then there exist $n_0 \in \mathbb{N}$ and $c > 0$, such that $A_n + P_n T^{-*}K|_{X_n} \in L(X_n)$ is invertible and*

$$\|(A_n + P_n T^{-*}K|_{X_n})^{-1}\|_{L(X_n)} \leq c$$

for all $n > n_0$. The constant c can be chosen as

$$c = \epsilon + \frac{1}{\|T\|_{L(X)} \inf_{u \in X \setminus \{0\}} |\langle (T^*A + K)u, u \rangle_X| / \|u\|_X^2}$$

for any $\epsilon > 0$.

Proof Let n be large enough such that T_n is bijective (see Lemma 1). We compute

$$\begin{aligned} & \inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle (A + T^{-*}K)u_n, v_n \rangle_X|}{\|u_n\|_X \|v_n\|_X} \\ & \geq \inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle (A + T^{-*}K)u_n, T_n v_n \rangle_X|}{\|T_n\|_{L(X_n)} \|u_n\|_X \|v_n\|_X} \end{aligned}$$

$$\begin{aligned}
&\geq \inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle (A + T^{-*}K)u_n, Tv_n \rangle_X|}{\|T_n\|_{L(X_n)} \|u_n\|_X \|v_n\|_X} \\
&\quad - \frac{\|A + T^{-*}K\|_{L(X)}}{\|T_n\|_{L(X_n)}} \|T - T_n\|_n \\
&= \inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle T^*(A + T^{-*}K)u_n, v_n \rangle_X|}{\|T_n\|_{L(X_n)} \|u_n\|_X \|v_n\|_X} \\
&\quad - \frac{\|A + T^{-*}K\|_{L(X)}}{\|T_n\|_{L(X_n)}} \|T - T_n\|_n \\
&= \inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle (T^*A + K)u_n, v_n \rangle_X|}{\|T_n\|_{L(X_n)} \|u_n\|_X \|v_n\|_X} \\
&\quad - \frac{\|A + T^{-*}K\|_{L(X)}}{\|T_n\|_{L(X_n)}} \|T - T_n\|_n \\
&\geq \tilde{c} \|T_n\|_{L(X_n)}^{-1} - \frac{\|A + T^{-*}K\|_{L(X)}}{\|T_n\|_{L(X_n)}} \|T - T_n\|_n
\end{aligned}$$

with coercivity constant

$$\tilde{c} := \inf_{u \in X \setminus \{0\}} |\langle (T^*A + K)u, u \rangle_X| / \|u\|_X^2 > 0.$$

Since $\|T_n\|_{L(X_n)}$ is uniformly bounded from above and below (see Lemma 1) and T_n converges to T in discrete norm by assumption, it follows the existence of $n_0 \in \mathbb{N}$ and $c > 0$ such that

$$\inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle (A + T^{-*}K)u_n, v_n \rangle_X|}{\|u_n\|_X \|v_n\|_X} \geq c$$

for all $n > n_0$. Hence $A_n + P_n T^{-*}K|_{X_n}$ is injective. Since A_n is Fredholm with index zero and K is compact, $A_n + P_n T^{-*}K|_{X_n}$ is Fredholm with index zero too. Thus $A_n + P_n T^{-*}K|_{X_n}$ is bijective. The norm estimate follows now from

$$\begin{aligned}
\inf_{u_n \in X_n \setminus \{0\}} \sup_{v_n \in X_n \setminus \{0\}} \frac{|\langle B_n u_n, v_n \rangle_X|}{\|u_n\|_X \|v_n\|_X} &= \inf_{u_n \in X_n \setminus \{0\}} \frac{\|B_n u_n\|_X}{\|u_n\|_X} \\
&= \left(\sup_{u_n \in X_n \setminus \{0\}} \frac{\|u_n\|_X}{\|B_n u_n\|_X} \right)^{-1} \\
&= \|B_n^{-1}\|_{L(X_n)}^{-1}
\end{aligned}$$

for any bijective $B_n \in L(X_n)$. □

Theorem 1 *Let $A \in L(X)$ be weakly T -coercive and $(A_n)_{n \in \mathbb{N}}$ be a T -compatible Galerkin approximation. Then $(A_n)_{n \in \mathbb{N}}$ is regular.*

Proof Without loss of generality let $(u_n \in L(X_n))_{n \in \mathbb{N}}$ be a bounded sequence, $(A_n u_n)_{n \in \mathbb{N}}$ and $u' \in X$ be such that $\lim_{n \rightarrow \infty} A_n u_n = u'$. Let $K \in L(X)$ be compact such that $T^*A + K$ is coercive. Let $\tilde{A} := A + T^{-*}K$ and $\tilde{A}_n := P_n \tilde{A}|_{X_n}$. Since K is compact and $(u_n)_{n \in \mathbb{N}}$ is bounded, there exist a subsequence $(u_{n(m)})_{m \in \mathbb{N}}$ and $u'' \in X$ such that $\lim_{m \rightarrow \infty} T^{-*}K u_{n(m)} = u''$. It follows

$$\lim_{m \rightarrow \infty} \tilde{A}_{n(m)} u_{n(m)} = u' + u''.$$

Due to Lemma 2 there exist $c > 0$ and $m_0 \in \mathbb{N}$, such that for all $m > m_0$ the operator $\tilde{A}_{n(m)}$ is invertible and $\|\tilde{A}_{n(m)}^{-1}\|_{L(X_{n(m)})} \leq c$. For $m > m_0$ we compute

$$\begin{aligned} \|u_{n(m)} - \tilde{A}^{-1}(u' + u'')\|_X &\leq \|u_{n(m)} - P_{n(m)} \tilde{A}^{-1}(u' + u'')\|_X + \|(I - P_{n(m)}) \tilde{A}^{-1}(u' + u'')\|_X \\ &\leq c \|\tilde{A}_{n(m)} u_{n(m)} - \tilde{A}_{n(m)} P_{n(m)} \tilde{A}^{-1}(u' + u'')\|_X \\ &\quad + \|(I - P_{n(m)}) \tilde{A}^{-1}(u' + u'')\|_X \\ &\leq c \|\tilde{A}_{n(m)} u_{n(m)} - (u' + u'')\|_X \\ &\quad + c \|(I - \tilde{A}_{n(m)} P_{n(m)} \tilde{A}^{-1})(u' + u'')\|_X \\ &\quad + \|(I - P_{n(m)}) \tilde{A}^{-1}(u' + u'')\|_X. \end{aligned}$$

The first term on the right hand side of the latter inequality converges to zero, as previously discussed. The third term converges to zero, because $(P_{n(m)})_{m \in \mathbb{N}}$ converges point-wise to the identity. The second term can be estimated as

$$\begin{aligned} \|(I - \tilde{A}_{n(m)} P_{n(m)} \tilde{A}^{-1})(u' + u'')\|_X &= \|(I - P_{n(m)} \tilde{A} P_{n(m)} \tilde{A}^{-1})(u' + u'')\|_X \\ &\leq \|(I - P_{n(m)})(u' + u'')\|_X \\ &\quad + \|\tilde{A}\|_{L(X)} \|(I - P_{n(m)}) \tilde{A}^{-1}(u' + u'')\|_X \end{aligned}$$

and converges to zero, because $(P_{n(m)})_{m \in \mathbb{N}}$ converges point-wise to the identity. Hence $\lim_{m \rightarrow \infty} u_{n(m)} = \tilde{A}^{-1}(u' + u'')$. $\square$

3 T-Compatible approximation of holomorphic eigenvalue problems

First let us recall the general theory on holomorphic (Fredholm) operator functions. We refer the reader e.g. to [17] and [28, Appendix]. Let $\Lambda \subset \mathbb{C}$ be an open, connected and non-empty subset of $\mathbb{C}$. Let $A(\cdot): \Lambda \rightarrow L(X)$ be an operator function. An operator function $A(\cdot)$ is called holomorphic, if it is complex differentiable. An operator function $A(\cdot)$ is called Fredholm, if $A(\lambda)$ is Fredholm for each $\lambda \in \Lambda$. We denote the resolvent set and spectrum of an operator function $A(\cdot): \Lambda \rightarrow L(X)$ as

$$\rho(A(\cdot)) := \{\lambda \in \Lambda : A(\lambda) \text{ is invertible}\} \quad \text{and} \quad \sigma(A(\cdot)) := \Lambda \setminus \rho(A(\cdot)).$$

For an operator function $A(\cdot): \Lambda \rightarrow L(X)$ we denote by $A^*(\cdot)$ the operator function defined by $A^*(\lambda) := A(\lambda)^*$ for each $\lambda \in \Lambda$ and by $A^{-1}(\cdot): \rho(A(\cdot)) \rightarrow L(X)$ the operator function defined by $A^{-1}(\lambda) := A(\lambda)^{-1}$ for each $\lambda \in \rho(A(\cdot))$. In addition, we note that for a holomorphic operator function $A(\cdot): \Lambda \rightarrow L(X)$ the operator function defined by $\lambda \mapsto A^*(\bar{\lambda})$ is holomorphic as well. Further, we denote by $A^{(j)}(\cdot): \Lambda \rightarrow L(X)$ the j^{th} derivative of a holomorphic operator function $A(\cdot): \Lambda \rightarrow L(X)$. It is well known (see e.g. [16, Theorem 8.2]) that for a holomorphic Fredholm operator function $A(\cdot): \Lambda \rightarrow L(X)$ such that $A(\lambda)$ is bijective for at least one $\lambda \in \Lambda$, the spectrum $\sigma(A(\cdot))$ is discrete, has no accumulation points in Λ and every $\lambda \in \sigma(A(\cdot))$ is an eigenvalue. That is, there exists $u \in X$ such that $A(\lambda)u = 0$. In this case we call u an eigenelement. An ordered collection of elements $(u_0, u_1, \dots, u_{m-1})$ in X is called a Jordan chain at λ if u_0 is an eigenelement corresponding to λ and if

$$\sum_{j=0}^l \frac{1}{j!} A^{(j)}(\lambda) u_{l-j} \quad \text{for } l = 0, 1, \dots, m-1.$$

The elements of a Jordan chain are called generalized eigenelements and the closed linear hull of all generalized eigenelements of $A(\cdot)$ at λ is called the generalized eigenspace $G(A(\cdot), \lambda)$ for $A(\cdot)$ at λ . For an eigenelement $u \in \ker A(\lambda) \setminus \{0\}$ we denote by $\varkappa(A(\cdot), \lambda, u)$ the maximal length of a Jordan chain at λ beginning with u and

$$\varkappa(A(\cdot), \lambda) := \max_{u \in \ker A(\lambda) \setminus \{0\}} \varkappa(A(\cdot), \lambda, u).$$

The maximal length of a Jordan chain $\varkappa(A(\cdot), \lambda)$ is always finite, see e.g. [28, Lemma A.8.3]. Next we generalize Definitions 1, 3, 5 and Theorem 1 to operator functions. For an operator function $A(\cdot): \Lambda \rightarrow L(X)$ we call $(A_n(\cdot) := P_n A(\cdot)|_{X_n}: \Lambda \rightarrow L(X_n))_{n \in \mathbb{N}}$ the Galerkin approximation of $A(\cdot)$.

Definition 6 Let $A(\cdot), T(\cdot): \Lambda \rightarrow L(X)$ be operator functions and $\rho(T(\cdot)) = \Lambda$. $A(\cdot)$ is (weakly) $(T(\cdot))$ -coercive, if $A(\lambda)$ is (weakly) $(T(\lambda))$ -coercive for each $\lambda \in \Lambda$.

Definition 7 Let $A(\cdot): \Lambda \rightarrow L(X)$ be weakly $T(\cdot)$ -coercive. The sequence of Galerkin approximations $(A_n(\cdot))_{n \in \mathbb{N}}$ is $T(\cdot)$ -compatible, if $(A_n(\lambda))_{n \in \mathbb{N}}$ is $T(\lambda)$ compatible for each $\lambda \in \Lambda$.

Definition 8 Let $A(\cdot): \Lambda \rightarrow L(X)$ be an operator function. The sequence of Galerkin approximations $(A_n(\cdot))_{n \in \mathbb{N}}$ is regular, if $(A_n(\lambda))_{n \in \mathbb{N}}$ is regular for each $\lambda \in \Lambda$.

Theorem 2 Let $A(\cdot): \Lambda \rightarrow L(X)$ be weakly $T(\cdot)$ -coercive and $(A_n(\cdot))_{n \in \mathbb{N}}$ be a $T(\cdot)$ -compatible Galerkin approximation. Then $(A_n(\cdot))_{n \in \mathbb{N}}$ is regular.

Proof Follows from Theorem 1. □

Our forthcoming results will heavily rely on [26,27], which's theory is formulated in a very generalized sense of approximations, i.e. *discrete approximation schemes*. However, we will need only the restricted case of Galerkin approximations. In particular the theory of [26,27] applies the following terms. Let $U, V, U_n, V_n, n \in \mathbb{N}$ be

Banach spaces and $p_n: U \rightarrow U_n, q_n: V \rightarrow V_n$. Note that it is neither required that $U_n \subset U$ nor $V_n \subset V$ and that these spaces are only connected through the mappings p_n and q_n . Further note that it is neither required that p_n and q_n are linear or bounded. Instead one assumes the following compatibility conditions

- (a1) $\lim_{n \rightarrow \infty} \|p_n u\|_{U_n} = \|u\|_U$ for all $u \in U$,
- (a2) $\lim_{n \rightarrow \infty} \|q_n v\|_{V_n} = \|v\|_V$ for all $v \in V$,
- (a3) $\lim_{n \rightarrow \infty} \|p_n(\alpha u + \alpha' u') - (\alpha p_n u + \alpha' p_n u')\|_{U_n} = 0$ for all $u, u' \in U, \alpha, \alpha' \in \mathbb{C}$,
- (a4) $\lim_{n \rightarrow \infty} \|q_n(\alpha v + \alpha' v') - (\alpha q_n v + \alpha' q_n v')\|_{V_n} = 0$ for all $v, v' \in V, \alpha, \alpha' \in \mathbb{C}$.

Then the convergence of the sequence $(u_n)_{n \in \mathbb{N}}, u_n \in U_n$ to $u \in U$ is defined as

$$\lim_{n \rightarrow \infty}^* u_n = u : \Leftrightarrow \lim_{n \rightarrow \infty} \|p_n u - u_n\|_{U_n} = 0,$$

and like-wise for $v_n \in V_n$ and $v \in V$

$$\lim_{n \rightarrow \infty}^* v_n = v : \Leftrightarrow \lim_{n \rightarrow \infty} \|q_n v - v_n\|_{V_n} = 0.$$

Similarly a sequence $(u_n \in U_n)_{n \in \mathbb{N}}$ is defined to be compact, if for every subsequence exists in turn a converging subsubsequence in the above sense. Consider now the approximation of $A \in L(U, V)$ by a sequence of operators $(B_n \in L(U_n, V_n))_{n \in \mathbb{N}}$. Then, by definition,

- (b1) $(B_n)_{n \in \mathbb{N}}$ approximates A , if for each $u \in U$ it holds $\lim_{n \rightarrow \infty} B_n p_n u = Au$,
- (b2) $(B_n)_{n \in \mathbb{N}}$ is regular, if for every bounded sequence $(u_n \in U_n)_{n \in \mathbb{N}}$ the compactness of $(B_n u_n)_{n \in \mathbb{N}}$ implies the compactness of $(u_n \in U_n)_{n \in \mathbb{N}}$.

Consider now a holomorphic operator function $A(\cdot): \Lambda \rightarrow L(U, V)$ and its approximation by a sequence $(B_n(\cdot): \Lambda \rightarrow L(U_n, V_n))_{n \in \mathbb{N}}$. Then, by definition,

- (c1) $(B_n(\cdot))_{n \in \mathbb{N}}$ is equibounded, if for every compact $\tilde{\Lambda} \subset \Lambda$ exists $c > 0$ such that $\|B_n(\lambda)\|_{L(U_n, V_n)} \leq c$ for all $\lambda \in \tilde{\Lambda}, n \in \mathbb{N}$.

In our particular case of interest U and V coincide and are, in addition, a Hilbert space X , the approximation spaces U_n and V_n coincide and are, in addition, a subspace $X_n \subset X$, p_n and q_n coincide and are, in addition, the orthogonal projection P_n , and $B_n(\cdot)$ is the Galerkin approximation $A_n(\cdot) = P_n A(\cdot)|_{X_n}$. It follows with basic properties of the orthogonal projection P_n that (a1)-(a4) and (b1) are satisfied. Since $A(\cdot)$ is holomorphic it is continuous and thus (c1) follows easily too.

Theorem 3 *Let $\Lambda \subset \mathbb{C}$ be open, connected and non-empty, X be a Hilbert space and $L(X)$ be the space of bounded linear operators from X to X . Let $A(\cdot): \Lambda \rightarrow L(X)$ be a holomorphic weakly $T(\cdot)$ -coercive operator function (see Definition 6) with non-empty resolvent set $\rho(A(\cdot)) \neq \emptyset$. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of closed subspaces of X with orthogonal projections P_n onto X_n , such that $(P_n)_{n \in \mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n \rightarrow \infty} \|u - P_n u\|_X = 0$ for each $u \in X$. Let $A_n(\cdot): \Lambda \rightarrow L(X_n)$ be the Galerkin approximation of $A(\cdot)$ defined by $A_n(\lambda) := P_n A(\lambda)|_{X_n}$ for each $\lambda \in \Lambda$. Assume that $A_n(\cdot)$ is $T(\cdot)$ -compatible (see Definition 7). Then the following results hold.*

- i) For every eigenvalue λ_0 of $A(\cdot)$ exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ converging to λ_0 with λ_n being an eigenvalue of $A_n(\cdot)$ for almost all $n \in \mathbb{N}$.
 ii) Let $(\lambda_n, u_n)_{n \in \mathbb{N}}$ be a sequence of normalized eigenpairs of $A_n(\cdot)$, i.e.

$$A_n(\lambda_n)u_n = 0,$$

and $\|u_n\|_X = 1$, so that $\lambda_n \rightarrow \lambda_0 \in \Lambda$, then

- a) λ_0 is an eigenvalue of $A(\cdot)$,
 b) $(u_n)_{n \in \mathbb{N}}$ is a compact sequence and its cluster points are normalized eigenelements of $A(\lambda_0)$.
 iii) For every compact $\tilde{\Lambda} \subset \rho(A)$ the sequence $(A_n(\cdot))_{n \in \mathbb{N}}$ is stable on $\tilde{\Lambda}$, i.e. there exist $n_0 \in \mathbb{N}$ and $c > 0$ such that $\|A_n(\lambda)^{-1}\|_{L(X_n)} \leq c$ for all $n > n_0$ and all $\lambda \in \tilde{\Lambda}$.
 iv) For every compact $\tilde{\Lambda} \subset \Lambda$ with $\tilde{\Lambda} \cap \sigma(A(\cdot)) = \{\lambda_0\}$ and rectifiable boundary $\partial \tilde{\Lambda} \subset \rho(A(\cdot))$ exists an index $n_0 \in \mathbb{N}$ such that

$$\dim G(A(\cdot), \lambda_0) = \sum_{\lambda_n \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}} \dim G(A_n(\cdot), \lambda_n).$$

for all $n > n_0$, whereby $G(B(\cdot), \lambda)$ denotes the generalized eigenspace of an operator function $B(\cdot)$ at $\lambda \in \Lambda$.

Let $\tilde{\Lambda} \subset \Lambda$ be a compact set with rectifiable boundary $\partial \tilde{\Lambda} \subset \rho(A(\cdot))$, $\tilde{\Lambda} \cap \sigma(A(\cdot)) = \{\lambda_0\}$ and

$$\delta_n := \max_{\substack{u_0 \in G(A(\cdot), \lambda_0) \\ \|u_0\|_X \leq 1}} \inf_{u_n \in X_n} \|u_0 - u_n\|_X,$$

$$\delta_n^* := \max_{\substack{u_0 \in G(A^*(\cdot), \lambda_0) \\ \|u_0\|_X \leq 1}} \inf_{u_n \in X_n} \|u_0 - u_n\|_X,$$

whereby $\overline{\lambda_0}$ denotes the complex conjugate of λ_0 and $A^*(\cdot)$ the adjoint operator function of $A(\cdot)$ defined by $A^*(\lambda) := A(\lambda)^*$ for each $\lambda \in \Lambda$. Then there exist $n_0 \in \mathbb{N}$ and $c > 0$ such that for all $n > n_0$

v)

$$|\lambda_0 - \lambda_n| \leq c(\delta_n \delta_n^*)^{1/\varkappa(A(\cdot), \lambda_0)}$$

for all $\lambda_n \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}$, whereby $\varkappa(A(\cdot), \lambda_0)$ denotes the maximal length of a Jordan chain of $A(\cdot)$ at the eigenvalue λ_0 .

vi)

$$|\lambda_0 - \lambda_n^{\text{mean}}| \leq c \delta_n \delta_n^*$$

whereby λ_n^{mean} is the weighted mean of all the eigenvalues of $A_n(\cdot)$ in $\tilde{\Lambda}$

$$\lambda_n^{\text{mean}} := \sum_{\lambda \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}} \lambda \frac{\dim G(A_n(\cdot), \lambda)}{\dim G(A(\cdot), \lambda_0)}.$$

vii)

$$\begin{aligned} \inf_{u_0 \in \ker A(\lambda_0)} \|u_n - u_0\|_X &\leq c \left(|\lambda_n - \lambda_0| + \max_{\substack{u'_0 \in \ker A(\lambda_0) \\ \|u'_0\|_X \leq 1}} \inf_{u'_n \in X_n} \|u'_0 - u'_n\|_X \right) \\ &\leq c \left(c(\delta_n \delta_n^*)^{1/\varkappa(A(\cdot), \lambda_0)} + \delta_n \right) \end{aligned}$$

for all $\lambda_n \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}$ and all $u_n \in \ker A_n(\lambda_n)$ with $\|u_n\|_X = 1$.

Proof *Proof of i)–iii).* The first three claims i)–iii) follow from [26, Theorem 2]. In particular the assumptions of [26, Theorem 2] are

- (d1) $\rho(A(\cdot))$ is non-empty,
- (d2) $A_n(\lambda)$ is Fredholm with index zero for each $\lambda \in \Lambda$, $n \in \mathbb{N}$,
- (d3) $(A_n(\cdot))_{n \in \mathbb{N}}$ is equibounded (i.e. (c1) holds),
- (d4) $(A_n(\lambda))_{n \in \mathbb{N}}$ approximates $A(\lambda)$ for each $\lambda \in \Lambda$ (i.e. (b1) holds),
- (d5) $(A_n(\lambda))_{n \in \mathbb{N}}$ is regular for each $\lambda \in \Lambda$ (i.e. (b2) holds).

The first two assumptions are also assumptions of this theorem. The second two assumptions are satisfied, because our particular discrete approximation scheme is a Galerkin scheme. The fifth assumption follows from Theorem 2. To enable a more self-sufficient reading of this article we include the explicit proofs of i)–iii) at this point.

Direct proof of ii). Since $\lim_{n \rightarrow \infty} \|A(\lambda_0) - A(\lambda_n)\|_{L(X)} = 0$ and by assumption $A_n(\lambda_n)u_n = 0$, it follows $\lim_{n \rightarrow \infty} A_n(\lambda_0)u_n = 0$. Due to the regularity of $A_n(\lambda_0)$, there exist $u \in X$ and a subsequence $(n(m))_{m \in \mathbb{N}}$ such that $\lim_{m \rightarrow \infty} \|u - u_{n(m)}\|_X = 0$. Since $\|u_{n(m)}\|_X = 1$ for each $m \in \mathbb{N}$, it follows $\|u\|_X = 1$. Together with the pointwise convergence of $A_{n(m)}(\lambda_0)$ we deduce

$$A(\lambda_0)u = \lim_{m \rightarrow \infty} A_{n(m)}(\lambda_{n(m)})u_{n(m)} = 0.$$

Hence $A(\lambda_0)u = 0$ and $\|u\|_X = 1$, i.e. (λ_0, u) is a normalized eigenpair of $A(\cdot)$. The very same technique allows to choose for any arbitrary subsequence $(n(m))_{m \in \mathbb{N}}$ a normalized $u \in X$ and a subsequence $(m(k))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} u_{n(m(k))} = u$ and $A(\lambda_0)u = 0$. Hence $(u_n)_{n \in \mathbb{N}}$ is a compact sequence its cluster points are normalized eigenelements.

Direct proof of iii). Assume the contrary. Then there exist a sequence $(\lambda_n \in \tilde{\Lambda})_{n \in \mathbb{N}}$ and a normalized sequence $(u_n \in X)_{n \in \mathbb{N}}$ with

$$\lim_{n \rightarrow \infty} \|A_n(\lambda_n)u_n\|_X = 0.$$

Since $\tilde{\Lambda}$ is compact there exist $\lambda_0 \in \tilde{\Lambda}$ and a subsequence $(n(m))_{m \in \mathbb{N}}$ such that $\lim_{m \rightarrow \infty} \lambda_{n(m)} = \lambda_0$. As $A_n(\lambda_0)$ is regular there exist normalized $u \in X$ and a subsequence $(m(k))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} u_{n(m(k))} = u$. Thus

$$A(\lambda_0)u = \lim_{k \rightarrow \infty} A_{n(m(k))}(\lambda_{n(m(k))})u_{n(m(k))} = 0$$

due to the continuity of $A(\cdot)$. Hence $\lambda_0 \in \tilde{\Lambda}$ is an eigenvalue to $A(\cdot)$, which is a contradiction to $\tilde{\Lambda} \subset \rho(A(\cdot))$.

Direct proof of i). Assume the contrary. Then there exist $\delta > 0$ and a subsequence $(n(m))_{m \in \mathbb{N}}$ such that $B_\delta := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\} \subset \rho(A_{n(m)}(\cdot))$. In addition we choose δ small enough such that $\Gamma := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \delta\} \subset \rho(A(\cdot))$. Due to 3) there exist constants $m_0, c > 0$ such that $\Gamma \subset \rho(A_{n(m)}(\cdot))$ and $\|A_{n(m)}^{-1}(\lambda)\|_{L(X_{n(m)})} \leq c$ for all $m > m_0, \lambda \in \Gamma$. Since $A_{n(m)}^{-1}(\cdot)$ is holomorphic in B_δ the principle of maximum of modulus yields $\|A_{n(m)}^{-1}(\lambda)\|_{L(X_{n(m)})} \leq c$ for all $\lambda \in B_\delta, m > m_0$. Thus for $u \in \ker A(\lambda_0)$, $\|u\|_X = 1$ and $m > m_0$ it follows

$$\|P_{n(m)}u\|_X = \|A_{n(m)}^{-1}(\lambda)A_{n(m)}(\lambda)P_{n(m)}u\|_X \leq c\|A_{n(m)}(\lambda)P_{n(m)}u\|_X$$

and hence $\lim_{m \rightarrow \infty} \|P_{n(m)}u\|_X = 0$, which is a contradiction to $\|u\|_X = 1$.

Proof of iv)-vi). The second three claims 3)-3) follow from [26, Theorem 3] and [27, Theorem 2, Theorem 3]. In particular the assumptions of [26, Theorem 3] are in addition to those of [26, Theorem 2] the existence of $c > 0$ and $r_n \in L(X_n, X)$, $n \in \mathbb{N}$ such that $\|r_n\|_{L(X_n, X)} \leq c$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} r_n P_n u = u$ for each $u \in X$. Since our particular discrete approximation scheme is a Galerkin scheme, we can simply choose r_n to be the inclusion operator $u_n \mapsto u_n$. [27, Theorem 2] assumes (a1)-(a4), (d1)-(d5) and two mappings p'_n, q'_n which we can choose as P_n . For [27, Theorem 3] we can again choose r_n as the inclusion map and $q'_n = P_n$. The proofs of [26, Theorem 3] and [27, Theorem 2, Theorem 3] are a bit technical and essentially consist of two steps. First one constructs operator functions which act on finite dimensional spaces and which locally have the same spectrum and Jordan chains as $A(\cdot)$ and $A_n(\cdot)$. In a second step one proves the desired claims for matrix functions.

Proof of vii). The last claim vii) is essentially Theorem 4.3.7 of [31]. However, [31, Theorem 4.3.7] considers only operator functions $A(\cdot)$ of a special category, which prohibits its direct application for our purpose. Thus we repeat the proof of [31, Theorem 4.3.7] whereby we achieve some simplification and completion. Without loss of generality we assume that $\tilde{\Lambda} \cap \sigma(A_n(\cdot)) \neq \emptyset$ for all $n \in \mathbb{N}$. Let $(\lambda_n, u_n)_{n \in \mathbb{N}}$ be a sequence of normalized eigenpairs of $A_n(\cdot)$ with $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ such that

$$\inf_{u \in \ker A(\lambda)} \|u_n - u\|_X = \max_{\tilde{\lambda}_n \in \tilde{\Lambda} \cap \sigma(A_n(\cdot)), \tilde{u}_n \in \ker A_n(\tilde{\lambda}_n)} \inf_{u \in \ker A(\lambda)} \|\tilde{u}_n - u\|_X. \quad (1)$$

Note that u_n exists since $\{\tilde{u}_n : \tilde{\lambda}_n \in \tilde{\Lambda} \cap \sigma(A_n(\cdot)), \tilde{u}_n \in \ker A_n(\tilde{\lambda}_n)\}$ is finite dimensional for each $n \in \mathbb{N}$. This explicit choice ensures the independence of the forthcoming constant c_0 on $\tilde{u}_n \in \{\tilde{u}_n : \tilde{\lambda}_n \in \tilde{\Lambda} \cap \sigma(A_n(\cdot)), \tilde{u}_n \in \ker A_n(\tilde{\lambda}_n)\}$. We

introduce two auxiliary sequences $(v_n \in \ker A(\lambda))_{n \in \mathbb{N}}$ and $(w_n \in \ker A(\lambda))_{n \in \mathbb{N}}$ which satisfy

$$\|u_n - v_n\|_X = \min_{v \in \ker A(\lambda)} \|u_n - v\|_X$$

and

$$\|u_n - P_n w_n\|_X = \min_{w \in P_n \ker A(\lambda)} \|u_n - w\|_X.$$

Note that v_n and w_n exist since $\ker A(\lambda)$ and $P_n \ker A(\lambda)$ are finite dimensional. The introduction of $(w_n)_{n \in \mathbb{N}}$ allows us to estimate

$$\inf_{u \in \ker A(\lambda)} \|u_n - u\|_X \leq \|u_n - w_n\|_X \leq \|u_n - P_n w_n\|_X + \|w_n - P_n w_n\|_X. \quad (2)$$

We estimate the two terms in the right hand side of (2) separately.

1. part: $(w_n)_{n \in \mathbb{N}}$ is bounded. It follows from ii) that

$$\lim_{n \rightarrow \infty} \|u_n - v_n\|_X = 0. \quad (3)$$

Further, we estimate

$$\|u_n - P_n w_n\|_X \leq \|u_n - P_n v_n\|_X \leq \|u_n - v_n\|_X + \|v_n - P_n v_n\|_X. \quad (4)$$

The first term in the right hand side of (4) tends to zero due to (3). Since $\ker A(\lambda)$ is finite dimensional it follows that $\lim_{n \rightarrow \infty} (I - P_n)|_{\ker A(\lambda)} = 0$ in operator norm and due to $v_n \in \ker A(\lambda)$ the second term in the right hand side of (4) tends to zero as well. Hence

$$\lim_{n \rightarrow \infty} \|u_n - P_n w_n\|_X = 0 \quad (5)$$

and since $(u_n)_{n \in \mathbb{N}}$ is bounded it follows that $(P_n w_n)_{n \in \mathbb{N}}$ is bounded too. Due to $w_n \in \ker A(\lambda)$ and $\lim_{n \rightarrow \infty} (I - P_n)|_{\ker A(\lambda)} = 0$ in operator norm it follows that $(w_n)_{n \in \mathbb{N}}$ is bounded too.

2. part: estimation of $\|w_n - P_n w_n\|_X$. Let $c_0 := \sup_{n \in \mathbb{N}} \|w_n\|_X$. Thence

$$\begin{aligned} \|w_n - P_n w_n\|_X &\leq \max_{\substack{u' \in \ker A(\lambda) \\ \|u'\|_X \leq c_0}} \|u' - P_n u'\|_X = \max_{\substack{u' \in \ker A(\lambda) \\ \|u'\|_X \leq c_0}} \inf_{u'_n \in X_n} \|u' - u'_n\|_X \\ &= c_0 \max_{\substack{u' \in \ker A(\lambda) \\ \|u'\|_X \leq 1}} \inf_{u'_n \in X_n} \|u' - u'_n\|_X. \end{aligned} \quad (6)$$

3. *part: estimate* $\|u_n - P_n w_n\|_X \leq c_1 \|A_n(\lambda_n)(u_n - P_n w_n)\|_X$. We show that there exists a constant $c_1 > 0$ such that

$$\|u_n - P_n w_n\|_X \leq c_1 \|A_n(\lambda_n)(u_n - P_n w_n)\|_X \quad (7)$$

holds for sufficiently large $n \in \mathbb{N}$. Assume the contrary, then there exists a subsequence $(n(m))_{m \in \mathbb{N}}$ such that

$$\|u_{n(m)} - P_{n(m)} w_{n(m)}\|_X > m \|A_{n(m)}(\lambda_{n(m)})(u_{n(m)} - P_{n(m)} w_{n(m)})\|_X \quad (8)$$

holds for all $m \in \mathbb{N}$. From (8) we deduce that

$$\lim_{m \rightarrow \infty} \left\| A_{n(m)}(\lambda_{n(m)}) \frac{u_{n(m)} - P_{n(m)} w_{n(m)}}{\|u_{n(m)} - P_{n(m)} w_{n(m)}\|_X} \right\|_X = 0.$$

Since $A_n(\cdot)$ is regular there exist $y \in \ker A(\lambda)$ and a subsequence $(m(k))_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \frac{u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}}{\|u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}\|_X} = y.$$

So we conclude with the abbreviation $\epsilon_k := \|u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}\|_X$

$$\begin{aligned} \epsilon_k &\leq \|u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))} - \epsilon_k P_{n(m(k))} y\|_X \\ &\leq \epsilon_k \left\| \frac{u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}}{\|u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}\|_X} - P_{n(m(k))} y \right\|_X \\ &\leq \epsilon_k \left(\left\| \frac{u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}}{\|u_{n(m(k))} - P_{n(m(k))} w_{n(m(k))}\|_X} - y \right\|_X + \|y - P_{n(m(k))} y\|_X \right) \\ &= \epsilon_k o(1) \end{aligned}$$

for $k \rightarrow \infty$ and hence $\epsilon_k = 0$ for sufficiently large k . However, $\epsilon_k \neq 0$ due to (8), which is a contradiction. Hence (7) holds.

4. *part: estimation of* $\|A_n(\lambda_n)(u_n - P_n w_n)\|_X$. Due to $u_n \in \ker A_n(\lambda_n)$ and $A_n(\cdot) = P_n A(\cdot)$ we can estimate

$$\begin{aligned} \|A_n(\lambda_n)(u_n - P_n w_n)\|_X &= \|A_n(\lambda_n) P_n w_n\|_X \\ &\leq \|A_n(\lambda) P_n w_n\|_X + \|(A_n(\lambda) - A_n(\lambda_n)) P_n w_n\|_X \quad (9) \\ &\leq \|A(\lambda) P_n w_n\|_X + \|(A(\lambda) - A(\lambda_n))\|_{L(X)} \|w_n\|_X. \end{aligned}$$

Since $A(\cdot)$ is holomorphic and $(\|w_n\|_X)_{n \in \mathbb{N}}$ is bounded, there exists a constant c_2 such that

$$\|(A(\lambda) - A(\lambda_n))\|_{L(X)} \|w_n\|_X \leq c_2 |\lambda - \lambda_n|. \quad (10)$$

Due to $w_n \in \ker A(\lambda)$ we can estimate the first term in the right hand side of (9) as

$$\|A(\lambda)P_n w_n\|_X = \|A(\lambda)(w_n - P_n w_n)\|_X \leq \|A(\lambda)\|_{L(X)} \|w_n - P_n w_n\|_X \quad (11)$$

5. part: conclusion. Finally the combination of (1), (2), (6), (7), (9), (10) and (11) yield the existence of a constant $c > 0$ such that

$$\inf_{u_0 \in \ker A(\lambda_0)} \|u_n - u_0\|_X \leq c \left(|\lambda_n - \lambda_0| + \max_{\substack{u'_0 \in \ker A(\lambda_0) \\ \|u'_0\|_X \leq 1}} \inf_{u'_n \in X_n} \|u'_0 - u'_n\|_X \right)$$

for all $\lambda_n \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}$ and all $u_n \in \ker A_n(\lambda_n)$ with $\|u_n\|_X = 1$. $\square$

Remark 1 We remark that the proof of Theorem 3 vii) actually requires only the regularity of $A_n(\cdot)$ and doesn't apply the $T(\cdot)$ -compatibility condition. Further, Theorem 3 assumes implicitly that the operators $A_n(\lambda)$ are Fredholm with index zero. However, for most applications X_n will be finite dimensional and hence in these cases this assumption will be trivial.

4 Example of application

In this section we present an example how to apply our theory to the approximation of an electromagnetic eigenvalue problem for a conductive material by means of Nédélec finite elements. Since the related stencil is quadratic in the eigenvalue parameter, the convergence of approximations is not covered by convenient theory such as [2,7,9,13]. To the best of our knowledge the obtained results are new. The applied technique essentially follows [21].

4.1 Maxwell eigenvalue problem for a conductive material

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedron with outward unit vector ν and $\omega \in \mathbb{C}$ be the temporal frequency. Let ϵ be the electric permittivity, μ be the magnetic permeability and γ be the conductivity of a heterogeneous and anisotropic material, i.e. $\epsilon, \mu^{-1}, \gamma \in (L^\infty(\Omega))^{3 \times 3}$ are real symmetric matrix functions such that

$$\begin{aligned} 0 < c_\epsilon &:= \inf_{\substack{\xi \in \mathbb{C}^3, |\xi|=1, \\ x \in \Omega}} \xi^* \epsilon(x) \xi, & 0 < c_\mu &:= \inf_{\substack{\xi \in \mathbb{C}^3, |\xi|=1, \\ x \in \Omega}} \xi^* \mu^{-1}(x) \xi, \\ 0 &\leq \inf_{\substack{\xi \in \mathbb{C}^3, |\xi|=1, \\ x \in \Omega}} \xi^* \gamma(x) \xi. \end{aligned}$$

We note that we label the conductivity with the symbol γ instead of the convenient symbol σ to avoid a conflict with the symbol $\sigma(A(\cdot))$ for the spectrum of an operator function. We denote the hermitian $L^2(\Omega)$ and $(L^2(\Omega))^3$ scalar products as $\langle \cdot, \cdot \rangle$. Let

∇ , div and curl be the differential operators defined by

$$\begin{aligned}\nabla u &:= (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)^\top, \\ \operatorname{curl} (u_1, u_2, u_3)^\top &:= (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)^\top, \\ \operatorname{div} (u_1, u_2, u_3)^\top &:= \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3.\end{aligned}$$

Then we introduce the Hilbert space $X = H_0(\operatorname{curl}; \Omega)$

$$\begin{aligned}X &:= \{u \in (L^2(\Omega))^3 : \operatorname{curl} u \in (L^2(\Omega))^3, v \times u = 0 \text{ at } \partial\Omega\}, \\ \langle u, u' \rangle_X &:= \langle \operatorname{curl} u, \operatorname{curl} u' \rangle + \langle u, u' \rangle \quad \text{for } u, u' \in X.\end{aligned}$$

Then we consider the Maxwell eigenvalue problem [29] to find $(\omega, u) \in \mathbb{C} \times X \setminus \{0\}$ such that

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u - \omega^2 \epsilon u - i\omega \gamma u = 0 \quad \text{in } \Omega. \quad (12)$$

Here the frequency ω takes the role of the spectral parameter λ . The weak formulation of (12) is to find $(\omega, u) \in \mathbb{C} \times X \setminus \{0\}$ such that $a(\omega; u, u') = 0$ for all $u' \in X$ with the sesquilinear form

$$a(\omega; u, u') := \langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle - \omega^2 \langle \epsilon u, u' \rangle - i\omega \langle \gamma u, u' \rangle, \quad \omega \in \mathbb{C}, u, u' \in X.$$

Let $A(\cdot): \mathbb{C} \rightarrow L(X)$ be defined by $\langle A(\omega)u, u' \rangle_X = a(\omega; u, u')$ for all $\omega \in \mathbb{C}, u, u' \in X$. Then the operator formulation of the eigenvalue problem is to find $(\omega, u) \in \mathbb{C} \times X \setminus \{0\}$ such that $A(\omega)u = 0$.

4.2 Weak $T(\cdot)$ -coercivity

In order to construct a suitable $T(\cdot)$ -operator function we follow e.g. [7] and introduce the orthogonal Helmholtz decomposition $X = V \oplus^{\perp_X} W$ with subspaces

$$V := \{u \in X : \operatorname{div} u = 0\} \quad \text{and} \quad W := \{\nabla w_0 : w_0 \in H_0^1(\Omega)\}$$

and respective orthogonal projections P_V and P_W . To achieve a more compact presentation we abbreviate $v := P_V u$, $w := P_W u$, $v' := P_V u'$ and $w' := P_W u'$ for $u, u' \in X$. A well known and important property of V is its compact embedding into $(L^2(\Omega))^3$, see, e.g. [35]. Subsequently we define

$$T(\omega) := P_V - \omega^{-2} P_W, \quad \omega \in \mathbb{C} \setminus \{0\}.$$

It easily follows that $T(\cdot)$ is bijective with inverse $T^{-1}(\omega) := P_V - \omega^2 P_W$. With the former notation it holds $\langle T^*(\omega)A(\omega)u, u' \rangle_X = a(\omega; v + w, v' - \omega^{-2}w')$. Then we compute $T^*(\omega)A(\omega) = A_1 + A_2$ with $A_1, A_2 \in L(X)$ defined by

$$\langle A_1 u, u' \rangle_X := \langle \mu^{-1} \operatorname{curl} v, \operatorname{curl} v' \rangle + \langle v, v' \rangle + \langle (\epsilon + i\omega^{-1}\gamma)w, w' \rangle,$$

$$\begin{aligned} \langle A_2 u, u' \rangle_X &:= \langle (\epsilon + i\omega^{-1}\gamma)v, w' \rangle - \omega^2 \langle (\epsilon + i\omega^{-1}\gamma)w, v' \rangle \\ &\quad - \omega^2 \langle (\epsilon + i\omega^{-1}\gamma)v, v' \rangle - \langle v, v' \rangle, \end{aligned}$$

for all $u, u' \in X$. Note that several terms in this decomposition vanished due to $\operatorname{div} w = \operatorname{div} w' = 0$. For $\omega \in \Lambda := \mathbb{C} \setminus \{-it : t \geq 0\}$ it holds $\Re(i\omega^{-1}) \geq 0$ and hence

$$\Re(\langle A_1 u, u \rangle_X) \geq \min(1, c_\epsilon, c_\mu)(\|v\|_X^2 + \|w\|_{(L^2(\Omega))^3}^2) = \min(1, c_\epsilon, c_\mu)\|u\|_X^2,$$

i.e. A_1 is coercive. The former equality holds since due to $\operatorname{div} w = 0$ we have $\|w\|_{(L^2(\Omega))^3} = \|w\|_X$ and since V and W are orthogonal it holds $\|v\|_X^2 + \|w\|_X^2 = \|u\|_X^2$. On the other hand it follows from the compact embedding $V \hookrightarrow (L^2(\Omega))^3$ that A_2 is coercive. Thus $A(\cdot) : \Lambda \rightarrow L(X)$ is weakly $T(\cdot)$ -coercive. For $\omega \in \{-it : t \in [0, \|\gamma\|_{(L^\infty(\Omega))^{3 \times 3}}/c_\epsilon]\}$ it is possible that $\epsilon + i\omega^{-1}\gamma$ is not uniformly coercive. For such frequencies we cannot expect that $A(\omega)$ is Fredholm. Further, note that $A(\omega)$ is already coercive and hence bijective for all $\omega \in \{it : t > 0\}$. Thus the resolvent set of $A(\cdot)$ is not empty.

4.3 $T(\cdot)$ -compatible approximation

In order to prove the convergence of discretizations by means of Theorem 3, it remains to discuss the $T(\cdot)$ -compatibility of approximations. It is well known that naive discretizations of Maxwell eigenvalue problems can lead to erroneous results and the application of special (Nédélec) finite element spaces is necessary. In this section we will show that if the Galerkin spaces X_n are chosen as Nédélec spaces, then indeed the corresponding approximations are $T(\cdot)$ -compatible. The research on the underlying theory of Nédélec elements led to many different conditions on the finite element spaces and an overview on their equivalence can be found in [13]. Here we choose the notion of L^2 -uniformly bounded cochain projections [2] to formulate our theory. To this end we introduce the de Rham Complex

$$Y^1 := H_0^1(\Omega) \xrightarrow{\nabla} Y^2 := X \xrightarrow{\operatorname{curl}} Y^3 := H_0(\operatorname{div}; \Omega) \xrightarrow{\operatorname{div}} Y^4 := L_0^2(\Omega) \quad (13)$$

and consider for each $n \in \mathbb{N}$ a subcomplex of (13)

$$Y_n^1 \xrightarrow{\nabla} Y_n^2 = X_n \xrightarrow{\operatorname{curl}} Y_n^3 \xrightarrow{\operatorname{div}} Y_n^4 \quad (14)$$

with finite dimensional spaces $Y_n^k \subset Y^k$, $k = 1, \dots, 4$ such that the corresponding orthogonal projections $P_{Y_n^k}$, $k = 1, \dots, 4$ converge point-wise to the identity for $n \rightarrow \infty$. Further, let $Z^1 := Z^4 := L^2(\Omega)$ and $Z^2 := Z^3 := (L^2(\Omega))^3$. We say that the sequence of subcomplexes (14) admits L^2 -uniformly bounded commuting cochain

projections $\pi_n^k, k = 1, \dots, 4$, if the following commuting diagram

$$\begin{array}{ccccccc} Y^1 & \xrightarrow{\nabla} & X = Y^2 & \xrightarrow{\text{curl}} & Y^3 & \xrightarrow{\text{div}} & Y^4 \\ \pi_n^1 \downarrow & & \pi_n^2 \downarrow & & \pi_n^3 \downarrow & & \pi_n^4 \downarrow \\ Y_n^1 & \xrightarrow{\nabla} & X_n = Y_n^2 & \xrightarrow{\text{curl}} & Y_n^3 & \xrightarrow{\text{div}} & Y_n^4 \end{array}$$

is satisfied, the projections are surjective and of the form $\pi_n^k = \tilde{\pi}_n^k E_k$ with embedding operators $E_k \in L(Y^k, Z^k)$ and projections $\tilde{\pi}_n^k \in L(Z^k, Y_n^k), k = 1, \dots, 4$, such that $c_{\tilde{\pi}} := \sup_{n \in \mathbb{N}, k=1, \dots, 4} \{ \|\tilde{\pi}_n^k\|_{L(Z^k)} \} < \infty$. We refer to [2] for the systematic construction of such finite element spaces $(Y_n^k)_{n \in \mathbb{N}}$ and the proof of existence of respective projections $(\pi_n^k)_{n \in \mathbb{N}}$. In this context the index n corresponds to the sequence of decreasing mesh width parameters $h_n \rightarrow 0+$. We note that $\tilde{\pi}_n^k \rightarrow I$ point-wise in Z^k for $n \rightarrow \infty$. Indeed, for $u \in Y^k$ it follows

$$\|(1 - \tilde{\pi}_n^k)u\|_{Z^k} = \|(1 - \tilde{\pi}_n^k)(u - P_{Y_n^k}u)\|_{Z^k} \leq (1 + c_{\tilde{\pi}})\|u - P_{Y_n^k}u\|_{Y^k}.$$

Further, $\pi_n^2 P_V$ converges to P_V in discrete norm. Indeed, we for $u \in X_n$

$$\text{curl}(1 - \pi_n^2)P_V u = (1 - \pi_n^3) \text{curl} P_V u = (1 - \pi_n^3) \text{curl} u = 0,$$

because $\text{curl} P_W = 0, P_V + P_W = I$ and $\text{curl} u \in Y_n^3$. Thus

$$\begin{aligned} \|(I - \pi_n^2)P_V\|_n &= \sup_{u \in X_n \setminus \{0\}} \|(1 - \pi_n^2)P_V u\|_{(L^2(\Omega))^3} / \|u\|_X \\ &\leq \|(I - \pi_n^2)P_V\|_{L(X, (L^2(\Omega))^3)} = \|(I - \tilde{\pi}_n^2)E_2 P_V\|_{L(X, (L^2(\Omega))^3)}. \end{aligned}$$

Since $E_2 P_V$ is compact and $\lim_{n \rightarrow \infty} \tilde{\pi}_n^2 = I$ point-wise in $(L^2(\Omega))^3$ it follows $\lim_{n \rightarrow \infty} \|P_V - \pi_n^2 P_V\|_n = 0$. Due to

$$\|P_V - \pi_n^2 P_V\|_n = \|(I - P_W) - \pi_n^2(I - P_W)\|_n = \|P_W - \pi_n^2 P_W\|_n$$

it follows for $T_n(\omega) := \pi_n^2 T(\omega)|_{X_n}$ with the triangle inequality that

$$\lim_{n \rightarrow \infty} \|T(\omega) - T_n(\omega)\|_n = 0.$$

Hence Theorem 3 can be applied for Nédélec approximations to the eigenvalue problem (12) and the respective convergence properties i)-vii) hold. Note that the obtained quadratic matrix eigenvalue problem can subsequently be linearized with the auxiliary variable $\tilde{u}_n := \omega u_n$ and the resulting linear eigenvalue can be solved with an Arnoldi algorithm.

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