Numerische
Mathematik

Error analysis of Nitsche's mortar method

Tom Gustafsson[1] · Rolf Stenberg[1] · Juha Videman[2]

Abstract

Optimal a priori and a posteriori error estimates are derived for three variants of Nitsche's mortar finite elements. The analysis is based on the equivalence of Nitsche's method and the stabilised mixed method. Nitsche's method is defined so that it is robust with respect to large jumps in the material and mesh parameters over the interface. Numerical results demonstrate the robustness of the a posteriori error estimators.

1 Introduction

Nitsche's method [23] is by now a well-established and successful method, e.g., for domain decomposition [4,18,25], elastic contact problems [7,8,10–12], and as a fictitious domain method [5,6,14]. However, its mathematical analysis has not, as yet, been entirely satisfactory. In fact, for an elliptic problem with a variational formulation in H^1, the existing a priori estimates require that the solution is in H^s, with $s > 3/2$; cf. [4,12]. Moreover, the a posteriori analysis has been based on a non-rigorous saturation assumption; cf. [4,9].

In our paper [24], we made the observation that there is a close connection between the Nitsche's method for Dirichlet conditions and a certain stabilised mixed finite element method, and we advocated the use of the former since it has the advantage that it directly yields a method with an optimally conditioned, symmetric, and positive-definite stiffness matrix. The a priori error analysis is also very straightforward but, as understood from above, not optimal.

✉ Rolf Stenberg
  rolf.stenberg@aalto.fi

  Tom Gustafsson
  tom.gustafsson@aalto.fi

  Juha Videman
  jvideman@math.tecnico.ulisboa.pt

[1] Department of Mathematics and Systems Analysis, Aalto University, 00076 Aalto, Finland

[2] CAMGSD/Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisbon, Portugal

The purpose of the present paper is to show that this connection can be used to improve the error analysis of the domain decomposition problem, i.e. we will derive optimal error estimates, both a priori and a posteriori. We consider three similar but distinct Nitsche's mortar methods. Two of the methods have appeared previously in the literature [19–21] and the third one is a simpler master–slave formulation where the stabilisation term is present only on the slave side of the interface. The methods are designed so that they are robust with respect to large jumps in the material and mesh parameters over the interface. The robustness is achieved by a proper scaling of the stabilisation/Nitsche terms; cf. [19,25]. For simplicity we consider the Poisson problem with two subdomains. The analysis, however, carries over to other problems such as the transmission problem for which the Nitsche's method was applied in [18]. In a forthcoming paper [16], we will extend these results, and our previous work on variational inequalities [17], to elastic contact problems including different material properties.

The plan of the paper is the following. In the next section, we present the model transmission problem, rewrite it in a mixed saddle point variational form and prove its stability in appropriately chosen continuous norms. In Sect. 3, we present three different stabilised mixed finite element methods and their respective Nitsche formulations. In Sect. 4, we prove the stability of the discrete saddle point formulations and derive optimal a priori error estimates. In Sect. 5, we perform the a posteriori error analysis and show that the residual estimators are both reliable and efficient. In Sect. 6, we report the results of our numerical computations.

2 The model problem

Suppose that the polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is divided into two non-overlapping parts Ω_i, $i = 1, 2$, and denote their common boundary by $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$. We assume that $\partial \Gamma \subset \partial \Omega$, with $\partial \Gamma$ being the boundary of the $n - 1$ dimensional manifold Γ.

We consider the problem: find functions u_i that satisfy

$$
\begin{aligned}
-\nabla \cdot k_i \nabla u_i &= f && \text{in } \Omega_i, \\
u_1 - u_2 &= 0 && \text{on } \Gamma, \\
k_1 \frac{\partial u_1}{\partial n_1} + k_2 \frac{\partial u_2}{\partial n_2} &= 0 && \text{on } \Gamma, \\
u_i &= 0 && \text{on } \partial \Omega_i \backslash \Gamma,
\end{aligned}
\tag{2.1}
$$

where $k_i > 0$, $i = 1, 2$, are material parameters, $f \in L^2(\Omega)$ is a load function and n_i denote the outer normal vectors to the subdomains Ω_i, $i = 1, 2$. In what follows we often write $n = n_1 = -n_2$. Throughout the paper we assume that $k_1 \geq k_2$.

The standard variational formulation of problem (2.1) reads as follows: find $u \in H_0^1(\Omega)$ such that

$$
(k \nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega),
\tag{2.2}
$$

where $k|_{\Omega_i} = k_i$ and $u|_{\overline{\Omega}_i} = u_i$. On the interface Γ, the restriction of the solution u lies in the Lions–Magenes space $H_{00}^{\frac{1}{2}}(\Gamma)$ (c.f. [22, Theorem 11.7, p. 66] or [26, Chapter 33]), with its intrinsic norm defined as

$$\|v\|_{\frac{1}{2},\Gamma}^2 = \|v\|_{0,\Gamma}^2 + \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x - y|^d} \, dx \, dy + \int_\Gamma \frac{v(x)^2}{\rho(x)} \, dx, \qquad (2.3)$$

where $\rho(x)$ is the distance from x to the boundary $\partial\Gamma$.

The mixed formulation follows from imposing the continuity condition on Γ in a weak form by using the normal flux as the Lagrange multiplier, viz.

$$\lambda = k_1 \frac{\partial u_1}{\partial n} = -k_2 \frac{\partial u_2}{\partial n}.$$

The Lagrange multiplier belongs to the dual space $Q = \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)'$, equipped with the norm

$$\|\xi\|_{-\frac{1}{2},\Gamma} = \sup_{v \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\langle v, \xi \rangle}{\|v\|_{\frac{1}{2},\Gamma}}, \qquad (2.4)$$

where $\langle \cdot, \cdot \rangle : Q' \times Q \to \mathbb{R}$ stands for the duality pairing.

Let

$$V_i = \{v \in H^1(\Omega_i) : v|_{\partial\Omega_i \backslash \Gamma} = 0\}, \quad V = V_1 \times V_2, \qquad (2.5)$$

and define the bilinear and linear forms, $\mathcal{B} : (V \times Q) \times (V \times Q) \to \mathbb{R}$ and $\mathcal{L} : V \to \mathbb{R}$ by

$$\mathcal{B}(w, \xi; v, \mu) = \sum_{i=1}^2 (k_i \nabla w_i, \nabla v_i)_{\Omega_i} - \langle [\![w]\!], \mu \rangle - \langle [\![v]\!], \xi \rangle, \qquad (2.6)$$

$$\mathcal{L}(v) = \sum_{i=1}^2 (f, v_i)_{\Omega_i}, \qquad (2.7)$$

where w and v denote the pair of functions $w = (w_1, w_2) \in V_1 \times V_2$ and $v = (v_1, v_2) \in V_1 \times V_2$. Furthermore, $[\![w]\!]|_\Gamma = (w_1 - w_2)|_\Gamma$ denotes the jump in the value of w over Γ. The mixed variational formulation of (2.1) reads as follows: find $(u, \lambda) \in V \times Q$ such that

$$\mathcal{B}(u, \lambda; v, \mu) = \mathcal{L}(v) \quad \forall (v, \mu) \in V \times Q. \qquad (2.8)$$

The norm in $V \times Q$ used in the analysis is scaled by the material parameters, viz.

$$\||(w, \xi)\||^2 = \sum_{i=1}^{2} \left(k_i \|\nabla w_i\|_{0,\Omega_i}^2 + \frac{1}{k_i} \|\xi\|_{-\frac{1}{2},\Gamma}^2 \right). \tag{2.9}$$

Theorem 1 (Continuous stability) *For every $(w, \xi) \in V \times Q$ there exists $(v, \mu) \in V \times Q$ such that*

$$\mathcal{B}(w, \xi; v, \mu) \gtrsim \||(w, \xi)\||^2 \tag{2.10}$$

and

$$\||(v, \mu)\|| \lesssim \||(w, \xi)\||. \tag{2.11}$$

Proof In both subdomains, we have the inf-sup condition (cf. [2])

$$\sup_{v_i \in V_i} \frac{\langle v_i, \xi \rangle}{\|\nabla v_i\|_{0,\Omega_i}} \geq C_i \|\xi\|_{-\frac{1}{2},\Gamma} \quad \forall \xi \in Q, \quad i = 1, 2. \tag{2.12}$$

Therefore

$$\sup_{v=(v_1,v_2)\in V} \frac{\langle [\![v]\!], \xi \rangle}{(\sum_{i=1}^{2} k_i \|\nabla v_i\|_{0,\Omega_i}^2)^{1/2}} \geq C \left(\frac{1}{k_1} + \frac{1}{k_2} \right)^{1/2} \|\xi\|_{-\frac{1}{2},\Gamma} \quad \forall \xi \in Q. \tag{2.13}$$

The stability follows now from the Babuška–Brezzi theory [2].   □

Remark 1 Given that $k_1 \geq k_2$, it holds with some constants $C_1, C_2 > 0$ that

$$C_1 \||(w, \xi)\||^2 \leq \sum_{i=1}^{2} k_i \|\nabla w_i\|_{0,\Omega_i}^2 + \frac{1}{k_2} \|\xi\|_{-\frac{1}{2},\Gamma}^2 \leq C_2 \||(w, \xi)\||^2. \tag{2.14}$$

This defines a norm that will be used in the following for defining and analysing a "master–slave" formulation.

3 The finite element methods

We start by defining the stabilised mixed method. The subdomains Ω_i are divided into sets of non-overlapping simplices $\mathcal{C}_h^i, i = 1, 2$, with h referring to the mesh parameter. The edges/facets of the elements in $\mathcal{C}_h^i$ are divided into two meshes: $\mathcal{E}_h^i$ consisting of those which are located in the interior of Ω_i, and $\mathcal{G}_h^i$ of those that lie on Γ. Furthermore, by $\mathcal{G}_h^\cap$ we denote the boundary mesh obtained by intersecting the edges/facets of $\mathcal{G}_h^1$

and $\mathcal{G}_h^2$. In particular, each $E \in \mathcal{G}_h^\cap$ corresponds to a pair $(E_1, E_2) \in \mathcal{G}_h^1 \times \mathcal{G}_h^2$ such that $E = E_1 \cap E_2$. In the subdomains, we define the finite element subspaces

$$V_{i,h} = \{v_{i,h} \in V_i : v_{i,h}|_K \in P_p(K) \,\forall K \in \mathcal{C}_h^i\}, \quad V_h = V_{1,h} \times V_{2,h}, \quad (3.1)$$

where $p \geq 1$. The finite element space for the dual variable consists of discontinuous piecewise polynomials, also of degree p, defined at the intersection mesh $\mathcal{G}_h^\cap$:

$$Q_h = \{\mu_h \in Q : \mu_h|_E \in P_p(E) \,\forall E \in \mathcal{G}_h^\cap\}. \quad (3.2)$$

We will now introduce three slightly different stabilised finite element methods and the corresponding Nitsche's formulations for problem (2.1).

3.1 Method I

We define a bilinear form $\mathcal{B}_h : (V_h \times Q_h) \times (V_h \times Q_h) \to \mathbb{R}$ through

$$\mathcal{B}_h(w, \xi; v, \mu) = \mathcal{B}(w, \xi; v, \mu) - \alpha \mathcal{S}_h(w, \xi; v, \mu), \quad (3.3)$$

where $\alpha > 0$ is a stabilisation parameter and

$$\mathcal{S}_h(w, \xi; v, \mu) = \sum_{i=1}^2 \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \left(\xi - k_i \frac{\partial w_i}{\partial n}, \mu - k_i \frac{\partial v_i}{\partial n} \right)_E, \quad (3.4)$$

a stabilising term, with h_E denoting the diameter of $E \in \mathcal{G}_h^i$. The first stabilised finite element method is written as: find $(u_h, \lambda_h) \in V_h \times Q_h$ such that

$$\mathcal{B}_h(u_h, \lambda_h; v_h, \mu_h) = \mathcal{L}(v_h) \quad \forall (v_h, \mu_h) \in V_h \times Q_h. \quad (3.5)$$

Note that testing with $(0, \mu_h) \in V_h \times Q_h$ in (3.5) yields the equation

$$\langle \llbracket u_h \rrbracket, \mu_h \rangle + \alpha \sum_{i=1}^2 \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \left(\lambda_h - k_i \frac{\partial v_{i,h}}{\partial n}, \mu_h \right)_E = 0 \quad \forall \mu_h \in Q_h. \quad (3.6)$$

Hence, denoting by $h_i : \Gamma \to \mathbb{R}$, $i = 1, 2$, a local mesh size function such that

$$h_i|_E = h_E \quad \forall E \in \mathcal{G}_h^i, \quad i = 1, 2, \quad (3.7)$$

equation (3.6) can be written as

$$\left(\llbracket u_h \rrbracket + \alpha \sum_{i=1}^2 \frac{h_i}{k_i} \left(\lambda_h - k_i \frac{\partial u_{i,h}}{\partial n} \right), \mu_h \right)_\Gamma = 0 \quad \forall \mu_h \in Q_h. \quad (3.8)$$

Now, since each $E \in \mathcal{G}_h^\cap$ is an intersection of a pair $(E_1, E_2) \in \mathcal{G}_h^1 \times \mathcal{G}_h^2$ and the polynomial degree is p for all variables, we obtain the following expression for the discrete Lagrange multiplier

$$\lambda_h = \left\{\!\!\left\{ k \frac{\partial u_h}{\partial n} \right\}\!\!\right\} - \beta \, [\![u_h]\!], \tag{3.9}$$

where

$$\beta = \frac{\alpha^{-1} k_1 k_2}{k_2 h_1 + k_1 h_2}, \tag{3.10}$$

and

$$\left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\} = \frac{k_2 h_1}{k_2 h_1 + k_1 h_2} k_1 \frac{\partial w_1}{\partial n} + \frac{k_1 h_2}{k_2 h_1 + k_1 h_2} k_2 \frac{\partial w_2}{\partial n}. \tag{3.11}$$

Substituting expression (3.9) into the discrete variational formulation leads to the *Nitsche formulation*: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \mathcal{L}(v_h) \quad \forall v_h \in V_h, \tag{3.12}$$

where the bilinear form a_h is defined through

$$a_h(w, v) = \sum_{i=1}^{2} (k_i \nabla w_i, \nabla v_i)_{\Omega_i} + b_h(w, v), \tag{3.13}$$

with

$$b_h(w, v) = \sum_{E \in \mathcal{G}_h^\cap} \left\{ (\beta \, [\![w]\!], [\![v]\!])_E - \left(\gamma \left[\!\!\left[k \frac{\partial w}{\partial n} \right]\!\!\right], \left[\!\!\left[k \frac{\partial v}{\partial n} \right]\!\!\right] \right)_E \right.$$
$$\left. - \left(\left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\}, [\![v]\!] \right)_E - \left([\![w]\!], \left\{\!\!\left\{ k \frac{\partial v}{\partial n} \right\}\!\!\right\} \right)_E \right\}, \tag{3.14}$$

and the jump term and the function γ are given by

$$\left[\!\!\left[k \frac{\partial w}{\partial n} \right]\!\!\right] = k_1 \frac{\partial w_1}{\partial n} - k_2 \frac{\partial w_2}{\partial n}, \quad \gamma = \frac{\alpha h_1 h_2}{k_2 h_1 + k_1 h_2}. \tag{3.15}$$

Note that (3.11) is a convex combination of two fluxes as in the method suggested in [25]. The formulation (3.12) corresponds to the method introduced in [21], and to the second method proposed for problem (2.1) in [19, pp. 468–470].

3.2 Method II: Master–slave formulation

Assume that $k_1 \gg k_2$. The norm equivalence (2.14) suggests using only the term from the "less rigid" subdomain Ω_2 for stabilisation in (3.4). Calling Ω_1 the master domain and Ω_2 the slave domain and stabilising from the slave side only, yields a mixed stabilised finite element as in (3.5) except that

$$\mathcal{S}_h(w, \xi; v, \mu) = \sum_{E \in \mathcal{G}_h^2} \frac{h_E}{k_2} \left(\xi - k_2 \frac{\partial w_2}{\partial n}, \mu - k_2 \frac{\partial v_2}{\partial n} \right)_E. \qquad (3.16)$$

Note that the space for the Lagrange multiplier is still defined by (3.2).

The corresponding Nitsche's formulation reads as in (3.12) with the bilinear form b_h defined simply as

$$b_h(w, v) = \sum_{E \in \mathcal{G}_h^\cap} \left\{ \left(\frac{k_2}{\alpha h_2} \llbracket w \rrbracket, \llbracket v \rrbracket \right)_E - \left(k_2 \frac{\partial w}{\partial n}, \llbracket v \rrbracket \right)_E - \left(\llbracket w \rrbracket, k_2 \frac{\partial v}{\partial n} \right)_E \right\}. \qquad (3.17)$$

3.3 Method III: Stabilisation using a convex combination of fluxes [4,19,20,25]

Let us reformulate (3.5) by considering the stabilising term

$$\alpha \mathcal{S}_h(w, \xi; v, \mu) = \left(\beta^{-1} \left(\xi - \left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\} \right), \mu - \left\{\!\!\left\{ k \frac{\partial v}{\partial n} \right\}\!\!\right\} \right)_\Gamma, \qquad (3.18)$$

where $\left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\}$ denotes the convex combination (3.11) and β is defined by (3.10). To derive the corresponding Nitsche's method, we proceed as above and obtain an equivalent expression for the discrete Lagrange multiplier:

$$\lambda_h = \left\{\!\!\left\{ k \frac{\partial u_h}{\partial n} \right\}\!\!\right\} - \beta \llbracket u_h \rrbracket. \qquad (3.19)$$

Substituting this back to the stabilised formulation leads to the method (3.12) with b_h given by

$$b_h(w, v) = \left(\beta \llbracket w \rrbracket, \llbracket v \rrbracket \right)_\Gamma - \left(\left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\}, \llbracket v \rrbracket \right)_\Gamma - \left(\llbracket w \rrbracket, \left\{\!\!\left\{ k \frac{\partial v}{\partial n} \right\}\!\!\right\} \right)_\Gamma. \qquad (3.20)$$

This exact method was discussed before in [20]. A similar method with a slightly different definition for the convex combination of fluxes was considered in [19].

Remark 2 *(On the choice of the method)* The performance of the different methods is equal by all practical measures when $k_1 \gg k_2$. The variational formulation of Method III has fewer terms and is therefore simpler to implement than Method I whereas the

master–slave formulation (Method II) is clearly the simplest of them all, both when it comes to the implementation and the analysis.

4 A priori error analysis

In this section, we perform a priori error analyses of the stabilised formulations which then, by construction, carry over to the Nitsche's formulations. We will perform the analysis in full detail for Method I and briefly indicate the differences in analysing the other two methods.

In order to prove the a priori estimate (Theorem 3), we need a stability estimate for the discrete bilinear form $\mathcal{B}_h$. The stability estimate is proven using Lemma 1 which follows from a scaling argument:

Lemma 1 (Discrete trace estimate) *There exists $C_I > 0$, independent of h, such that*

$$C_I \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \left\| k_i \frac{\partial v_{i,h}}{\partial n} \right\|_{0,E}^2 \leq k_i \|\nabla v_{i,h}\|_{0,\Omega_i}^2 \quad \forall v_{i,h} \in V_{i,h}, \quad i = 1, 2.$$

The discrete stability of Method I will be established in the mesh-dependent norm

$$\|(w_h, \xi_h)\|_h^2 = \|(w_h, \xi_h)\|^2 + \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2. \tag{4.1}$$

Note, however, that trivially we have

$$\|(w_h, \xi_h)\|_h \geq \|(w_h, \xi_h)\|. \tag{4.2}$$

Theorem 2 (Discrete stability) *Suppose that $0 < \alpha < C_I$. Then for every $(w_h, \xi_h) \in V_h \times Q_h$ there exists $(v_h, \mu_h) \in V_h \times Q_h$ such that*

$$\mathcal{B}_h(w_h, \xi_h; v_h, \mu_h) \gtrsim \|(w_h, \xi_h)\|_h^2 \tag{4.3}$$

and

$$\|(v_h, \mu_h)\|_h \lesssim \|(w_h, \xi_h)\|_h. \tag{4.4}$$

Proof Applying the discrete trace estimate leads to stability in the mesh-dependent part of the norm

$$\mathcal{B}_h(w_h, \xi_h; w_h, -\xi_h) \geq (1 - \alpha C_I^{-1}) \sum_{i=1}^{2} k_i \|\nabla w_{i,h}\|_{0,\Omega_i}^2 + \alpha \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2$$

$$\geq C_1 \left(\sum_{i=1}^{2} k_i \|\nabla w_{i,h}\|_{0,\Omega_i}^2 + \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2 \right).$$

(4.5)

Next, we recall the steps (cf. [15]) for extending the result to the continuous part of the norm. By the continuous inf-sup condition (2.13), for any $\xi_h \in Q_h$ there exists $v \in V$ such that

$$\frac{\langle [\![v]\!], \xi_h \rangle}{\left(\sum_{i=1}^{2} k_i \|\nabla v_i\|_{0,\Omega_i}^2 \right)^{1/2}} \geq C \left(\frac{1}{k_1} + \frac{1}{k_2} \right)^{1/2} \|\xi_h\|_{-\frac{1}{2},\Gamma}.$$

(4.6)

Consequently, there exist positive constants C_2, C_3, C_4, such that for the Clément interpolant $I_h v \in V_h$ of v it holds

$$\langle [\![I_h v]\!], \xi_h \rangle \geq C_2 \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|\xi_h\|_{-\frac{1}{2},\Gamma}^2 - C_3 \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2,$$

(4.7)

$$\sum_{i=1}^{2} k_i \|\nabla v_{i,h}\|_{0,\Omega_i}^2 \leq C_4 \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|\xi_h\|_{-\frac{1}{2},\Gamma}^2.$$

(4.8)

Using the Cauchy–Schwarz inequality, the arithmetic-geometric mean inequality, and the discrete trace estimate (Lemma 1), we then see that

$$\mathcal{B}_h(w_h, \xi_h; -I_h v, 0) = -\sum_{i=1}^{2} (k_i \nabla w_{i,h}, \nabla I_h v_i) + \langle [\![I_h v]\!], \xi_h \rangle$$

$$- \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} h_E \left(\xi_h - k_i \frac{\partial w_{i,h}}{\partial n}, \frac{\partial I_h v_i}{\partial n} \right)_E$$

$$\geq -C_5 \left(\sum_{i=1}^{2} k_i \|\nabla w_{i,h}\|_{0,\Omega_i}^2 + \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2 \right)$$

$$+ C_6 \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|\xi_h\|_{-\frac{1}{2},\Gamma}^2.$$

(4.9)

Combining estimates (4.5) and (4.9), we finally obtain

$$\mathcal{B}_h(w_h, \xi_h; w_h - \delta I_h v, -\xi_h)$$

$$\geq (C_1 - \delta C_5) \left(\sum_{i=1}^{2} k_i \|\nabla w_{i,h}\|_{0,\Omega_i}^2 + \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2 \right)$$

$$+ \delta C_6 \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|\xi_h\|_{-\frac{1}{2},\Gamma}^2$$

$$\geq C_7 \left(\sum_{i=1}^{2} k_i \|\nabla w_{i,h}\|_{0,\Omega_i}^2 + \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2 \right.$$

$$\left. + \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|\xi_h\|_{-\frac{1}{2},\Gamma}^2 \right),$$

where the last bound follows from choosing $0 < \delta < C_1/C_5$.

In order to obtain (4.4), we first use the triangle inequality and (4.8) to get

$$\||(w_h - \delta I_h v, -\xi_h)|\| \leq \||(w_h, \xi_h)|\|.$$

The claim follows by adding

$$\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \|\xi_h\|_{0,E}^2$$

to both sides of the inequality.   □

We will need one more lemma before we can establish an optimal a priori estimate. Let $f_h \in V_h$ be an approximation of f and define

$$\mathrm{osc}_K(f) = h_K \|f - f_h\|_{0,K}. \tag{4.10}$$

Moreover, for each $E \in \mathcal{G}_h^i$, denote by $K(E) \in \mathcal{C}_h^i$ the element satisfying $\partial K(E) \cap E = E$.

Lemma 2 *For an arbitrary $(v_h, \mu_h) \in V_h \times Q_h$ it holds*

$$\left(\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \frac{h_E}{k_i} \left\| \mu_h - k_i \frac{\partial v_{i,h}}{\partial n} \right\|_{0,E}^2 \right)^{1/2}$$

$$\lesssim \||(u - v_h, \lambda - \mu_h)|\| + \left(\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \mathrm{osc}_{K(E)}(f)^2 \right)^{1/2}. \tag{4.11}$$

Proof Let $b_E \in P_d(E) \cap H_0^1(E)$, $E \in \mathcal{G}_h^1$, be the edge/facet bubble function with maximum value one. Define σ_E as the polynomial defined on $K(E)$ through

$$\sigma_E\big|_E = \frac{h_E b_E}{k_1}\left(\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right) \quad \text{and} \quad \sigma_E\big|_{\partial K(E)\setminus E} = 0.$$

We have by the norm equivalence in polynomial spaces

$$\frac{h_E}{k_1}\left\|\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right\|_{0,E}^2 \lesssim \frac{h_E}{k_1}\left\|\sqrt{b_E}\left(\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right)\right\|_{0,E}^2$$

$$= \left(\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}, \sigma_E\right)_E.$$

Let $\sigma = \sum_{E \in \mathcal{G}_h^1} \sigma_E$. Testing the continuous variational problem with $(v_1, v_2, \mu) = (\sigma, 0, 0)$ gives $(k_1\nabla u_1, \nabla\sigma)_{\Omega_1} - \langle\sigma, \lambda\rangle - (f, \sigma)_{\Omega_1} = 0$. This leads to

$$\sum_{E \in \mathcal{G}_h^1} \frac{h_E}{k_1}\left\|\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right\|_{0,E}^2$$

$$\lesssim \langle\sigma, \mu_h - \lambda\rangle + (k_1\nabla u_1, \nabla\sigma)_{\Omega_1} - (f, \sigma)_{\Omega_1} - \sum_{E \in \mathcal{G}_h^1}\left(k_1\frac{\partial v_{1,h}}{\partial n}, \sigma_E\right)_E$$

$$= \langle\sigma, \mu_h - \lambda\rangle + (k_1\nabla u_1, \nabla\sigma)_{\Omega_1} - (f, \sigma)_{\Omega_1}$$

$$- \sum_{E \in \mathcal{G}_h^1}\left((\nabla \cdot k_1\nabla v_{1,h}, \sigma_E)_{K(E)} + (k_1\nabla v_{1,h}, \nabla\sigma_E)_{K(E)}\right)$$

$$= \langle\sigma, \mu_h - \lambda\rangle + (k_1\nabla(u_1 - v_{1,h}), \nabla\sigma)_{\Omega_1} + \sum_{E \in \mathcal{G}_h^1}(-\nabla \cdot k_1\nabla v_{1,h} - f, \sigma_E)_{K(E)}.$$

By inverse estimates

$$k_1\|\sigma\|_{1,\Omega_1}^2 \lesssim k_1\sum_{E \in \mathcal{G}_h^1} h_E^{-2}\|\sigma_E\|_{0,K(E)}^2 \lesssim \sum_{E \in \mathcal{G}_h^1}\frac{h_E}{k_1}\left\|\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right\|_{0,E}^2. \quad (4.12)$$

Using the Cauchy–Schwarz and the trace inequalities, it then follows that

$$\sum_{E \in \mathcal{G}_h^1}\frac{h_E}{k_1}\left\|\mu_h - k_1\frac{\partial v_{1,h}}{\partial n}\right\|_{0,E}^2$$

$$\lesssim \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma}\|\sigma\|_{\frac{1}{2},\Gamma} + k_1\|\nabla(u_1 - v_{1,h})\|_{0,\Omega_1}\|\nabla\sigma\|_{0,\Omega_1}$$

$$+ \sum_{E \in \mathcal{G}_h^1}\|\nabla \cdot k_1\nabla v_{1,h} + f\|_{0,K(E)}\|\sigma_E\|_{0,K(E)}$$

$$\lesssim \frac{1}{\sqrt{k_1}} \|\lambda - \mu_h\|_{-\frac{1}{2}, \Gamma} \sqrt{k_1} \|\sigma\|_{1, \Omega_1} + \sqrt{k_1} \|\nabla(u_1 - v_{1,h})\|_{0, \Omega_1} \sqrt{k_1} \|\sigma\|_{1, \Omega_1}$$

$$+ \left(\sum_{E \in \mathcal{G}_h^1} \frac{h_E^2}{k_1} \|\nabla \cdot k_1 \nabla v_{1,h} + f\|_{0, K(E)}^2 \right)^{1/2} \left(k_1 \sum_{E \in \mathcal{G}_h^1} h_E^{-2} \|\sigma_E\|_{0, K(E)}^2 \right)^{1/2}.$$

In view of the standard lower bound for interior residuals [27] and the discrete inequalities (4.12), we conclude that

$$\left(\sum_{E \in \mathcal{G}_h^1} \frac{h_E}{k_1} \left\| \mu_h - k_1 \frac{\partial v_{1,h}}{\partial n} \right\|_{0, E}^2 \right)^{1/2}$$

$$\lesssim \sqrt{k_1} \|\nabla(u_1 - v_{1,h})\|_{0, \Omega_1} + \frac{1}{\sqrt{k_1}} \|\lambda - \mu_h\|_{-\frac{1}{2}, \Gamma} \tag{4.13}$$

$$+ \left(\sum_{E \in \mathcal{G}_h^1} \text{osc}_{K(E)}(f)^2 \right)^{1/2}.$$

The estimate in Ω_2 is proven similarly. Adding the estimates in Ω_1 and Ω_2 leads to (4.11). □

The proof of the a priori estimate is now straightforward.

Theorem 3 (A priori estimate) *The exact solution* $(u, \lambda) \in V \times Q$ *of* (2.8) *and the discrete solution* $(u_h, \lambda_h) \in V_h \times Q_h$ *of* (3.5) *satisfy*

$$\|\|(u - u_h, \lambda - \lambda_h)\|\|$$

$$\lesssim \inf_{(v_h, \mu_h) \in V_h \times Q_h} \|\|(u - v_h, \lambda - \mu_h)\|\| + \left(\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \text{osc}_{K(E)}(f)^2 \right)^{1/2}. \tag{4.14}$$

Proof The discrete stability estimate guarantees the existence of $(w_h, \xi_h) \in V_h \times Q_h$, with $\|\|(w_h, \xi_h)\|\|_h = 1$, such that for any $(v_h, \mu_h) \in V_h \times Q_h$ it holds

$$\|\|(u_h - v_h, \lambda_h - \mu_h)\|\| \leq \|\|(u_h - v_h, \lambda_h - \mu_h)\|\|_h \lesssim \mathcal{B}_h(u_h - v_h, \lambda_h - \mu_h; w_h, \xi_h).$$

We have

$$\mathcal{B}_h(u_h - v_h, \lambda_h - \mu_h; w_h, \xi_h) = \mathcal{B}(u - v_h, \lambda - \mu_h; w_h, \xi_h) + \alpha \mathcal{S}_h(v_h, \mu_h; w_h, \xi_h).$$

The first term above is estimated using the continuity of $\mathcal{B}$ in the continuous norm

$$\mathcal{B}(u - v_h, \lambda - \mu_h; w_h, \xi_h) \lesssim \|\|(u - v_h, \lambda - \mu_h)\|\| \cdot \|\|(w_h, \xi_h)\|\|$$
$$\lesssim \|\|(u - v_h, \lambda - \mu_h)\|\|.$$

For the second term, the Cauchy–Schwarz inequality, Lemma 2 and the discrete trace estimate yield

$$|\mathcal{S}_h(v_h, \mu_h; w_h, \xi_h)|$$

$$\lesssim \left(\|\|(u - v_h, \lambda - \mu_h)\|\| + \left(\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \mathrm{osc}_{K(E)}(f)^2 \right)^{1/2} \right) \cdot \|\|(w_h, \xi_h)\|\|_h$$

$$\lesssim \|\|(u - v_h, \lambda - \mu_h)\|\| + \left(\sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} \mathrm{osc}_{K(E)}(f)^2 \right)^{1/2}.$$

$\square$

Remark 3 *(Method II)* The discrete stability can be established in the norm

$$\left(\sum_{i=1}^{2} k_i \|\nabla w_i\|_{0,\Omega_i}^2 + \frac{1}{k_2} \|\xi\|_{-\frac{1}{2},\Gamma}^2 + \sum_{E \in \mathcal{G}_h^2} \frac{h_E}{k_2} \|\xi_h\|_{0,E}^2 \cdot \right)^{1/2}$$

and, as seen from its proof, Lemma 2 is valid individually for both stabilising terms. We thus obtain the a priori estimate

$$\|\|(u - u_h, \lambda - \lambda_h)\|\|$$

$$\lesssim \inf_{(v_h, \mu_h) \in V_h \times Q_h} \|\|(u - v_h, \lambda - \mu_h)\|\| + \left(\sum_{E \in \mathcal{G}_h^2} \mathrm{osc}_{K(E)}(f)^2 \right)^{1/2}, \tag{4.15}$$

where

$$\|\|(w, \xi)\|\| = \left(\sum_{i=1}^{2} k_i \|\nabla w_i\|_{0,\Omega_i}^2 + \frac{1}{k_2} \|\xi\|_{-\frac{1}{2},\Gamma}^2 \right)^{1/2}. \tag{4.16}$$

Remark 4 *(Method III)* The analysis of the third method is similar, albeit a bit more cumbersome. The crucial observation is that we can write

$$\xi - \left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\} = \alpha_1 \left(\xi - k_1 \frac{\partial w_1}{\partial n} \right) + \alpha_2 \left(\xi - k_2 \frac{\partial w_2}{\partial n} \right),$$

where

$$\alpha_1 = \frac{k_2 h_1}{k_2 h_1 + k_1 h_2}, \quad \alpha_2 = \frac{k_1 h_2}{k_2 h_1 + k_1 h_2}.$$

Given that $0 \leq \alpha_i(x) \leq 1 \ \forall x \in \Gamma, i = 1, 2$, and $\alpha_1 + \alpha_2 = 1$, it can be verified using the triangle inequality that the a priori estimate of Theorem 3 holds also for Method III.

5 A posteriori estimate

Let us first define local residual estimators corresponding to the finite element solution (u_h, λ_h) through

$$\eta_K^2 = \frac{h_K^2}{k_i} \|\nabla \cdot k_i \nabla u_{i,h} + f\|_{0,K}^2, \quad K \in \mathcal{C}_h^i, \tag{5.1}$$

$$\eta_{E,\Omega}^2 = \frac{h_E}{k_i} \left\| \left[\!\!\left[k_i \frac{\partial u_{i,h}}{\partial n} \right]\!\!\right] \right\|_{0,E}^2, \quad E \in \mathcal{E}_h^i, \tag{5.2}$$

$$\eta_{E,\Gamma}^2 = \frac{h_E}{k_i} \left\| \lambda_h - k_i \frac{\partial u_{i,h}}{\partial n} \right\|_{0,E}^2 + \frac{k_i}{h_E} \| [\![u_h]\!] \|_{0,E}^2, \quad E \in \mathcal{G}_h^i. \tag{5.3}$$

with $i = 1, 2$. The global error estimator is then denoted by

$$\eta^2 = \sum_{i=1}^{2} \left(\sum_{K \in \mathcal{C}_h^i} \eta_K^2 + \sum_{E \in \mathcal{E}_h^i} \eta_{E,\Omega}^2 + \sum_{E \in \mathcal{G}_h^i} \eta_{E,\Gamma}^2 \right). \tag{5.4}$$

In the following theorem we show that the error estimator η is both efficient and reliable.

Theorem 4 (A posteriori estimate) *It holds that*

$$\||(u - u_h, \lambda - \lambda_h)\|| \lesssim \eta \tag{5.5}$$

and

$$\eta \lesssim \||(u - u_h, \lambda - \lambda_h)\|| + \left(\sum_{i=1}^{2} \sum_{K \in \mathcal{C}_h^i} \mathrm{osc}_K(f)^2 \right)^{1/2}. \tag{5.6}$$

Proof The continuous stability estimate of Theorem 1 guarantees the existence of a pair $(v, \mu) \in V \times Q$, with $\||(v, \mu)\|| = 1$, that satisfies

$$\||(u - u_h, \lambda - \lambda_h)\|| \lesssim \mathcal{B}(u - u_h, \lambda - \lambda_h; v, \mu). \tag{5.7}$$

Let $I_h v \in V_h$ be the Clément interpolant of $v \in V$. The stabilised method is consistent, thus

$$\mathcal{B}_h(u - u_h, \lambda - \lambda_h; I_h v, 0) = 0.$$

Therefore, we can write

$$\mathcal{B}(u - u_h, \lambda - \lambda_h; v, \mu) = \mathcal{B}(u - u_h, \lambda - \lambda_h; v - I_h v, \mu) + \alpha \mathcal{S}_h(u_h, \lambda_h; I_h v, 0).$$

After integration by parts, the first term yields

$$
\begin{aligned}
\mathcal{B}(u &- u_h, \lambda - \lambda_h; v - I_h v, \mu) \\
&= \mathcal{L}(v - I_h v) - \mathcal{B}(u_h, \lambda_h; v - I_h v, \mu) \\
&= \sum_{i=1}^{2} \left[\sum_{K \in \mathcal{C}_h^i} (\nabla \cdot k_i \nabla u_{i,h} + f_i, v_i - I_h v_i)_K + \sum_{E \in \mathcal{E}_h^i} \left(\left[\!\!\left[k_i \frac{\partial u_{i,h}}{\partial n} \right]\!\!\right], v_i - I_h v_i \right)_E \right. \\
&\qquad \left. + \sum_{E \in \mathcal{G}_h^i} \left(\lambda_h - k_i \frac{\partial u_{i,h}}{\partial n}, v_i - I_h v_i \right)_E \right] - \langle [\![u_h]\!], \mu \rangle.
\end{aligned}
\tag{5.8}
$$

On the other hand, for the Clément interpolant it holds

$$
\begin{aligned}
k_i \| \nabla I_h v_i \|_{0, \Omega_i}^2 + \sum_{K \in \mathcal{C}_h^i} \frac{k_i}{h_K^2} \| v_i - I_h v_i \|_{0, K}^2 + \sum_{E \in \mathcal{E}_h^i} \frac{k_i}{h_E} \| v_i - I_h v_i \|_{0, E}^2 \\
\lesssim k_i \| \nabla v_i \|_{0, \Omega_i}^2.
\end{aligned}
\tag{5.9}
$$

For the first three terms in (5.8), we thus get

$$
\begin{aligned}
\sum_{K \in \mathcal{C}_h^i} (\nabla \cdot k_i \nabla u_{i,h} + f_i, v_i - I_h v_i)_K + \sum_{E \in \mathcal{E}_h^i} \left(\left[\!\!\left[k_i \frac{\partial u_{i,h}}{\partial n} \right]\!\!\right], v_i - I_h v_i \right)_E \\
+ \sum_{E \in \mathcal{G}_h^i} \left(\lambda_h - k_i \frac{\partial u_{i,h}}{\partial n}, v_i - I_h v_i \right)_E \lesssim \eta \, \| k_i \nabla v_i \|_{0, \Omega_i} \lesssim \eta.
\end{aligned}
\tag{5.10}
$$

The last term in (5.8) is estimated using the following discrete inverse inequality for the $H_{00}^{1/2}(\Gamma)$ norm (cf. [1,13])

$$\| v_h \|_{1/2, \Gamma}^2 \lesssim \sum_{E \in \mathcal{G}_h^i} h_E^{-1} \| v_h \|_{0, E}^2 \quad \forall v_h \in V_h,$$

viz.

$$
\begin{aligned}
-\langle [\![u_h]\!], \mu \rangle &\leq \| [\![u_h]\!] \|_{1/2, \Gamma} \| \mu \|_{-1/2, \Gamma} \\
&\lesssim \sum_{i=1}^{2} \left(\sum_{E \in \mathcal{G}_h^i} \frac{k_i}{h_E} \| [\![u_h]\!] \|_{0, E}^2 \right)^{1/2} \sqrt{k_i} \| \mu \|_{-1/2, \Gamma}.
\end{aligned}
$$

On the other hand, from the Cauchy–Schwarz and the discrete trace inequalities and from (5.9), it follows that

$$\mathcal{S}_h(u_h, \lambda_h; I_h v, 0) \lesssim \eta. \tag{5.11}$$

The upper bound (5.5) can now be established by joining the above estimates.

The lower bound (5.6) follows from Lemma 2 together with standard lower bounds, cf. [27]. □

We end this section by reiterating that the purpose of the mixed stabilised formulation is to perform the error analysis. We advocate the use of Nitsche's formulation for computations and note that substituting the discrete Lagrange multiplier (3.9) in the error indicators, we obtain for $E \in \mathcal{G}_h^1$

$$\frac{h_E}{k_1} \left\| \lambda_h - k_1 \frac{\partial u_{1,h}}{\partial n} \right\|_{0,E}^2 = \frac{h_E}{k_1} \left\| \alpha_2 \left[\!\!\left[k \frac{\partial u_h}{\partial n} \right]\!\!\right] + \beta \left[\!\!\left[u_h \right]\!\!\right] \right\|_{0,E}^2 \tag{5.12}$$

and for $E \in \mathcal{G}_h^2$

$$\frac{h_E}{k_2} \left\| \lambda_h - k_2 \frac{\partial u_{2,h}}{\partial n} \right\|_{0,E}^2 = \frac{h_E}{k_2} \left\| \alpha_1 \left[\!\!\left[k \frac{\partial u_h}{\partial n} \right]\!\!\right] - \beta \left[\!\!\left[u_h \right]\!\!\right] \right\|_{0,E}^2. \tag{5.13}$$

Remark 5 *(Method II)* The local estimators $\eta_K, \eta_{E,\Omega}$ and $\eta_{E,\Gamma}$ are defined through (5.1)–(5.3) and the estimates (5.5) and (5.6) hold true in the norm (4.16). After substituting the discrete Lagrange multiplier

$$\lambda_h = k_2 \frac{\partial u_{2,h}}{\partial n} - \alpha^{-1} \frac{k_2}{h_2} \left[\!\!\left[u_h \right]\!\!\right]$$

into the error indicators, we for the corresponding Nitsche's formulation obtain

$$\frac{h_E}{k_1} \left\| \lambda_h - k_1 \frac{\partial u_{1,h}}{\partial n} \right\|_{0,E}^2 = \frac{h_E}{k_1} \left\| \left[\!\!\left[k \frac{\partial u_h}{\partial n} \right]\!\!\right] + \alpha^{-1} \frac{k_2}{h_2} \left[\!\!\left[u_h \right]\!\!\right] \right\|_{0,E}^2 \tag{5.14}$$

for $E \in \mathcal{G}_h^2$, and

$$\frac{h_E}{k_2} \left\| \lambda_h - k_2 \frac{\partial u_{2,h}}{\partial n} \right\|_{0,E}^2 = \alpha^{-2} \frac{k_2}{h_E} \left\| \left[\!\!\left[u_h \right]\!\!\right] \right\|_{0,E}^2 \tag{5.15}$$

for $E \in \mathcal{G}_h^2$.

Remark 6 *(Method III)* Once again the local estimators for the stabilised method are defined as in (5.1)–(5.3) and the a posteriori estimates (5.5) and (5.6) hold true. In the Nitsche's formulation, the error indicators depending on the Lagrange multiplier are given by (5.12) and (5.13).
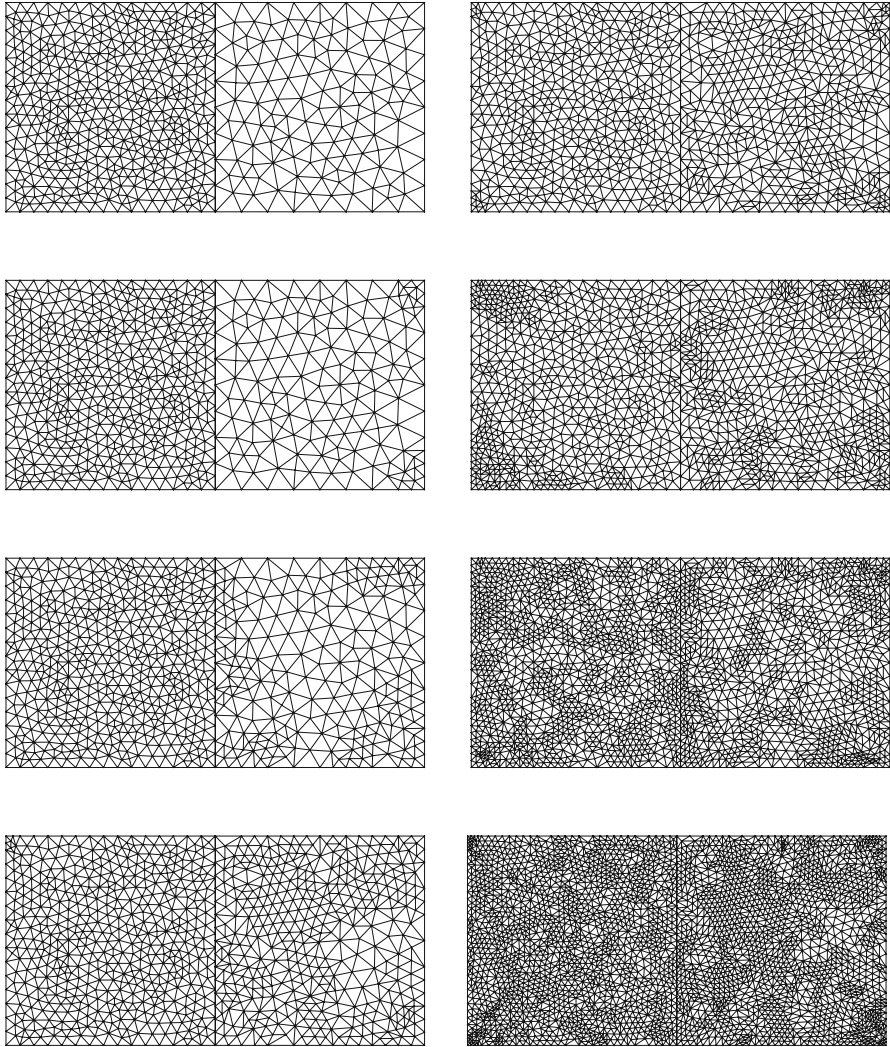
Fig. 1 The sequence of adaptive meshes with $k_1 = k_2 = 1$

6 Numerical results

We experiment with the proposed method by solving the domain decomposition problem adaptively with $\Omega_1 = (0, 1)^2$, $\Omega_2 = (1, 2) \times (0, 1)$, $f = 1$, $\alpha = 10^{-2}$ and linear elements. After each solution we mark a triangle $K \in \mathcal{C}_h^i$, $i = 1, 2$, for refinement if it satisfies $\mathcal{E}_K > \theta \max_{K' \in \mathcal{C}_h^1 \cup \mathcal{C}_h^2} \mathcal{E}_{K'}$ where $\theta = \frac{1}{\sqrt{2}}$ and
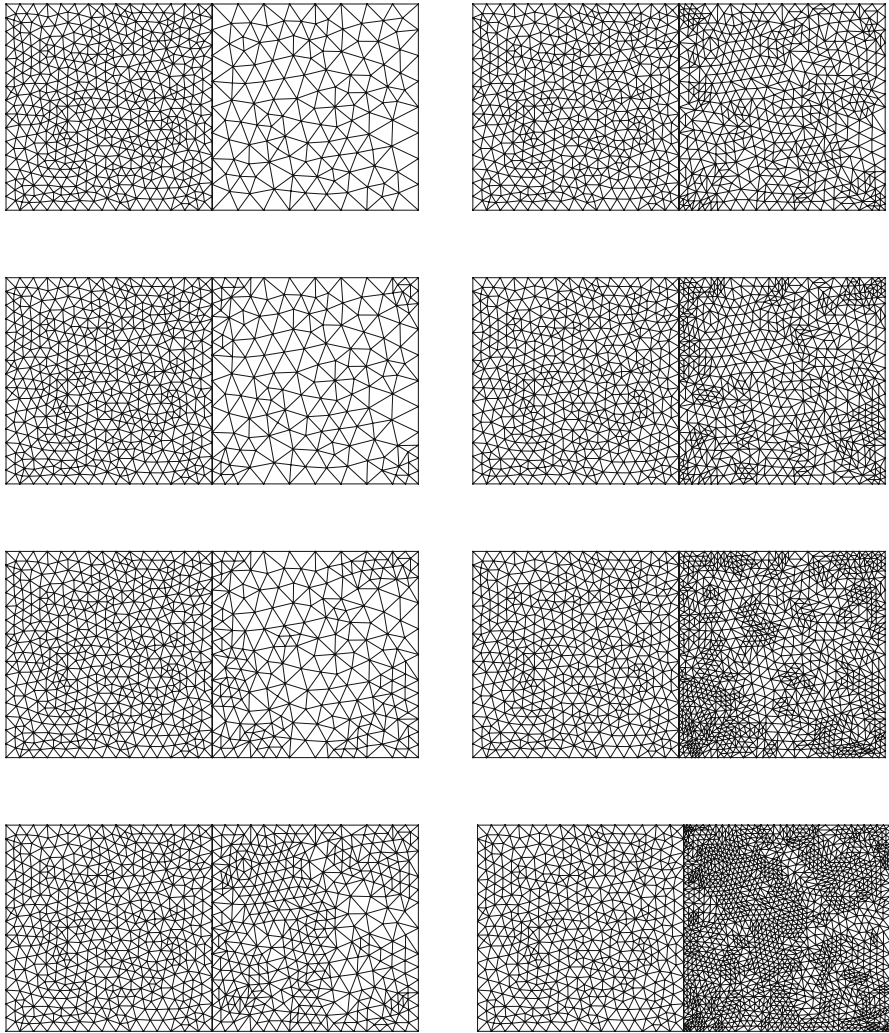
Fig. 2 The sequence of adaptive meshes with $k_1 = 10$ and $k_2 = 0.1$

$$\mathcal{E}_K^2 = \frac{h_K^2}{k_i} \|\nabla \cdot k_i \nabla u_{i,h} + f\|_{0,K}^2 + \frac{1}{2} \sum_{E \subset \partial K \backslash \Gamma} \frac{h_E}{k_i} \left\| \left[\!\!\left[k_i \frac{\partial u_{i,h}}{\partial n} \right]\!\!\right] \right\|_{0,E}^2$$

$$+ \sum_{E \subset \partial K \cap \Gamma} \left\{ \frac{h_E}{k_i} \left\| \lambda_h - k_i \frac{\partial u_{i,h}}{\partial n} \right\|_{0,E}^2 + \frac{k_i}{h_E} \| [\![u_h]\!] \|_{0,E}^2 \right\} \quad \forall K \in \mathcal{C}_h^i.$$

The set of marked elements is refined using the red-green-blue strategy, see e.g. Bartels [3].
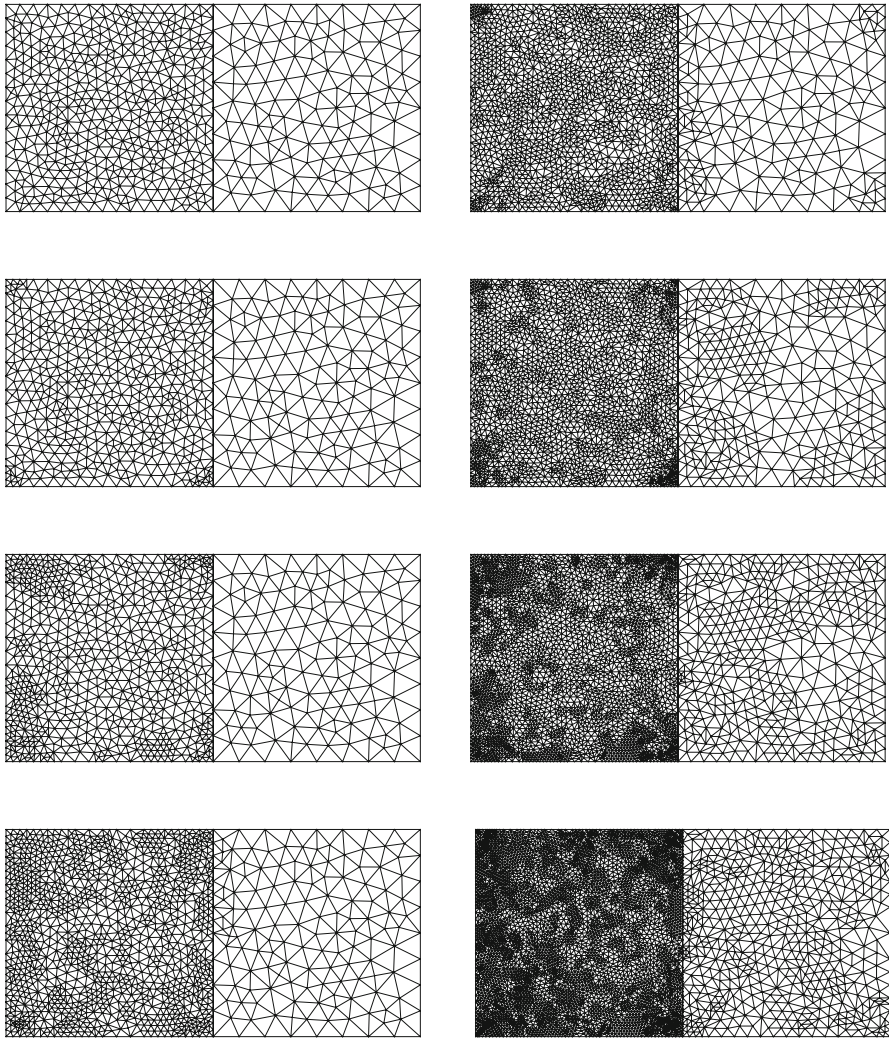
Fig. 3 The sequence of adaptive meshes with $k_1 = 0.1$ and $k_2 = 10$

Changing the material parameters from $(k_1, k_2) = (1, 1)$ to $(k_1, k_2) = (10, 0.1)$, and finally to $(k_1, k_2) = (0.1, 10)$ produces adaptive meshes where the domain with a smaller material parameter receives more elements, see Figs. 1, 2 and 3. This is in accordance with results on adaptive methods for linear elastic contact problems, see e.g. Wohlmuth [28] where it is demonstrated that softer the material, more the respective domain is refined.

Next we solve the domain decomposition problem in an L-shaped domain with $\Omega_1 = (0, 1)^2$, $\Omega_2 = (1, 2) \times (0, 2)$ and $k_1 = k_2 = 1$. The resulting sequence of meshes is depicted in Fig. 4 and the global error estimator as a function of the number of degrees-of-freedom N is given in Fig. 5. Note that the exact solution is
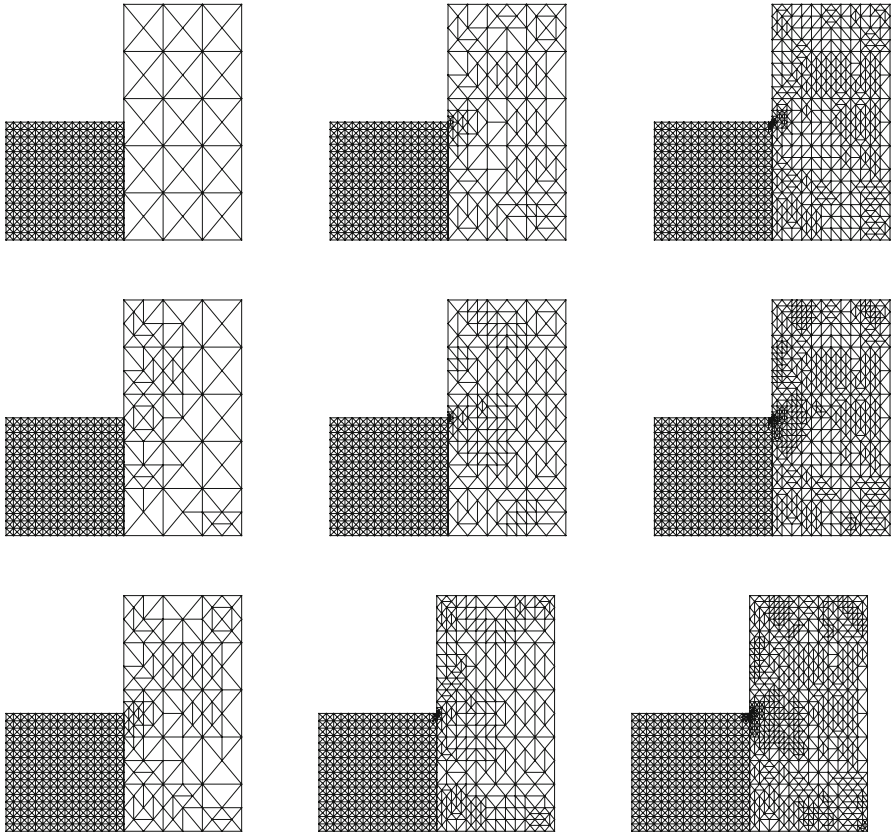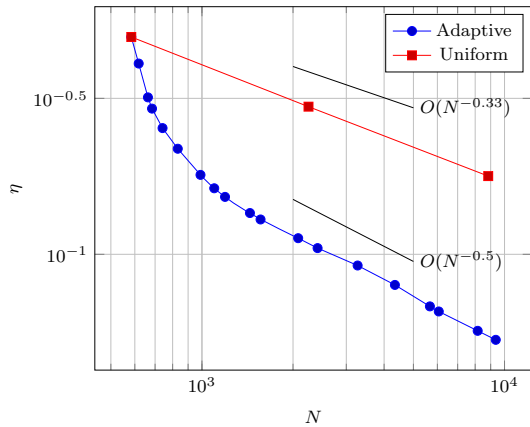
Fig. 4 The sequence of adaptive meshes on an L-shaped domain with $k_1 = k_2 = 1$

Fig. 5 The global error estimator η as a function of the number of degrees-of-freedom N in the L-shaped domain case

in $H^{5/3-\varepsilon}$, $\varepsilon > 0$, in the neighbourhood of the reentrant corner which limits the convergence rate of uniform refinements to $O(N^{-1/3})$.

We finally remark that Methods II and III yield very similar numerical results.

References

1. Ainsworth, M., Kelly, D.W.: A posteriori error estimators and adaptivity for finite element approximation of the non-homogeneous Dirichlet problem. Adv. Comput. Math. **15**(2001), 3–23 (2002)
2. Babuška, I.: The finite element method with Lagrangian multipliers. Numer. Math. **20**, 179–192 (1973)
3. Bartels, S.: Numerical Approximation of Partial Differential Equations, vol. 64 of Texts in Applied Mathematics. Springer, Berlin (2016)
4. Becker, R., Hansbo, P., Stenberg, R.: A finite element method for domain decomposition with non-matching grids. ESAIM Math. Model. Numer. Anal. **37**, 209–225 (2003)
5. Burman, E., Hansbo, P.: Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method. Appl. Numer. Math. **62**, 328–341 (2012)
6. Burman, E., Hansbo, P.: Fictitious domain methods using cut elements: III. A stabilized Nitsche method for Stokes' problem. ESAIM Math. Model. Numer. Anal. **48**, 859–874 (2014)
7. Burman, E., Hansbo, P., Larson, M.G.: The penalty-free Nitsche method and nonconforming finite elements for the Signorini problem. SIAM J. Numer. Anal. **55**, 2523–2539 (2017)
8. Chouly, F., Fabre, M., Hild, P., Mlika, R., Pousin, J., Renard, Y.: An overview of recent results on Nitsche's method for contact problems. In: Bordas, S., Burman, E., Larson, M., Olshanskii, M. (eds.) Geometrically Unfitted Finite Element Methods and Applications, vol. 121 of Lecture Notes in Computational Science and Engineering, pp. 93–141. Springer, Berlin (2017)
9. Chouly, F., Fabre, M., Hild, P., Pousin, J., Renard, Y.: Residual-based a posteriori error estimation for contact problems approximated by Nitsche's method. IMA J. Numer. Anal. **38**, 921–954 (2018)
10. Chouly, F., Hild, P., Renard, Y.: A Nitsche finite element method for dynamic contact: 1. Space semi-discretization and time-marching schemes. ESAIM Math. Model. Numer. Anal. **49**, 481–502 (2015)
11. Chouly, F., Hild, P., Renard, Y.: A Nitsche finite element method for dynamic contact: 2. Stability of the schemes and numerical experiments. ESAIM Math. Model. Numer. Anal. **49**, 503–528 (2015)
12. Chouly, F., Hild, P., Renard, Y.: Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments. Math. Comput. **84**, 1089–1112 (2015)
13. Dahmen, W., Faermann, B., Graham, I.G., Hackbusch, W., Sauter, S.A.: Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method. Math. Comput. **73**, 1107–1138 (2004)
14. Fabre, M., Pousin, J., Renard, Y.: A fictitious domain method for frictionless contact problems in elasticity using Nitsche's method. SMAI J. Comput. Math. **2**, 19–50 (2016)
15. Franca, L.P., Stenberg, R.: Error analysis of Galerkin least squares methods for the elasticity equations. SIAM J. Numer. Anal. **28**, 1680–1697 (1991)
16. Gustafsson, T., Stenberg, R., Videman, J.: On Nitsche's method for elastic contact problems. Preprint. arXiv:1902.09312
17. Gustafsson, T., Stenberg, R., Videman, J.: Mixed and stabilized finite element methods for the obstacle problem. SIAM J. Numer. Anal. **55**, 2718–2744 (2017)
18. Heinrich, B., Nicaise, S.: The Nitsche mortar finite-element method for transmission problems with singularities. IMA J. Numer. Anal. **23**, 331–358 (2003)
19. Juntunen, M.: On the connection between the stabilized Lagrange multiplier and Nitsche's methods. Numer. Math. **131**, 453–471 (2015)
20. Juntunen, M.: On the local mesh size of Nitsche's method for discontinuous material parameters. In: Abdulle, A., Deparis, S., Kressner, D., Nobile, F., Picasso, M. (eds.) Numerical Mathematics and

Advanced Applications ENUMATH 2013, vol. 103 of Lecture Notes in Computational Science and Engineering, pp. 57–63. Springer, Berlin (2015)

21. Juntunen, M., Stenberg, R.: Nitsche's method for discontinuous material parameters. In: Persson, K., Revstedt, J., Sandberg, G., Wallin, M. (eds.) Proceedings of the 25th Nordic Seminar on Computational Mechanics, Lund University (pp. 95–98) (2012)

22. Lions, J.-L., Magenes, E.: Non-homogeneous Boundary Value Problems and Applications, vol. 1. Springer-Verlag, Berlin, Heidelberg (1972)

23. Nitsche, J.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Sem. Univ. Hamburg **36**, 9–15 (1971)

24. Stenberg, R.: On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**, 139–148 (1995)

25. Stenberg, R.: Mortaring by a method of J. A. Nitsche. In: Idelsohn, S., Oñate, E., Dvorkin, E. (eds.) Computational Mechanics—New Trends and Applications. CIMNE, Barcelona (1998)

26. Tartar, L.: An Introduction to Sobolev Spaces and Interpolation Spaces, vol. 3 of Lecture Notes of the Unione Matematica Italiana. Springer, Berlin (2007)

27. Verfürth, R.: A Posteriori Error Estimation Techniques for Finite Element Methods. Oxford University Press, Oxford (2013)

28. Wohlmuth, B.: Variationally consistent discretization schemes and numerical algorithms for contact problems. Acta Numerica **20**, 569–734 (2011)