Numer. Math. (2018) 139:709–741
https://doi.org/10.1007/s00211-018-0946-6

Numerische Mathematik

Stability and error analysis for a diffuse interface approach to an advection–diffusion equation on a moving surface

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Received: 2 December 2016 / Revised: 18 September 2017 / Published online: 25 January 2018

Abstract In this paper we analyze a fully discrete numerical scheme for solving a parabolic PDE on a moving surface. The method is based on a diffuse interface approach that involves a level set description of the moving surface. Under suitable conditions on the spatial grid size, the time step and the interface width we obtain stability and error bounds with respect to natural norms. Furthermore, we present test calculations that confirm our analysis.

Mathematics Subject Classification 65M60 · 65M15 · 58J32

1 Introduction

Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a family of closed hypersurfaces in $\mathbb{R}^{n+1}(n = 1, 2)$ evolving in time. In this paper we consider a finite element approach for solving the parabolic surface PDE equation

$$\partial_t^\bullet u + u\nabla_\Gamma \cdot \boldsymbol{v} - \Delta_\Gamma u = f \qquad \text{on } S_T \tag{1}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma(0), \tag{2}$$

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which models advection and diffusion of a surface quantity u with $u(\cdot,t) : \Gamma(t) \to \mathbb{R}$. Here, $S_T = \bigcup_{t\in(0,T)} \big(\Gamma(t) \times \{t\}\big)$ and $\boldsymbol{v} : \overline{S_T} \to \mathbb{R}^{n+1}$ denotes a given velocity field. Furthermore, ∇_Γ is the tangential gradient, $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ the Laplace Beltrami operator and $\partial_t^\bullet = \partial_t + \boldsymbol{v} \cdot \nabla$ denotes the material derivative.

Parabolic surface PDEs of the form (1) have applications in fluid dynamics and materials science, such as the transport and diffusion of surfactants on a fluid/fluid interface, [25] or diffusion-induced grain boundary motion, [5]. In these as in several other applications the velocity $\boldsymbol{v}$ is not given but determined through an additional equation so that (1) becomes a subproblem of a more complicated system in which the variable u is coupled to other variables. The analysis and the numerical solution of such systems then naturally requires the development of corresponding methods for (1). We refer to [13] for a comprehensive overview of finite element methods for solving PDEs on stationary and evolving surfaces.

Concerning the numerical methods that have been proposed for (1) one may distinguish between Lagrangian and Eulerian type schemes. The first approach has been pursued by Dziuk and Elliott within their evolving surface finite element method, [8], which uses polyhedral approximations of the evolving hypersurfaces $\Gamma(t)$. While [8] contains an error analysis in the spatially discrete case, the fully discrete case is investigated in [11,14] and [19]. Optimal L^2-error bounds are obtained in [12] and a corresponding finite volume approach is proposed and analyzed in [18]. Since the mesh for the discretization of (1) is fitted to the hypersurface $\Gamma(t)$, a coupling to a bulk equation is not straightforward. This difficulty is not present in Eulerian type schemes, in which $\Gamma(t)$ is typically described via a level set function defined in an open neighbourhood of $\Gamma(t)$. In order to discretize the surface PDE in this setting it has been proposed in [1,3] and [27] to extend the surface quantity u to a band around $\Gamma(t)$ and to solve a suitable (weakly) parabolic PDE in that bulk region using a finite difference method. In [9] and [10], the same idea is used in a finite element context for which the underlying variational formulation is derived with the help of a transport identity. An Eulerian finite element approach that doesn't use an extended PDE is proposed and analyzed in [20] and [21]. The method is based on a weak formulation on the space-time manifold and the finite element space is obtained by taking traces of the corresponding bulk finite elements. The approximation of $\Gamma(t)$ on which these spaces are defined usually arises from a suitable interpolation of the given level set function describing $\Gamma(t)$. The resulting discrete hypersurface will in general cut arbitrarily through the background mesh and its location forms one of the main difficulties in implementing the scheme. A different approach of generating the discrete hypersurfaces is pursued in [17], where a discretization of (4) below is combined with the cut finite element technique. Finally, Section 5 in [7] proposes a hybrid method that employs the above–mentioned idea of trace finite elements together with a narrow band technique for the elliptic part of the PDE.

In this paper we are concerned with the diffuse interface approach for solving (1), which was introduced in [22] for a stationary surface and in [16,23] and [26] for evolving surfaces. As in some of the methods described above, the surface quantity u is extended to a bulk quantity satisfying a suitable parabolic PDE in a neighbourhood of $\Gamma(t)$ and the bulk equation is then localized to a thin layer of thickness ϵ with the help of a phase field function (see [15] for a corresponding convergence analysis). Since we are interested in using finite elements, the localized PDE needs to be written in a suitable variational form. Following [16] this is achieved with the help of a transport identity and results in a discretization by linear finite elements in space and a backward Euler scheme in time. The detailed derivation along with an existence result for the discrete solution will be given in Sect. 3. As the main new contribution of our paper we shall derive conditions relating the interface width ϵ, the spatial grid size h and the time step τ which allow for a rigorous stability and error analysis. More precisely, we shall prove that the numerical solution is bounded uniformly in $L^\infty(L^2)$ and $L^2(H^1)$ over the diffuse interface (see Theorem 1 in Sect. 4) and that it converges with respect to these norms both over the diffuse interface and on the sharp interface with an order $O(\epsilon)$ provided that

$$h \le c_1\epsilon, \;\; \tau \le c_2\epsilon^2,$$

see Theorem 2 and Corollary 1 in Sect. 5 respectively. In Sect. 6 we report on results of numerical tests both for $n = 1$ and $n = 2$.

An advantage of our approach is that in the implementation the evolution of the hypersurfaces is easily incorporated by evaluating the phase field function. We shall employ a function with compact support, namely $\rho(x,t) := g(\frac{\phi(x,t)}{\epsilon})$, where $\Gamma(t)$ is the zero level set of $\phi(\cdot,t)$ and

$$g(r) = \begin{cases} \cos^2(r), & |r| \le \frac{\pi}{2}, \\ 0, & |r| > \frac{\pi}{2}. \end{cases}$$

In view of the evolution of the hypersurfaces the numerical scheme then naturally contains terms in which ρ is evaluated at different times. One of the main challenges in the analysis is to handle the corresponding differences, for which one has to bound integrals that are multiplied by a negative power of ϵ (arising from derivatives of ρ) as well as integrals that are not weighted with ρ. We shall deal with these difficulties by introducing an additional stabilization term with extended support that is also used for proving the well–posedness of the scheme.

Let us finally remark that a phase field approach involving a phase field function with noncompact support and finite elements has been proposed in [4] for an elliptic surface PDE. Theorem 7 in [4] provides an error estimate in terms of an approximation error and an error due to the phase field representation. The latter decays at a rate $O(\epsilon^p)$ for some $p < 1$, while a coupling between ϵ and the grid size h is not discussed.

2 Preliminaries

2.1 Surface representation and surface derivatives

For each $t \in [0,T]$ let $\Gamma(t) \subset \mathbb{R}^{n+1}$ $(n = 1, 2)$ be a connected, compact and orientable hypersurface without boundary. We suppose that $\boldsymbol{v} : \overline{S_T} \to \mathbb{R}^{n+1}$ is a prescribed velocity field of the form

$$\boldsymbol{v} = V\nu + \boldsymbol{v_\tau}, \quad \text{with } (\boldsymbol{v_\tau}, \nu) = 0. \tag{3}$$

Here, ν is a unit normal and V the corresponding normal velocity of $\Gamma(t)$ and $(\cdot, \cdot)$ denotes the Euclidian scalar product in $\mathbb{R}^{n+1}$. Note that the normal part $V\nu$ is responsible for the geometric motion of $\Gamma(t)$, while the tangential part $\boldsymbol{v_\tau}$ is associated with the transport of material along the surface. We assume that there exists a smooth map $\Psi : \Gamma(0) \times [0, T] \to \mathbb{R}^{n+1}$ such that $\Psi(\cdot, t)$ is a diffeomorphism from $\Gamma(0)$ onto $\Gamma(t)$ for every $t \in [0, T]$ satisfying

$$\frac{\partial \Psi}{\partial t}(P, t) = \boldsymbol{v}(\Psi(P, t), t), \quad P \in \Gamma(0), t \in (0, T]; \tag{4}$$

$$\Psi(P, 0) = P, \qquad P \in \Gamma(0). \tag{5}$$

Let us next introduce the differential operators which are required to formulate our PDE. To begin, for fixed t and a function $\eta : \Gamma(t) \to \mathbb{R}$ we denote by $\nabla_\Gamma \eta = (\underline{D}_1 \eta, \ldots, \underline{D}_{n+1} \eta)$ its tangential gradient. If $\bar{\eta}$ is an extension of η to an open neighbourhood of $\Gamma(t)$ then

$$\nabla_\Gamma \eta(x) = \big(I - \nu(x, t) \otimes \nu(x, t)\big)\nabla \bar{\eta}(x), \quad x \in \Gamma(t). \tag{6}$$

Furthermore, $\Delta_\Gamma \eta = \nabla_\Gamma \cdot \nabla_\Gamma \eta = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i \eta$ denotes the Laplace-Beltrami operator.

Next, for a smooth function η on S_T we define the material derivative of η at $(x, t) = (\Psi(P, t), t)$ by $\partial_t^\bullet \eta(x, t) := \frac{d}{dt}[\eta(\Psi(P, t), t)]$. If $\bar{\eta}$ is an extension of η to an open space-time neighbourhood, then

$$\partial_t^\bullet \eta(x, t) = \bar{\eta}_t(x, t) + (\boldsymbol{v}(x, t), \nabla \bar{\eta}(x, t)), \quad (x, t) \in S_T.$$

Our numerical approach will be based on an implicit representation of $\Gamma(t)$, so that we suppose in what follows that there exists a smooth function $\phi : \Omega \times [0, T] \to \mathbb{R}$ such that for $0 \le t \le T$

$$\Gamma(t) = \{x \in \Omega \mid \phi(x, t) = 0\} \quad \text{and} \quad \nabla \phi(x, t) \neq 0, \; x \in \Gamma(t). \tag{7}$$

Here, $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain with $\Gamma(t) \subset \Omega$ for all $t \in [0, T]$. For later use we introduce for $t \in [0, T]$, $r > 0$ the sets

$$U_r(t) := \{x \in \Omega \mid |\phi(x, t)| < r\} \quad \text{and} \quad \mathcal{U}_{r,T} := \bigcup_{t \in [0,T]} \big(U_r(t) \times \{t\}\big).$$

In view of (7) there exist $\delta_0 > 0$, $0 < c_0 \le c_1$, $c_2 > 0$ such that $\overline{U_{\delta_0}(t)} \subset \Omega$, $0 \le t \le T$ and

$$c_0 \le |\nabla \phi(x, t)| \le c_1, \; |D^2 \phi(x, t)|, \; |\phi_t(x, t)|, \; |\phi_{tt}(x, t)| \le c_2, \quad (x, t) \in \mathcal{U}_{\delta_0, T}. \tag{8}$$

2.2 Extension

Our next aim is to extend functions defined on S_T to a space-time neighbourhood. A common approach which is well suited to a description of $\Gamma(t)$ via the signed distance function consists in extending constantly in the normal direction. In what follows we shall introduce a suitable generalization to the case (7). Consider for $P \in \Gamma(0)$ and $t \in [0, T]$ the parameter-dependent system of ODEs

$$\gamma_{P,t}'(s) = \frac{\nabla\phi(\gamma_{P,t}(s), t)}{|\nabla\phi(\gamma_{P,t}(s), t)|^2}, \quad \gamma_{P,t}(0) = \Psi(P, t). \tag{9}$$

Using a compactness argument it can be shown that there exists $0 < \delta < \delta_0$ so that the solution $\gamma_{P,t}$ of (9) exists uniquely on $(-\delta, \delta)$ uniformly in $P \in \Gamma(0), t \in [0, T]$. Thus we can define the smooth mapping $F_t : \Gamma(0) \times (-\delta, \delta) \to \mathbb{R}^{n+1}$ by

$$F_t(P, s) := \gamma_{P,t}(s), \quad P \in \Gamma(0), |s| < \delta. \tag{10}$$

In view of the chain rule and (9) we immediately see that $\frac{d}{ds}\phi(\gamma_{P,t}(s), t) = 1$, which implies that $\phi(\gamma_{P,t}(s), t) = s$, $|s| < \delta$ since $\gamma_{P,t}(0) = \Psi(P, t) \in \Gamma(t)$. In particular, $x = F_t(P, s)$ yields that $|\phi(x, t)| < \delta$ and it is not difficult to verify that F_t is a diffeomorphism of $\Gamma(0) \times (-\delta, \delta)$ onto $U_\delta(t)$ for $t \in [0, T]$, whose inverse has the form

$$F_t^{-1}(x) = (p(x, t), \phi(x, t)), \quad x \in U_\delta(t). \tag{11}$$

Here, $p : \mathcal{U}_{\delta,T} \to \mathbb{R}^{n+1}$ satisfies $p(x, t) \in \Gamma(0), x \in U_\delta(t)$. Furthermore, since $\phi(F_t(P, s), t) = s$ we deduce from (11) that

$$p(x, t) = P, \quad \text{if } x = F_t(P, s) \in U_\delta(t). \tag{12}$$

The function $\tilde{p} : \mathcal{U}_{\delta,T} \to \mathbb{R}^{n+1}$, $\tilde{p}(x, t) := \Psi(p(x, t), t)$ then is smooth and satisfies $\tilde{p}(x, t) \in \Gamma(t), 0 \le t \le T$. In addition we claim that

$$\tilde{p}(x, t) = x, \quad x \in \Gamma(t). \tag{13}$$

To see this, let $x \in \Gamma(t)$, say $x = \Psi(P, t) = \gamma_{P,t}(0) = F_t(P, 0)$ for some $P \in \Gamma(0)$. Using (12) with $s = 0$ we deduce that

$$\tilde{p}(x, t) = \Psi(p(x, t), t) = \Psi(P, t) = x,$$

proving (13). Let us next use $\tilde{p}$ in order to extend a function $z : \overline{S_T} \to \mathbb{R}$ to $\mathcal{U}_{\delta,T}$ by setting

$$z^e(x, t) := z(\tilde{p}(x, t), t), \quad (x, t) \in \mathcal{U}_{\delta,T}. \tag{14}$$

Clearly, $z^e(\cdot, t) = z(\cdot, t)$ on $\Gamma(t)$ by (13). Moreover, (12) implies for $P \in \Gamma(0)$, $|s| < \delta$

$$z^e(F_t(P, s), t) = z\big(\tilde{p}(F_t(P, s), t), t\big) = z\big(\Psi(p(F_t(P, s), t), t), t\big) = z(\Psi(P, t), t),$$

from which we obtain by differentiating with respect to s and using (9), (10) that

$$\big(\nabla z^e(x,t), \nabla \phi(x,t)\big) = 0, \quad (x,t) \in \mathcal{U}_{\delta,T}. \tag{15}$$

Lemma 1 *Let z^e be defined by (14). Then we have for $t \in [0,T]$, $0 < r < \delta$ and $|\alpha| = k \in \{0,1,2\}$:*

$$\|D^\alpha_x z^e(\cdot,t)\|_{L^2(U_r(t))} \le C\sqrt{r}\|z(\cdot,t)\|_{H^k(\Gamma(t))}; \tag{16}$$

$$\|D^\alpha_x z^e_t(\cdot,t)\|_{L^2(U_r(t))} \le C\sqrt{r}\big(\|\partial^\bullet_t z(\cdot,t)\|_{H^k(\Gamma(t))} + \|z(\cdot,t)\|_{H^{k+1}(\Gamma(t))}\big). \tag{17}$$

Proof Let us recall that F_t is a diffeomorphism from $\Gamma(0) \times (-r,r)$ onto $U_r(t)$ while $\Psi(\cdot,t)$ is a diffeomorphism from $\Gamma(0)$ onto $\Gamma(t)$. We deduce from (12) and the definition of $\tilde{p}$ that $\tilde{p}(F_t(P,s),t) = \Psi(P,t)$, $P \in \Gamma(0)$, $|s| < r$ so that we obtain with the help of the transformation rule

$$\begin{aligned}\int_{U_r(t)} |z^e(x,t)|^2 dx &= \int_{U_r(t)} |z(\tilde{p}(x,t),t)|^2 dx \le c\int_{-r}^{r}\int_{\Gamma(0)} |z(\Psi(P,t),t)|^2 do_P\, ds \\ &\le cr\int_{\Gamma(t)} |z(Q,t)|^2 do_Q \end{aligned}\tag{18}$$

which is (16) for $k = 0$. Next, differentiating the identity $\phi(\tilde{p}(x,t),t) = 0$ with respect to x_i we infer that $(\nabla\phi(\tilde{p}(x,t),t), \tilde{p}_{x_i}(x,t)) = 0$, $i = 1,\ldots,n+1$. Hence we obtain from (14) and (6) that

$$z^e_{x_i}(x,t) = \sum_{k=1}^{n+1} z^e_{x_k}(\tilde{p}(x,t),t)\tilde{p}_{k,x_i}(x,t) = \sum_{k=1}^{n+1} \underline{D}_k z(\tilde{p}(x,t),t)\tilde{p}_{k,x_i}(x,t), \tag{19}$$

$$\begin{aligned} z^e_{x_i x_j}(x,t) &= \sum_{k,l=1}^{n+1} \underline{D}_l\underline{D}_k z(\tilde{p}(x,t),t)\tilde{p}_{k,x_i}(x,t)\tilde{p}_{l,x_j}(x,t) \\ &\quad + \sum_{k=1}^{n+1} \underline{D}_k z(\tilde{p}(x,t),t)\tilde{p}_{k,x_i x_j}(x,t). \end{aligned}\tag{20}$$

Similarly, $(\nabla\phi(\tilde{p}(x,t),t), \tilde{p}_t(x,t)) = -\phi_t(\tilde{p}(x,t),t) = (\nabla\phi(\tilde{p}(x,t),t), \boldsymbol{v}(\tilde{p}(x,t),t))$ by (24) below, so that

$$\begin{aligned} z^e_t(x,t) &= z^e_t(\tilde{p}(x,t),t) + (\nabla z^e(\tilde{p}(x,t),t), \tilde{p}_t(x,t)) \\ &= \partial^\bullet_t z(\tilde{p}(x,t),t) + \sum_{k=1}^{n+1} \underline{D}_k z(\tilde{p}(x,t),t)\big(\tilde{p}_{k,t}(x,t) - \boldsymbol{v}_k(\tilde{p}(x,t),t)\big). \end{aligned}\tag{21}$$

Combining (19), (20) with the argument in (18) we obtain (16). The estimate (17) follows in a similar way if one starts from (21). □

Let us next extend the surface differential operators ∇_Γ and $\partial_t^\bullet$. By reversing the orientation of $\Gamma(t)$ if necessary we may assume that the functions $\nu : \mathcal{U}_{\delta,T} \to \mathbb{R}^{n+1}$, $V : \mathcal{U}_{\delta,T} \to \mathbb{R}$ defined by

$$\nu(x,t) := \frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}, \quad V(x,t) := -\frac{\phi_t(x,t)}{|\nabla\phi(x,t)|}, \quad (x,t) \in \mathcal{U}_{\delta,T}$$

are extensions of the unit normal and the normal velocity respectively. In particular, we define for a function $\eta \in C^1(U_\delta(t))$ its Eulerian tangential gradient by

$$\nabla_\phi \eta(x) := \big(I - \nu(x,t) \otimes \nu(x,t)\big)\nabla\eta(x), \quad x \in U_\delta(t) \tag{22}$$

and remark that $(\nabla_\phi \eta)_{|\Gamma(t)} = \nabla_\Gamma[\eta_{|\Gamma(t)}]$. Furthermore, it follows from Lemma 2 in [10] that for $\eta \in C_0^1(\Omega)$ with supp$\eta \subset U_\delta(t)$

$$\int_\Omega \nabla_\phi \eta \, |\nabla\phi| = -\int_\Omega \eta H \nu \, |\nabla\phi|, \quad \text{where } H = -\nabla \cdot \nu. \tag{23}$$

Note that $H_{|\Gamma(t)}$ is the mean curvature of $\Gamma(t)$.

Let us also extend the velocity field $\boldsymbol{v}$ to $\mathcal{U}_{\delta,T}$. We first extend its tangential part by setting

$$\tilde{\mathbf{v}}_\tau(x,t) := (I - \nu(x,t) \otimes \nu(x,t))\boldsymbol{v_\tau}^e(x,t), \quad (x,t) \in \mathcal{U}_{\delta,T}.$$

In view of (3) the function $\boldsymbol{v}(x,t) := V(x,t)\nu(x,t) + \tilde{\mathbf{v}}_\tau(x,t)$ extends the given velocity field from $\overline{S_T}$ to $\mathcal{U}_{\delta,T}$ and satisfies

$$\phi_t + (\boldsymbol{v}, \nabla\phi) = 0 \quad \text{in } \mathcal{U}_{\delta,T}. \tag{24}$$

In particular, we can use the extended velocity $\boldsymbol{v}$ to define the material derivative for a function η on $\mathcal{U}_{\delta,T}$ by setting

$$\partial_t^\bullet \eta(x,t) := \eta_t(x,t) + (\boldsymbol{v}(x,t), \nabla\eta(x,t)), \quad (x,t) \in \mathcal{U}_{\delta,T}.$$

3 Weak formulation and numerical scheme

3.1 Phase field approach

Consider for $0 < \epsilon < \frac{2\delta}{\pi}$ the function

$$\rho(x,t) := g\left(\frac{\phi(x,t)}{\epsilon}\right),$$

where $g \in C^{1,1}(\mathbb{R})$ is given by

$$g(r) = \begin{cases} \cos^2(r), & |r| \le \frac{\pi}{2}, \\ 0, & |r| > \frac{\pi}{2}. \end{cases}$$

Note that $\text{supp}[\rho(\cdot, t)] = \overline{U_{\frac{\epsilon\pi}{2}}(t)} \subset U_\delta(t)$. Furthermore, we obtain from the definition of ∇_ϕ and (24)

$$\nabla_\phi \rho = \frac{1}{\epsilon} g'\left(\frac{\phi}{\epsilon}\right) \nabla_\phi \phi = 0, \tag{25}$$

$$\partial_t^\bullet \rho = \frac{1}{\epsilon} g'\left(\frac{\phi}{\epsilon}\right) \big(\phi_t + (\boldsymbol{v}, \nabla\phi)\big) = 0. \tag{26}$$

The phase field function ρ allows us to approximate the integration over a surface $\Gamma(t)$ in terms of a volume integral over the diffuse interface. More precisely, for fixed $t \in [0, T]$, the coarea formula implies for $\eta \in L^1(\Omega)$

$$\int_\Omega \eta\, \rho(\cdot, t)\, |\nabla\phi(\cdot, t)|\, dx = \int_{-\frac{\epsilon\pi}{2}}^{\frac{\epsilon\pi}{2}} g\left(\frac{s}{\epsilon}\right) \int_{\{\phi(\cdot,t)=s\}} \eta\, d\mathcal{H}^n ds \approx \frac{\epsilon\pi}{2} \int_{\{\phi(\cdot,t)=0\}} \eta\, d\mathcal{H}^n$$

for small $\epsilon > 0$, so that we can view $\frac{2}{\epsilon\pi} \int_\Omega \eta\, \rho(\cdot, t)\, |\nabla\phi(\cdot, t)|\, dx$ as an approximation of $\int_{\Gamma(t)} \eta\, d\mathcal{H}^n$. This formula explains the appearance of the weight $\rho(\cdot, t)\, |\nabla\phi(\cdot, t)|$ in subsequent volume integrals.

In what follows we shall make use of the following continuity properties of ρ.

Lemma 2 *Let $s, t \in [0, T]$ with $|s - t| < \frac{\pi}{4c_2}\epsilon$, c_2 as in (8). Then supp$[\rho(\cdot, s)] \subset U_{\frac{3\epsilon\pi}{4}}(t)$ and*

$$|\rho(\cdot, t) - \rho(\cdot, s)| \le C \frac{|t - s|}{\epsilon} \sqrt{\rho(\cdot, t)} + C \frac{(t - s)^2}{\epsilon^2} \chi_{U_{\frac{3\epsilon\pi}{4}}(t)} \quad \text{in } \Omega; \tag{27}$$

$$|\rho_t(\cdot, t) - \rho_t(\cdot, s)| \le C \frac{|t - s|}{\varepsilon^2} \chi_{U_{\frac{3\epsilon\pi}{4}}(t)} \quad \text{in } \Omega. \tag{28}$$

Proof Let $s, t \in [0, T]$ with $|s - t| < \frac{\pi}{4c_2}\epsilon$ and $x \in \text{supp}[\rho(\cdot, s)] = \overline{U_{\frac{\epsilon\pi}{2}}(s)}$. Using the mean value theorem and (8) we then have

$$|\phi(x, t)| \le |\phi(x, s)| + |\phi_t(x, \xi)|\, |t - s| \le \frac{\epsilon\pi}{2} + c_2|t - s| < \frac{3\epsilon\pi}{4},$$

i.e. $x \in U_{\frac{3\epsilon\pi}{4}}(t)$. In order to prove (27) and (28) we first observe that it is enough to verify the estimates for $x \in U_{\frac{3\epsilon\pi}{4}}(t)$ in view of what we have just shown. There exists ξ between s and t such that

$$|\rho(x, t) - \rho(x, s)| = |\rho_t(x, \xi)|\, |t - s| = \frac{1}{\epsilon} |\phi_t(x, \xi)| \left| g'\left(\frac{\phi(x, \xi)}{\varepsilon}\right) \right| |t - s|$$

$$\leq \frac{c_2|t-s|}{\epsilon} \left| g' \left(\frac{\phi(x,\xi)}{\varepsilon} \right) \right| \tag{29}$$

by (8). Furthermore, since

$$g'(r) = \begin{cases} -2\sin(r)\cos(r), & |r| \leq \frac{\pi}{2}, \\ 0, & |r| > \frac{\pi}{2} \end{cases}$$

we see immediately that

$$|g'(r)| \leq 2\sqrt{g(r)}, \quad |g'(r) - g'(\tilde{r})| \leq 2|r - \tilde{r}|, \quad r, \tilde{r} \in \mathbb{R}. \tag{30}$$

As a result,

$$\left| g' \left(\frac{\phi(x,\xi)}{\varepsilon} \right) \right| \leq \left| g' \left(\frac{\phi(x,t)}{\varepsilon} \right) \right| + \frac{2}{\epsilon} |\phi(x,\xi) - \phi(x,t)| \leq 2\sqrt{\rho(x,t)} + \frac{2c_2|t-s|}{\epsilon}.$$

Inserting this bound into (29) yields (27). Finally, using again (30) and (8) we obtain for $x \in U_{\frac{3\epsilon\pi}{4}}(t)$

$$\begin{aligned} |\rho_t(x,t) - \rho_t(x,s)| &\leq \frac{1}{\epsilon} \left| g' \left(\frac{\phi(x,t)}{\epsilon} \right) - g' \left(\frac{\phi(x,s)}{\epsilon} \right) \right| |\phi_t(x,t)| \\ &\quad + \frac{1}{\varepsilon} \left| g' \left(\frac{\phi(x,s)}{\epsilon} \right) \right| |\phi_t(x,t) - \phi_t(x,s)| \\ &\leq \frac{C}{\epsilon^2} |\phi(x,t) - \phi(x,s)| + \frac{C}{\varepsilon} |\phi_t(x,t) - \phi_t(x,s)| \leq \frac{C}{\varepsilon^2} |t-s|. \end{aligned}$$

□

Remark 1 Our analysis will work for other profile functions g than the one chosen above as long as they satisfy $g \in C^{1,1}(\mathbb{R})$ and $g(r) > 0$ if $|r| < R$, $g(r) = 0$ if $|r| \geq R$ as well as $|g'(r)| \leq C\sqrt{g(r)}$ for suitable $R, C > 0$. Profile functions with noncompact support have been used in [4,22] and [26]. However it is not obvious how to extend the analysis presented below to that setting.

3.2 Discretization

Suppose that u is a smooth solution of (1). It is shown in Lemma 8 of the "Appendix" that its extension u^e satisfies the strictly parabolic PDE

$$\partial_t^\bullet u^e + u^e \nabla_\phi \cdot \boldsymbol{v} - \frac{1}{|\nabla\phi|} \nabla \cdot \left(|\nabla\phi| \nabla u^e \right) = f^e + \phi R \quad \text{in } \mathcal{U}_{\delta,T}, \tag{31}$$

where

$$R(x,t) = \sum_{k,l=1}^{n+1} b_{lk}(x,t)\underline{D}_l\underline{D}_k u(\tilde{p}(x,t),t) + \sum_{k=1}^{n+1} c_k(x,t)\underline{D}_k u(\tilde{p}(x,t),t) + d(x,t)u(\tilde{p}(x,t),t) \tag{32}$$

and b_{lk}, c_k, d are smooth functions depending on ϕ and $\boldsymbol{v}$.

In order to associate with (31) a suitable variational formulation we adapt an idea from [16], which uses an Eulerian transport identity. More precisely, we infer with the help of Lemma 3 in [10], (26) and (31) that for every $\eta \in H^1(\Omega)$

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^e\eta\,\rho\,|\nabla\phi| &= \int_\Omega \big(\partial_t^\bullet(u^e\eta\rho) + u^e\eta\rho\,\nabla_\phi\cdot\boldsymbol{v}\big)|\nabla\phi| \\ &= \int_\Omega \eta\big(\partial_t^\bullet u^e + u^e\,\nabla_\phi\cdot\boldsymbol{v}\big)\rho\,|\nabla\phi| + \int_\Omega u^e\partial_t^\bullet\eta\,\rho\,|\nabla\phi| \\ &= \int_\Omega \eta\,\nabla\cdot\big(|\nabla\phi|\,\nabla u^e\big)\rho + \int_\Omega \eta\big(f^e + \phi R\big)\rho\,|\nabla\phi| \\ &\quad + \int_\Omega u^e\partial_t^\bullet\eta\,\rho\,|\nabla\phi| = -\int_\Omega (\nabla u^e,\nabla\eta)\rho\,|\nabla\phi| + \int_\Omega f^e\eta\,\rho\,|\nabla\phi| \\ &\quad + \int_\Omega u^e(\boldsymbol{v},\nabla\eta)\rho\,|\nabla\phi| + \int_\Omega \phi\,R\,\eta\,\rho\,|\nabla\phi|. \end{aligned} \tag{33}$$

Here, the last equality follows from integration by parts together with the fact that $(\nabla u^e, \nabla\rho) = \frac{1}{\epsilon}g'\left(\frac{\phi}{\epsilon}\right)(\nabla u^e, \nabla\phi) = 0$ in view of (15).

Let us first discretize with respect to time and denote by $0 = t_0 < t_1 < \cdots < t_M = T$ a partioning of $[0, T]$ with time steps $\tau_m := t_m - t_{m-1}$ and $\tau := \max_{m=1,\ldots,M}\tau_m$. For a function $f = f(x,t)$ we shall write $f^m(x) = f(x,t_m)$. Integrating (33) with respect to $t \in (t_{m-1}, t_m)$ we obtain for $\eta \in H^1(\Omega)$

$$\begin{aligned}&\int_\Omega u^{e,m}\eta\rho^m|\nabla\phi^m| - \int_\Omega u^{e,m-1}\eta\rho^{m-1}|\nabla\phi^{m-1}| + \int_{t_{m-1}}^{t_m}\int_\Omega (\nabla u^e,\nabla\eta)\rho\,|\nabla\phi| \\ &\quad - \int_{t_{m-1}}^{t_m}\int_\Omega u^e(\boldsymbol{v},\nabla\eta)\rho\,|\nabla\phi| = \int_{t_{m-1}}^{t_m}\int_\Omega f^e\,\eta\,\rho\,|\nabla\phi| + \int_{t_{m-1}}^{t_m}\int_\Omega \phi\,R\,\eta\,\rho\,|\nabla\phi|.\end{aligned} \tag{34}$$

Under a suitable regularity assumption on u we have that $|\phi\,R| \leq C\epsilon$ on suppρ so that we neglect the corresponding term when now deriving the spatial discretization from (34).

In what follows we assume that Ω is polyhedral and consider a family $(\mathcal{T}_h)_{0<h\leq h_0}$ of triangulations of Ω with mesh size $h = \max_{T\in\mathcal{T}_h} h_T,\ h_T = \mathrm{diam}(T)$. We assume that the family is regular in the sense that there exists $\sigma > 0$ with

$$r_T \geq \sigma h_T \qquad \forall T \in \mathcal{T}_h \quad \forall 0 < h \leq h_0, \tag{35}$$

where r_T is the radius of the largest ball contained in T. Let us denote by $\mathcal{N}_h$ the set of vertices of the triangulation $\mathcal{T}_h$. In order to formulate our scheme we require a second phase field function with a slightly larger support, namely

$$\tilde{\rho}(x,t) = g\left(\frac{\phi(x,t)}{2\epsilon}\right), \quad 0 < \epsilon < \frac{\delta}{\pi}.$$

For $0 \le m \le M$ we then define

$$\mathcal{T}_h^m := \{T \in \mathcal{T}_h \mid \tilde{\rho}^m(x) > 0 \text{ for some } x \in T \cap \mathcal{N}_h\} \quad \text{and} \quad D_h^m := \bigcup_{T \in \mathcal{T}_h^m} T$$

as well as the finite element space

$$V_h^m := \{v_h \in C^0(D_h^m) \mid v_{h|T} \text{ is a linear polynomial on each } T \in \mathcal{T}_h^m\}.$$

We denote by $I_h^m : C^0(D_h^m) \to V_h^m$ the standard Lagrange interpolation operator, i.e. $[I_h^m f](x) = f(x), x \in D_h^m \cap \mathcal{N}_h$. Note that $D_h^m = \mathrm{supp} I_h^m \tilde{\rho}^m$.

Lemma 3 *Suppose that*

$$h \le \frac{\cos^2\left(\frac{3\pi}{8}\right)}{2c_1}\epsilon, \; \tau \le \frac{\cos^2\left(\frac{3\pi}{8}\right)}{2c_2}\epsilon. \tag{36}$$

Then

a) $U_{\frac{3\epsilon\pi}{4}}(t) \subset D_h^m \subset U_{\frac{3\epsilon\pi}{2}}(s)$ *for all* $s, t \in [\max(t_{m-1}, 0), \min(t_{m+1}, T)], 0 \le m \le M$;

b) $[I_h^m \tilde{\rho}^m](x) \ge \frac{1}{2}\cos^2\left(\frac{3\pi}{8}\right), \; x \in U_{\frac{3\epsilon\pi}{4}}(t_m), 0 \le m \le M$.

Proof a) Let $x \in D_h^m$, so that there exists $y \in \mathcal{N}_h$ such that $|y - x| \le h$ and $\tilde{\rho}^m(y) > 0$. Hence $|\phi^m(y)| < \epsilon\pi$ and the mean value theorem together with (8) yields for $s \in [\max(t_{m-1}, 0), \min(t_{m+1}, T)]$

$$\begin{aligned} |\phi(x,s)| &\le |\phi(x,s) - \phi^m(x)| + |\phi^m(x) - \phi^m(y)| + |\phi^m(y)| \\ &< |\phi_t(x,\xi)|\,|s - t_m| + |\nabla\phi^m(\eta)|\,|x - y| + \epsilon\pi \\ &\le c_2\tau + c_1 h + \epsilon\pi \le \cos^2\left(\frac{3\pi}{8}\right)\epsilon + \epsilon\pi \le \frac{3\epsilon\pi}{2} \end{aligned}$$

in view of (36). Hence, $x \in U_{\frac{3\epsilon\pi}{2}}(s)$. Next, let $x \in U_{\frac{3\epsilon\pi}{4}}(t)$ for some $t \in [\max(t_{m-1}, 0), \min(t_{m+1}, T)]$. Then $\tilde{\rho}(x,t) \ge \cos^2(\frac{3\pi}{8})$ and we obtain similarly as above

$$\begin{aligned} [I_h^m \tilde{\rho}^m](x) &\ge \tilde{\rho}(x,t) - |\tilde{\rho}(x,t) - \tilde{\rho}^m(x)| - |\tilde{\rho}^m(x) - [I_h^m \tilde{\rho}^m](x)| \\ &\ge \cos^2\left(\frac{3\pi}{8}\right) - |\tilde{\rho}_t(x,\xi)|\,|t - t_m| - h \max_{y \in U_\delta(t_m)} |\nabla\tilde{\rho}^m(y)| \end{aligned}$$

$$\geq \cos^2\left(\frac{3\pi}{8}\right) - c_2\frac{\tau}{2\epsilon} - c_1\frac{h}{2\epsilon} \geq \frac{1}{2}\cos^2\left(\frac{3\pi}{8}\right).$$

In particular, $[I_h^m \tilde{\rho}^m](x) > 0$, so that $x \in D_h^m$. Using the above inequality for $t = t_m$ implies b). □

Our finite element approximation of (1), (2) now reads: For $m = 1, 2, \ldots, M$ find $u_h^m \in V_h^m$ such that for all $v_h \in V_h^m$

$$\begin{aligned}
&\int_\Omega u_h^m\, v_h\, \rho^m\, |\nabla\phi^m| - \int_\Omega u_h^{m-1}\, v_h\, \rho^{m-1}\, |\nabla\phi^{m-1}| + \tau_m \int_\Omega (\nabla u_h^m, \nabla v_h)\, \rho^m\, |\nabla\phi^m| \\
&\quad -\tau_m \int_\Omega u_h^m\, (\boldsymbol{v}^m, \nabla v_h)\, \rho^m\, |\nabla\phi^m| + \gamma\tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m (\nabla u_h^m, \nabla v_h) \\
&= \tau_m \int_\Omega f^{e,m}\, v_h\, \rho^m\, |\nabla\phi^m|. \qquad (37)
\end{aligned}$$

Here, $u_h^0 \in V_h^0$ is defined as an L^2 projection of $u_0^e(x) := u_0(\tilde{p}(x,0)), x \in U_\delta(0)$, more precisely

$$\int_{D_h^0} u_h^0\, v_h = \int_{D_h^0} u_0^e\, v_h \qquad \forall v_h \in V_h^0. \qquad (38)$$

Furthermore, $f^{e,m}(x) := f(\tilde{p}(x, t_m), t_m), x \in U_\delta(t_m), 1 \leq m \leq M$. The parameter $\gamma > 0$ will be chosen in such a way as to ensure existence and stability for the scheme, see Lemma 5 and Theorem 1 below.

Remark 2 a) Lemma 3 a) implies that $\mathrm{supp}\rho^m, \mathrm{supp}\rho^{m-1} \subset D_h^m = \mathrm{supp} I_h^m \tilde{\rho}^m$, so that all integrals appearing in (37) are taken only over D_h^m. In particular, if $f \equiv 0$ we see from the choice $v_h \equiv 1$ on D_h^m that the scheme is mass conserving in the sense that

$$\int_\Omega u_h^m\, \rho^m\, |\nabla\phi^m| = \int_\Omega u_h^0\, \rho^0\, |\nabla\phi^0|, \qquad m = 1, \ldots, M.$$

b) The term $\gamma\tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m (\nabla u_h^m, \nabla v_h)$ introduces artificial diffusion into the scheme and will play a crucial role in our analyis. A different form of stabilization is used in [16], Section 2.5.
c) Unlike the schemes introduced in [16] our method is not fully practical because we assume that the integrals are evaluated exactly. In Sect. 6 we shall follow [16] in using numerical integration to obtain a fully practical scheme. A nice feature of the resulting method is that the evolution of the hypersurfaces is tracked in a simple way via the evaluation of ρ.

In what follows we shall be concerned with the existence, stability and error bounds for (37). The extension of our analysis to the fully practical method mentioned above is currently out of reach and left for future research. However, the test calculations in Sect. 6 show that the parameter choices suggested by the analysis work well also for the fully practical scheme.

Lemma 4 *There exists* $0 < h_1 \leq h_0$ *such that* D_h^m *is connected for all* $0 < h \leq h_1$ *and* $0 \leq m \leq M$.

Proof To begin, we remark that there exists $0 < h_1 \leq h_0$ and $\mu > 0$ only depending on σ, c_0, c_1, c_2 such that for every $a \in \mathcal{N}_h \cap \overline{U_\delta(t)}$ there exists a neighbour $b \in \mathcal{N}_h$ with

$$|\phi(a,t) - \phi(b,t)| \geq \mu h_T \qquad \text{where } a, b, \in T \tag{39}$$

for all $t \in [0, T]$, $0 < h \leq h_1$. Since $\Gamma(t_m)$ is connected it is sufficient to show that for every $y \in D_h^m$ there exists $z \in \Gamma(t_m)$ and a path in D_h^m connecting y to z. Let us fix $y \in D_h^m$, say $y \in T$, where $\tilde{\rho}^m(x) > 0$ for some $x \in T \cap \mathcal{N}_h$. We assume w.l.o.g. that $0 < \phi^m(x) < \epsilon\pi$. In view of (39) there exists a neighbour $x_1 \in \mathcal{N}_h$ of x such that $\phi^m(x_1) \leq \phi^m(x) - \mu h_{\tilde{T}}$, where $x, x_1 \in \tilde{T}$. If $\phi^m(x_1) \leq 0$ then there is $z \in [x, x_1]$ with $\phi^m(z) = 0$. Hence, $z \in \Gamma(t_m)$ and the union of the segments $[y, x]$ and $[x, z]$ is a path in D_h^m connecting y to z. If $\phi^m(x_1) > 0$, then $\tilde{\rho}^m(x_1) > 0$ so that $[x, x_1] \subset D_h^m$ and we may repeat the above argument with x replaced by x_1 and so on, until we reach $\Gamma(t_m)$ in a finite number of steps. □

Lemma 5 *(Existence) Let* $0 < h \leq h_1$. *There exists* $\tau_0 > 0$ *such that the scheme (37) has a unique solution* $u_h^m \in V_h^m$ *provided that* $0 < \tau \leq \tau_0$.

Proof Since (37) is equivalent to solving a linear system with a quadratic coefficient matrix, it is sufficient to prove that the following problem only has the trivial solution: find $u_h \in V_h^m$ such that for all $v_h \in V_h^m$

$$\int_\Omega u_h\, v_h\, \rho^m\, |\nabla\phi^m| + \tau_m \int_\Omega (\nabla u_h, \nabla v_h)\, \rho^m\, |\nabla\phi^m| - \tau_m \int_\Omega u_h\, (\boldsymbol{v}^m, \nabla v_h)\, \rho^m\, |\nabla\phi^m|$$
$$+\gamma\tau_m^2 \int_\Omega I_h^m\, \tilde{\rho}^m (\nabla u_h, \nabla v_h) = 0.$$

Inserting $v_h = u_h$ we infer

$$\int_\Omega (u_h)^2 \rho^m\, |\nabla\phi^m| + \tau_m \int_\Omega |\nabla u_h|^2\, \rho^m\, |\nabla\phi^m| + \gamma\tau_m^2 \int_\Omega I_h^m\, \tilde{\rho}^m |\nabla u_h|^2$$
$$= \tau_m \int_\Omega u_h\, (\boldsymbol{v}^m, \nabla u_h)\, \rho^m\, |\nabla\phi^m| \leq \tau_m \max_{x \in \overline{U_\delta(t_m)}} |\boldsymbol{v}^m(x)| \int_\Omega |u_h|\, |\nabla u_h|\, \rho^m\, |\nabla\phi^m|$$
$$\leq \frac{1}{2} \int_\Omega (u_h)^2\, \rho^m\, |\nabla\phi^m| + \frac{1}{2}\tau \left(\max_{x \in \overline{U_\delta(t_m)}} |\boldsymbol{v}^m(x)| \right)^2 \tau_m \int_\Omega |\nabla u_h|^2\, \rho^m\, |\nabla\phi^m|.$$

If we choose $\tau_0 > 0$ so small that $\frac{1}{2}\tau \left(\max_{x \in \overline{U_\delta(t_m)}} |\boldsymbol{v}^m(x)|\right)^2 \leq 1$ we deduce that

$$\int_\Omega (u_h)^2\, \rho^m\, |\nabla\phi^m| = \int_\Omega I_h^m\, \tilde{\rho}^m |\nabla u_h|^2 = 0,$$

which implies that $u_h \equiv 0$ on $\Gamma(t_m)$ and $\nabla u_h \equiv 0$ in D_h^m. According to Lemma 4, D_h^m is connected, so that we conclude that $u_h \equiv 0$. □

4 Stability bound

The following lemma will be useful in estimating L^2-integrals that are not weighted by ρ.

Lemma 6 *There exists $C \geq 0$ such that for $t \in [0, T]$:*

$$\int_{U_{\frac{3\epsilon\pi}{4}}(t)} f^2 \leq C \int_\Omega f^2 \rho(\cdot, t)|\nabla\phi(\cdot, t)| + C\varepsilon^2 \int_{U_{\frac{3\epsilon\pi}{4}}(t)} |\nabla f|^2 \qquad \textit{for all } f \in H^1(\Omega). \tag{40}$$

Remark 3 Note that Lemma 3 b) implies that

$$\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |\nabla f|^2 \leq \frac{2}{\cos^2(\frac{3\pi}{8})} \int_\Omega I_h^m \tilde{\rho}^m |\nabla f|^2, \quad f \in H^1(\Omega), m = 0, \ldots, M. \tag{41}$$

Proof We may assume that f is smooth, the general case then follows with the help of an approximation argument. Since F_t is a diffeomorphism from $\Gamma(0) \times (-\frac{3\epsilon\pi}{4}, \frac{3\epsilon\pi}{4})$ onto $U_{\frac{3\epsilon\pi}{4}}(t)$, the transformation rule yields

$$c_1 \int_{U_{\frac{3\epsilon\pi}{4}}(t)} f(x)^2 dx \leq \int_{-\frac{3\varepsilon\pi}{4}}^{\frac{3\varepsilon\pi}{4}} \int_{\Gamma(0)} f(F_t(P, s))^2 do_P ds \leq c_2 \int_{U_{\frac{3\epsilon\pi}{4}}(t)} f(x)^2 dx. \tag{42}$$

The definition of F_t together with (9) implies for $|s| \leq \frac{3\varepsilon\pi}{4}$, $|\tilde{s}| \leq \frac{\varepsilon\pi}{4}$

$$\begin{aligned} f(F_t(P, s)) &= f(F_t(P, \tilde{s})) + \int_{\tilde{s}}^{s} \left(\nabla f(F_t(P, r)), \frac{\partial F_t}{\partial r}(P, r)\right) dr \\ &= f(F_t(P, \tilde{s})) + \int_{\tilde{s}}^{s} \left(\nabla f(F_t(P, r)), \frac{\nabla\phi(F_t(P, r), t)}{|\nabla\phi(F_t(P, r), t)|^2}\right) dr \end{aligned}$$

and therefore

$$\begin{aligned} f(F_t(P, s))^2 &\leq 2f(F_t(P, \tilde{s}))^2 + C\varepsilon \int_{-\frac{3\varepsilon\pi}{4}}^{\frac{3\varepsilon\pi}{4}} |\nabla f(F_t(p, r))|^2 dr \\ &\leq Cf(F_t(P, \tilde{s}))^2 \rho(F_t(P, \tilde{s}), t) + C\varepsilon \int_{-\frac{3\varepsilon\pi}{4}}^{\frac{3\varepsilon\pi}{4}} |\nabla f(F_t(p, r))|^2 dr, \end{aligned}$$

since $\rho(F_t(P, \tilde{s}), t) = \cos^2\left(\frac{\phi(F_t(P, \tilde{s}), t)}{\epsilon}\right) = \cos^2(\frac{\tilde{s}}{\epsilon}) \geq \cos^2(\frac{\pi}{4})$, $|\tilde{s}| \leq \frac{\epsilon\pi}{4}$. Integrating with respect to $P \in \Gamma(0)$, $s \in (-\frac{3\varepsilon\pi}{4}, \frac{3\varepsilon\pi}{4})$ and recalling (42) we obtain for $|\tilde{s}| \leq \frac{\varepsilon\pi}{4}$

$$\int_{U_{\frac{3\epsilon\pi}{4}}(t)} f(x)^2 dx \leq C\epsilon \int_{\Gamma(0)} f(F_t(P, \tilde{s}))^2 \rho(F_t(P, \tilde{s}), t) do_P$$

$$
\begin{aligned}
&+C\epsilon^2 \int_{-\frac{3\varepsilon\pi}{4}}^{\frac{3\varepsilon\pi}{4}} \int_{\Gamma(0)} |\nabla f(F_t(p,r))|^2 do_P dr \\
&\leq C\epsilon \int_{\Gamma(0)} f(F_t(P,\tilde{s}))^2 \rho(F_t(P,\tilde{s}),t) do_P \\
&+C\varepsilon^2 \int_{U_{\frac{3\epsilon\pi}{4}}(t)} |\nabla f(x)|^2 dx.
\end{aligned}
$$

If we integrate with respect to $\tilde{s} \in (-\frac{\varepsilon\pi}{4}, \frac{\varepsilon\pi}{4})$, divide by ϵ and recall (8) we obtain the assertion. □

It follows from Theorem 4.4 in [8] (extended in a straightforward way to the case of a nontrivial f) that (1), (2) has a unique solution u which satisfies

$$
\begin{aligned}
&\sup_{(0,T)} \|u(\cdot,t)\|^2_{L^2(\Gamma(t))} + \int_0^T \|\nabla_\Gamma u(\cdot,t)\|^2_{L^2(\Gamma(t))} dt \leq c \Big(\|u_0\|^2_{L^2(\Gamma(0))} \\
&+ \int_0^T \|f(\cdot,t)\|^2_{L^2(\Gamma(t))} dt \Big).
\end{aligned}
$$

The following theorem gives a discrete version of this estimate in the phase field setting.

Theorem 1 *Suppose that (36) holds. There exist $\gamma_1 > 0$ and $\tau_1 \leq \tau_0$ such that*

$$
\begin{aligned}
&\max_{m=1,\dots,M} \frac{2}{\epsilon\pi} \int_\Omega (u_h^m)^2 \rho^m |\nabla\phi^m| + \sum_{m=1}^M \tau_m \frac{2}{\epsilon\pi} \int_\Omega |\nabla u_h^m|^2 \rho^m |\nabla\phi^m| \\
&\leq C \left(\int_{\Gamma(0)} (u_0)^2 + \sum_{m=1}^M \tau_m \int_{\Gamma(t_m)} (f^m)^2 \right),
\end{aligned}
$$

provided that $\gamma \geq \gamma_1$ and $\tau \leq \min(\tau_1, \epsilon^2)$.

Proof Setting $v_h = u_h^m$ in (37) we find after a straighforward calculation

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (u_h^m)^2 \rho^m |\nabla\phi^m| - \frac{1}{2}\int_\Omega (u_h^{m-1})^2 \rho^{m-1} |\nabla\phi^{m-1}| \\
&\quad + \frac{1}{2}\int_\Omega (u_h^m - u_h^{m-1})^2 \rho^{m-1} |\nabla\phi^{m-1}| \\
&\quad + \tau_m \int_\Omega |\nabla u_h^m|^2 \rho^m |\nabla\phi^m| + \gamma\tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m |\nabla u_h^m|^2 \\
&= -\frac{1}{2}\int_\Omega (u_h^m)^2 (\rho^m - \rho^{m-1}) |\nabla\phi^{m-1}| + \frac{1}{2}\int_\Omega (u_h^m)^2 \rho^m (|\nabla\phi^{m-1}| - |\nabla\phi^m|) \\
&\quad + \tau_m \int_\Omega u_h^m (\boldsymbol{v}^m, \nabla u_h^m) \rho^m |\nabla\phi^m| + \tau_m \int_\Omega f^{e,m} u_h^m \rho^m |\nabla\phi^m| \\
&:= I + II + III + IV. \qquad (43)
\end{aligned}
$$

Clearly,

$$I = -\frac{1}{2}\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2\,\rho_t(\cdot,s)\,|\nabla\phi^{m-1}|ds, \tag{44}$$

while

$$\begin{aligned} II &= -\frac{1}{2}\tau_m\int_\Omega (u_h^m)^2\,\rho^m\,(\nabla\phi_t^m,\nu^m) \\ &\quad+\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m\left(|\nabla\phi^{m-1}|-|\nabla\phi^m|+\tau_m(\nabla\phi_t^m,\nu^m)\right) = II_1+II_2. \end{aligned} \tag{45}$$

Integrating by parts and abbreviating $H^m = -\nabla\cdot\nu^m$ we obtain

$$\begin{aligned} II_1 &= \frac{1}{2}\tau_m\int_\Omega (u_h^m)^2\,(\nabla\rho^m,\nu^m)\,\phi_t^m + \tau_m\int_\Omega u_h^m(\nabla u_h^m,\nu^m)\rho^m\,\phi_t^m \\ &\quad+\frac{1}{2}\tau_m\int_\Omega (u_h^m)^2\,\nabla\cdot\nu^m\,\rho^m\,\phi_t^m \\ &= \frac{1}{2}\tau_m\int_\Omega (u_h^m)^2\,\rho_t^m\,|\nabla\phi^m| + \tau_m\int_\Omega u_h^m(\nabla u_h^m,\nu^m)\rho^m\,\phi_t^m \\ &\quad-\frac{1}{2}\tau_m\int_\Omega (u_h^m)^2 H^m\,\rho^m\,\phi_t^m, \end{aligned}$$

since

$$(\nabla\rho^m,\nu^m)\phi_t^m = \frac{1}{\epsilon}g'\left(\frac{\phi^m}{\epsilon}\right)\phi_t^m\,(\nabla\phi^m,\nu^m) = \rho_t^m\,|\nabla\phi^m|.$$

In order to rewrite III we first observe that in view of (22) and (24)

$$\begin{aligned} (\boldsymbol{v}^m,\nabla u_h^m) &= (\boldsymbol{v}^m,\nabla_{\phi^m}u_h^m) + (\nabla u_h^m,\nu^m)(\boldsymbol{v}^m,\nu^m) \\ &= (\boldsymbol{v}^m,\nabla_{\phi^m}u_h^m) - (\nabla u_h^m,\nu^m)\frac{\phi_t^m}{|\nabla\phi^m|}, \end{aligned}$$

so that (23), (25) and again (24) imply

$$\begin{aligned} III &= \frac{1}{2}\tau_m\int_\Omega (\boldsymbol{v}^m,\nabla_{\phi^m}(u_h^m)^2)\rho^m\,|\nabla\phi^m| - \tau_m\int_\Omega u_h^m(\nabla u_h^m,\nu^m)\rho^m\,\phi_t^m \\ &= -\frac{1}{2}\tau_m\int_\Omega \nabla_{\phi^m}\cdot\boldsymbol{v}^m(u_h^m)^2\,\rho^m\,|\nabla\phi^m| - \frac{1}{2}\tau_m\int_\Omega H^m(\boldsymbol{v}^m,\nu^m)(u_h^m)^2\,\rho^m\,|\nabla\phi^m| \\ &\quad-\tau_m\int_\Omega u_h^m(\nabla u_h^m,\nu^m)\rho^m\,\phi_t^m = -\frac{1}{2}\tau_m\int_\Omega \nabla_{\phi^m}\cdot\boldsymbol{v}^m(u_h^m)^2\,\rho^m\,|\nabla\phi^m| \\ &\quad+\frac{1}{2}\tau_m\int_\Omega (u_h^m)^2 H^m\,\rho^m\,\phi_t^m - \tau_m\int_\Omega u_h^m(\nabla u_h^m,\nu^m)\,\rho^m\,\phi_t^m. \end{aligned} \tag{46}$$

Inserting (44)–(46) into (43) we infer that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m\,|\nabla\phi^m| - \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1}\,|\nabla\phi^{m-1}| + \tau_m\int_\Omega |\nabla u_h^m|^2\,\rho^m\,|\nabla\phi^m| \\
&+\gamma\tau_m^2\int_\Omega I_h^m\tilde{\rho}^m|\nabla u_h^m|^2 \quad \le \frac{1}{2}\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2\left(\rho_t^m|\nabla\phi^m| - \rho_t(.,s)|\nabla\phi^{m-1}|\right) \\
&-\frac{1}{2}\tau_m\int_\Omega \nabla_{\phi^m}\cdot \boldsymbol{v}^m(u_h^m)^2\,\rho^m\,|\nabla\phi^m| + \frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m\,\big(|\nabla\phi^{m-1}| \\
&-|\nabla\phi^m| + \tau_m(\nabla\phi_t^m,\nu^m)\big) + \tau_m\int_\Omega f^{e,m}u_h^m\,\rho^m\,|\nabla\phi^m|. \qquad (47)
\end{aligned}$$

We deduce from (28), Lemma 6, (41) and the assumption $\tau\le\epsilon^2$ that

$$\begin{aligned}
&\left|\frac{1}{2}\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2(\rho_t^m|\nabla\phi^m| - \rho_t(.,s)|\nabla\phi^{m-1}|)\right| \\
&\le C\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2\big(|\rho_t^m - \rho_t(.,s)| + |\rho_t(.,s)|\,||\nabla\phi^m| - |\nabla\phi^{m-1}||\big) \\
&\le C\frac{\tau_m^2}{\epsilon^2}\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} (u_h^m)^2 \le C\frac{\tau_m^2}{\epsilon^2}\int_\Omega (u_h^m)^2\,\rho^m|\nabla\phi^m| + C\tau_m^2\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |\nabla u_h^m|^2 \\
&\le C\tau_m\int_\Omega (u_h^m)^2\,\rho^m|\nabla\phi^m| + (\gamma-1)\tau_m^2\int_\Omega I_h\tilde{\rho}^m\,|\nabla u_h^m|^2
\end{aligned}$$

if we choose $\gamma\ge\gamma_1 := C+1$. Finally, using Taylor expansion and (8) we infer that

$$\begin{aligned}
&\Bigg|-\frac{1}{2}\tau_m\int_\Omega \nabla_{\phi^m}\cdot \boldsymbol{v}^m(u_h^m)^2\,\rho^m\,|\nabla\phi^m| \\
&\quad+\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m\,\big(|\nabla\phi^{m-1}| - |\nabla\phi^m| + \tau_m(\nabla\phi_t^m,\nu^m)\big)\Bigg| \\
&\quad\le C\tau_m\int_\Omega (u_h^m)^2\,\rho^m|\nabla\phi^m| + C\tau_m^2\int (u_h^m)^2\rho^m \le C\tau_m\int_\Omega (u_h^m)^2\rho^m|\nabla\phi^m|.
\end{aligned}$$

Inserting the above estimates into (47) we find

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m\,|\nabla\phi^m| + \tau_m\int_\Omega |\nabla u_h^m|^2\,\rho^m\,|\nabla\phi^m| + \tau_m^2\int_\Omega I_h^m\tilde{\rho}^m|\nabla u_h^m|^2 \\
&\quad\le \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1}\,|\nabla\phi^{m-1}| + C\tau_m\int_\Omega (u_h^m)^2\,\rho^m\,|\nabla\phi^m| \\
&\quad+\tau_m\int_\Omega (f^{e,m})^2\,\rho^m\,|\nabla\phi^m|. \qquad (48)
\end{aligned}$$

If $\tau_1 \le \tau_0$ is sufficiently small we therefore deduce for $\tau \le \tau_1$

$$\int_\Omega (u_h^m)^2 \, \rho^m \, |\nabla \phi^m| + \tau_m \int_\Omega |\nabla u_h^m|^2 \, \rho^m \, |\nabla \phi^m| \\ \le (1 + C\tau_m) \int_\Omega (u_h^{m-1})^2 \, \rho^{m-1} \, |\nabla \phi^{m-1}| + C\tau_m \int_\Omega (f^{e,m})^2 \, \rho^m \, |\nabla \phi^m|,$$

from which we obtain after summation from $m = 1, \ldots, l$ and division by ϵ that

$$\frac{1}{\epsilon} \int_\Omega (u_h^l)^2 \, \rho^l \, |\nabla \phi^l| + \sum_{m=1}^{l} \tau_m \frac{1}{\epsilon} \int_\Omega |\nabla u_h^m|^2 \, \rho^m \, |\nabla \phi^m| \\ \le \frac{1}{\epsilon} \int_\Omega (u_h^0)^2 \, \rho^0 \, |\nabla \phi^0| + C \sum_{m=0}^{l-1} \tau_{m+1} \frac{1}{\epsilon} \int_\Omega (u_h^m)^2 \, \rho^m \, |\nabla \phi^m| \\ + C \sum_{m=1}^{l} \tau_m \frac{1}{\epsilon} \int_\Omega (f^{e,m})^2 \, \rho^m \, |\nabla \phi^m|.$$

Using Lemma 3 a), (38) and (16) we may estimate

$$\frac{1}{\epsilon} \int_\Omega (u_h^0)^2 \, \rho^0 \, |\nabla \phi^0| \le \frac{C}{\epsilon} \int_{D_h^0} (u_h^0)^2 \le \frac{C}{\epsilon} \int_{D_h^0} (u_0^e)^2 \le \frac{C}{\epsilon} \int_{U_{\frac{3\epsilon\pi}{2}}(0)} (u_0^e)^2 \\ \le C \int_{\Gamma(0)} (u_0)^2.$$

Arguing in a similar way for the term involving $f^{e,m}$ we derive

$$\frac{1}{\epsilon} \int_\Omega (u_h^l)^2 \, \rho^l \, |\nabla \phi^l| + \sum_{m=1}^{l} \tau_m \frac{1}{\epsilon} \int_\Omega |\nabla u_h^m|^2 \, \rho^m \, |\nabla \phi^m| \\ \le C \sum_{m=0}^{l-1} \tau_{m+1} \frac{1}{\epsilon} \int_\Omega (u_h^m)^2 \, \rho^m \, |\nabla \phi^m| + C \left(\int_{\Gamma(0)} (u_0)^2 \right. \\ \left. + \sum_{m=1}^{l} \tau_m \int_{\Gamma(t_m)} (f^m)^2 \right). \tag{49}$$

The discrete Gronwall inequality yields the bound on $\max_{m=1,\ldots,M} \frac{1}{\epsilon} \int_\Omega (u_h^m)^2 \, \rho^m$ $|\nabla \phi^m|$, which combined with (49) implies the second inequality. □

5 Error estimate

Before we formulate our error bound we derive interpolation estimates that are adapted to our setting.

Lemma 7 *Suppose that* (36) *holds and let* z^e *be defined by* (14)*. Then we have for* $m = 1, \ldots, M$ *and* $t \in [t_{m-1}, t_m]$*:*

$$\int_{D_h^m} |(z^e - I_h^m z^e)(\cdot,t)|^2 + h^2 \int_{D_h^m} |\nabla(z^e - I_h^m z^e)(\cdot,t)|^2 \le C\epsilon h^4 \|z(\cdot,t)\|^2_{H^2(\Gamma(t))},$$
$$\int_{D_h^m} |(z_t^e - I_h^m z_t^e)(\cdot,t)|^2 \le C\epsilon h^4 \big(\|\partial_t^\bullet z(\cdot,t)\|^2_{H^2(\Gamma(t))} + \|z(\cdot,t)\|^2_{H^3(\Gamma(t))}\big).$$

Proof Let $t \in [t_{m-1}, t_m]$. Standard interpolation theory together with Lemma 3 a) and (16) implies that

$$\int_{D_h^m} |(z^e - I_h^m z^e)(\cdot,t)|^2 + h^2 \int_{D_h^m} |\nabla(z^e - I_h^m z^e)(\cdot,t)|^2$$
$$\le ch^4 \int_{D_h^m} |D^2 z^e(\cdot,t)|^2 \le ch^4 \int_{U_{\frac{3\epsilon\pi}{2}}(t)} |D^2 z^e(\cdot,t)|^2 \le C\epsilon h^4 \|z(\cdot,t)\|^2_{H^2(\Gamma(t))}.$$

The second bound follows in the same way using (17). □

Theorem 2 *Suppose that the solution of* (1), (2) *satisfies*

$$\max_{t\in[0,T]} \|u(\cdot,t)\|^2_{W^{2,\infty}(\Gamma(t))} + \int_0^T \big(\|u(\cdot,t)\|^2_{H^3(\Gamma(t))} + \|\partial_t^\bullet u(\cdot,t)\|^2_{H^2(\Gamma(t))}\big)dt < \infty. \tag{50}$$

Then there exists $0 < \tau_2 \le \tau_1$ *and a constant* $C \ge 0$ *such that*

$$\max_{m=1,\ldots,M} \frac{2}{\epsilon\pi} \int_\Omega |u^{e,m} - u_h^m|^2 \rho^m |\nabla\phi^m|$$
$$+ \sum_{m=1}^M \tau_m \frac{2}{\epsilon\pi} \int_\Omega |\nabla(u^{e,m} - u_h^m)|^2 \rho^m |\nabla\phi^m| \le C\epsilon^2,$$

provided that $\tau \le \min(\epsilon^2, \tau_2)$*,* $\gamma \ge \gamma_1$ *and* (36) *hold.*

Proof Let us write

$$u^{e,m} - u_h^m = (u^{e,m} - I_h^m u^{e,m}) + (I_h^m u^{e,m} - u_h^m) =: d^m + e_h^m.$$

If we combine (34) for $\eta = v_h \in V_h^m$ with (37) we find

$$\int_\Omega e_h^m v_h\, \rho^m |\nabla\phi^m| - \int_\Omega e_h^{m-1} v_h\, \rho^{m-1} |\nabla\phi^{m-1}| + \tau_m \int_\Omega (\nabla e_h^m, \nabla v_h) \rho^m |\nabla\phi^m|$$
$$-\tau_m \int_\Omega e_h^m (\boldsymbol{v}^m, \nabla v_h) \rho^m |\nabla\phi^m| + \gamma\tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m (\nabla e_h^m, \nabla v_h)$$
$$= \Big[-\int_\Omega d^m v_h\, \rho^m |\nabla\phi^m| + \int_\Omega d^{m-1} v_h\, \rho^{m-1} |\nabla\phi^{m-1}|\Big]$$

$$
\begin{aligned}
&-\tau_m \int_\Omega (\nabla d^m, \nabla v_h)\rho^m |\nabla \phi^m| + \tau_m \int_\Omega d^m (\boldsymbol{v}^m, \nabla v_h)\rho^m |\nabla \phi^m| \\
&+\gamma \tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m (\nabla I_h u^{e,m}, \nabla v_h) + \int_{t_{m-1}}^{t_m} \int_\Omega \big[(\nabla u^{e,m}, \nabla v_h)\rho^m |\nabla \phi^m| \\
&-(\nabla u^e, \nabla v_h)\rho\, |\nabla \phi|\big] + \int_{t_{m-1}}^{t_m} \int_\Omega \big[u^e (\boldsymbol{v}, \nabla v_h)\rho\, |\nabla \phi| - u^{e,m}(\boldsymbol{v}^m, \nabla v_h)\rho^m |\nabla \phi^m|\big] \\
&+ \int_{t_{m-1}}^{t_m} \int_\Omega \big[f^e v_h\, \rho\, |\nabla \phi| - f^{e,m} v_h\, \rho^m |\nabla \phi^m|\big] + \int_{t_{m-1}}^{t_m} \int_\Omega \phi\, R\, v_h\, \rho\, |\nabla \phi| \\
=:& \sum_{i=1}^{8} \langle S_i^m, v_h \rangle.
\end{aligned}
$$

Inserting $v_h = e_h^m$ and following the argument in the proof of Theorem 1 leading to (48) we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (e_h^m)^2 \rho^m |\nabla \phi^m| + \tau_m \int_\Omega |\nabla e_h^m|^2 \rho^m |\nabla \phi^m| + \tau_m^2 \int_\Omega I_h^m \tilde{\rho}^m |\nabla e_h^m|^2 \\
&\quad \le \frac{1}{2}\int_\Omega (e_h^{m-1})^2 \rho^{m-1} |\nabla \phi^{m-1}| + C\tau_m \int_\Omega (e_h^m)^2 \rho^m |\nabla \phi^m| + \sum_{i=1}^{8} \langle S_i^m, e_h^m \rangle.
\end{aligned} \tag{51}
$$

We now deal individually with the terms $\langle S_i^m, e_h^m \rangle$, $i = 1, \ldots, 8$ in (51). Clearly,

$$
\begin{aligned}
|\langle S_1^m, e_h^m \rangle| \le\, & C \int_\Omega |d^m - d^{m-1}|\, |e_h^m|\, \rho^m + C \int_\Omega |d^{m-1}|\, |e_h^m|\, |\nabla(\phi^m - \phi^{m-1})|\, \rho^m \\
&+ C \int_\Omega |d^{m-1}|\, |e_h^m|\, |\rho^m - \rho^{m-1}| \equiv I + II + III.
\end{aligned}
$$

In order to estimate I we first deduce from Lemma 3 a) that every $T \in \mathcal{T}_h$ with $T \cap \mathrm{supp}\rho^m \neq \emptyset$ satisfies $T \in \mathcal{T}_h^{m-1} \cap \mathcal{T}_h^m$. Therefore $I_h^{m-1} u^{e,m-1} = I_h^m u^{e,m-1}$ on $\mathrm{supp}\rho^m$, which yields

$$
\begin{aligned}
d^m - d^{m-1} &= [u^{e,m} - u^{e,m-1}] - I_h^m [u^{e,m} - u^{e,m-1}] \\
&= \int_{t_{m-1}}^{t_m} (u_t^e - I_h^m u_t^e)(\cdot, t) \quad \text{on } \mathrm{supp}\rho^m.
\end{aligned}
$$

Hence, Lemma 3 a), Lemma 7 and (8) imply that

$$
\begin{aligned}
|I| &\le C \int_\Omega \int_{t_{m-1}}^{t_m} |u_t^e - I_h^m u_t^e|\, |e_h^m|\, \rho^m \\
&\le C\sqrt{\tau_m} \left(\int_\Omega (e_h^m)^2 \rho^m \right)^{\frac{1}{2}} \left(\int_{t_{m-1}}^{t_m} \int_{D_h^m} |u_t^e - I_h^m u_t^e|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

$$\le \tau_m \int_\Omega (e_h^m)^2 \rho^m |\nabla\phi^m| + C\epsilon h^4 \int_{t_{m-1}}^{t_m} \Big(\|\partial_t^\bullet u(\cdot,t)\|^2_{H^2(\Gamma(t))} + \|u(\cdot,t)\|^2_{H^3(\Gamma(t))} \Big)\, dt$$

and similarly,

$$|II| \le C\tau_m \int_\Omega |d^{m-1}|\,|e_h^m|\,\rho^m \le C\tau_m \left(\int_\Omega (e_h^m)^2\rho^m\right)^{\frac12} \left(\int_{D_h^{m-1}} |d^{m-1}|^2\right)^{\frac12}$$
$$\le \tau_m \int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + C\epsilon h^4 \tau_m \|u^{m-1}\|^2_{H^2(\Gamma(t_{m-1}))}.$$

Next, we deduce from (27), Lemma 3 a), (8), Lemma 7, Lemma 6 and (41) that

$$|III| \le C\frac{\tau_m}{\epsilon}\int_\Omega |d^{m-1}\,|e_h^m|\sqrt{\rho^m} + C\frac{\tau_m^2}{\epsilon^2}\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |d^{m-1}|\,|e_h^m|$$
$$\le \tau_m \int_\Omega (e_h^m)^2\rho^m + C\frac{\tau_m}{\epsilon^2}\|d^{m-1}\|^2_{L^2(D_h^{m-1})}$$
$$+C\frac{\tau_m^2}{\epsilon^2}\left(\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} (e_h^m)^2\right)^{\frac12} \|d^{m-1}\|_{L^2(D_h^{m-1})}$$
$$\le C\tau_m \int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + C\frac{\tau_m h^4}{\epsilon}\|u^{m-1}\|^2_{H^2(\Gamma(t_{m-1}))}$$
$$+C\frac{\tau_m^2 h^2}{\epsilon^{\frac32}}\|u^{m-1}\|_{H^2(\Gamma(t_{m-1}))}\left(\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \epsilon^2\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |\nabla e_h^m|^2\right)^{\frac12}$$
$$\le C\tau_m \int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \frac{\tau_m^2}{8}\int_\Omega I_h^m\tilde\rho^m|\nabla e_h^m|^2 + C\frac{\tau_m h^4}{\epsilon}\|u^{m-1}\|^2_{H^2(\Gamma(t_{m-1}))},$$

where we used that $\tau \le \epsilon^2$. Again by Lemma 7 we have

$$|\langle S_2^m, e_h^m\rangle| \le \tau_m \left(\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m|\right)^{\frac12}\left(\int_{D_h^m} |\nabla d^m|^2\right)^{\frac12}$$
$$\le \frac18\tau_m \int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m| + C\tau_m\epsilon h^2\|u^m\|^2_{H^2(\Gamma(t_m))},$$

while

$$|\langle S_3^m, e_h^m\rangle| \le C\tau_m \int_\Omega |d^m|\,|\nabla e_h^m|\,\rho^m|\nabla\phi^m|$$

$$\leq C\tau_m \left(\int_\Omega |\nabla e_h^m|^2 \rho^m |\nabla \phi^m|\right)^{\frac{1}{2}} \left(\int_{D_h^m} |d^m|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{8}\tau_m \int_\Omega |\nabla e_h^m|^2 \rho^m |\nabla \phi^m| + C\tau_m \epsilon h^4 \|u^m\|^2_{H^2(\Gamma(t_m))}.$$

Lemma 3 a), (16) and Lemma 7 yield

$$|\langle S_4^m, e_h^m\rangle| \leq C\tau_m^2 \left(\int_\Omega I_h^m \tilde{\rho}^m |\nabla e_h^m|^2\right)^{\frac{1}{2}} \left(\int_{D_h^m} |\nabla I_h^m u^{e,m}|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{\tau_m^2}{8} \int_\Omega I_h^m \tilde{\rho}^m |\nabla e_h^m|^2 + C\tau_m^2 \int_{D_h^m} \left(|\nabla u^{e,m}|^2 + |\nabla d^m|^2\right)$$
$$\leq \frac{\tau_m^2}{8} \int_\Omega I_h^m \tilde{\rho}^m |\nabla e_h^m|^2 + C\tau_m^2 \epsilon \|u^m\|^2_{H^2(\Gamma(t_m))}.$$

We deduce from (27), Lemma 3 a), Lemma 1 and (41) that

$$|\langle S_5^m, e_h^m\rangle| \leq C \int_{t_{m-1}}^{t_m} \int_\Omega \big[|\nabla(u^{e,m} - u^e)|\, \rho^m$$
$$+|\nabla u^e|\, |\nabla(\phi^m - \phi)|\, \rho^m + |\nabla u^e|\, |\rho^m - \rho|\big]\, |\nabla e_h^m|$$
$$\leq C\tau_m \int_{t_{m-1}}^{t_m} \int_\Omega |\nabla u_t^e|\, |\nabla e_h^m| \rho^m + C\tau_m \int_{t_{m-1}}^{t_m} \int_\Omega |\nabla u^e|\, |\nabla e_h^m| \rho^m$$
$$+C\frac{\tau_m}{\epsilon} \int_{t_{m-1}}^{t_m} \int_\Omega |\nabla u^e|\, |\nabla e_h^m| \sqrt{\rho^m} + C\frac{\tau_m^2}{\epsilon^2} \int_{t_{m-1}}^{t_m} \int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |\nabla u^e|\, |\nabla e_h^m|$$
$$\leq C\left[\tau_m^{\frac{3}{2}} \left(\int_{t_{m-1}}^{t_m} \|u_t^e(\cdot,t)\|^2_{H^1(U_{\frac{3\epsilon\pi}{4}}(t))}\right)^{\frac{1}{2}}\right.$$
$$\left. + \frac{\tau_m^2}{\epsilon} \max_{t_{m-1}\leq t\leq t_m} \|u^e(\cdot,t)\|_{H^1(U_{\frac{3\epsilon\pi}{4}}(t))}\right] \left(\int_\Omega |\nabla e_h^m|^2 \rho^m\right)^{\frac{1}{2}}$$
$$+C\frac{\tau_m^3}{\epsilon^2} \max_{t_{m-1}\leq t\leq t_m} \|u^e(\cdot,t)\|_{H^1(U_{\frac{3\epsilon\pi}{2}}(t))} \left(\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)} |\nabla e_h^m|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{\tau_m}{8} \int_\Omega |\nabla e_h^m|^2 \rho^m |\nabla \phi^m| + C\tau_m^2 \epsilon \int_{t_{m-1}}^{t_m} \big(\|\partial_t^\bullet u(\cdot,t)\|^2_{H^1(\Gamma(t))}$$
$$+\|u(\cdot,t)\|^2_{H^2(\Gamma(t))}\big) dt + \frac{\tau_m^2}{8} \int_\Omega I_h^m \tilde{\rho}^m |\nabla e_h^m|^2 + C\tau_m^2 \epsilon \max_{t_{m-1}\leq t\leq t_m} \|u(\cdot,t)\|^2_{H^1(\Gamma(t))}.$$

Here we have used again that $\tau_m \le \tau \le \epsilon^2$. In a similar way we obtain

$$\begin{aligned}|\langle S_6^m, e_h^m\rangle| &\le C\int_{t_{m-1}}^{t_m}\int_\Omega \big[|u^{e,m}-u^e|\,\rho^m + |u^e|\,|\boldsymbol{v}^m|\nabla\phi^m| \\ &-\boldsymbol{v}|\nabla\phi||\,\rho^m + |u^e|\,|\rho^m-\rho|\big]\,|\nabla e_h^m| \le \frac{\tau_m}{8}\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m| \\ &+C\tau_m^2\epsilon\int_{t_{m-1}}^{t_m}\Big(\|\partial_t^\bullet u(\cdot,t)\|^2_{L^2(\Gamma(t))}+\|u(\cdot,t)\|^2_{H^1(\Gamma(t))}\Big)\,dt \\ &+\frac{\tau_m^2}{8}\int_\Omega I_h^m\tilde\rho^m|\nabla e_h^m|^2 + C\tau_m^2\epsilon\max_{t_{m-1}\le t\le t_m}\|u(\cdot,t)\|^2_{L^2(\Gamma(t))}\end{aligned}$$

as well as

$$\begin{aligned}|\langle S_7^m, e_h^m\rangle| &\le C\int_{t_{m-1}}^{t_m}\int_\Omega \big[|f^e|\nabla\phi| - f^{e,m}|\nabla\phi^m||\,|e_h^m|\,\rho^m + |f^{e,m}|\,|e_h^m|\,|\rho-\rho^m|\big] \\ &\le C\tau_m^2\int_\Omega |e_h^m|\,\rho^m + C\frac{\tau_m^2}{\epsilon}\int_\Omega |e_h^m|\sqrt{\rho^m} + C\frac{\tau_m^3}{\epsilon^2}\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)}|e_h^m| \\ &\le \tau_m\int_\Omega (e_h^m)^2\rho^m + C\frac{\tau_m^3}{\epsilon} + C\frac{\tau_m^3}{\epsilon^{\frac32}}\left(\int_\Omega (e_h^m)^2\rho^m + \epsilon^2\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)}|\nabla e_h^m|^2\right)^{\frac12} \\ &\le C\tau_m\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \frac{\tau_m^2}{8}\int_\Omega I_h^m\tilde\rho^m|\nabla e_h^m|^2 + C\frac{\tau_m^3}{\epsilon},\end{aligned}$$

where we have used that $|U_{\frac{3\epsilon\pi}{4}}(t_m)| \le C\epsilon$ and again the fact that $\tau \le \epsilon^2$. Finally, (32) and the definition of ρ imply that

$$|R(\cdot,t)| \le C\|u(\cdot,t)\|_{W^{2,\infty}(\Gamma(t))} \text{ and } |\phi(\cdot,t)| \le c\epsilon \text{ a.e. on } \operatorname{supp}\rho(\cdot,t),\ t\in[t_{m-1},t_m]$$

so that we may estimate with the help of (27), (50) and Lemma 6

$$\begin{aligned}|\langle S_8^m, e_h^m\rangle| &\le C\int_{t_{m-1}}^{t_m}\int_\Omega \big[|\phi|\,|e_h^m|\,\rho^m + |\phi|\,|e_h^m|\,|\rho-\rho^m|\big] \\ &\le C\epsilon\tau_m\int_\Omega |e_h^m|\,\rho^m + C\tau_m^2\int_\Omega |e_h^m|\sqrt{\rho^m} + C\frac{\tau_m^3}{\epsilon}\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)}|e_h^m| \\ &\le \tau_m\int_\Omega (e_h^m)^2\rho^m + C\tau_m\epsilon^3 + C\tau_m^3\epsilon \\ &\quad +C\frac{\tau_m^3}{\sqrt\epsilon}\left(\int_\Omega (e_h^m)^2\rho^m + \epsilon^2\int_{U_{\frac{3\epsilon\pi}{4}}(t_m)}|\nabla e_h^m|^2\right)^{\frac12} \\ &\le C\tau_m\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \frac{\tau_m^2}{8}\int_\Omega I_h^m\tilde\rho^m|\nabla e_h^m|^2 + C\tau_m\epsilon^3 + C\tau_m^3\epsilon.\end{aligned}$$

Inserting the above estimates into (51) we obtain

$$\frac{1}{2}\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \frac{\tau_m}{2}\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m| + \frac{\tau_m^2}{4}\int_\Omega I_h^m\tilde{\rho}^m|\nabla e_h^m|^2$$
$$\le \frac{1}{2}\int_\Omega (e_h^{m-1})^2\rho^{m-1}|\nabla\phi^{m-1}| + C\tau_m\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + C\left(\frac{\tau_m^3}{\epsilon}+\tau_m\epsilon^3\right)$$
$$+C\tau_m\,\epsilon\left(h^2+\frac{h^4}{\epsilon^2}+\tau\right)\max_{t_{m-1}\le t\le t_m}\|u(\cdot,t)\|^2_{H^2(\Gamma(t))}$$
$$+C\epsilon\big(h^4+\tau^2\big)\int_{t_{m-1}}^{t_m}\Big(\|\partial_t^\bullet u(\cdot,t)\|^2_{H^2(\Gamma(t))}+\|u(\cdot,t)\|^2_{H^3(\Gamma(t))}\Big)\,dt.$$

Choosing $\tau_2 \le \tau_1$ small enough and using (36) as well as $\tau \le \epsilon^2$ we infer

$$\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + \tau_m\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m|$$
$$\le (1+C\tau_m)\int_\Omega (e_h^{m-1})^2\rho^{m-1}|\nabla\phi^{m-1}| + C\epsilon^3\tau_m\max_{t_{m-1}\le t\le t_m}\|u(\cdot,t)\|^2_{H^2(\Gamma(t))}$$
$$+C\epsilon^5\int_{t_{m-1}}^{t_m}\big(\|\partial_t^\bullet u(\cdot,t)\|^2_{H^2(\Gamma(t))}+\|u(\cdot,t)\|^2_{H^3(\Gamma(t))}\big)dt + C\tau_m\epsilon^3.$$

Summing from $m = 1,\ldots,l$, dividing by ϵ and recalling (50) we derive

$$\frac{1}{\varepsilon}\int_\Omega (e_h^l)^2\rho^l|\nabla\phi^l| + \sum_{m=1}^{l}\tau_m\frac{1}{\epsilon}\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m|$$
$$\le \frac{1}{\varepsilon}\int_\Omega (e_h^0)^2\rho^0\,|\nabla\phi^0| + C\sum_{m=0}^{l-1}\tau_{m+1}\frac{1}{\varepsilon}\int_\Omega (e_h^m)^2\rho^m\,|\nabla\phi^m| + C\epsilon^2.$$

In order to estimate the first term on the right hand side we write $e_h^0 = (I_h^0 u_0^e - u_0^e) + (u_0^e - u_h^0)$ and recall the definition (38) of u_h^0 as an L^2 projection:

$$\int_\Omega (e_h^0)^2\rho^0\,|\nabla\phi^0| \le C\int_{D_h^0}(e_h^0)^2 \le C\int_{D_h^0}|u_0^e - I_h^0 u_0^e|^2 \le C\epsilon h^4\|u_0\|^2_{H^2(\Gamma(0))}$$

by Lemma 7. Thus

$$\frac{1}{\varepsilon}\int_\Omega (e_h^l)^2\rho^l|\nabla\phi^l| + \sum_{m=1}^{l}\tau_m\frac{1}{\epsilon}\int_\Omega |\nabla e_h^m|^2\rho^m|\nabla\phi^m|$$
$$\le C\sum_{m=0}^{l-1}\tau_{m+1}\frac{1}{\varepsilon}\int_\Omega (e_h^m)^2\rho^m|\nabla\phi^m| + C\epsilon^2 \tag{52}$$

and the discrete Gronwall lemma gives

$$\max_{m=1,\dots,M} \frac{1}{\varepsilon} \int_\Omega (e_h^m)^2 \rho^m |\nabla \phi^m| \le C\epsilon^2. \tag{53}$$

The remainder of the proof follows from (52) and Lemma 7. □

Using the result of Theorem 2 we can now also derive an error bound on the surface.

Corollary 1 *In addition to the assumptions of Theorem 2 suppose that there exists* $\alpha > 0$ *such that* $h_T \ge \alpha\epsilon$ *for all* $T \in \mathcal{T}_h$ *with* $|T \cap \Gamma(t)| > 0$, $t \in [0, T]$. *Then*

$$\max_{m=1,\dots,M} \int_{\Gamma(t_m)} |u^m - u_h^m|^2 + \sum_{m=1}^M \tau_m \int_{\Gamma(t_m)} |\nabla_\Gamma (u^m - u_h^m)|^2 \le C\epsilon^2.$$

Proof Let us fix $m \in \{1, \dots, M\}$ and define $\mathcal{T}_{\Gamma,h}^m := \{T \in \mathcal{T}_h \mid |T \cap \Gamma(t_m)| > 0\}$. Hence, given $T \in \mathcal{T}_{\Gamma,h}^m$, there exists $x_T \in \Gamma(t_m)$ with $\phi^m(x_T) = 0$. We infer from (8) and (36) that for arbitrary $x \in T$

$$|\phi^m(x)| = |\phi^m(x) - \phi^m(x_T)| \le c_1 |x - x_T| \le c_1 h_T \le \frac{\epsilon}{2} \cos^2\left(\frac{3\pi}{8}\right) \le \frac{\epsilon\pi}{4},$$

and therefore

$$\rho^m(x) \ge \frac{1}{2} \quad \text{for all } x \in T,\ T \in \mathcal{T}_{\Gamma,h}^m. \tag{54}$$

We now argue in a similar way as in [6], page 368. Using an interpolation inequality and an inverse estimate we infer that

$$\begin{aligned}
\int_{\Gamma(t_m)} |u^m - u_h^m|^2 &= \sum_{T \in \mathcal{T}_{\Gamma,h}^m} \int_{T \cap \Gamma(t_m)} |u^m - u_h^m|^2 \\
&\le 2 \sum_{T \in \mathcal{T}_{\Gamma,h}^m} |T \cap \Gamma(t_m)| \left(\|d^m\|_{L^\infty(T)}^2 + \|e_h^m\|_{L^\infty(T)}^2 \right) \\
&\le C \sum_{T \in \mathcal{T}_{\Gamma,h}^m} |T \cap \Gamma(t_m)|\, h_T^2 \|\nabla u^{e,m}\|_{W^{1,\infty}(T)}^2 \\
&\quad + C \sum_{T \in \mathcal{T}_{\Gamma,h}^m} h_T^n h_T^{-(n+1)} \|e_h^m\|_{L^2(T)}^2 \\
&\le Ch^2 |\Gamma(t_m)| \|u^m\|_{W^{1,\infty}(\Gamma(t_m))}^2 + C\epsilon^{-1} \sum_{T \in \mathcal{T}_{\Gamma,h}^m} \int_T |e_h^m|^2 \rho^m |\nabla \phi^m|,
\end{aligned}$$

where the last inequality follows from (54), (8) and the assumption that $h_T \ge \alpha\epsilon$, $T \in \mathcal{T}_{\Gamma,h}^m$. In a similar way we obtain

$$\int_{\Gamma(t_m)} |\nabla_\Gamma (u^m - u_h^m)|^2 \le Ch^2 |\Gamma(t_m)| \|u^m\|_{W^{2,\infty}(\Gamma(t_m))}^2$$

$$+C\epsilon^{-1}\sum_{T\in\mathcal{T}^m_{\Gamma,h}}\int_T|\nabla e^m_h|^2\rho^m\,|\nabla\phi^m|.$$

Thus,

$$\max_{m=1,\dots,M}\int_{\Gamma(t_m)}|u^m-u^m_h|^2+\sum_{m=1}^M\tau_m\int_{\Gamma(t_m)}|\nabla_\Gamma(u^m-u^m_h)|^2$$
$$\le Ch^2\max_{t\in[0,T]}\|u(\cdot,t)\|^2_{W^{2,\infty}(\Gamma(t))}$$
$$+C\epsilon^{-1}\max_{m=1,\dots,M}\int_\Omega|e^m_h|^2\rho^m|\nabla\phi^m|+C\epsilon^{-1}\sum_{m=1}^M\tau_m\int_\Omega|\nabla e^m_h|^2\rho^m|\nabla\phi^m|\le C\epsilon^2,$$

by (36), (53) and (52). □

6 Numerical results

As already mentioned in Remark 2 c), the scheme (37), (38) is not fully practical. Therefore, our implementation uses the following modification: Find $u^m_h\in V^m_h$, such that

$$\begin{aligned}&\int_\Omega u^m_h\,v_h\,I^m_h\rho^m\,|\nabla I^m_h\phi^m|-\int_\Omega u^{m-1}_h\,v_h\,I^{m-1}_h\rho^{m-1}\,|\nabla I^{m-1}_h\phi^{m-1}|\\&+\tau_m\int_\Omega(\nabla u^m_h,\nabla v_h)\,I^m_h\rho^m\,|\nabla I^m_h\phi^m|-\tau_m\int_\Omega u^m_h\,(I^m_h\hat{\boldsymbol{v}}^m,\nabla v_h)\,I^m_h\rho^m\,|\nabla I^m_h\phi^m|\\&+\gamma\tau_m^2\int_\Omega I^m_h\tilde{\rho}^m(\nabla u^m_h,\nabla v_h)=\tau_m\int_\Omega I^m_h\hat{f}^m\,v_h\,I^m_h\rho^m\,|\nabla I^m_h\phi^m|\end{aligned}\tag{55}$$

for all $v_h\in V^m_h$ and $1\le m\le M$. Here, $\hat{\boldsymbol{v}}^m(x):=\boldsymbol{v}(\hat{p}(x,t_m),t_m)$, $\hat{f}^m(x)=f(\hat{p}(x,t_m),t_m)$, where $\hat{p}(x,t)$ denotes the closest point projection of a point x onto $\Gamma(t)$. Setting $\hat{u}_0(x)=u_0(\hat{p}(x,0))$ we define the initial data $\hat{u}^0_h\in V^0_h$ by

$$\int_{D^0_h}\hat{u}^0_h\,v_h=\int_{D^0_h}I^0_h\hat{u}_0\,v_h\qquad\forall v_h\in V^0_h.\tag{56}$$

Let us remark that the evaluation of $\hat{p}(x,t)$ is easier compared to $\tilde{p}(x,t)$, which has been used to extend the data for the scheme (37), (38). However, we claim that

$$\tilde{p}(x,t)-\hat{p}(x,t)=O(\phi(x,t)^2).\tag{57}$$

To see this, we first observe that $\hat{p}(x,t)$ is characterized by the conditions

$$\phi(\hat{p}(x,t),t)=0\quad\text{and}\quad x-\hat{p}(x,t)\perp\Gamma(t)\text{ at }\hat{p}(x,t).$$

Therefore, it is not difficult to verify with the help of Taylor expansion that

$$x-\hat{p}(x,t)=\lambda(x,t)\nabla\phi(\hat{p}(x,t),t), \quad \text{with } \lambda(x,t)=\frac{\phi(x,t)}{|\nabla\phi(\hat{p}(x,t),t)|^2}+O(\phi(x,t)^2).$$

Combining this relation with (58) in the "Appendix" we find that

$$\tilde{p}(x,t)-\hat{p}(x,t)=\phi(x,t)\left[\frac{\nabla\phi(\hat{p}(x,t),t)}{|\nabla\phi(\hat{p}(x,t),t)|^2}-\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|^2}\right] +O(\phi(x,t)^2)=O(\phi(x,t)^2).$$

In particular, we infer from (57) that replacing $\tilde{p}$ by $\hat{p}$ in the extension of $\boldsymbol{v}$, f and u_0 will not affect the result of Theorem 2. In contrast, it is not straightforward to handle the interpolation terms $I_h^m\rho^m$ and $I_h^{m-1}\rho^{m-1}$ in (55). Applying a standard interpolation estimate to $\rho^m - I_h^m\rho^m$ will result in a term of the form $h^2\|\rho^m\|_{H^2}\approx\frac{h^2}{\epsilon^2}$, which we are currently not able to analyze. The results of our test calculations below however show that the use of the interpolation operator in (55), (56) does not lead to reduced convergence rates. More precisely we investigate the experimental order of convergence (eoc) for the following errors:

$$\mathcal{E}_1=\max_{m=1,\dots,M}\frac{2}{\epsilon\pi}\int_\Omega |I_h^m\hat{u}^m-u_h^m|^2\, I_h^m\rho^m\,|\nabla I_h^m\phi^m|,$$
$$\mathcal{E}_2=\frac{2}{\epsilon\pi}\sum_{m=1}^M\tau_m\int_\Omega |\nabla(I_h^m\hat{u}^m-u_h^m)|^2 I_h^m\rho^m\,|\nabla I_h^m\phi^m|,$$

where $\hat{u}^m(x)=u(\hat{p}(x,t_m),t_m)$. We use the finite element toolbox Alberta 2.0, [24], and implement a similar mesh refinement strategy to that in [2] with a fine mesh constructed in D_h^m and a coarser mesh in $\Omega\setminus D_h^m$. The linear systems appearing in each time step were solved using GMRES together with diagonal preconditioning. The values of h given below are such that $h:=\max_{T\in D_h^m}h_T$, $h_T=\text{diam}(T)$.

Remark 4 Although the analysis requires $\gamma>0$, the method works with $\gamma=0$ and produces very similar eocs to the ones displayed in the tables below for $\gamma=0.01$.

6.1 2D examples

We set $\Omega=(-2.4,2.4)^2$, $T=0.1$, and choose $\gamma=0.01$, $\epsilon=85.33\,h$ as well as a uniform time step $\tau_m=0.0025\varepsilon^2$, $m=1,\dots,M$. In all our examples below $\Gamma(t)$ will be a circle $\Gamma(t)=\{x\in\mathbb{R}^2\,|\,|x-m(t)|=1\}$ of radius 1 with center $m(t)\in\mathbb{R}^2$. In addition to $\mathcal{E}_1,\mathcal{E}_2$ we shall also investigate the errors appearing in Corollary 1. To do so we choose $L>0$ and define the following quadrature points

$$x_l(t):=m(t)+\left(\cos\left(\frac{2\pi l}{L}\right),\sin\left(\frac{2\pi l}{L}\right)\right)^T,\quad l=0,\dots,L-1$$

Table 1 Errors and experimental orders of convergence for Example 1

h	ε	$\mathcal{E}_1$	eoc_1	$\mathcal{E}_2$	eoc_2
4.6875e−03	0.4	2.0565e−04	–	1.0763e−03	–
3.3146e−03	$0.2\sqrt{2}$	3.2822e−05	5.295	2.7030e−04	3.987
2.3437e−03	0.2	6.5608e−06	4.645	6.7864e−05	3.988
1.6573e−03	$0.1\sqrt{2}$	1.4513e−06	4.353	1.7017e−05	3.991
1.1719e−03	0.1	3.4022e−07	4.186	4.2668e−06	3.991

Table 2 Errors and experimental orders of convergence for Example 1

h	ε	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4$	eoc_4
4.6875e−03	0.4	2.7651e−05	–	4.3137e−06	–
3.3146e−03	$0.2\sqrt{2}$	8.1077e−06	3.540	1.6031e−06	2.856
2.3437e−03	0.2	2.1848e−06	3.784	5.9541e−07	2.858
1.6573e−03	$0.1\sqrt{2}$	5.6637e−07	3.895	2.3962e−07	2.626
1.1719e−03	0.1	1.4412e−07	3.949	9.6590e−08	2.622

as well as

$$\mathcal{E}_3 = \max_{m=1,\ldots,M} \sum_{l=0}^{L-1} \frac{2\pi}{L} |u(x_l(t_m), t_m) - u_h^m(x_l(t_m))|^2,$$

$$\mathcal{E}_4 = \sum_{m=1}^{M} \tau_m \sum_{l=0}^{L-1} \frac{2\pi}{L} |\nabla_\Gamma u(x_l(t_m), t_m) - \nabla_\Gamma u_h^m(x_l(t_m))|^2.$$

In our computations $L = 200$ turned out to be sufficient.

Example 1 For our first example we consider the stationary unit circle $\Gamma(t) = \Gamma = S^1, t \in [0, T]$ described as the zero level set of the function $\phi(x) := x_1^2 + x_2^2 - 1$.

The function $u(x,t) := e^{-4t}\left[x_1x_2\cos(\pi t) + \frac{1}{2}(x_1^2 - x_2^2)\sin(\pi t)\right]$ is a solution of (1), (2) for the velocity field $\boldsymbol{v}(x) = \frac{\pi}{2}(x_2, -x_1)^T$, $f = 0$ and the initial data $u_0(x) = x_1x_2$. A similar choice of velocity appears in Example 3 in [10]. In Tables 1 and 2 we display the values of $\mathcal{E}_i$, $i = 1 \to 4$, together with the eocs.

Example 2 (cf. [16, Section 3.1], [26], Example 5.2) We consider the family of unit circles $\Gamma(t) = \{x \in \mathbb{R}^2 \,|\, (x_1 + \frac{1}{2} - 2t)^2 + x_2^2 = 1\}$ described as the zero level set of $\phi(x,t) = (x_1 + \frac{1}{2} - 2t)^2 + x_2^2 - 1$. The function $u : S_T \to \mathbb{R}$, $u(x,t) = e^{-4t}(x_1 + \frac{1}{2} - 2t)x_2$ is a solution of (1), (2) for the velocity field $\boldsymbol{v}(x,t) = (2,0)^T$, $f = 0$ and the initial data $u_0(x) = (x_1 + \frac{1}{2})x_2$. The results are displayed in Tables 3 and 4 where we see eocs that are similar to the ones in Tables 1 and 2.

Table 3 Errors and experimental orders of convergence for Example 2

h	ε	$\mathcal{E}_1$	eoc_1	$\mathcal{E}_2$	eoc_2
4.6875e−03	0.4	1.5537e−04	–	9.3201e−04	–
3.3146e−03	$0.2\sqrt{2}$	2.5206e−05	5.248	2.3280e−04	4.002
2.3437e−03	0.2	4.8726e−06	4.742	5.8500e−05	3.985
1.6573e−03	$0.1\sqrt{2}$	1.0558e−06	4.413	1.4776e−05	3.970
1.1719e−03	0.1	2.4507e−07	4.214	3.7865e−06	3.929

Table 4 Errors and experimental orders of convergence for Example 2

h	ε	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4$	eoc_4
4.6875e−03	0.4	1.8431e−05	–	3.0082e−06	–
3.3146e−03	$0.2\sqrt{2}$	5.6312e−06	3.421	1.2489e−06	2.537
2.3437e−03	0.2	1.5443e−06	3.733	4.8015e−07	2.758
1.6573e−03	$0.1\sqrt{2}$	4.0396e−07	3.869	1.9389e−07	2.616
1.1719e−03	0.1	1.0350e−07	3.929	8.1747e−08	2.492

We see that the eoc for $\mathcal{E}_1$ is reducing towards 4, the eocs for $\mathcal{E}_2$ and $\mathcal{E}_3$ are close to 4 and the eoc for $\mathcal{E}_4$ is between 2 and 3 which is better than Theorem 2 predicts. Since $\mathcal{E}_1$ and $\mathcal{E}_3$ approximate L^2–errors, higher eocs can be expected although a corresponding proof is by no means straightforward and beyond the scope of this paper. The higher eoc for $\mathcal{E}_2$ presumably reflects a superconvergence effect because we consider $\nabla(I_h^m \hat{u}^m - u_h^m)$ rather than $\nabla(\hat{u}^m - u_h^m)$. We expect that $\mathcal{E}_4$ will tend towards 2 if ε, h and τ are reduced further.

6.2 3D example

Example 3 Here we consider the first example in Section 7 of [18] in which a family of expanding and collapsing spheres is considered such that $\Gamma(t) = \{x \in \mathbb{R}^2 \mid |x| = r(t)\}$ where $r(t) = 1 + \sin^2(\pi t)$, described as the zero level set of $\phi(x,t) = x_1^2 + x_2^2 + x_3^2 - r(t)^2$. The function $u : S_T \to \mathbb{R}$, $u(x,t) = \frac{2}{r(t)^2|x|^2} e^{-6\int_0^t \frac{1}{r^2(t)}} x_1 x_3$ is a solution of (1), (2) for the velocity field $\boldsymbol{v}(x,t) = \frac{r'(t)}{|x|} x$, $f = 0$ and the initial data $u_0(x) = \frac{2}{|x|^2} x_1 x_3$. We set $\Omega = (-4,4)^3$, $T = 0.1$ and choose $\gamma = 0.01$, $\epsilon = 1.85\,h$ as well as a uniform time step $\tau_m = 0.5h^2$, $m = 1, \ldots, M$. For this example we only display the errors on the surfaces which are in this case approximated by the quadrature rules

$$\mathcal{E}_3 = \max_{m=1,\ldots,M} \sum_{k=0}^{2L-1} \sum_{l=0}^{L-1} \left(\frac{\pi}{L}\right)^2 |u(x_{k,l}(t_m), t_m) - u_h^m(x_{k,l}(t_m))|^2 \sin\left(\frac{l\pi}{L}\right),$$

$$\mathcal{E}_4 = \sum_{m=1}^{M} \tau_m \sum_{k=0}^{2L-1} \sum_{l=0}^{L-1} \left(\frac{\pi}{L}\right)^2 |\nabla_\Gamma u(x_{k,l}(t_m), t_m) - \nabla_\Gamma u_h^m(x_{k,l}(t_m))|^2 \sin\left(\frac{l\pi}{L}\right),$$

Table 5 Errors and experimental orders of convergence for Example 3

h	ε	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4$	eoc_4
2.1651e−01	0.4	5.2016e−05	–	2.5203e−03	–
1.5309e−01	$0.2\sqrt{2}$	1.1008e−05	4.481	1.3058e−03	1.897
1.0825e−01	0.2	2.8535e−06	3.896	6.8447e−04	1.864
7.6547e−02	$0.1\sqrt{2}$	6.9422e−07	4.079	3.4543e−04	1.973

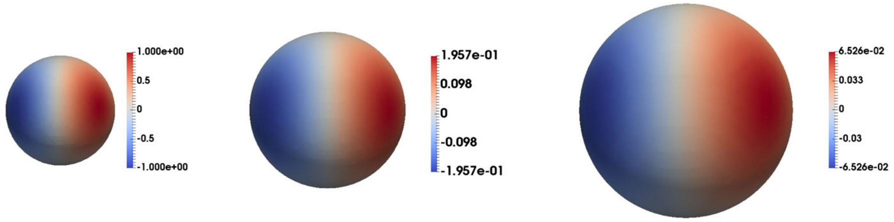

Fig. 1 Computational results from Example 3: u_h^m at times $t_m = 0, 0.2, 0.4$ plotted on the zero level surface of ϕ_h^m

where

$$x_{k,l}(t) = r(t)\left(\cos\left(\frac{k\pi}{L}\right)\sin\left(\frac{l\pi}{L}\right), \sin\left(\frac{k\pi}{L}\right)\sin\left(\frac{l\pi}{L}\right), \cos\left(\frac{l\pi}{L}\right)\right)^T,$$
$$k = 0, \ldots, 2L-1, l = 0, \ldots, L-1.$$

For the choice $L = 200$ the results are displayed in Table 5, where we see eocs close to 4 for $\mathcal{E}_3$ and eocs close to 2 for $\mathcal{E}_4$.

We conclude with Fig. 1 in which we present the approximate u_h^m at times $t_m = 0, 0.2, 0.4$ plotted on the zero level surface of ϕ_h^m.

Acknowledgements The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for its hospitality during the programme *Coupling Geometric PDEs with Physics for Cell Morphology, Motility and Pattern Formation* supported by EPSRC Grant Number EP/K032208/1 and VS gratefully acknowledges the support of the Leverhulme Trust Research Project Grant (RPG-2014-149).

Appendix

Lemma 8 *Suppose that u is a smooth solution of* (1) *and denote by u^e the extension defined in* (14)*. Then u^e is a solution of* (31) *with R satisfying* (32)*.*

Proof We use the notation introduced in Sect. 2.2 and begin by deriving a formula for $\tilde{p}(x,t)$ for $x \in U_\delta(t)$, $t \in [0,T]$. Define

$$\eta(\tau) := F_t(p(x,t),(1-\tau)\phi(x,t)), \quad \tau \in [0,1].$$

Recalling (9) and the definition of F_t we have

$$\eta'(\tau) = -\phi(x,t)\frac{\nabla\phi(\gamma_{p(x,t),t}((1-\tau)\phi(x,t)),t)}{\left|\nabla\phi(\gamma_{p(x,t),t}((1-\tau)\phi(x,t)),t)\right|^2}.$$

Observing that $\gamma_{p(x,t),t}(\phi(x,t)) = F_t(p(x,t),\phi(x,t)) = x$ and using similar arguments to calculate $\eta''(\tau)$ we find for $k = 1,\ldots,n+1$ that

$$\eta_k'(0) = -\phi(x,t)\frac{\phi_{x_k}(x,t)}{|\nabla\phi(x,t)|^2},$$
$$\eta_k''(0) = \phi(x,t)^2\sum_{l,r=1}^{n+1}\left(\delta_{kr} - \frac{2\phi_{x_k}(x,t)\phi_{x_r}(x,t)}{|\nabla\phi(x,t)|^2}\right)\frac{\phi_{x_l}(x,t)\phi_{x_lx_r}(x,t)}{|\nabla\phi(x,t)|^4}.$$

Since $\eta(1) = F_t(p(x,t),0) = \Psi(p(x,t),t) = \tilde{p}(x,t)$, $\eta(0) = x$ we deduce with the help of Taylor's theorem that for $k = 1,\ldots,n+1$

$$\tilde{p}_k(x,t) = x_k - \phi(x,t)\frac{\phi_{x_k}(x,t)}{|\nabla\phi(x,t)|^2} + \frac{1}{2}\phi(x,t)^2\sum_{l,r=1}^{n+1}\Big(\delta_{kr} - \frac{2\phi_{x_k}(x,t)\phi_{x_r}(x,t)}{|\nabla\phi(x,t)|^2}\Big)\frac{\phi_{x_l}(x,t)\phi_{x_lx_r}(x,t)}{|\nabla\phi(x,t)|^4} + \phi(x,t)^3 r_k(x,t), \tag{58}$$

where r_k are smooth functions. Starting from (58) it is not difficult to derive formulae for $\tilde{p}_{x_i}$, $\tilde{p}_{x_ix_j}$ (cf. (2.9), (2.10) in [7]) and hence to deduce from (19) and (20) that

$$\nabla u^e(x,t) = (I + \phi(x,t)A(x,t))\nabla_\Gamma u(\tilde{p}(x,t),t) \tag{59}$$

$$\frac{1}{|\nabla\phi(x,t)|}\nabla\cdot\big(|\nabla\phi(x,t)|\nabla u^e(x,t)\big) = (\Delta_\Gamma u)(\tilde{p}(x,t),t) + \phi(x,t)\left(\sum_{k,l=1}^{n+1} b_{lk}(x,t)\underline{D}_l\underline{D}_k u(\tilde{p}(x,t),t) + \sum_{k=1}^{n+1}\tilde{c}_k(x,t)\underline{D}_k u(\tilde{p}(x,t),t)\right), \tag{60}$$

where $A = (a_{ik})$, b_{lk} and $\tilde{c}_k$ are smooth. Furthermore, differentiating (58) with respect to t we find that

$$\tilde{p}_t(x,t) = -\frac{\phi_t(x,t)}{|\nabla\phi(x,t)|^2}\nabla\phi(x,t) + \phi(x,t)q(x,t)$$
$$= V(x,t)\nu(x,t) + \phi(x,t)q(x,t), \quad q \text{ smooth}$$

so that we infer from (59), (21) and (6) that

$$\begin{aligned}\partial_t^\bullet u^e(x,t) &= u_t^e(x,t) + \big(\boldsymbol{v}(x,t), \nabla u^e(x,t)\big) = u_t^e(x,t)\\ &\quad + \big(\boldsymbol{v}(x,t), (I + \phi(x,t)A(x,t))\nabla_\Gamma u(\tilde{p}(x,t),t)\big)\\ &= \partial_t^\bullet u(\tilde{p}(x,t),t) + \big((\boldsymbol{v}(x,t) - \boldsymbol{v}(\tilde{p}(x,t),t)), \nabla_\Gamma u(\tilde{p}(x,t),t)\big)\\ &\quad + V(x,t)\big((\nu(x,t) - \nu(\tilde{p}(x,t),t)), \nabla_\Gamma u(\tilde{p}(x,t),t)\big)\\ &\quad + \phi(x,t)\big(q(x,t) + A(x,t)^T \boldsymbol{v}(x,t), \nabla_\Gamma u(\tilde{p}(x,t),t)\big).\end{aligned} \tag{61}$$

The fundamental theorem of calculus together with (58) implies that

$$\begin{aligned}\boldsymbol{v}(x,t) - \boldsymbol{v}(\tilde{p}(x,t),t) &= \int_0^1 D\boldsymbol{v}(sx + (1-s)\tilde{p}(x,t),t)ds\,(x - \tilde{p}(x,t))\\ &= \phi(x,t)\,\tilde{q}(x,t)\end{aligned}$$

for some smooth $\tilde{q}$. Arguing in the same way for the corresponding difference involving ν we infer from (61)

$$\partial_t^\bullet u^e(x,t) = \partial_t^\bullet u(\tilde{p}(x,t),t) + \phi(x,t)\sum_{k=1}^{n+1} \hat{c}_k(x,t)\underline{D}_k u(\tilde{p}(x,t),t), \tag{62}$$

where $\hat{c}_k$ are smooth. Finally, since $\nabla_\phi \cdot \boldsymbol{v}(\tilde{p}(x,t),t) = \nabla_\Gamma \cdot \boldsymbol{v}(\tilde{p}(x,t),t)$ we have

$$\begin{aligned}u^e(x,t)\nabla_\phi \cdot \boldsymbol{v}(x,t) &= u(\tilde{p}(x,t),t)\nabla_\phi \cdot \boldsymbol{v}(\tilde{p}(x,t),t)\\ &\quad + u(\tilde{p}(x,t),t)\big(\nabla_\phi \cdot \boldsymbol{v}(x,t) - \nabla_\phi \cdot \boldsymbol{v}(\tilde{p}(x,t),t)\big)\\ &= u(\tilde{p}(x,t),t)\nabla_\Gamma \cdot \boldsymbol{v}(\tilde{p}(x,t),t) + u(\tilde{p}(x,t),t)\phi(x,t)\bar{r}(x,t).\end{aligned} \tag{63}$$

Here, the second term has been rewritten in a similar way as above for some smooth $\bar{r}$. Combining (60)–(63) we deduce that the extension u^e of a function u solving (1) satisfies (31), where R has the form (32). □

References

1. Adalsteinsson, D., Sethian, J.A.: Transport and diffusion of material quantities on propagating interfaces via level set methods. J. Comput. Phys. **185**, 271–288 (2003)
2. Barrett, J.W., Nürnberg, R., Styles, V.: Finite element approximation of a phase field model for void electromigration. SIAM J. Numer. Anal. **46**, 738–772 (2004)
3. Bertalmio, M., Cheng, L.T., Osher, S., Sapiro, G.: Variational problems and partial differential equations on implicit surfaces: the framework and examples in image processing and pattern formation. J. Comput. Phys. **174**, 759–780 (2001)
4. Burger, M.: Finite element approximation of elliptic partial differential equations on implicit surfaces. Comput. Vis. Sci. **12**, 87–100 (2009)
5. Cahn, J.W., Fife, P., Penrose, O.: A phase field model for diffusion induced grain boundary motion. Acta Mater. **45**, 4397–4413 (1997)

6. Deckelnick, K., Dziuk, G., Elliott, C.M., Heine, C.-J.: An h-narrow band finite element method for elliptic equations on implicit surfaces. IMA J. Numer. Anal. **30**, 351–376 (2010)
7. Deckelnick, K., Elliott, C.M., Ranner, T.: Unfitted finite element methods using bulk meshes for surface partial differential equations. SIAM J. Numer. Anal. **52**, 2137–2162 (2014)
8. Dziuk, G., Elliott, C.M.: Finite elements on evolving surfaces. IMA J. Numer. Anal. **27**, 262–292 (2007)
9. Dziuk, G., Elliott, C.M.: Eulerian finite element method for parabolic PDEs on implicit surfaces. Interfaces Free Bound. **10**, 119–138 (2008)
10. Dziuk, G., Elliott, C.M.: An Eulerian approach to transport and diffusion on evolving implicit surfaces. Comput. Vis. Sci. **13**, 17–28 (2010)
11. Dziuk, G., Elliott, C.M.: A fully discrete evolving surface finite element method. SIAM J. Numer. Anal. **50**, 2677–2694 (2012)
12. Dziuk, G., Elliott, C.M.: L^2-estimates for the evolving surface finite element method. Math. Comp. **82**, 1–24 (2013)
13. Dziuk, G., Elliott, C.M.: Finite element methods for surface PDEs. Acta Numer. **22**, 289–396 (2013)
14. Dziuk, G., Lubich, C., Mansour, D.: Runge–Kutta time discretization of parabolic differential equations on evolving surfaces. IMA J. Numer. Anal. **32**, 394–416 (2012)
15. Elliott, C.M., Stinner, B.: Analysis of a diffuse interface approach to an advection diffusion equation on a moving surface. Math. Models Methods Appl. Sci. **19**, 787–802 (2009)
16. Elliott, C.M., Stinner, B., Styles, V., Welford, R.: Numerical computation of advection and diffusion on evolving diffuse interfaces. IMA J. Numer. Anal. **31**, 786–812 (2011)
17. Hansbo, P., Larson, M.G., Zahedi, S.: Characteristic cut finite element methods for convection-diffusion problems on time dependent surfaces. Comput. Methods Appl. Mech. Eng. **293**, 431–461 (2015)
18. Lenz, M., Nemadjieu, S.F., Rumpf, M.: A convergent finite volume scheme for diffusion on evolving surfaces. SIAM J. Numer. Anal. **49**, 15–37 (2011)
19. Lubich, C., Mansour, D., Venkataraman, C.: Backward difference time discretization of parabolic differential equations on evolving surfaces. IMA J. Numer. Anal. **33**, 1365–1385 (2013)
20. Olshanskii, M.A., Reusken, A., Xu, X.: An Eulerian space-time finite element method for diffusion problems on evolving surfaces. SIAM J. Numer. Anal. **52**, 1354–1377 (2014)
21. Olshanskii, M.A., Reusken, A.: Error analysis of a space-time finite element method for solving PDEs on evolving surfaces. SIAM J. Numer. Anal. **52**, 2092–2120 (2014)
22. Rätz, A., Voigt, A.: PDE's on surfaces—a diffuse interface approach. Comm. Math. Sci. **4**, 575–590 (2006)
23. Rätz, A., Voigt, A.: A diffuse-interface approximation for surface diffusion including adatoms. Nonlinearity **20**, 177–192 (2007)
24. Schmidt, A., Siebert, K.G.: Design of Adaptive Finite Element Software. The Finite Element Toolbox ALBERTA. Lecture Notes in Computational Science and Engineering 42. Springer, Berlin (2005)
25. Stone, H.A.: A simple derivation of the time-dependent convective–diffusive equation for surfactant transport along a deforming interface. Phys. Fluids A **2**, 111–112 (1990)
26. Teigen, K.E., Li, X., Lowengrub, J., Wang, F., Voigt, A.: A diffuse-interface approach for modeling transport, diffusion and adsorption/desorption of material quantities on a deformable interface. Commun. Math. Sci. **7**, 1009–1037 (2009)
27. Xu, J.J., Zhao, H.K.: An Eulerian formulation for solving partial differential equations along a moving interface. J. Sci. Comput. **19**, 573–594 (2003)