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A unified approach to inequalities for K-functionals and moduli of smoothness

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Abstract
The paper provides a detailed study of crucial inequalities for smoothness and interpolation characteristics in rearrangement invariant Banach function spaces. We present a unified approach based on Holmstedt formulas to obtain these estimates. As examples, we derive new inequalities for moduli of smoothness and K-functionals in various Lorentz spaces.

Keywords Moduli of smoothness · K-functionals · Holmstedt formulas · Weighted Lorentz spaces · Lorentz–Karamata spaces

Mathematics Subject Classification Primary 41A17 · 46B70; Secondary 46E30 · 46E35

1 Introduction

Some, nowadays well-known, inequalities between moduli of continuity, or more general, between moduli of smoothness are attached to the names of Marchaud, Ul'yanov, and Kolyada. These inequalities play an important role in approximation theory as well as in the theory of function spaces, in particular, they can be used to derive embedding properties of function spaces with fixed degree of smoothness, see, e.g., [6, Section 5.4], [8], [15].

The purpose of this paper is to consider crucial inequalities (Marchaud, Ul'yanov, etc.) from an abstract point of view. To this end, in Sect. 4 we assume suitable embeddings between interpolation and potential spaces (the interpolation spaces may be interpreted as abstract Besov spaces). Simultaneously, abstract versions of the Holmstedt formulas are developed,

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which allow also to cover limiting cases. In Sect. 5 applications are given in the case of general weighted Lorentz spaces. Finally, Sect. 6 deals with applications to Lorentz–Karamata spaces.

To illustrate our results, we start in Sect. 1.1 with the formulation of the aforesaid basic inequalities adapted to Lebesgue spaces $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Their improvements and extensions in the framework of Lorentz spaces $L_{p,r}(\mathbb{R}^n)$ (note that $L_{p,p} = L_p$) are described in Sect. 1.2, proofs are given in Sect. 2.

1.1 Some basic results

A detailed study of inequalities between different moduli of smoothness on $L_p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, can be naturally divided into two parts: inequalities for moduli of smoothness of different orders in L_p and inequalities in different metrics (L_p, L_{p*}). In the paper a modulus of smoothness of order $\kappa > 0$ on an r.i. function space X (defined in Sect. 3, e.g., $X = L_p$) is given by

$$\omega_\kappa(f,t)_X = \sup_{|h|\leqslant t} \left\| \Delta_h^\kappa f(x) \right\|_X, \quad \text{where} \quad \Delta_h^\kappa f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\kappa}{\nu} f\left(x+\nu h\right). \tag{1.1}$$

Let us begin with the key inequalities on $L_p(\mathbb{R}^n)$. Trivially, if $k, m, n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, then

$$\omega_{k+m}(f,t)_{L_p} \lesssim \omega_k(f,t)_{L_p} \quad \text{for all } t > 0 \text{ and } f \in L_p(\mathbb{R}^n). \tag{1.2}$$

In 1927 Marchaud [46] proved his famous inequality (being a weak inverse of (1.2)): Given $k, m, n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, then

$$\omega_k(f,t)_{L_p} \lesssim t^k \int_t^\infty u^{-k} \omega_{k+m}(f,u)_{L_p} \frac{du}{u} \quad \text{for all } t > 0 \text{ and } f \in L_p(\mathbb{R}^n). \tag{1.3}$$

Using geometric properties of the L_p spaces when $1 < p < \infty$, in 1958 M. F. Timan improved (1.3) (see, e.g., [15, Chapter 2, Theorem 8.4]): If $k, m, n \in \mathbb{N}$, $1 < p < \infty$, and $q = \min\{2, p\}$, then

$$\omega_k(f,t)_{L_p} \lesssim t^k \left(\int_t^\infty \left[u^{-k} \omega_{k+m}(f,u)_{L_p} \right]^q \frac{du}{u} \right)^{1/q} \tag{1.4}$$

for all $t > 0$ and $f \in L_p(\mathbb{R}^n)$.
Observe the natural formal passage from (1.4) to (1.3) when $p \to 1+$.
In 2008 F. Dai, Z. Ditzian and S. Tikhonov [18] derived an improvement of (1.2): If $k, m, n \in \mathbb{N}$, $1 < p < \infty$, and $r = \max\{2, p\}$, then

$$t^k \left(\int_t^\infty \left[u^{-k} \omega_{k+m}(f,u)_{L_p} \right]^r \frac{du}{u} \right)^{1/r} \lesssim \omega_k(f,t)_{L_p} \tag{1.5}$$

for all $t > 0$ and $f \in L_p(\mathbb{R}^n)$.
Observe again the natural formal passage from (1.5) to (1.2), this time when $p \to \infty$. We call (1.5) a reverse Marchaud inequality (in [18] it is called a sharp Jackson inequality).

Consider now inequalities for moduli of smoothness in different Lebesgue metrics. In 1968 P.L. Ul'yanov [66] proved such an inequality for periodic functions in $L_p(\mathbb{T})$. Its $\mathbb{R}^n$-counterpart reads as follows (see, e.g., [9]): If $k, n \in \mathbb{N}$, $1 \leqslant p < \infty$, $0 < \delta < \min\{n/p, k\}$, and $1/p^* = 1/p - \delta/n$, then

$$\omega_k(f,t)_{L_{p^*}} \lesssim \left(\int_0^t \left[u^{-\delta}\omega_k(f,u)_{L_p}\right]^{p^*}\frac{du}{u}\right)^{1/p^*} \quad \text{as} \quad t \to 0+ \tag{1.6}$$

holds for all $f \in L_p(\mathbb{R}^n)$ (for which the right-hand side of(1.6) is finite).[1]

In 1988 V.I. Kolyada [42] gave a definite strengthening of (1.6) on $L_p(\mathbb{T}^n)$. In the $\mathbb{R}^n$-setting his result is the following (see [33]):
Suppose that $k, n \in \mathbb{N}$, and either $1 < p < \infty$ and $n \geqslant 1$, or $p = 1$ and $n \geqslant 2$. If $0 < \delta < \min\{n/p, k\}$ and $1/p^* = 1/p - \delta/n$, then, for all $f \in L_p(\mathbb{R}^n)$,

$$t^{k-\delta}\left(\int_t^\infty [u^{\delta-k}\omega_k(f,u)_{L_{p^*}}]^p \frac{du}{u}\right)^{1/p} \lesssim \left(\int_0^t [u^{-\delta}\omega_k(f,u)_{L_p}]^{p^*}\frac{du}{u}\right)^{1/p^*} \tag{1.7}$$

as $t \to 0+$.

Another extension of (1.6), which is not comparable with inequality (1.7), is the so-called sharp Ul'yanov inequality proved in 2010 independently in [58] and [63]:
If $k, n \in \mathbb{N}$, $1 < p < \infty$, $0 < \delta < n/p$, and $1/p^* = 1/p - \delta/n$, then, for all $f \in L_p(\mathbb{R}^n)$,

$$\omega_k(f,t)_{L_{p^*}} \lesssim \left(\int_0^t [u^{-\delta}\omega_{k+\delta}(f,u)_{L_p}]^{p^*}\frac{du}{u}\right)^{1/p^*} \quad \text{as} \quad t \to 0+. \tag{1.8}$$

In the case $p = 1$ (1.8) does not hold in general [60, Theorem 1(B)] and it requires some modifications [24, Rem. 6.20] (see also [60, Theorem 1(A)]). If $k, n \in \mathbb{N}$, $0 < \delta < n$, and $1/p^* = 1 - \delta/n$, then, for all $f \in L_1(\mathbb{R}^n)$,

$$\omega_k(f,t)_{L_{p^*}} \lesssim \left(\int_0^{t(|\ln t|)^{1/(kp^*)}} [u^{-\delta}\omega_{k+\delta}(f,u)_{L_1}]^{p^*}\frac{du}{u}\right)^{1/p^*} \quad \text{as} \quad t \to 0+.$$

The importance of these inequalities instigated much research in various areas of analysis (theory of function spaces, approximation theory, interpolation theory) and led to numerous publications. We mention only a few recent papers: [20–23, 34, 37, 38, 40, 43, 52, 60, 63]. Basic properties of moduli of smoothness of functions from $L_p(\mathbb{R}^n)$, $0 < p \leqslant \infty$, are given in [41].

1.2 Inequalities for moduli of smoothness on Lorentz spaces

We say that a measurable function f belongs to the Lorentz space $L_{p,r} = L_{p,r}(\mathbb{R}^n)$, $1 \leqslant p, r \leqslant \infty$, if (see, e.g., [6, Section 4.4])

$$\|f\|_{p,r} := \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty & \text{if} \quad r < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t) < \infty & \text{if} \quad r = \infty, \end{cases}$$

[1] One can show that if $f \in L_p(\mathbb{R}^n)$ and the right-hand side of (1.6) is finite for some $t > 0$, then $f \in L_{p^*}(\mathbb{R}^n)$ and so the modulus of smoothness appearing on the left-hand side of (1.6) is well defined. Note that we always look at inequalities involving moduli of smoothness in different metrics at this way. One can also show that if $f \in L_p(\mathbb{R}^n)$ and the right-hand side of (1.6) is finite for some $t > 0$, then it is finite for all $t > 0$ - cf. Remark 6.8 mentioned below).

where f^* denotes the non-increasing rearrangement of f. Thus $L_p = L_{p,p}$ and $\|f\|_p = \|f\|_{p,p}$.

The next statements extend the inequalities mentioned above to Lorentz spaces.

Proposition 1.1 *If $n \in \mathbb{N}$, $1 < p < \infty$, $1 \leqslant q_0, q_1, r_0, r_1 \leqslant \infty$, $r_0 \leqslant r_1$, and $\beta > 0$, then, for all $f \in L_{p,r_0}(\mathbb{R}^n)$:*

(A) Marchaud-type inequality.

$$\omega_\beta(f,t)_{L_{p,r_1}} \lesssim t^\beta \left(\int_t^\infty \left[u^{-\beta} \omega_{\beta+\sigma}(f,u)_{L_{p,r_0}} \right]^{q_0} \frac{du}{u} \right)^{1/q_0} \quad \text{as } t \to 0+ \tag{1.9}$$

provided $\sigma > 0$ and $q_0 \leqslant \min\{p, 2, r_1\}$ if $p \neq 2$. If $p = 2$ and $r_0 \leqslant 2$, then take $q_0 \leqslant \min\{2, r_1\}$, and in the case $p = 2$, $r_0 > 2$ one has to take $q_0 < 2$.

(B) Reverse Marchaud-type inequality.

$$t^\beta \left(\int_t^\infty \left[u^{-\gamma} \omega_{\beta+\gamma}(f,u)_{L_{p,r_1}} \right]^{q_1} \frac{du}{u} \right)^{1/q_1} \lesssim \omega_\beta(f,t)_{L_{p,r_0}} \quad \text{as } t \to 0+ \tag{1.10}$$

(with usual modification if $q_1 = \infty$) provided $\gamma > 0$ and $q_1 \geqslant \max\{p, 2, r_0\}$ if $p \neq 2$. If $p = 2$ and $r_1 \geqslant 2$, then take $q_1 \geqslant \max\{2, r_0\}$, and in the case $p = 2$, $r_1 < 2$ one has to take $q_1 > 2$.

Denote by $W_p^k(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $k \in \mathbb{N}$, the Sobolev space of order k, i.e., $f \in W_p^k(\mathbb{R}^n)$ if f and all its (weak) derivatives up to the order k belong to $L_p(\mathbb{R}^n)$. It is well known that, by Taylor's formula,

$$\omega_{m+k}(f,t)_{L_p} \lesssim t^k \sum_{|\mu|=k} \omega_m(D^\mu f, t)_{L_p}, \quad m \in \mathbb{N},\ \mu \in \mathbb{N}_0^n, \text{ for all } f \in W_p^k(\mathbb{R}^n) \text{ and } t > 0.$$

Here we use the multi-index notation $|\mu| := \sum_{j=1}^n \mu_j$, $D^\mu = \prod_{j=1}^n (\partial/\partial x_j)^{\mu_j}$. We want to state an improvement and some type of reverse of this inequality in the case $1 < p < \infty$. To this end, we need Besov spaces and Riesz potential spaces, both modelled upon Lorentz spaces.

We make use of the Fourier analytical approach in $\mathcal{S}'$ (cf. [7]):

Take a C^∞-function φ such that

$$\operatorname{supp} \varphi \subset \{x \in \mathbb{R}^n : |x| \leq 7/4\} \quad \text{and} \quad \varphi(x) = 1 \text{ if } |x| \leq 3/2. \tag{1.11}$$

For $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let

$$\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x). \tag{1.12}$$

The sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$ is a smooth dyadic resolution of unity, i.e., $1 = \sum_{j=-\infty}^\infty \varphi_j(x)$ for all $x \in \mathbb{R}^n$, $x \neq 0$.

Let $1 \leq p, q, r \leq \infty$ and $\sigma > 0$. The Besov space $B^\sigma_{(p,r),q}(\mathbb{R}^n)$ consists of all $f \in L_{p,r}(\mathbb{R}^n)$ such that

$$|f|_{B^\sigma_{(p,r),q}} = \left(\sum_{j=-\infty}^\infty \left[2^{j\sigma} \|\mathcal{F}^{-1}[\varphi_j] * f\|_{L_{p,r}} \right]^q \right)^{1/q} < \infty \tag{1.13}$$

(the sum should be replaced by the supremum if $q = \infty$). Here the symbol $\mathcal{F}^{-1}$ is used for the inverse Fourier transform. An equivalent characterization of this semi-norm in terms of moduli of smoothness is given by

$$|f|^*_{B^\sigma_{(p,r),q}} = \left(\int_0^\infty \left[t^{-\sigma}\omega_k(f,t)_{L_{p,r}}\right]^q \frac{dt}{t}\right)^{1/q}, \qquad 0 < \sigma < k. \tag{1.14}$$

The Riesz potential space $H^\sigma_{p,r}(\mathbb{R}^n)$, $\sigma \geqslant 0$, consists of all $f \in L_{p,r}(\mathbb{R}^n)$ for which

$$|f|_{H^\sigma_{p,r}} := \|D^\sigma_R f\|_{L_{p,r}} < \infty, \quad \text{where} \quad D^\sigma_R f := \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[|\xi|^\sigma \varphi_j] * f \tag{1.15}$$

(the σ-th Riesz derivative) converges in $\mathcal{S}'$ to an $L_{p,r}(\mathbb{R}^n)$-function. Note that $W^k_p = H^k_{p,p}$ if $1 < p < \infty$.

Proposition 1.2 *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, $1 \leqslant r_0 = r_1 = r \leqslant \infty$, and $\beta, \sigma > 0$.*

(A) *If $f \in L_{p,r}(\mathbb{R}^n)$ then, under the assumptions on the parameters q_0 and r of Proposition 1.1 (A), for all $t > 0$,*

$$\omega_\sigma(D^\beta_R f, t)_{L_{p,r}} \lesssim \left(\int_0^t \left[u^{-\beta}\omega_{\beta+\sigma}(f,u)_{L_{p,r}}\right]^{q_0} \frac{du}{u}\right)^{1/q_0}. \tag{1.16}$$

In particular, if $\beta = m$ and $\sigma = k \in \mathbb{N}$, then, for all $\mu \in \mathbb{N}^n_0$ with $|\mu| = m$,

$$\omega_k(D^\mu f, t)_{L_{p,r}} \lesssim \left(\int_0^t \left[u^{-m}\omega_{k+m}(f,u)_{L_{p,r}}\right]^{q_0} \frac{du}{u}\right)^{1/q_0}. \tag{1.17}$$

(B) *If $f \in H^\beta_{p,r}(\mathbb{R}^n)$ then, under the assumptions on the parameters q_1 and r of Proposition 1.1 (B), for all $t > 0$,*

$$\left(\int_0^t \left[u^{-\beta}\omega_{\beta+\sigma}(f,u)_{L_{p,r}}\right]^{q_1} \frac{du}{u}\right)^{1/q_1} \lesssim \omega_\sigma(D^\beta_R f, t)_{L_{p,r}}. \tag{1.18}$$

In particular, if $\beta = m$ and $\sigma = k \in \mathbb{N}$, then

$$\left(\int_0^t \left[u^{-m}\omega_{m+k}(f,u)_{L_{p,r}}\right]^{q_1} \frac{du}{u}\right)^{1/q_1} \lesssim \sup_{j=1,\dots,n} \omega_k\Big(\frac{\partial^m f}{\partial x_j^m}, t\Big)_{L_{p,r}}.$$

Finally consider inequalities between moduli of smoothness in different metrics.

Proposition 1.3 *Suppose $n \in \mathbb{N}$, $1 < p < \infty$, $0 < \delta < n/p$, $1/p^* = 1/p - \delta/n$, $1 \leqslant q_0, q_1, r_0, r_1 \leqslant \infty$, and $\beta > 0$.*

(A) Sharp Ul'yanov inequality. *If $r_0, q_1 \leqslant r_1$, then, for all $t > 0$ and $f \in L_{p,r_0}(\mathbb{R}^n)$,*

$$\omega_\beta(f,t)_{L_{p^*,r_1}} \lesssim \left(\int_0^t [u^{-\delta}\,\omega_{\beta+\delta}(f,u)_{L_{p,r_0}}]^{q_1} \frac{du}{u}\right)^{1/q_1}. \tag{1.19}$$

(B) Kolyada-type inequality. *If $r_0 \leqslant q_0$, $q_1 \leqslant r_1$, then, for all $t > 0$ and $f \in L_{p,r_0}(\mathbb{R}^n)$,*

$$\begin{aligned} t^\beta &\left(\int_t^\infty \left[u^{-\beta}\omega_{\beta+\delta}(f,u)_{L_{p^*,r_1}}\right]^{q_0} \frac{du}{u}\right)^{1/q_0} \\ &\lesssim \left(\int_0^t [u^{-\delta}\omega_{\beta+\delta}(f,u)_{L_{p,r_0}}]^{q_1} \frac{du}{u}\right)^{1/q_1} \end{aligned} \tag{1.20}$$

2 Remarks and proofs in outlines

Peetre's K-functional K_0 for the compatible couple $(L_{p,r}, H^{\sigma}_{p,r})$ plays a decisive role in the proofs of Propositions 1.1–1.3. It is defined by

$$K_0(f, t; L_{p,r}, H^{\sigma}_{p,r}) = \inf_{g \in H^{\sigma}_{p,r}} (\|f - g\|_{p,r} + t|g|_{H^{\sigma}_{p,r}}), \qquad f \in L_{p,r}, \quad t > 0.$$

We also need the characterization, for $1 < p < \infty$, $\sigma > 0$, $1 \leqslant r \leqslant \infty$,

$$K_0(f, t^{\sigma}; L_{p,r}, H^{\sigma}_{p,r}) \approx \omega_{\sigma}(f, t)_{L_{p,r}}, \quad f \in L_{p,r}, \quad t > 0, \tag{2.1}$$

(see [67] and its extension in [30, (1.13)]) and the identification of the interpolation space given by

$$(L_{p,r}, H^{\sigma}_{p,r})_{\theta,q} = B^{\theta\sigma}_{(p,r),q}, \qquad \sigma > 0,\ 0 < \theta < 1,\ 1 < p < \infty,\ 1 \leqslant r, q \leqslant \infty,$$

where $(\cdot, \cdot)_{\theta,q}$ denotes Peetre's real interpolation method. The improvements and extensions of inequalities (1.3)–(1.8) can be easily proved via the Holmstedt formulas [6, Section 5.2]. One only needs to exchange in [63] the embeddings between Besov and potential spaces modelled on Lebesgue spaces by the corresponding ones modelled on Lorentz spaces. Therefore, we only sketch the proofs of the propositions stated in Sect. 1.2.

Concerning (1.9) and (1.10), note that, under the restrictions on q_0 and q_1 given in Proposition 1.1, the following embeddings are true:

$$B^{\sigma}_{(p,r_0),q_0} \hookrightarrow H^{\sigma}_{p,r_1} \tag{2.2}$$

if $1 \leqslant p < \infty$, $1 \leqslant q_0, r_0, r_1 \leqslant \infty$, $r_0 \leqslant r_1$ (see Theorem 1.1, (iv)–(vi) in [57]) and

$$H^{\gamma}_{p,r_0} \hookrightarrow B^{\gamma}_{(p,r_1),q_1} \tag{2.3}$$

if $1 \leqslant p < \infty$, $1 \leqslant q_1, r_0, r_1 \leqslant \infty$, $r_0 \leqslant r_1$ (see Theorem 1.2, (iv)–(vi) in [57]).

Remark 2.1 In parts (i) and (ii) of this remark we assume the same restrictions on the parameters under which (1.9) and (1.10) hold, respectively.

(i) Divide equation (1.9) by $t^{-\beta}$ and let $t \to 0+$. Then on the right-hand side one gets $|f|^{*}_{B^{\beta}_{(p,r_0),q_0}}$. One way how to handle the left-hand side is to introduce the generalized Weierstraß means $W^{\beta}_t f = \mathcal{F}^{-1}[e^{(t|\xi|)^{\beta}}] * f$. By [30, (1.11)], one has

$$K_0(f, t^{\beta}; L_{p,r_1}, H^{\beta}_{p,r_1}) \approx \|f - W^{\beta}_t f\|_{p,r_1}, \quad f \in L_{p,r_1}, \quad t > 0.$$

Also, by [13, Corollary 3.4.11],

$$\lim_{t \to 0+} t^{-\beta} \|f - W^{\beta}_t f\|_{p,r_1} \approx |f|_{H^{\beta}_{p,r_1}}.$$

Hence, in view of (2.1), (1.9) implies (2.2). In particular, (1.9) and (2.2) are equivalent assertions. This means the following: if inequality (1.9) holds under certain range of parameters, then embedding (2.2) is valid for such parameters and vice versa.

(ii) If (1.10) is true, then its right-hand side is equivalent to $K_0(f, t^{\beta}; L_{p,r_0}, H^{\beta}_{p,r_0})$, which trivially is smaller than $t^{\beta}|f|_{H^{\beta}_{p,r_0}}$. Dividing inequality (1.10) by t^{β}, one gets

$$\left(\int_t^{\infty} \left[u^{-\gamma} \omega_{\beta+\gamma}(f, u)_{L_{p,r_1}}\right]^{q_1} \frac{du}{u}\right)^{1/q_1} \lesssim |f|_{H^{\beta}_{p,r_0}}$$

uniformly in $t > 0$, and (2.3) follows. Thus, (1.10) and (2.3) are again equivalent statements. □

Concerning Proposition 1.2 (A), let $f \in B^{\beta}_{(p,r),q_0}$. Then, by (2.2), $f \in H^{\beta}_{p,r}$, hence $D^{\beta}_R f \in L_{p,r}$, and

$$\omega_\sigma(D^{\beta}_R f, t)_{L_{p,r}} \lesssim \|D^{\beta}_R f - h\|_{L_{p,r}} + t^\sigma |h|_{H^{\sigma}_{p,r}} \quad \text{for all } h \in H^{\sigma}_{p,r}. \tag{2.4}$$

If $g \in H^{\sigma+\beta}_{p,r}$, then $D^{\beta}_R g \in H^{\sigma}_{p,r}$, $|D^{\beta}_R g|_{H^{\sigma}_{p,r}} = |g|_{H^{\sigma+\beta}_{p,r}}$ and $\|D^{\beta}_R(f-g)\|_{L_{p,r}} \lesssim |f-g|_{B^{\beta}_{(p,r),q_0}}$. Now choose $h = D^{\beta}_R g$ in (2.4) to obtain

$$\omega_\sigma(D^{\beta}_R f, t)_{L_{p,r}} \lesssim |f-g|_{B^{\beta}_{(p,r),q_0}} + t^\sigma |D^{\beta}_R g|_{H^{\sigma}_{p,r}} \approx |f-g|_{(L_{p,r}, H^{\beta+\sigma}_{p,r})_{\theta,q_0}} + t^\sigma |g|_{H^{\sigma+\beta}_{p,r}},$$

where in Peetre's $(\cdot,\cdot)_{\theta,q_0}$-interpolation method one has to put $\theta = \beta/(\beta+\sigma)$. Taking the minimum over all $g \in H^{\sigma+\beta}_{p,r}$ in the last display and using the appropriate Holmstedt formula ([6, p. 310]), we arrive at (1.16).

Regarding (1.17), observe that the j-th Riesz transform R_j, $1 \leqslant j \leqslant n$, (with the Fourier symbol $\xi_j/|\xi|$, $\xi \in \mathbb{R}^n$) is a bounded operator from L_p into L_p, $1 < p < \infty$, hence also bounded from $L_{p,r}$ into $L_{p,r}$, $1 < p < \infty$, $1 \leqslant r \leqslant \infty$. Now set $\mathcal{R}^\mu := \prod_{j=1}^n R_j^{\mu_j}$ to obtain $\|D^\mu f\|_{L_{p,r}} = \|\mathcal{R}^\mu D^{|\mu|}_R f\|_{L_{p,r}} \lesssim \|D^{|\mu|}_R f\|_{L_{p,r}}$. Hence,

$$\omega_k(D^\mu f, t)_{L_{p,r}} = \sup_{|y|\leqslant t} \|\Delta^k_y \mathcal{R}^\mu D^{|\mu|}_R f\|_{L_{p,r}} = \sup_{|y|\leqslant t} \|\mathcal{R}^\mu \Delta^k_y D^m_R f\|_{L_{p,r}} \lesssim \omega_k(D^m_R f, t)_{L_{p,r}}$$

and (1.17) follows from (1.16).

Concerning Proposition 1.2 (B), we follow the argument starting with (12.13) in [23]. Thus, by [30, Lemma 1.4 with $\alpha = 0$],

$$\omega_\sigma(D^{\beta}_R f, t)_{L_{p,r}} \approx K_0(D^{\beta}_R f, t^\sigma; L_{p,r}, H^{\sigma}_{p,r}) \approx \|D^{\beta}_R(f - V_t f)\|_{p,r} + t^\sigma |D^{\beta}_R V_t f|_{H^{\sigma}_{p,r}},$$

where $V_t f$ are the de la Vallée-Poussin means of f. Now use Theorem 1.2 (iv) - (vi) in [57], subsequently, the lifting property of Besov spaces, and again [30, Lemma 1.4] to obtain

$$\begin{aligned} \omega_\sigma(D^{\beta}_R f, t)_{L_{p,r}} &\gtrsim |D^{\beta}_R(f - V_t f)|_{B^{0}_{(p,r),q_1}} + t^\sigma |D^{\beta}_R V_t f|_{H^{\sigma}_{p,r}} \\ &\approx |f - V_t f|_{B^{\beta}_{(p,r),q_1}} + t^\sigma |V_t f|_{H^{\beta+\sigma}_{p,r}} \approx K_0(f, t^\sigma; B^{\beta}_{(p,r),q_1}, H^{\beta+\sigma}_{p,r}). \end{aligned}$$

Since $B^{\beta}_{(p,r),q_1} = (L_{p,r}, H^{\beta+\sigma}_{p,r})_{\theta,q_1}$, $\beta = \theta(\beta+\sigma)$ (see, e.g., [7, Theorem 6.3.1]), hence $1-\theta = \sigma/(\beta+\sigma)$ and, therefore, by the Holmstedt formula, we finally derive

$$\left(\int_0^t [u^{-\beta}\omega_{\beta+\sigma}(f,u)_{L_{p,r}}]^{q_1} \frac{du}{u}\right)^{1/q_1} \lesssim \omega_\sigma(D^{\beta}_R f, t)_{L_{p,r}}.$$

In particular, if $\beta = m \in \mathbb{N}$, then, for even m and hence $\gamma_j \in 2\mathbb{N}_0$,

$$D^{\beta}_R f = \mathcal{F}^{-1}\Big[(\xi_1^2 + \cdots + \xi_n^2)^{m/2}\mathcal{F}[f]\Big] = \sum_{|\gamma|=m} \mathcal{F}^{-1}\Big[\prod_{j=1}^n \xi_j^{\gamma_j}\mathcal{F}[f]\Big], \qquad \gamma \in \mathbb{N}_0{}^n.$$

If γ_j is odd, observe that

$$|\xi|^m = |\xi|^{m-1}(\xi_1^2 + \cdots + \xi_n^2)/|\xi| = |\xi|^{m-1}\left(\xi_1 \cdot \frac{\xi_1}{|\xi|} + \cdots + \xi_n \cdot \frac{\xi_n}{|\xi|}\right)$$

and that $\xi_j/|\xi|$ is the symbol of the j-th Riesz transform being a bounded operator on L^p, $1<p<\infty$, and hence also on the Lorentz spaces under consideration. Therefore, when $\sigma=k\in\mathbb{N}$,

$$\omega_k(D_R^m f,t)_{L_{p,r}} \lesssim \sup_{|\gamma|=m} \omega_k\left(\frac{\partial^\gamma f}{\partial x^\gamma},t\right)_{L_{p,r}}$$

and hence the assertion follows along the lines of the paper [23]. □

For the proof of Proposition 1.3, suppose that $\beta,\delta>0$ and that p, p^* and δ satisfy the assumptions. By Theorem 1.1 (iii) in [57],

$$B^{\delta}_{(p,r_0),q_1} \hookrightarrow L_{p^*,r_1} \qquad \text{if} \quad 1\leqslant q_1\leqslant r_1\leqslant\infty,\quad 1\leqslant r_0\leqslant\infty. \tag{2.5}$$

Moreover, Theorem 1.6 (i, iii) in [57] contains a version of the Hardy-Littlewood-Sobolev theorem on fractional integration, which states that

$$H^{\beta+\delta}_{p,r_0} \hookrightarrow H^{\beta}_{p^*,r_1} \qquad \text{if} \quad 1\leqslant r_0\leqslant r_1\leqslant\infty,\quad \beta\geqslant 0. \tag{2.6}$$

The use of Holmstedt's formula completes the proof of (1.19).

Concerning the proof of (1.20), we need the embedding

$$H^{\sigma+\delta}_{p,r_0} \hookrightarrow B^{\sigma}_{(p^*,r_1),q_0}, \quad \text{if } 1\leqslant r_0\leqslant q_0\leqslant\infty,\quad 1\leqslant r_1\leqslant\infty, \tag{2.7}$$

which holds by [57, Theorem 1.2 (iii)], and also embedding (2.5), which requires the additional restriction $q_1\leqslant r_1$. □

Remark 2.2 Similarly to Remark 2.1, we may derive that each of inequalities (1.16)–(1.20) implies the corresponding embedding. For example, let (1.19) be true. Since $H^{\beta+\delta}_{p,r_0}=\{f\in L_{p,r_0}:\omega_{\beta+\delta}(f,u)_{L_{p,r_0}}\leqslant Cu^{\beta+\delta}\}$, inequality (1.19) implies $H^{\beta+\delta}_{p,r_0}\hookrightarrow H^{\beta}_{p^*,r_1}$, which is (2.6). Likewise, (1.20) yields (2.7).

Remark 2.3 Let $n\in\mathbb{N}$, $1<p<\infty$, $0<\delta<n/p$, $1/p^*=1/p-\delta/n$, and $\beta>0$.

(a) The combination of the Kolyada inequality with the Marchaud inequality leads to a special case of the Ul'yanov inequality. If $1\leqslant r:=r_0=r_1=q_0=q_1\leqslant\infty$ and $r\leqslant\min\{p^*,2\}$, then, for all $f\in L_{p,r}(\mathbb{R}^n)$,

$$\omega_\beta(f,t)_{L_{p^*,r}} \lesssim \left(\int_0^t [u^{-\delta}\,\omega_{\beta+\delta}(f,u)_{L_{p,r}}]^r\frac{du}{u}\right)^{1/r} \quad \text{as } t\to 0+. \tag{2.8}$$

This follows on applying to the left-hand side of (1.20) Marchaud inequality (1.9), where we replace p by p^*.

(b) Similarly, if $1\leqslant r:=r_0=r_1=q_1\leqslant\infty$, $r\geqslant\max\{p^*,2\}$, and $\gamma>0$, then the combination of Ul'yanov inequality (1.19) and reverse Marchaud inequality (1.10) (where p is replaced by p^*) yields a special case of the Kolyada inequality, namely, for all $f\in L_{p,r}(\mathbb{R}^n)$,

$$t^\beta\left(\int_t^\infty\Big[u^{-\beta}\omega_{\beta+\gamma}(f,u)_{L_{p^*,r}}\Big]^r\frac{du}{u}\right)^{1/r} \lesssim \left(\int_0^t [u^{-\delta}\,\omega_{\beta+\delta}(f,u)_{L_{p,r}}]^r\frac{du}{u}\right)^{1/r} \tag{2.9}$$

as $t\to 0+$.

Note that in the case $0<\gamma<\delta$ the order of the modulus of smoothness on the left-hand side is smaller than the one on the right-hand side.

2.1 Sharp Ul'yanov and Kolyada inequalities for $p = 1$

As it was mentioned above both (1.19) and (1.20) do not hold in general when $p = 1$. However, under some additional conditions on parameters both results are still valid even in the Lorentz space setting.

Proposition 1.3'. *Suppose* $n \in \mathbb{N}$, $n \geqslant 2$, $1 \leqslant \delta < n$, $1/p^* = 1 - \delta/n$, $1 \leqslant q_1 \leqslant r_1 \leqslant \infty$ *and* $\beta > 0$, $\beta + \delta \in \mathbb{N}$.

(A) *Then, for all* $t > 0$ *and for all* $f \in L_1(\mathbb{R}^n)$,

$$\omega_\beta(f,t)_{L_{p^*,r_1}} \lesssim \left(\int_0^t [u^{-\delta}\omega_{\beta+\delta}(f,u)_{L_1}]^{q_1}\frac{du}{u}\right)^{1/q_1}. \tag{2.10}$$

(B) *If* $1 \leqslant q_0 \leqslant \infty$ *then, for all* $t > 0$ *and for all* $f \in L_1(\mathbb{R}^n)$,

$$t^\beta \left(\int_t^\infty [u^{-\beta}\omega_{\beta+\delta}(f,u)_{L_{p^*,r_1}}]^{q_0}\frac{du}{u}\right)^{1/q_0} \lesssim \left(\int_0^t [u^{-\delta}\omega_{\beta+\delta}(f,u)_{L_1}]^{q_1}\frac{du}{u}\right)^{1/q_1}.$$

Proof of Proposition 1.3$'$(A). If $g \in H^\beta_{p^*,r_1}$, then in light of (2.1), for all $f \in L_{p^*,r_1}$ and all positive t,

$$\omega_\beta(f,t)_{L_{p^*,r_1}} \approx K_0(f,t^\beta; L_{p^*,r_1}, H^\beta_{p^*,r_1}) \lesssim \|f-g\|_{p^*,r_1} + t^\beta |g|_{H^\beta_{p^*,r_1}}. \tag{2.11}$$

Now we take into account the following result by Alvino [3] (appeared in 1977, rediscovered by Poornima [54] in 1983 and by Tartar [59] in 1998)

$$\|h\|_{n/(n-1),1} \lesssim \sum_{j=1}^n \left\|\frac{\partial h}{\partial x_j}\right\|_1, \qquad n \geqslant 2. \tag{2.12}$$

Together with Hörmander's multiplier criterion and [57, Theorem 1.6 (iii)], this yields

$$W_1^{\beta+\delta} \hookrightarrow W^{\beta+\delta-1}_{n/(n-1),1} = H^{\beta+\delta-1}_{n/(n-1),1} \hookrightarrow H^\beta_{p^*,1} \hookrightarrow H^\beta_{p^*,r_1} \qquad \text{if} \quad r_1 \geqslant 1 \tag{2.13}$$

and for the corresponding seminorms we have, for all $g \in W_1^{\beta+\delta}$,

$$|g|_{H^\beta_{p^*,r_1}} \lesssim |g|_{H^\beta_{p^*,1}} \lesssim |g|_{W_1^{\beta+\delta}}, \qquad 0 < \frac{1}{p^*} = 1 - \frac{\delta}{n}, \quad r_1 \geqslant 1. \tag{2.14}$$

Note that using Alvino's result, we necessarily assume $\delta \geqslant 1$.

By [57, Theorem 1.1 (iii)], the first embedding below is valid, the second one is elementary and, therefore,

$$B^\delta_{(1,1),q_1} \hookrightarrow L_{p^*,q_1} \hookrightarrow L_{p^*,r_1}, \qquad \|f\|_{p^*,r_1} \lesssim |f|_{B^\delta_{(1,1),q_1}} \tag{2.15}$$

for all $f \in B^\delta_{(1,1),q_1}$ if $1 \leqslant q_1 \leqslant r_1 \leqslant \infty$.

Applying estimates (2.14), (2.15), and (2.11), we arrive at

$$\omega_\beta(f,t)_{L_{p^*,r_1}} \lesssim |f-g|_{B^\delta_{(1,1),q_1}} + t^\beta |g|_{W_1^{\beta+\delta}}$$

for all $g \in W_1^{\beta+\delta}$. Together with Holmstedt's formula, this yields

$$\omega_\beta(f,t)_{L_{p^*,r_1}} \lesssim K_0(f,t^\beta; B^\delta_{(1,1),q_1}, W_1^{\beta+\delta})$$

$$\approx \left(\int_0^{t^{\beta+\delta}} [u^{-\delta/(\beta+\delta)} K_0(f, u; L_1, W_1^{\beta+\delta})]^{q_1} \frac{du}{u} \right)^{1/q_1}$$

$$\approx \left(\int_0^t [u^{-\delta} \omega_{\beta+\delta}(f, u)_{L_1}]^{q_1} \frac{du}{u} \right)^{1/q_1},$$

where the condition $\beta + \delta \in \mathbb{N}$ allows us to identify the resulting K_0-functional with the classical modulus of smoothness in L_1. □

Proof of Proposition 1.3$'$(B). Following the proof of (1.20), we need analogues of (2.5) and (2.7) for $p = 1$. In fact, in this case (2.5) holds whenever $1 \leqslant q_1 \leqslant r_1 \leqslant \infty$ (see (2.15)). Concerning (2.7), we modify it by repeating the argument in (2.13) to get

$$W_1^{\beta+\delta} \hookrightarrow W_{n/(n-1),1}^{\beta+\delta-1} = H_{n/(n-1),1}^{\beta+\delta-1}.$$

Hence, applying (2.7) upon $H_{n/(n-1),1}^{\beta+\delta-1}$, under our assumptions, we arrive at

$$W_1^{\beta+\delta} \hookrightarrow B_{(p^*,r_1),q_0}^{\beta}, \quad \frac{1}{p^*} = 1 - \frac{\delta}{n} > 0,\ \delta \geqslant 1,\ \beta > 0,\ \delta + \beta \in \mathbb{N},\ 1 \leqslant q_0 \leqslant \infty.$$

By the Holmstedt formula,

$$\begin{aligned} I_{p^*} &:= t^{\beta/(\beta+\delta)} \left(\int_t^{\infty} [u^{-\beta/(\beta+\delta)} K_0(f, u; L_{p^*,r_1}, H_{p^*,r_1}^{\beta+\delta})]^{q_0} \frac{du}{u} \right)^{1/q_0} \\ &\approx K_0(f, t^{\beta/(\beta+\delta)}; L_{p^*,r_1}, (L_{p^*,r_1}, H_{p^*,r_1}^{\beta+\delta})_{\beta/(\beta+\delta), q_0}) \\ &\lesssim \|f - g\|_{p^*,r_1} + t^{\beta/(\beta+\delta)} |g|_{B_{(p^*,r_1),q_0}^{\beta}} \\ &\lesssim |f - g|_{B_{(1,1),q_1}^{\delta}} + t^{\beta/(\beta+\delta)} |g|_{W_1^{\beta+\delta}}, \qquad 1 \leqslant q_1 \leqslant r_1 \leqslant \infty. \end{aligned}$$

Since this estimate holds for all $g \in W_1^{\beta+\delta}$, we have

$$\begin{aligned} I_{p^*} &\lesssim K_0(f, t^{\beta/(\beta+\delta)}; (L_1, W_1^{\beta+\delta})_{\delta/(\beta+\delta), q_1}, W_1^{\beta+\delta}) \\ &\approx \left(\int_0^t [u^{-\delta/(\beta+\delta)} K_0(f, u; L_1, W_1^{\beta+\delta})]^{q_1} \frac{du}{u} \right)^{1/q_1}. \end{aligned}$$

Now simple substitutions, the characterizations of the K_0-functionals via moduli of smoothness of integer order give the assertion. □

Remark 2.4 Proposition 1.3' contains the corresponding results for Lebesgue spaces (for part (A), take $p^* = q_1 = r_1$ and see [40, 41, 63], for part (B), take $p^* = q_1 = r_1$, $q_0 = 1$ and see [40–42, 63]). We also note that even though (1.8) does not hold in general for $p = 1$ and $p^* < \infty$, it is still valid for $p = 1$ and $p^* = \infty$ ([40, Corollary 8.3]), i.e., there holds

$$\omega_k(f, t)_{L_\infty} \lesssim \int_0^t u^{-n} \omega_{k+n}(f, u)_{L_1} \frac{du}{u}, \qquad k \in \mathbb{N}.$$

3 Notation and preliminaries

Throughout the paper, we write $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) if $\mathcal{A} \leq c\,\mathcal{B}$ (or $c\,\mathcal{A} \geq \mathcal{B}$) for some positive constant c, which depends only on nonessential variables involved in the expressions $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A} \approx \mathcal{B}$ if $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$.

In the whole paper the symbol $(\mathfrak{R}, \mu)$ denotes a totally σ-finite measurable space with a non-atomic measure μ, and $\mathcal{M}(\mathfrak{R}, \mu)$ is the set of all extended complex-valued μ-measurable functions on $\mathfrak{R}$. By $\mathcal{M}^+(\mathfrak{R}, \mu)$ we mean the family of all non-negative functions from $\mathcal{M}(\mathfrak{R}, \mu)$. When $\mathfrak{R}$ is an interval $(a, b) \subseteq \mathbb{R}$ and μ is the Lebesgue measure on (a, b), we denote these sets by $\mathcal{M}(a, b)$ and $\mathcal{M}^+(a, b)$, respectively. Moreover, by $\mathcal{M}^+(a, b; \downarrow)$ (and $\mathcal{M}^+(a, b; \uparrow)$) we mean the subset of $\mathcal{M}^+(a, b)$ consisting of all non-increasing (non-decreasing) functions on (a, b). We denote by λ_n the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

For two normed spaces X and Y, we will use the notation $Y \hookrightarrow X$ if $Y \subset X$ and $\|f\|_X \lesssim \|f\|_Y$ for all $f \in Y$.

A normed linear space X of functions from $\mathcal{M}(\mathfrak{R}, \mu)$, equipped with the norm $\|\cdot\|_X$, is said to be a *Banach function space* if the following four axioms hold:

1. $0 \leqslant g \leqslant f$ μ-a.e. implies $\|g\|_X \leqslant \|f\|_X$;
2. $0 \leqslant f_n \nearrow f$ μ-a.e. implies $\|f_n\|_X \nearrow \|f\|_X$;
3. $\|\chi_E\|_X < \infty$ for every $E \subset \mathfrak{R}$ of finite measure;[2]
4. if $\mu(E) < \infty$, then there is a constant C_E such that $\int_E |f(x)|\, d\mu(x) \leqslant C_E \|f\|_X$ for every $f \in X$.

Given a Banach function space X, which satisfies

(5) $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$,[3]

we obtain a *rearrangement-invariant Banach function space* (shortly *r.i. space*). Note that, by [6, Chapter 2, Theorem 6.6] and [6, Chapter 2, Theorem 2.7], $L_1 \cap L_\infty \hookrightarrow X \hookrightarrow L_1 + L_\infty$ for any r.i. space X.

Given a Banach function space X on $(\mathfrak{R}, \mu)$, the set

$$X' = \left\{ f \in \mathcal{M}(\mathfrak{R}, \mu) : \int_{\mathfrak{R}} |f(x)g(x)|\, d\mu < \infty \text{ for every } g \in X \right\},$$

equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_{\mathfrak{R}} |f(x)g(x)|\, d\mu,$$

is called the *associate space* of X. It turns out that X' is again a Banach function space and that $X'' = X$. Furthermore, the *Hölder inequality*

$$\int_{\mathfrak{R}} |f(x)g(x)|\, d\mu \le \|f\|_X \|g\|_{X'}$$

holds for every f and g in $\mathcal{M}(\mathfrak{R}, \mu)$. It will be useful to note that

$$\|f\|_X = \sup_{\|g\|_{X'} \le 1} \int_{\mathfrak{R}} |f(x)g(x)|\, d\mu. \tag{3.1}$$

For every r.i. space X on $(\mathfrak{R}, \mu)$, there exists an r.i. space $\overline{X}$ over $((0, \infty), dt)$ such that

$$\|f\|_X = \|f^*\|_{\overline{X}} \quad \text{for every } f \in X$$

2 The symbol χ_E stands for the characteristic function of the set E.

3 Recall that f^* and g^* denote the non-increasing rearrangements of functions f and g.

(cf. [6, Chapter 2, Theorem 4.10]). This space, equipped with the norm

$$\|f\|_{\overline{X}} = \sup_{\|g\|_{X'} \leq 1} \int_0^\infty f^*(t) g^*(t)\, dt,$$

is called the *representation space* of X.

A Banach space F of real valued measurable functions defined on the measurable space $(\mathfrak{R}, \mu)$ is called *a Banach function lattice* if its norm has the following property:

$$|f(x)| \leqslant |g(x)| \quad \mu\text{-a.e.}, \quad g \in F \quad \Rightarrow \quad f \in F \quad \text{and} \quad \|f\|_F \leqslant \|g\|_F.$$

In this paper we will consider a Banach lattice F over a measurable space $((0,\infty), dt/t)$, satisfying the condition

$$\Phi(1) < \infty, \tag{3.2}$$

where $\Phi(x) := \|\min(x, \cdot)\|_F$ for all $x \in (0,\infty)$. (The function Φ is sometimes called the fundamental function of the lattice F.) Note that Φ is a quasiconcave function on $(0,\infty)$, which means that $\Phi \in \mathcal{M}^+((0,\infty); \uparrow)$ and $\frac{\Phi}{Id} \in \mathcal{M}^+((0,\infty); \downarrow)$ (here Id stands for the identity map on $(0,\infty)$). Condition (3.2) implies that $\Phi(x) < \infty$ for any $x \in (0,\infty)$, moreover, $\Phi \in C(0,\infty)$ (cf. [27, Remark 2.1.2]).

Let (X, Y) be a compatible couple of Banach spaces (cf., [6, p. 310]). The K-functional is defined for each $f \in X + Y$ and $t > 0$ by

$$K(f, t; X, Y) := \inf_{f = f_1 + f_2} \Big(\|f_1\|_X + t\|f_2\|_Y \Big), \tag{3.3}$$

where the infimum extends over all representation $f = f_1 + f_2$ with $f_1 \in X$ and $f_2 \in Y$. As a function of t, $K(f, t; X, Y)$ is *quasiconcave* on $(0,\infty)$.

Similarly, we define, for each $f \in X + Y$ and $t > 0$,

$$K_0(f, t; X, Y) := \inf_{f = f_1 + f_2} \Big(\|f_1\|_X + t|f_2|_Y \Big) \tag{3.4}$$

and

$$K_1(f, t; X, Y) := \inf_{f = f_1 + f_2} \Big(|f_1|_X + t|f_2|_Y \Big), \tag{3.5}$$

where $|\cdot|_X$ and $|\cdot|_Y$ are seminorms on X and Y.

If (X, Y) is a compatible couple of Banach spaces and F is a Banach lattice, then we define the space $(X, Y)_F$ to be the set of all $f \in X + Y$ for which the norm

$$\|f\|_{(X,Y)_F} = \|K(f, \cdot; X, Y)\|_F$$

is finite. Note that if $1 \leqslant r < \infty$, $\theta \in (0, 1)$ and the Banach lattice F is the set of all functions $h \in \mathcal{M}(0,\infty)$ such that

$$\|h\|_F := \left(\int_0^\infty \Big(s^{-\theta} |h(s)| \Big)^r \frac{ds}{s} \right)^{1/r} < \infty,$$

then the space $(X, Y)_F$ coincides with the classical space $(X, Y)_{\theta,r}$ defined, e.g., in [6, p. 299].

We will also work with more general classes of functions, which are not linear. Let ρ be a functional on $\mathcal{M}^+(\mathbb{R}^n, \lambda_n)$ satisfying

(N1) $\rho(f) \geqslant 0$ for any $f \in \mathcal{M}^+(\mathbb{R}^n, \lambda_n)$ and $\rho(f) = 0 \Leftrightarrow f = 0$, λ_n-a.e.,

(N2) $\rho(\alpha f) = \alpha \rho(f)$ for any $f \in \mathcal{M}^+(\mathbb{R}^n, \lambda_n)$ and $\alpha \geqslant 0$.

Such a functional is called a gage and the collection

$$X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \lambda_n) = \{f \in \mathcal{M}^+(\mathbb{R}^n, \lambda_n) : \rho(f) < \infty\}$$

is said *a (function) gaged cone* (cf. [17]). Moreover, we put

$$\|f\|_X := \rho(f), \quad f \in X.$$

An associate space of a gaged cone X is defined in the same way as for Banach function spaces.

If X is a gaged cone, then the functional $|\cdot|_X : X \to \mathbb{R}$ is called a semi-gage on X provided that the functional $|\cdot|_X$ is non-negative and positively homogeneous on X.

Given two function gaged cones X and Y, the embedding $Y \hookrightarrow X$ means that $Y \subset X$ and $\|f\|_X \lesssim \|f\|_Y$ for all $f \in Y$.

A pair of function gaged cones (X, Y) is said a compatible couple of function gaged cones if there is some Hausdorff topological vector space, say Z, in which each of X and Y is continuously embedded. Given a compatible couple (X, Y) of function gaged cones, the K-functionals $K(f, t; X, Y)$, $K_0(f, t; X, Y)$, and $K_1(f, t; X, Y)$ are defined analogously to (3.3)–(3.5). Moreover, if F is a Banach lattice over a measure space $((0, \infty), dt/t)$ satisfying (3.2), then the space $(X, Y)_F$ is defined analogously to the case when (X, Y) is a compatible couple of Banach spaces.

In this paper we work with function gaged cones being the subsets of $L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$.

Given $k \in \mathbb{N}$ and a Banach function space $X = X(\mathbb{R}^n)$, we denote by $W^k X$ the corresponding Sobolev space, that is, the space of all functions on $\mathbb{R}^n$ whose distributional derivatives $D^\alpha f$, $|\alpha| \leqslant k$, belong to X. This space is equipped with the norm

$$\|f\|_{W^k X} := \|f\|_X + |f|_{W^k X} := \|f\|_X + \sum_{k=|\alpha|} \|D^\alpha f\|_X.$$

Note that $W^k X = A^k X$, where A is the Sobolev integral operator; see, for example, the representation theorem in [12, Section 3.4]. If X is a function gaged cone, then the Sobolev class $W^k X$ is defined similarly.

We are going to use the classical equivalence between the K-functional K_0 and modulus of smoothness: for any $k \in \mathbb{N}$ and an r.i. Banach function space X, one has

$$\omega_k(f, t)_X \approx K_0(f, t^k; X, W^k X) \qquad \text{for all } t > 0 \text{ and } f \in X \tag{3.6}$$

provided that in the space $W^k X$ we choose the seminorm $|f|_{W^k X} := \sum_{k=|\alpha|} \|D^\alpha f\|_X$. The proof follows the same reasoning as the one given for $X = L_p$ in [6, pp. 339–341].

Let $-\infty \leqslant a < b \leqslant +\infty$ and let $\xi : (a, b) \to \mathbb{R}$ be a non-decreasing function on (a, b). Put $\xi(a) = \lim_{t \to a+} \xi(t)$ and $\xi(b) = \lim_{t \to b-} \xi(t)$. *The generalized reverse function* $\mathrm{R}\xi$ of ξ is defined by

$$(\mathrm{R}\xi)(t) := \inf\Big\{\tau \in (a, b) : \xi(\tau) > t\Big\} \quad \text{for all } t \in (\xi(a), \xi(b)).$$

The following properties of the generalized reverse function can be easily verified.

Lemma 3.1 *If the function ξ given above is left continuous on (a, b), then*

$$\xi\big((\mathrm{R}\xi)(t)\big) \leqslant t \quad \textit{for any } t \in (\xi(a), \xi(b))$$

and

$$t \leqslant (\mathrm{R}\xi)(\xi(t)) \quad \textit{for any } t \in (a, b).$$

Moreover, if $\xi \in C((a,b))$, *then*

$$\xi\big((\mathrm{R}\xi)(t)\big) = t \quad \textit{for any} \;\; t \in (\xi(a), \xi(b)).$$

We note that Lemma 3.1 does not hold without the assumption that ξ is left continuous. Moreover, an analogue of $\xi\big((\mathrm{R}\xi)(t)\big) = t$, namely $(\mathrm{R}\xi)(\xi(t)) = t$ for any $t \in (a,b)$, need not hold even if $\xi \in C((a,b))$.

If $(a,b) \subset \mathbb{R}$ and $p \in (0,\infty]$, then the symbol $\|\cdot\|_{p,(a,b)}$ stands for the quasinorm in the Lebesgue space $L_p((a,b))$.

As usual, for $p \in [1,\infty]$, we define p' by $1/p + 1/p' = 1$. Throughout the paper we use the abbreviations LHS$(*)$ (RHS$(*)$) for the left- (right-) hand side of the relation $(*)$.

4 General inequalities for K-functionals

4.1 Holmstedt-type formulas

The next theorem is a folklore in some way and it can be considered as an abstract form of the limiting cases of the Holmstedt-type formulas (see, e.g., [6, Corollary 2.3, p. 310 and p. 430] and [11, p. 466]). Since we have not been able to find an explicit reference of the needed general form (cf. [2, 48]), we prove it below. The importance of this result can be seen in, e.g., [55].

Theorem 4.1 *Let* (X_0, X_1) *be a compatible couple of Banach function spaces.*

(A) *Let* F_0 *be a Banach lattice over* $((0,\infty), dt/t)$. *Assume that the function* $\Xi(t) := \|\min(\cdot, t)\|_{F_0}$, $t \in (0,\infty)$, *satisfies* $\Xi(1) < \infty$. *If* ϕ *is the generalized reverse function of* Ξ, *then*

$$K(f,t;(X_0,X_1)_{F_0},X_1) \approx \|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0} + K(f,\phi(t);X_0,X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0} \tag{4.1}$$

for all $t \in (\Xi(0), \Xi(\infty))$ *and* $f \in (X_0,X_1)_{F_0} + X_1$.

(B) *Let* F_1 *be a Banach lattice over* $((0,\infty), dt/t)$. *Assume that the function* $\Theta(t) := t/\|\min(\cdot,t)\|_{F_1}$, $t \in (0,\infty)$, *satisfies* $\Theta(1) < \infty$. *If* ψ *is the generalized reverse function of* Θ, *then*

$$K(f,t;X_0,(X_0,X_1)_{F_1}) \approx t\,\frac{K(f,\psi(t);X_0,X_1)}{\psi(t)}\|s\,\chi_{(0,\psi(t))}(s)\|_{F_1} + t\|K(f,s;X_0,X_1)\chi_{(\psi(t),\infty)}(s)\|_{F_1} \tag{4.2}$$

for all $t \in (\Theta(0), \Theta(\infty))$ *and* $f \in X_0 + (X_0,X_1)_{F_1}$.

Remark 4.2 (i) Formulas (4.1) and (4.2) remain valid for K-functionals given by (3.4) and (3.5).

(ii) By Theorem 4.1, estimate (4.1) holds for all $t \in (\Xi(0), \Xi(\infty))$ and $f \in (X_0,X_1)_{F_0} + X_1$, or equivalently, for all $t \in (\Xi(0), \Xi(\infty))$ and $f \in X$ for which RHS(4.1) is finite. Similar remark can be made about equivalence (4.2).

Proof of Theorem 4.1 We start with (A). As the function Ξ is quasiconcave, it is continuous and hence $\Xi(\phi(t)) = \|\min(s,\phi(t))\|_{F_0} = t$ for any $t \in (\Xi(0), \Xi(\infty))$ by Lemma 3.1. If $f = f_0 + f_1$, where $f_0 \in (X_0,X_1)_{F_0}$ and $f_1 \in X_1$, then, for all $t \in (\Xi(0), \Xi(\infty))$,

$$\|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0} \leqslant \|K(f_0,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}$$

$$
\begin{aligned}
&+\|K(f_1,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}\\
&\leqslant \|K(f_0,s;X_0,X_1)\|_{F_0}+\|s\chi_{(0,\phi(t))}(s)\|_{F_0}\|f_1\|_{X_1}\\
&\leqslant \|f_0\|_{(X_0,X_1)_{F_0}}+\|\min(s,\phi(t))\|_{F_0}\|f_1\|_{X_1}\\
&=\|f_0\|_{(X_0,X_1)_{F_0}}+t\|f_1\|_{X_1}
\end{aligned}
$$

and

$$
\begin{aligned}
K(f,\phi(t);X_0,X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0} &\leqslant K(f_0,\phi(t);X_0,X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}\\
&\quad+K(f_1,\phi(t);X_0,X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}\\
&\leqslant \|K(f_0,s;X_0,X_1)\|_{F_0}+\phi(t)\|f_1\|_{X_1}\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}\\
&\leqslant \|f_0\|_{(X_0,X_1)_{F_0}}+\|\min(s,\phi(t))\|_{F_0}\|f_1\|_{X_1}\\
&=\|f_0\|_{(X_0,X_1)_{F_0}}+t\|f_1\|_{X_1}.
\end{aligned}
$$

Thus, taking the infimum over all decompositions $f=f_0+f_1$ of the function f, with $f_0\in(X_0,X_1)_{F_0}$ and $f_1\in X_1$, we arrive at the estimate LHS(4.1)$\gtrsim$RHS(4.1).

To prove the opposite estimate, take $t\in(\Xi(0),\Xi(\infty))$ and suppose that $f=f_0+f_1$, with $f_0\in X_0$, $f_1\in X_1$, be such a representation that

$$
\|f_0\|_{X_0}+\phi(t)\|f_1\|_{X_1}\leqslant 2K(f,\phi(t);X_0,X_1).
$$

Since, for all $s>0$,

$$
K(f_0,s;X_0,X_1)\leqslant\|f_0\|_{X_0}\leqslant 2K(f,\phi(t);X_0,X_1)
$$

and

$$
\frac{K(f_1,s;X_0,X_1)}{s}\leqslant\|f_1\|_{X_1}\leqslant\frac{2}{\phi(t)}K(f,\phi(t);X_0,X_1),
$$

we get, for all $f\in(X_0,X_1)_{F_0}+X_1$,

$$
\begin{aligned}
K(f,t;(X_0,X_1)_{F_0},X_1)&\leqslant\|f_0\|_{(X_0,X_1)_{F_0}}+t\|f_1\|_{X_1}\\
&\lesssim\|K(f_0,s;X_0,X_1)\|_{F_0}+t\frac{K(f,\phi(t);X_0,X_1)}{\phi(t)}\\
&\lesssim\|K(f_0,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}\\
&\quad+\|K(f_0,s;X_0,X_1)\chi_{(\phi(t),\infty)}(s)\|_{F_0}\\
&\quad+t\frac{K(f,\phi(t);X_0,X_1)}{\phi(t)}=:J_1+J_2+J_3.
\end{aligned}
$$

As $f_0=f-f_1$, we obtain

$$
\begin{aligned}
J_1&\leqslant\|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}+\|K(f_1,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}\\
&\leqslant\|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}+\|f_1\|_{X_1}\|s\chi_{(0,\phi(t))}(s)\|_{F_0}\\
&\lesssim\|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}+\frac{K(f,\phi(t);X_0,X_1)}{\phi(t)}\|s\chi_{(0,\phi(t))}(s)\|_{F_0}\\
&\lesssim\|K(f,s;X_0,X_1)\chi_{(0,\phi(t))}(s)\|_{F_0}
\end{aligned}
$$

and

$$
J_2\lesssim\|f_0\|_{X_0}\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}\lesssim K(f,\phi(t);X_0,X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}.
$$

Since $t = \Xi(\phi(t)) \leqslant \|s\, \chi_{(0,\phi(t))}(s)\|_{F_0} + \phi(t)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}$, we get

$$\begin{aligned} J_3 &\leqslant \Big(\|s\, \chi_{(0,\phi(t))}(s)\|_{F_0} + \phi(t)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}\Big) \frac{K(f,\phi(t); X_0, X_1)}{\phi(t)} \\ &\leqslant \|K(f,s; X_0, X_1)\chi_{(0,\phi(t))}(s)\|_{F_0} + K(f,\phi(t); X_0, X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}. \end{aligned}$$

Consequently, for all $t \in (\Xi(0), \Xi(\infty))$ and $f \in (X_0, X_1)_{F_0} + X_1$,

$$\begin{aligned} K(f,t; (X_0, X_1)_{F_0}, X_1) \; &\lesssim \|K(f,s; X_0, X_1)\chi_{(0,\phi(t))}(s)\|_{F_0} \\ &\quad + K(f,\phi(t); X_0, X_1)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0}. \end{aligned}$$

To prove part (B), we notice that (cf. [6, Chapter V, Prop. 1.2])

$$K(f,t; X_0, (X_0, X_1)_{F_1}) = tK(f, 1/t; (X_0, X_1)_{F_1}, X_0) = tK(f, 1/t; (X_1, X_0)_{\widetilde{F_1}}, X_0),$$

where $\widetilde{F_1} = \{f : tf(1/t) \in F_1\}$ and $\|f\|_{\widetilde{F_1}} = \|tf(1/t)\|_{F_1}$. Now we apply part (A) and the reverse preceding substitutions to arrive at the statement. □

4.2 Inequalities for *K*-functionals involving the potential-type operators

Let $\{A^\tau\}_{\tau \in \mathfrak{M}}$, where $\mathfrak{M} = \{\tau : 0 \leqslant \tau < \tau_0\}$ or $\mathfrak{M} = \{k \in \mathbb{N}_0 : k < \tau_0\}$ for some $\tau_0 \in (0, \infty)$, be a family of linear operators defined on $L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ satisfying

(P1) $A^\tau : X \to X$ for any $\tau \in \mathfrak{M}$ and for any function gaged cone $X \subset L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$;
(P2) $A^0 X = X$ for any function gaged cone $X \subset L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$;
(P3) $A^\tau(A^\sigma X) = A^\sigma(A^\tau X) = A^{\tau+\sigma} X$ for any function gaged cone $X \subset L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ and for any $\sigma, \tau, \sigma + \tau \in \mathfrak{M}$.

Here $A^\tau X$, $\tau \in \mathfrak{M}$, is the range of A^τ equipped with the gage (or norm)

$$\|f\|_{A^\tau X} = \|f\|_X + |f|_{A^\tau X}, \quad \text{where} \quad |f|_{A^\tau X} = \inf\{\|g\|_X : f = A^\tau g\}$$

Theorem 4.3 (Ul'yanov-type inequalities) *Assume that X is an r.i. Banach function space and $Y, Z \subset L_1 + L_\infty$ are function gaged cones.*
Let

$$A^{\sigma+\tau} X \hookrightarrow Y, \qquad A^\sigma X \hookrightarrow Z, \quad \textit{for some } \tau > 0 \textit{ and } \sigma \geqslant 0. \tag{4.3}$$

(A) *Let F_0 be a Banach lattice over* $((0,\infty), dt/t)$ *satisfying*

$$(X, Y)_{F_0} \hookrightarrow Z. \tag{4.4}$$

Assume that the function $\Xi_0(t) := \|\min(\cdot, t)\|_{F_0}$, $t \in (0, \infty)$, is such that $\Xi_0(1) < \infty$. If ϕ_0 is the generalized reverse function of Ξ_0, then

$$\begin{aligned} K(f,t; Z, A^\tau Z) \lesssim \|K(f,s; X, A^{\sigma+\tau} X)\chi_{(0,\phi_0(t))}(s)\|_{F_0} \\ + K(f,\phi_0(t); X, A^{\sigma+\tau} X)\|\chi_{(\phi_0(t),\infty)}(s)\|_{F_0} \end{aligned} \tag{4.5}$$

for all $t \in (\Xi_0(0), \Xi_0(\infty))$ and $f \in X$ (for which RHS (4.5) *is finite).*
(B) *Let F_1 be a Banach lattice over* $((0,\infty), dt/t)$ *satisfying*

$$Z \hookrightarrow (X, Y)_{F_1} =: V. \tag{4.6}$$

Assume that the function $\Xi_1(t) := \|\min(\cdot, t)\|_{F_1}$, $t \in (0, \infty)$, *is such that* $\Xi_1(1) < \infty$. *If* ϕ_1 *is the generalized reverse function of* Ξ_1, *then*

$$K(f, t; V, A^\tau V) \lesssim \|K(f, s; X, A^{\sigma+\tau} X)\chi_{(0,\phi_1(t))}(s)\|_{F_1} + K(f, \phi_1(t); X, A^{\sigma+\tau} X)\|\chi_{(\phi_1(t),\infty)}(s)\|_{F_1} \tag{4.7}$$

for all $t \in (\Xi_1(0), \Xi_1(\infty))$ *and* $f \in X$ (*for which* RHS (4.7) *is finite*).

Remark 4.4 (i) It is clear from the proof that inequality (4.5) holds provided that

$$A^{\sigma+\tau} X \hookrightarrow Y \quad \text{for some } \tau > 0 \text{ and } \sigma \geqslant 0$$

and

$$A^\sigma X \hookrightarrow (X, Y)_{F_0} =: Z.$$

(ii) A different, abstract approach to Ul'yanov inequalities, based on semi-groups of linear equibounded operators in Banach spaces, is given in [65].

Proof of Theorem 4.3 The property (P3) of operators A^τ and the second embedding in (4.3) imply that

$$A^{\sigma+\tau} X \hookrightarrow A^\tau Z. \tag{4.8}$$

Further, using the first embedding in (4.3) and (4.4), we get

$$(X, A^{\sigma+\tau} X)_{F_0} \hookrightarrow (X, Y)_{F_0} \hookrightarrow Z.$$

This, (4.8), and Theorem 4.1 (A) (see also Remark 4.2 (ii)) yield, for all $t \in (\Xi_0(0), \Xi_0(\infty))$ and $f \in X$,

$$\begin{aligned} K(f, t; Z, A^\tau Z) &\lesssim K(f, t; (X, A^{\sigma+\tau} X)_{F_0}, A^{\sigma+\tau} X) \\ &\approx \|K(f, s; X, A^{\sigma+\tau} X)\chi_{(0,\phi_0(t))}(s)\|_{F_0} \\ &\quad + K(f, \phi_0(t); X, A^{\sigma+\tau} X)\|\chi_{(\phi_0(t),\infty)}(s)\|_{F_0}, \end{aligned}$$

and (4.5) is proved.

To obtain (4.7), using (4.8) and (4.6), we arrive at

$$A^{\sigma+\tau} X \hookrightarrow A^\tau V.$$

Moreover, applying the first embedding in (4.3) and definition of V, we obtain

$$(X, A^{\sigma+\tau} X)_{F_1} \hookrightarrow (X, Y)_{F_1} = V.$$

Consequently, for all $t > 0$,

$$K(f, t; V, A^\tau V) \lesssim K(f, t; (X, A^{\sigma+\tau} X)_{F_1}, A^{\sigma+\tau} X),$$

which, together with Theorem 4.1 (A) (and Remark 4.2 (ii)), yields (4.7). □

Using part (B) of Theorem 4.1, one can prove the following results (Marchaud and reverse Marchaud-type inequalities).

Theorem 4.5 *Assume that* X *is an r.i. Banach function space. Let* F_1 *be a Banach lattice over* $((0, \infty), dt/t)$. *Assume that the function* $\Theta(t) := t/\|\min(\cdot, t)\|_{F_1}$, $t \in (0, \infty)$, *satisfies* $\Theta(1) < \infty$ *and that* ψ *is the generalized reverse function of* Θ.

(A) (Marchaud-type inequality) *If*

$$(X, A^{\sigma+\tau}X)_{F_1} \hookrightarrow A^{\tau}X, \quad \text{with some } \tau, \sigma > 0, \tag{4.9}$$

then

$$K(f, t; X, A^{\tau}X) \lesssim t\frac{K(f, \psi(t); X, A^{\sigma+\tau}X)}{\psi(t)}\|s\,\chi_{(0,\psi(t))}(s)\|_{F_1} + t\|K(f, s; X, A^{\sigma+\tau}X)\chi_{(\psi(t),\infty)}(s)\|_{F_1} \tag{4.10}$$

for all $t \in (\Theta(0), \Theta(\infty))$ *and* $f \in X$ *(for which* RHS*(4.10) is finite).*

(B) (Reverse Marchaud-type inequality) *If*

$$A^{\tau}X \hookrightarrow (X, A^{\sigma+\tau}X)_{F_1}, \quad \text{with some } \tau, \sigma > 0, \tag{4.11}$$

then

$$t\frac{K(f, \psi(t); X, A^{\sigma+\tau}X)}{\psi(t)}\|s\,\chi_{(0,\psi(t))}(s)\|_{F_1} + t\|K(f, s; X, A^{\sigma+\tau}X)\chi_{(\psi(t),\infty)}(s)\|_{F_1} \lesssim K(f, t; X, A^{\tau}X) \tag{4.12}$$

for all $t \in (\Theta(0), \Theta(\infty))$ *and* $f \in X$ *(for which* RHS (4.12) *is finite).*

Proof To prove (A), we obtain, by (4.9) and Theorem 4.1 (B) (see also Remark 4.2 (ii)),

$$\begin{aligned} K(f, t; X, A^{\tau}X) &\lesssim K(f, t; X, (X, A^{\sigma+\tau}X)_{F_1}) \\ &\approx t\frac{K(f, \psi(t); X, A^{\sigma+\tau}X)}{\psi(t)}\|s\,\chi_{(0,\psi(t))}(s)\|_{F_1} \\ &\quad + t\|K(f, s; X, A^{\sigma+\tau}X)\chi_{(\psi(t),\infty)}(s)\|_{F_1} \end{aligned}$$

for any $t \in (\Theta(0), \Theta(\infty))$ and $f \in X$.

In part (B) embedding (4.11) is reverse to (4.9), therefore the above inequality sign is also reverse. □

Combining parts (A) and (B) of Theorem 4.1, we obtain the following result.

Theorem 4.6 (Kolyada-type inequality) *Assume that* X *and* Z, $Z \subset X$, *are r.i. Banach function spaces. Let* F_0, F_1 *be Banach lattices over* $((0, \infty), dt/t)$ *satisfying, for some* $\tau > 0$ *and* $\sigma \geqslant 0$,

$$(X, A^{\tau+\sigma}X)_{F_0} \hookrightarrow Z \tag{4.13}$$

and

$$A^{\tau+\sigma}X \hookrightarrow (Z, A^{\tau}Z)_{F_1}. \tag{4.14}$$

Assume that the functions $\Xi(t) := \|\min(\cdot, t)\|_{F_0}$ *and* $\Theta(t) := t/\|\min(\cdot, t)\|_{F_1}$, $t \in (0, \infty)$, *satisfy* $\Xi(1) < \infty$ *and* $\Theta(1) < \infty$. *If* ϕ *and* ψ *are the generalized reverse functions of* Ξ *and* Θ, *respectively, then*

$$\begin{aligned} & t\frac{K(f, \psi(t); Z, A^{\tau}Z)}{\psi(t)}\|s\,\chi_{(0,\psi(t))}(s)\|_{F_1} + t\|K(f, s; Z, A^{\tau}Z)\chi_{(\psi(t),\infty)}(s)\|_{F_1} \\ &\quad \lesssim \|K(f, s; X, A^{\tau+\sigma}X)\chi_{(0,\phi(t))}(s)\|_{F_0} \\ &\qquad + K(f, \phi(t); X, A^{\tau+\sigma}X)\|\chi_{(\phi(t),\infty)}(s)\|_{F_0} \end{aligned} \tag{4.15}$$

for all $t \in (\Xi(0), \Xi(\infty)) \cap (\Theta(0), \Theta(\infty))$ *and* $f \in X$ *(for which* RHS *(4.15) is finite).*

Proof Taking into account (4.13) and (4.14), we get

$$K(f,t;Z,(Z,A^{\tau}Z)_{F_1}) \lesssim K(f,t;(X,A^{\tau+\sigma}X)_{F_0},A^{\tau+\sigma}X) \quad \text{for all } t>0.$$

To complete the proof, note that, by Theorem 4.1 (B),

$$K(f,t;Z,(Z,A^{\tau}Z)_{F_1}) \approx \text{LHS}(4.15) \quad \text{for all } t \in (\Theta(0),\Theta(\infty))$$

and, by Theorem 4.1 (A),

$$K(f,t;(X,A^{\tau+\sigma}X)_{F_0},A^{\tau+\sigma}X) \approx \text{RHS}(4.15) \quad \text{for all } t \in (\Xi(0),\Xi(\infty)).$$

□

Remark 4.7 (i) Note that Theorems 4.3, 4.5, and 4.6 remain true if the K-functional K is replaced by the K-functional K_0 or by the K-functional K_1 given by (3.4) or by (3.5).

(ii) Theorems 4.1, 4.3, 4.5, and 4.6 are true if the Banach function spaces are replaced by function gaged cones.

To give a flavor of how to use Theorems 4.3, 4.5, and 4.6, we present the following examples on the classical Ul'yanov inequality (1.6) and sharp Ul'yanov inequality in the Lorentz setting, cf. Proposition 1.3.

Example 4.8 We obtain the following extension of the classical Ul'yanov inequality (1.6):

If $1 \leqslant p < \infty$, $k,n \in \mathbb{N}$, $0<\delta<\min(k,n/p)$, and $1/p^* = 1/p - \delta/n$, then

$$t^{k-\delta} \sup_{t\leqslant u<1} \frac{\omega_k(f,u)_{L_{p^*}}}{u^{k-\delta}} \lesssim \left(\int_0^t \left[u^{-\delta}\omega_k(f,u)_{L_p}\right]^{p^*} \frac{du}{u}\right)^{1/p^*} \quad \text{as } t \to 0+ \tag{4.16}$$

holds for all $f \in L_p(\mathbb{R}^n)$.

Note that, since LHS(1.6)$\leqslant$ LHS(4.16), inequality (1.6) follows from (4.16). Moreover, (4.16) provides a sharper bound from below. Indeed, considering $f \in C^\infty$ implies $\omega_k(f,u)_{L_r} \approx u^k$, $0<u<1$, for any $1 \leqslant r \leqslant \infty$. Thus, inequality (1.6) even for smooth functions gives only the rough estimate $t^k \lesssim t^{k-\delta}$ while (4.16) becomes an equivalence.

To prove (4.16), first, we apply Sobolev's embedding $\dot{W}^k L_p \hookrightarrow L^c_{\bar{p}}$ with $1 \leqslant p < n/k$ and $1/\bar{p} = 1/p - k/n$ (here, as usual, $\dot{W}^k L_p$ is the homogeneous Sobolev space and $L^c_{\bar{p}} = L_{\bar{p}}/\{constants\}$ is the factor space with the norm $\|f\|_{L^c_{\bar{p}}} = \inf_{c\in\mathbb{R}^1}\|f-c\|_{\bar{p}}$). See the book [45, 1.77, 1.78] for the case $k=1$. For $k>1$, it follows from the Poincaré inequality, namely,

$$\|f-c\|_{L_{\bar{p}}} \lesssim \|f^{\#}\|_{L_{\bar{p}}} \lesssim \|f^{\#}_k\|_{L_p} \lesssim |f|_{W^k L_p}, \quad 1<p<n/k,$$

where $c = \lim_{t\to\infty} f^*(t)$ and $f^{\#}_k$ is the maximal function given by $f^{\#}_k(x) = \sup\limits_{x\in Q} \frac{1}{|Q|^{1+\frac{k}{n}}} \int_Q |f - P_k f|$, $P_k f$ is a linear projection mapping L_1 onto the space of polynomials of degree at most k, and $f^{\#} = f^{\#}_0$. The first estimate follows from [5, Corollary 4.3] and Hardy type inequalities, the second and third estimates from [19, Theorem 9.3, Theorem 5.6, and Corollary 2.2].

For $p=1$ we obtain by same way

$$\|f-c\|_{L_{\frac{n}{n-k},\infty}} \lesssim \|f^{\#}\|_{L_{\frac{n}{n-k},\infty}} \lesssim \|f^{\#}_k\|_{L_{\frac{n}{n-k},\infty}} \lesssim |f|_{W^k L_1}.$$

By truncated method ([1, Theorem 7.2.1]), we can obtain

$$\|f-c\|_{L_{\frac{n}{n-k}}} \lesssim |f|_{W^k L_1}.$$

By interpolation (see [53]),

$$(L_p, \dot{W}^k L_p)_{\alpha,p^*} = \dot{B}^\delta_{p,p*} \hookrightarrow L^c_{p^*} = (L_p, L^c_{\bar{p}})_{\alpha,p^*} \tag{4.17}$$

with $\alpha := \delta/k$ and $1/p^* = 1/p - \delta/n$.

On the other hand, since $L_p \hookrightarrow L_{p,\infty} = (L_{p^*n/(n+p^*k)}, L^c_{p^*})_{1-\alpha,\infty}$ and $\dot{W}^k L_{p^*n/(n+p^*k)} \hookrightarrow L_{p^*}$, we obtain

$$\begin{aligned} \dot{W}^k L_p &\hookrightarrow \dot{W}^k (L_{p^*n/(n+p^*k)}, L_{p^*})_{1-\alpha,\infty} \\ &= (\dot{W}^k L_{p^*n/(n+p^*k)}, \dot{W}^k L_{p^*})_{1-\alpha,\infty} \hookrightarrow (L_{p^*}, \dot{W}^k L_{p^*})_{1-\alpha,\infty}, \end{aligned} \tag{4.18}$$

where the equality follows from [51] and [50].

Embeddings (4.17), (4.18), Theorem 4.6 (with $\sigma = 0$ and K_0 instead of K), and the known relation $\omega_k(f, t^{1/k})_{L_p} \approx K_0(f, t; L_p, W^k L_p)$ give

$$t^{1-\alpha} \sup_{t \leqslant s} \frac{\omega_k(f, s^{1/k})_{L_{p^*}}}{s^{1-\alpha}} \lesssim \left(\int_0^t \left[u^{-\alpha} \omega_k(f, u^{1/k})_{L_p} \right]^{p^*} \frac{du}{u} \right)^{1/p^*} \tag{4.19}$$

for $f \in L_p$ and $t > 0$ if $0 < \alpha < 1$ and $1/p^* = 1/p - \alpha k/n$. Finally, (4.19) and the change of variables yield (4.16).

Note that in the previous example, we did not use optimal Sobolev embeddings and thus did not obtain the sharp Ul'yanov inequality (1.19). The optimal embeddings require to use Lorentz spaces.

Example 4.9 Let $n \in \mathbb{N}$, $1 < p < \infty$, $0 < \delta < n/p$, $0 < \beta$, and $1 \leqslant r_0 \leqslant r_1 \leqslant \infty$. Take θ satisfying $\max\left\{\frac{\delta}{\delta+\beta}, \frac{p\delta}{n}\right\} < \theta < 1$ and set

$$X = L_{p,r_0}, Y = L_{\bar{p},r_0}, Z = L_{p^*,r_1} \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{\delta}{n}, \ \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\delta}{n\theta}.$$

Then, in light of (2.6), one has $H^\delta_{p,r_0} \hookrightarrow L_{p^*,r_1}$. Thus, the second embedding in (4.3) holds with $A^\delta X = H^\delta_{p,r_0}$ and $\sigma = \delta$, where A is the Sobolev integral operator.

Since

$$H^{\delta+\beta}_{p,r_0} \hookrightarrow L_{\bar{p},r_0},$$

which holds by [57, Theorem 1.6 (i)] (note that $\delta + \beta > \frac{\delta}{\theta} = n\left(\frac{1}{p} - \frac{1}{\bar{p}}\right)$), and

$$L_{p^*,r_1} = (L_{p,r_0}, L_{\bar{p},r_0})_{\theta,r_1} \quad \text{with} \quad \frac{1}{p^*} = \frac{1-\theta}{p} + \frac{\theta}{\bar{p}}$$

(see [7, Theorem 5.3.1]), we derive that (4.3) and (4.4) hold with $\sigma = \delta$, $\tau = \beta$, and the Banach lattice F_0 defined as the set of all functions $g \in \mathcal{M}(0, \infty)$ such that $\|g\|_{F_0} = \|u^{-\theta-1/r_1} g(u)\|_{r_1,(0,\infty)}$. Finally, Theorem 4.3 (A) with $\phi_0(t) \approx t^{1/(1-\theta)} = t^{\frac{\beta+\delta}{\beta}}$ implies

$$\begin{aligned} \omega_\beta(f, t^{1/\beta})_{L_{p^*,r_1}} &\approx K_0(f, t; L_{p^*,r_1}, H^\beta L_{p^*,r_1}) \\ &\times \left(\int_0^{t^{\frac{\beta+\delta}{\beta}}} u^{-\theta r_1 - 1} \omega_{\beta+\delta}(f, u^{1/(\beta+\delta)})^{r_1}_{L_{p,r_0}} du \right)^{1/r_1} \\ &+ \omega_{\beta+\delta}(f, t^{1/(\beta+\delta)})_{L_{p,r_0}} \left(\int_{t^{\frac{\beta+\delta}{\beta}}}^\infty u^{-\theta r_1 - 1} du \right)^{1/r_1} \end{aligned}$$

$$\approx \Big(\int_0^{t^{\frac{\beta+\delta}{\beta}}} u^{-\theta r_1 - 1} \omega_{\beta+\delta}(f, u^{1/(\beta+\delta)})^{r_1}_{L_{p,r_0}} du \Big)^{1/r_1},$$

with the usual modifications for $r_1 = \infty$. The latter is equivalent to the sharp Ul'yanov inequality $\omega_\beta(f,t)_{L_{p^*,r_1}} \lesssim \Big(\int_0^t [u^{-\delta}\, \omega_{\beta+\delta}(f,u)_{L_{p,r_0}}]^{r_1} \frac{du}{u} \Big)^{1/r_1}$ as $t \to 0+$ for $f \in L_{p,r_0}$; see (1.19).

5 The Ul'yanov inequality between weighted Lorentz spaces

5.1 Definitions and preliminaries

The following definition is motivated by the known result on the equivalence between the classical Lorentz space norm and the one involving $f^{**}(t) - f^*(t)$, namely,

$$\|f\|_{L_{p,r}} \approx \left(\int_0^\infty \Big(t^{1/p-1/r} \big(f^{**}(t) - f^*(t) \big) \Big)^r dt \right)^{1/r}, \qquad 1 < p, r < \infty,$$

where $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$ provided that $f^{**}(\infty) = 0$, see [6, Proposition 7.12, p. 384].

Let X be an r.i. space over $(\mathbb{R}^n, \lambda_n)$ and let w be a *weight*, that is, a nonnegative measurable function on $(0,\infty)$. We define the function gaged cone

$$S_X(w)(\mathbb{R}^n, \lambda_n) := \{ f \in \mathcal{M}(\mathbb{R}^n, \lambda_n) : \ f^*(\infty) = 0, \ \|f\|_{S_X(w)} := \|(f^{**} - f^*)w\|_{\overline{X}} < \infty \},$$

where $\overline{X}$ is a representation space of X.

We will also need weighted Lorentz spaces defined as follows (cf., e.g., [16]): If $1 \leqslant r < \infty$, we put

$$\Lambda_r(w)(\mathbb{R}^n, \lambda_n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n, \lambda_n) : \ \|f\|_{\Lambda_r(w)} := \Big(\int_0^\infty \big(f^*(s)\big)^r w(s)\, ds \Big)^{1/r} < \infty \right\},$$

$$\Gamma_r(w)(\mathbb{R}^n, \lambda_n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n, \lambda_n) : \ \|f\|_{\Gamma_r(w)} := \Big(\int_0^\infty \big(f^{**}(s)\big)^r w(s)\, ds \Big)^{1/r} < \infty \right\},$$

$$S_r(w)(\mathbb{R}^n, \lambda_n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n, \lambda_n) : \ f^*(\infty) = 0, \quad \|f\|_{S_r(w)} < \infty \right\},$$

where

$$\|f\|_{S_r(w)} := \Big(\int_0^\infty \big(f^{**}(s) - f^*(s) \big)^r w(s)\, ds \Big)^{1/r}.$$

We will use the following conditions on weights:

- $w \in B_r$ (i.e., w satisfies the B_r condition) if there is $c > 0$ such that

$$t^r \int_t^\infty s^{-r} w(s)\, ds \leqslant c \int_0^t w(s)\, ds \quad \text{for every} \quad t > 0;$$

- $w \in B_r^*$ (i.e., w satisfies the B_r^* condition) if there is $c > 0$ such that

$$t^r \int_0^t s^{-r} w(s)\, ds \leqslant c \int_0^t w(s)\, ds \quad \text{for every} \quad t > 0;$$

- $w \in B_\infty^*$ (i.e., w satisfies the B_∞^* condition) if there is $c > 0$ such that

$$\int_0^t \log \frac{t}{s}\, w(s)\,ds \leqslant c \int_0^t w(s)\,ds \quad \text{for every} \quad t > 0.$$

In general, $\Lambda_r(w)(\mathbb{R}^n, \lambda_n)$ and $S_r(w)(\mathbb{R}^n, \lambda_n)$ are not r.i. spaces, they are not even linear. On the other hand, $\Gamma_r(w)(\mathbb{R}^n, \lambda_n)$ is always an r.i. space for $1 \leqslant r < \infty$ and in this case the representation space of $\Gamma_r(w)(\mathbb{R}^n, \lambda_n)$ is $\Gamma_r(w)((0,\infty), dt)$.

If $\Lambda_r(w)(\mathbb{R}^n, \lambda_n)$ is an r.i. space (e.g., if $1 < r < \infty$ and $w \in B_r$, see Lemma 5.1 below), then the representation space of $\Lambda_r(w)(\mathbb{R}^n, \lambda_n)$ is the space $\Lambda_r(w)((0,\infty), dt)$.

Similarly, if $S_r(w)(\mathbb{R}^n, \lambda_n)$ is an r.i. space (e.g., if $1 < r < \infty$ and $w \in RB_r$, i.e. $w(1/t)t^{r-2} \in B_r$; see [16, Theorem 3.3]), then the representation space of $S_r(w)(\mathbb{R}^n, \lambda_n)$ is the space $S_r(w)((0,\infty), dt)$. Moreover, if $w \in RB_r$, $1 < r < \infty$, then $S_r(w)(\mathbb{R}^n, \lambda_n)$ coincides with $\Gamma_r(w)(\mathbb{R}^n, \lambda_n)$.

The *dilation operator* E_t, $t \in (0,\infty)$, is defined on $\mathcal{M}^+(0,\infty)$ by

$$(E_t f)(s) := f(ts) \quad \text{for all } \ s \in (0,\infty).$$

Given an r.i. space X and $t \in (0,\infty)$, the operator E_t is bounded from $\overline{X}$ to $\overline{X}$ (cf. [6, p. 148]). If h_X denotes the *dilation function*, i.e.,

$$h_X(t) := \|E_{1/t}\|_{\overline{X}\to\overline{X}} \quad \text{for all } \ t \in (0,\infty),$$

then the *lower and upper Boyd index* of the space X is given by

$$\underline{\alpha}_X := \lim_{t\to 0+} \frac{\log h_X(t)}{\log t} \qquad \text{and} \qquad \overline{\alpha}_X := \lim_{t\to\infty} \frac{\log h_X(t)}{\log t},$$

respectively. The Boyd indices satisfy (cf. [6, p. 149])

$$0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le 1.$$

The *Hardy averaging operator* P and its *dual* Q are defined on $\mathcal{M}^+(0,\infty)$, for each $t \in (0,\infty)$, by

$$(Pf)(t) := \frac{1}{t}\int_0^t f(s)\,ds \ \text{ and } \ (Qf)(t) := \int_t^\infty \frac{f(s)}{s}\,ds,$$

respectively. Recall that (cf. [6, p. 150]) given an r.i. space X, the operator P is bounded on $\overline{X}$ if and only if $\overline{\alpha}_X < 1$, while the operator Q is bounded on $\overline{X}$ if and only if $0 < \underline{\alpha}_X$.

We will need the following result, which is partially known but the present formulation seems to be new.

Lemma 5.1 *Let w be a weight, $1 < r < \infty$, and $X := \Lambda_r(w)(\mathbb{R}^n, \lambda_n)$.*
1. *The following conditions are equivalent:*

(a) $w \in B_r$,
(b) *X is an r.i. space,*
(c) *the operator P is bounded on $\overline{X}$,*
(d) $\overline{\alpha}_X < 1$,
(e) $X = \Gamma_r(w)(\mathbb{R}^n, \lambda_n)$.

2. *If $w \in B_r$ and $\eta \in (0,1)$, then the following conditions are equivalent:*

(a) *$w \in B_q^*$ with $q = \eta r$,*

(b) *the operator*

$$(Q_\eta f)(t) = t^{-\eta} \int_t^\infty s^\eta f(s) \frac{ds}{s}, \qquad t \in (0, \infty),$$

is bounded on $\overline{X}$,

(c) $\eta < \underline{\alpha}_X$.

3. *If* $w \in B_r$, *then the following conditions are equivalent:*

(a) $w \in B_\infty^*$,
(b) *the operator* Q *is bounded on* $\overline{X}$,
(c) $0 < \underline{\alpha}_X$.

Proof Part 1 is known; in more detail, for $(a) \Leftrightarrow (b)$ see [56, Theorem 4], for $(a) \Leftrightarrow (c)$ see [4, Theorem 1.7], for $(c) \Leftrightarrow (d)$ see [6, p. 150], and $(c) \Leftrightarrow (e)$ is clear.

The proof of part 2 easily follows from the paper [49, Theorem 3.1]. The condition $w \in B_{\eta r}^*$ is equivalent (cf. [49, Theorem 3.1]) to the fact that the operator Q_η is bounded in

$$L_r^\downarrow(w) := \left\{ f \in \mathcal{M}^+(0,\infty;\downarrow) : \|f\|_{L_r^\downarrow(w)} := \left(\int_0^\infty |f(t)|^r w(t)\, dt \right)^{1/r} < \infty \right\}.$$

It remains to show that the operator Q_η is bounded on $L_r^\downarrow(w)$ if and only if it is bounded in $\Lambda_r(w)$. Part "if" is clear. To prove the part "only if", we first note that, by Fubini's theorem and the Hardy–Littlewood rearrangement inequality (see [6, p. 44]),

$$\begin{aligned} \int_0^t (Q_\eta f)(x)\, dx &= \frac{1}{1-\eta} \int_0^\infty \min\left(1, \frac{t}{u}\right)^{1-\eta} f(u)\, du \\ &\leqslant \frac{1}{1-\eta} \int_0^\infty \min\left(1, \frac{t}{u}\right)^{1-\eta} f^*(u)\, du = \int_0^t (Q_\eta f^*)(x)\, dx. \end{aligned} \tag{5.1}$$

Therefore, the fact that $Q_\eta f \in \mathcal{M}^+(0,\infty;\downarrow)$, the B_r condition, the first part of this lemma, inequality (5.1), and the boundedness of Q_η on $L_r^\downarrow(w)$ imply, for any $f \in \mathcal{M}^+(0,\infty)$,

$$\begin{aligned} &\left(\int_0^\infty \left((Q_\eta f)^*(s)\right)^r w(s)\, ds \right)^{1/r} = \left(\int_0^\infty \left((Q_\eta f)(s)\right)^r w(s)\, ds \right)^{1/r} \\ &\approx \left(\int_0^\infty \left(\frac{1}{s} \int_0^s (Q_\eta f)(u)\, du \right)^r w(s)\, ds \right)^{1/r} \leqslant \left(\int_0^\infty \left(\frac{1}{s} \int_0^s (Q_\eta f^*)(u)\, du \right)^r w(s)\, ds \right)^{1/r} \\ &\lesssim \left(\int_0^\infty \left((Q_\eta f^*)(s)\right)^r w(s)\, ds \right)^{1/r} \lesssim \left(\int_0^\infty \left(f^*(s)\right)^r w(s)\, ds \right)^{1/r}. \end{aligned}$$

The proof of part 3 is similar, one makes use of the fact that the condition $w \in B_\infty^*$ is equivalent to the boundedness of the operator Q on the space $L_r^\downarrow(w)$ (cf. [49, Theorem 3.3]). □

In the rest of this section we work with spaces over $(\mathbb{R}^n, \lambda_n)$ and sometimes we omit the symbol $(\mathbb{R}^n, \lambda_n)$ from the notation of spaces in question.

Lemma 5.2 *Let* $1 < r < \infty$, $w \in B_r$, $\beta \in \mathbb{R}$, *and let* $v(t) := t^\beta$ *for all* $t \in (0, \infty)$. *If* $X := \Lambda_r(w)(\mathbb{R}^n, \lambda_n)$, *then*

$$S_X(v)(\mathbb{R}^n, \lambda_n) \hookrightarrow S_r(wv^r)(\mathbb{R}^n, \lambda_n).$$

Proof Let $\beta \in \mathbb{R}$, $1 < r < \infty$, and $f \in \mathcal{M}^+(\mathbb{R}^n, \lambda_n)$. Since $\int_t^{2t} s^{\beta-1}\,ds \approx t^\beta$ for all $t > 0$, and since

$$\text{the function } t \mapsto t(f^{**}(t) - f^*(t)) \text{ is non-decreasing on } (0,\infty) \tag{5.2}$$

(cf. [14, Prop. 4.2]), on putting

$$g(s) := (f^{**}(s) - f^*(s))s^\beta, \qquad s \in (0,\infty),$$

we obtain that

$$(f^{**}(t) - f^*(t))t^\beta \lesssim \frac{1}{t}\int_t^{2t} g(s)\,ds \leqslant (Pg^*)(t) \quad \text{for all } t > 0.$$

Therefore,

$$\left(\int_0^\infty (f^{**}(t) - f^*(t))^r (v(t))^r w(t)\,dt\right)^{1/r} \lesssim \left(\int_0^\infty ((Pg^*)(t))^r w(t)\,dt\right)^{1/r}.$$

Together with the condition $w \in B_r$ and the first part of Lemma 5.1 (recall that in our case $\overline{X} = \Lambda_r(w)((0,\infty), dt)$), this implies that

$$\|f\|_{S_r(v^r w)} \lesssim \left(\int_0^\infty (g^*(t))^r w(t)\,dt\right)^{1/r} = \|g\|_{\overline{X}} = \|f\|_{S_X(v)(\mathbb{R}^n,\lambda_n)},$$

the required result. □

In what follows, given $\gamma \geqslant 0$ and $n \in \mathbb{N}$, we define the weight $v_{\gamma,n}$ by

$$v_{\gamma,n}(t) := t^{-\frac{\gamma}{n}} \quad \text{for all } t > 0. \tag{5.3}$$

The next lemma represents a key step in the proof of Proposition 5.4 below. It was proved in [32, Theorem 1.1] for $k = 1$, the proof for $k \in \mathbb{N}$ is analogous.

Lemma 5.3 *If $k, n \in \mathbb{N}$ and X is an r.i. space, satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, then, for all $t > 0$ and $f \in X + S_X(v_{k,n})$,*

$$\begin{aligned}
&K(f,t;X,S_X(v_{k,n}))\\
&\quad\approx \|(f^*(s) - f^*(t))\chi_{(0,t^{\frac{n}{k}})}(s)\|_{\overline{X}} + t\|s^{-\frac{k}{n}}(f^{**}(s) - f^*(s))\chi_{(t^{\frac{n}{k}},\infty)}(s)\|_{\overline{X}}\\
&\quad\approx \|(f^{**}(s) - f^*(s))\chi_{(0,t^{\frac{n}{k}})}(s)\|_{\overline{X}} + t\|s^{-\frac{k}{n}}(f^{**}(s) - f^*(s))\chi_{(t^{\frac{n}{k}},\infty)}(s)\|_{\overline{X}}.
\end{aligned}$$

Proposition 5.4 *If $k, m, n \in \mathbb{N}$, $1 < r < \infty$, and $w \in B_r \cap B_\infty^*$, then*

$$\left(\Lambda_r(w), S_{\Lambda_r(w)}(v_{k+m,n})\right)_{\frac{m}{k+m},r} = S_{\Lambda_r(wv_{mr,n})}(v_{0,n}).$$

Proof Let $X := \Lambda_r(w) = \Lambda_r(w)(\mathbb{R}^n, \lambda_n)$. Then the space $\Lambda_r(w)((0,\infty), dt)$ is the representation space of X. By Lemma 5.1, our assumptions guarantee that $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Therefore, using Lemma 5.3 (with $k+m$ instead of k), we obtain, for all $t > 0$ and $f \in X + S_X(v_{k+m,n})$,

$$\begin{aligned}
K(f,t;X,S_X(v_{k+m,n})) &\approx \left(\int_0^{t^{\frac{n}{k+m}}} \left([f^{**}(s) - f^*(s)]^*\right)^r w(s)ds\right)^{1/r}\\
&\quad + t\left(\int_{t^{\frac{n}{k+m}}}^\infty \left(\left[\left(f^{**}(s) - f^*(s)\right)s^{-\frac{k+m}{n}}\right]^*\right)^r w(s)ds\right)^{1/r}.
\end{aligned}$$

If

$$Y := \left(\Lambda_r(w), S_{\Lambda_r(w)}(v_{k+m,n})\right)_{\frac{m}{k+m},r},$$

then

$$\begin{aligned}\|f\|_Y &= \left(\int_0^\infty \left(t^{-\frac{m}{k+m}} K(f,t;\Lambda_r(w), S_{\Lambda_r(w)}(v_{k+m,n}))\right)^r \frac{dt}{t}\right)^{1/r}\\ &\approx \left(\int_0^\infty t^{-\frac{mr}{k+m}} \int_0^{t^{\frac{n}{k+m}}} \left([f^{**}(s)-f^*(s)]^*\right)^r w(s)ds\frac{dt}{t}\right)^{1/r}\\ &\quad + \left(\int_0^\infty t^{\frac{kr}{k+m}} \int_{t^{\frac{n}{k+m}}}^\infty \left(\left[\left(f^{**}(s)-f^*(s)\right)s^{-\frac{k+m}{n}}\right]^*\right)^r w(s)ds\frac{dt}{t}\right)^{1/r}\\ &=: I_1 + I_2.\end{aligned}$$

Applying Fubini's theorem, we arrive at

$$I_1 \approx \left(\int_0^\infty \left([f^{**}(t)-f^*(t)]^*\right)^r t^{-\frac{mr}{n}} w(t)dt\right)^{1/r} = \|f\|_{S_{\Lambda_r(wv_{mr,n})}}(v_{0,n}) \tag{5.4}$$

and

$$I_2 \approx \left(\int_0^\infty t^{\frac{kr}{n}} \left(\left[\left(f^{**}(t)-f^*(t)\right)t^{-\frac{k+m}{n}}\right]^*\right)^r w(t)\,dt\right)^{1/r}. \tag{5.5}$$

Thus, it remains to show that RHS(5.5)$\lesssim$ RHS(5.4).

Let $f \in \mathcal{M}^+(\mathbb{R}^n, \lambda_n)$ and $g(s) := f^{**}(s) - f^*(s)$ for all $s > 0$. Making use of (5.2) and the estimate $\quad t^{-\frac{k+m}{n}-1} \approx \int_t^\infty s^{-\frac{k+m}{n}-2}\,ds \quad$ for all $t > 0$, we obtain that

$$(f^{**}(t)-f^*(t))t^{-\frac{k+m}{n}} \lesssim \int_t^\infty g(s)s^{-\frac{k+m}{n}-1}\,ds \quad \text{for all } t > 0.$$

Together with the fact that the function $t \mapsto t^{\frac{kr}{n}}$ is non-decreasing on $(0,\infty)$, this implies that

$$\begin{aligned}\text{RHS(5.5)} &\lesssim \left(\int_0^\infty t^{\frac{kr}{n}}\left(\int_t^\infty g(s)s^{-\frac{k+m}{n}-1}\,ds\right)^r w(t)\,dt\right)^{1/r}\\ &\lesssim \left(\int_0^\infty \left(\int_t^\infty g(s)s^{-\frac{m}{n}-1}\,ds\right)^r w(t)\,dt\right)^{1/r}.\end{aligned}$$

Given $t > 0$, we define the non-increasing function h_t by

$$h_t(s) := \min\{s^{-\frac{m}{n}-1}, t^{-\frac{m}{n}-1}\} \quad \text{for all } s > 0.$$

Then

$$\int_t^\infty g(s)s^{-\frac{m}{n}-1}\,ds \leqslant \int_0^\infty g(s)h_t(s)\,ds \quad \text{for all } t > 0$$

and, on applying the Hardy-Littlewood-Pólya rearrangement inequality, we arrive at

$$\begin{aligned}\int_t^\infty g(s)s^{-\frac{m}{n}-1}\,ds &\leqslant \int_0^\infty g^*(s)h_t(s)\,ds\\ &= t^{-\frac{m}{n}-1}\int_0^t g^*(s)\,ds + \int_t^\infty g^*(s)s^{-\frac{m}{n}-1}\,ds\end{aligned}$$

$$\leqslant (P(g^*(s)s^{-\frac{m}{n}}))(t) + (Q(g^*(s)s^{-\frac{m}{n}}))(t) \quad \text{for all } t > 0.$$

Consequently,

$$\begin{aligned}\mathrm{RHS}(5.5) &\lesssim \left(\int_0^\infty [(P(g^*(s)s^{-\frac{m}{n}}))(t)]^r w(t)\,dt\right)^{1/r} + \left(\int_0^\infty [(Q(g^*(s)s^{-\frac{m}{n}}))(t)]^r w(t)\,dt\right)^{1/r}\\ &=: N_1 + N_2.\end{aligned}$$

Making use of the assumption $w \in B_r \cap B_\infty^*$ and Lemma 5.1, the fact that the function $t \mapsto g^*(t)t^{-\frac{m}{n}}$ is non-increasing on $(0,\infty)$ and the definition of g, we get

$$\begin{aligned}N_1 &\lesssim \left(\int_0^\infty \left([g^*(t)t^{-\frac{m}{n}}]^*\right)^r w(t)\,dt\right)^{1/r}\\ &= \left(\int_0^\infty \left(g^*(t)t^{-\frac{m}{n}}\right)^r w(t)\,dt\right)^{1/r}\\ &= \left(\int_0^\infty \left([f^{**}(t) - f^*(t)]^* t^{-\frac{m}{n}}\right)^r w(t)\,dt\right)^{1/r}\\ &= \mathrm{RHS}(5.4)\end{aligned}$$

and, similarly,

$$N_2 \lesssim \mathrm{RHS}(5.4).$$

□

Lemma 5.5 *If $m, n \in \mathbb{N}$, $1 < r < \infty$, and $w \in B_r \cap B_\infty^*$, then*

$$S_{\Lambda_r(wv_{mr,n})}(v_{0,n}) \hookrightarrow S_{\Lambda_r(w)}(v_{m,n}). \tag{5.6}$$

Proof Put $X := \Lambda_r(wv_{mr,n})$, $Y := \Lambda_r(w)$. Embedding (5.6) means that, for all $f \in S_X(v_{0,n})$,

$$\|(f^{**} - f^*)v_{m,n}\|_{\overline{Y}} \lesssim \|f^{**} - f^*\|_{\overline{X}},$$

i.e.,

$$\left(\int_0^\infty \left([(f^{**} - f^*)v_{m,n}]^*(t)\right)^r w(t)\,dt\right)^{1/r} \lesssim \left(\int_0^\infty \left([f^{**} - f^*]^*(t)\right)^r w(t)v_{mr,n}(t)\,dt\right)^{1/r}.$$

This can be proved quite analogously as the estimate RHS(5.5)$\lesssim$ RHS(5.4). □

To prove the needed embeddings for Sobolev spaces modelled upon weighted Lorentz spaces given in Proposition 5.7 below, we make use of the following lemma, which is closely related to the results from [47] and can be seen as a Sobolev-Gagliardo-Nirenberg type inequality.

Lemma 5.6 *Suppose that $X(\mathbb{R}^n)$ is an r.i. space such that $\frac{k-1}{n} < \underline{\alpha}_X$, $k \in \mathbb{N}$, $k < n$, and the set of bounded functions is dense in X. Then*

$$\|t^{-\frac{k}{n}}(f^{**}(t) - f^*(t))\|_{\overline{X}} \lesssim \big\||D^k f|^*\big\|_{\overline{X}}, \qquad f \in W^k X,$$

where

$$|D^k f| = \left(\sum_{|\alpha|=k} |D^\alpha f|^2\right)^{1/2}.$$

Proof First, from Theorem 2 in [47], we have

$$\| \left(t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t))\right)^{*} \|_{\overline{X}} \lesssim \left\| |D^k f|^* \right\|_{\overline{X}}, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{5.7}$$

Further, we show that the condition $\frac{k-1}{n} < \underline{\alpha}_X$ implies $\lim_{t\to 0+} \varphi_X(t) = 0$, where φ_X is the fundamental function of X. Indeed, for $t \in (0, \frac{1}{2})$, it follows that $t^{\frac{1-k}{n}} \lesssim Q_{\frac{k-1}{n}}\left(\chi_{(0,1)}\right)(t)$ for $k > 1$ and $\log\frac{1}{t} \lesssim Q_{\frac{k-1}{n}}\left(\chi_{(0,1)}\right)(t)$ for $k = 1$ and using boundedness of $Q_{\frac{k-1}{n}}$ in $\overline{X}$ (see [6, Theorem 5.15, p. 150]), we derive

$$\varphi_X(t) = \|\chi_{(0,t)}\|_{\overline{X}} \lesssim t^{\frac{k-1}{n}} \|Q_{\frac{k-1}{n}}\left(\chi_{(0,1)}\right)\|_{\overline{X}} \lesssim t^{\frac{k-1}{n}} \|\chi_{(0,1)}\|_{\overline{X}} \qquad \text{if} \quad k > 1,$$

$$\varphi_X(t) = \|\chi_{(0,t)}\|_{\overline{X}} \lesssim \left(\log\frac{1}{t}\right)^{-1} \|Q_{\frac{k-1}{n}}\left(\chi_{(0,1)}\right)\|_{\overline{X}} \lesssim \left(\log\frac{1}{t}\right)^{-1} \|\chi_{(0,1)}\|_{\overline{X}} \qquad \text{if} \quad k = 1.$$

Thus, $\lim_{t\to 0+} \varphi_X(t) = 0$. Using [6, Theorem 5.5, Chapter 2, p. 67], we obtain that $X_a = X_b$ and X_b is separable, where X_a is the subset of functions $f \in X$ which have absolutely continuous norms and X_b is the closure in X of the set of simple functions. By our assumption $X = X_b$. Thus $X = X_a = X_b$. Then in light of Semenov's theorem (see [44, Theorem 8, Chapter II]), it follows that continuous functions are dense in X_b. Further, by standard density argument, one can see that $C_0^\infty(\mathbb{R}^n)$ is dense in X_b. (For another proof see Remark 3.13 in [25]. Somewhat similar argument can be found in [39].)

By Lorentz-Shimogaki result [6, Theorem 7.4, p. 169] and [6, Theorem 4.6, p. 61], if $\|f_k - f\|_X \to 0$, then $\|f_k^* - f^*\|_{\overline{X}} \to 0$ as $k \to \infty$. Thus, using a limiting argument, we may extend the validity of (5.7) from functions in $C_0^\infty(\mathbb{R}^n)$ to all functions in $W^k X$.

Since $t(f^{**}(t) - f^*(t))$ is an increasing function, we have

$$\begin{aligned} t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t)) &\lesssim t(f^{**}(t)-f^{*}(t)) \int_t^{2t} x^{-\frac{k}{n}-2}\,dx \\ &\lesssim Q_{\frac{k-1}{n}}\left(t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t))\right)(t). \end{aligned}$$

Taking into account the condition $\frac{k-1}{n} < \underline{\alpha}_X$, the operator $Q_{\frac{k-1}{n}}$ is bounded on $\overline{X}$ and therefore

$$\begin{aligned} \|t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t))\|_{\overline{X}} &\lesssim \|Q_{\frac{k-1}{n}}\left(t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t))\right)\|_{\overline{X}} \\ &\lesssim \|\left(t^{-\frac{k}{n}}(f^{**}(t)-f^{*}(t))\right)^{*}\|_{\overline{X}} \lesssim \left\||D^k f|^*\right\|_{\overline{X}}. \end{aligned}$$

□

Proposition 5.7 *If $k, m, n \in \mathbb{N}$, $k + m < n$, $1 < r < \infty$, and $w \in B_r \cap B^*_{\frac{r(k+m-1)}{n}}$, then*

$$W^m \Lambda_r(w) \hookrightarrow S_{\Lambda_r(w)}(v_{m,n}) \tag{5.8}$$

and

$$W^{k+m} \Lambda_r(w) \hookrightarrow S_{\Lambda_r(w)}(v_{k+m,n}). \tag{5.9}$$

Proof Set $X := \Lambda_r(w)(\mathbb{R}^n, \lambda_n)$. By Lemma 5.1, part 2, the assumption $w \in B_r \cap B^*_{\frac{r(k+m-1)}{n}}$ implies that $\frac{k+m-1}{n} < \underline{\alpha}_X$, which, in turn, gives $\frac{m-1}{n} < \underline{\alpha}_X$. Consequently, if $l \in \{m, k+m\}$, then, by Lemma 5.6,

$$\left\||D^l f|^*\right\|_{\overline{X}} \gtrsim \|t^{-\frac{l}{n}}(f^{**}(t)-f^{*}(t))\|_{\overline{X}},$$

and embeddings (5.8) and (5.9) follow. □

5.2 The Ul'yanov inequality between weighted Lorentz spaces

The next theorem provides an estimate of the K-functional $K(f, t; S_r(w), W^k S_r(w))$. Note that in general, the function gaged cone $S_r(w)$ is not linear (cf., e.g., [16]). For the definition of the K-functional for the couple $(S_r(w), W^k S_r(w))$ see the discussion in Sect. 3.

Theorem 5.8 *If $k, m, n \in \mathbb{N}$, $k+m<n$, $1<r<\infty$, $v_{mr,n}(t)=t^{-\frac{mr}{n}}$, and $w \in B_r \cap B^*_{\frac{r(k+m-1)}{n}}$, then*

$$K(f, t^k; S_r(wv_{mr,n}), W^k S_r(wv_{mr,n})) \lesssim \left(\int_0^t \left(s^{-m} K(f, s^{k+m}; \Lambda_r(w), W^{k+m}\Lambda_r(w)) \right)^r \frac{ds}{s} \right)^{\frac{1}{r}} \tag{5.10}$$

for all $t>0$ and $f \in \Lambda_r(w)$ (for which RHS *(5.10) is finite).*

Proof By Lemma 5.2,

$$Z := S_{\Lambda_r(w)}(v_{m,n}) \hookrightarrow S_r(wv_{mr,n}),$$

which implies that

$$K(f, t; S_r(wv_{mr,n}), W^k S_r(wv_{mr,n})) \lesssim K(f, t; Z, W^k Z) \tag{5.11}$$

for all $f \in Z$ and all $t>0$.

To estimate RHS(5.11), we are going to apply Theorem 4.3 (A), with $X := \Lambda_r(w)$, $Y := S_{\Lambda_r(w)}(v_{k+m,n})$, the function gaged cone Z mentioned above, with the Sobolev integral operator as the potential operator A, and the Banach lattice F_0 defined as the set of all functions $h \in \mathcal{M}(0,\infty)$ such that

$$\|h\|_{F_0} := \left(\int_0^\infty \left(s^{-\frac{m}{k+m}} |h(s)| \right)^r \frac{ds}{s} \right)^{1/r} < \infty.$$

Note also that the assumption $w \in B_r \cap B^*_{\frac{r(k+m-1)}{n}}$ and Lemma 5.1 imply that $w \in B_r \cap B^*_\infty$.

Embeddings (5.9) and (5.8) of Proposition 5.7 show that assumption (4.3) of Theorem 4.3 (A) is satisfied with $\sigma := m$ and $\tau := k$. Using Proposition 5.4, we arrive at

$$\left(\Lambda_r(w), S_{\Lambda_r(w)}(v_{k+m,n}) \right)_{\frac{m}{k+m}, r} = S_{\Lambda_r(wv_{mr,n})}(v_{0,n}).$$

Since, by Lemma 5.5,

$$S_{\Lambda_r(wv_{mr,n})}(v_{0,n}) \hookrightarrow S_{\Lambda_r(w)}(v_{m,n}),$$

we obtain that

$$\left(\Lambda_r(w), S_{\Lambda_r(w)}(v_{k+m,n}) \right)_{\frac{m}{k+m}, r} \hookrightarrow S_{\Lambda_r(w)}(v_{m,n}),$$

which means that assumption (4.4) of Theorem 4.3 (A) is also satisfied. Consequently, estimate (4.5) of Theorem 4.3 (A) implies that

$$K(f, t; Z, W^k Z) \lesssim \left(\int_0^{t^{\frac{k+m}{k}}} \left(s^{-\frac{m}{k+m}} K(f, s; \Lambda_r(w), W^{k+m}\Lambda_r(w)) \right)^r \frac{ds}{s} \right)^{\frac{1}{r}}$$

$$+ K(f, t^{\frac{k+m}{k}}; \Lambda_r(w), W^{k+m}\Lambda_r(w)) \left(\int_{t^{\frac{k+m}{k}}}^{\infty} \left(s^{-\frac{m}{k+m}} \right)^r \frac{ds}{s} \right)^{\frac{1}{r}}$$

$$\approx \left(\int_0^{t^{\frac{k+m}{k}}} \left(s^{-\frac{m}{k+m}} K(f, s; \Lambda_r(w), W^{k+m}\Lambda_r(w)) \right)^r \frac{ds}{s} \right)^{\frac{1}{r}}. \tag{5.12}$$

Combining estimates (5.11) and (5.12), we obtain (5.10). □

Remark 5.9 Note that Theorem 5.8 remains true if the K-functionals K are replaced by the K-functionals K_0 (cf. Remark 4.7).

Since $S_r(w)$ is not a linear space, the calculation of the K-functional $K(f, t; S_r(wv_{mr,n}), W^k S_r(wv_{mr,n}))$ may cause additional difficulties. In order to use the previous theorem, we would like to find a Banach function space Y such that $S_r(wv_{mr,n}) \hookrightarrow Y$. The smallest such space Y is the second associate space $(S_r(wv_{mr,n}))''$.

By [16, Theorem 4.1], if

$$\int_0^{\infty} t^{-\frac{mr}{n}-r} w(t)dt = \infty, \tag{5.13}$$

then

$$(S_r(wv_{mr,n}))' = \Gamma_{r'}(\overline{w}),$$

where

$$\Gamma_r(\overline{w}) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \quad \|f\|_{\Gamma_r(\overline{w})} := \left(\int_0^{\infty} \left(f^{**}(s) \right)^r \overline{w}(s)\, ds \right)^{1/r} < \infty \right\}$$

and

$$\overline{w}(t) = t^{-\frac{mr}{n}-r} w(t) \left(\int_t^{\infty} s^{-\frac{mr}{n}-r} w(s)ds \right)^{-r'} \quad \text{for all } t > 0. \tag{5.14}$$

Now we can use [28, Theorem A] to get that

$$(S_r(wv_{mr,n}))'' = (\Gamma_{r'}(\overline{w}))' = \Gamma_r(\nu),$$

where

$$\nu(t) = \frac{t^{r+r'-1} \int_0^t \overline{w}(s)ds \int_t^{\infty} s^{-r'}\overline{w}(s)ds}{\left(\int_0^t \overline{w}(s)ds + t^{r'} \int_t^{\infty} s^{-r'}\overline{w}(s)ds \right)^{r+1}} \quad \text{for all } t > 0. \tag{5.15}$$

Consequently, $S_r(wv_{mr,n}) \hookrightarrow \Gamma_r(\nu)$. Hence, for all $t > 0$ and $f \in S_r(wv_{mr,n})$,

$$K(f, t; \Gamma_r(\nu), W^k\Gamma_r(\nu)) \lesssim K(f, t; S_r(wv_{mr,n}), W^k S_r(wv_{mr,n})).$$

Thus, using Theorem 5.8 (together with Remark 5.9), the facts that $\Gamma_r(\nu)$ and $\Lambda_r(w)$ are r.i. spaces and that, for all $t > 0$,

$$K_0(f, t^k; \Gamma_r(\nu), W^k\Gamma_r(\nu)) \approx \omega_k(f, t)_{\Gamma_r(\nu)}$$

and

$$K_0(f, t^{k+m}; \Lambda_r(w), W^{k+m}\Lambda_r(w)) \approx \omega_{k+m}(f, t)_{\Lambda_r(w)}$$

(cf. (3.6)), we arrive at the following result.

Corollary 5.10 *Let $k, m, n \in \mathbb{N}$, $k+m<n$, and $1<r<\infty$. If $w \in B_r \cap B^*_{\frac{r(k+m-1)}{n}}$ satisfies (5.13) and ν is given by (5.15), then*

$$\omega_k(f,t)_{\Gamma_r(\nu)} \lesssim \left(\int_0^t \left(s^{-m}\omega_{k+m}(f,s)_{\Lambda_r(w)}\right)^r \frac{ds}{s}\right)^{\frac{1}{r}} \tag{5.16}$$

for all $t>0$ and $f \in \Lambda_r(w)$ (for which RHS(5.16) is finite). Equivalently (see Lemma 5.1, Part 1), we have

$$\omega_k(f,t)_{\Gamma_r(\nu)} \lesssim \left(\int_0^t \left(s^{-m}\omega_{k+m}(f,s)_{\Gamma_r(w)}\right)^r \frac{ds}{s}\right)^{\frac{1}{r}}.$$

As an important example, we obtain Ul'yanov's inequalities between the Lorentz-Karamata spaces. To define the Lorentz-Karamata spaces $L_{p,r;b}(\mathbb{R}^n)$, $1 \leqslant p, r \leqslant \infty$, we introduce slowly varying functions.

Definition 5.11 A measurable function $b:(0,\infty)\to(0,\infty)$ is said to be *slowly varying* on $(0,\infty)$, notation $b \in SV(0,\infty)$ if, for each $\varepsilon>0$, there is a non-decreasing function g_ε and a non-increasing function $g_{-\varepsilon}$ such that $t^\varepsilon b(t) \approx g_\varepsilon(t)$ and $t^{-\varepsilon}b(t) \approx g_{-\varepsilon}(t)$, for all $t \in (0,\infty)$.

Clearly, $\ell^\beta(\ell \circ \ell)^\gamma \in SV(0,\infty)$, etc., where $\beta, \gamma \in \mathbb{R}$ and $\ell(t)=(1+|\log t|)$, $t>0$.
Convention. For the sake of simplicity, in the following we assume that $t^{\pm\varepsilon}b(t)$ are already monotone.

5.3 The Ul'yanov inequality for Lorentz-Karamata spaces: a first look

We introduce the *Lorentz-Karamata space* $L_{p,r;b}(\mathbb{R}^n)$, $p, r \in [1,\infty]$, $b \in SV(0,\infty)$, as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,r;b} := \left(\int_0^\infty [t^{1/p}b(t)f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty$$

(with the usual modification for $r=\infty$).
If

$$w(t)=t^{\frac{r}{p}-1}b^r(t), \quad 1<p<\infty, \quad 1\leqslant r<\infty, \quad b\in SV(0,\infty),$$

then $\Lambda_r(w)=L_{p,r;b}$. Let $1<r<\infty$. First we note that the condition $1<p<\infty$ implies that $w \in B_r$. Moreover, if $p<n/(k+m-1)$, then $w \in B^*_{\frac{r(k+m-1)}{n}}$. It is easy to see that the function given by (5.14) satisfies

$$\overline{w}(t) \approx t^{\frac{mr'}{n}+\frac{r'}{p'}-1}b^{-r'}(t) \qquad \text{for all } t>0.$$

Furthermore, if $p<n/(k+m-1)$, then

$$\int_0^t \overline{w}(s)ds + t^{r'}\int_t^\infty s^{-r'}\overline{w}(s)ds \approx t^{\frac{mr'}{n}+\frac{r'}{p'}}b^{-r'}(t) \quad \text{for all } t>0,$$

which, together with (5.15), implies that

$$\nu(t) \approx t^{r(\frac{1}{p}-\frac{m}{n})-1}b^r(t) \qquad \text{for all } t>0,$$

and

$$\Gamma_r(\nu) = L_{p^*,r;b} \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}.$$

Therefore, by Corollary 5.10, for all $t > 0$ and $f \in L_{p,r;b}$,

$$\omega_k(f,t)_{L_{p^*,r;b}} \lesssim \left(\int_0^t \left(u^{-m} \omega_{k+m}(f,u)_{L_{p,r;b}} \right)^r \frac{du}{u} \right)^{\frac{1}{r}}, \quad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}.$$

Using the estimate $\omega_{k+m}(f,u)_{L_{p,r;b}} \lesssim \omega_{k+m}(f,u)_{L_{p,\overline{r};b}}$ with $\overline{r} \leqslant r$, we immediately obtain the following corollary.

Corollary 5.12 *If $k, m, n \in \mathbb{N}$, $k+m < n$, $1 < p < n/(k+m-1)$, $1 < \overline{r} \leqslant r < \infty$, $b \in SV(0,\infty)$, and $1/p^* = 1/p - m/n$, then*

$$\omega_k(f,t)_{L_{p^*,r;b}} \lesssim \left(\int_0^t \left(u^{-m} \omega_{k+m}(f,u)_{L_{p,\overline{r};b}} \right)^r \frac{du}{u} \right)^{\frac{1}{r}} \tag{5.17}$$

for all $t > 0$ and $f \in L_{p,\overline{r};b}$ (for which RHS (5.17) *is finite).*

In particular, if $b \equiv 1$, then (5.17) yields the known estimate (1.19) for integer parameters k and m satisfying $k+m < n$. Note that the restriction $\overline{r} \leqslant r$ is natural since (5.17) does not hold in general for $\overline{r} > r$, see [30, Theorem 1.1(iii)].

In the next section we will investigate inequalities of type (5.17) in more details.

6 Sharp Ul'yanov inequality between the Lorentz–Karamata spaces

In the previous section we obtained the Ul'yanov-type inequalities for K-functionals and moduli of smoothness between the general weighted Lorentz spaces, which causes restrictions on the parameters. In particular, we assumed that $k, m \in \mathbb{N}$. On the other hand, it is clear that, when dealing with more specific Lorentz spaces, one could get better results, i.e., sharp Ul'yanov inequalities for a wider range of parameters.

Our main goal in this section is to establish new sharp Ul'yanov inequalities between the Lorentz–Karamata spaces introduced in the previous subsection.

First we mention some simple properties of slowly varying functions (recall that slowly varying functions have been introduced in Definition 5.11 at the end of Sect. 5.2). In what follows we write only SV instead of $SV(0,\infty)$.

Lemma 6.1 (cf. [31, Prop. 2.2]) *Let $b, b_1, b_2 \in SV$.*
(i) *Then $b_1 b_2 \in SV$, $b^r \in SV$ and $b(t^r) \in SV$ for each $r \in \mathbb{R}$.*
(ii) *If ε and κ are positive numbers, then there are positive constants c_ε and C_ε such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\} b(t) \leqslant b(\kappa t) \leqslant C_\varepsilon \max\{\kappa^{\varepsilon}, \kappa^{-\varepsilon}\} b(t) \quad \text{for every } t > 0.$$

(iii) *If $\alpha > 0$ and $q \in (0,\infty]$, then, for all $t > 0$,*

$$\|\tau^{\alpha - 1/q} b(\tau)\|_{q,(0,t)} \approx t^\alpha b(t) \quad \text{and} \quad \|\tau^{-\alpha-1/q} b(\tau)\|_{q,(t,\infty)} \approx t^{-\alpha} b(t).$$

If $b \in SV$, then also $b^{-1} := 1/b \in SV$. We will show that these functions have comparable, sufficiently smooth regularizations:

(a) Given $N \in \mathbb{N}$, following [35, Lemma 6.3], we set

$$a_0(t) := b^{-1}(t) \equiv \frac{1}{b(t)}\,, \qquad a_\ell(t) = \frac{1}{t}\int_0^t a_{\ell-1}(u)\,du\,, \tag{6.1}$$
$$t > 0,\ \ \ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant N.$$

Then, by direct computation, we obtain, for all $t > 0$ and $\ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant N$, that

$$a_\ell(t) \approx b^{-1}(t) \quad \text{and} \quad a'_\ell(t) = -\frac{1}{t}[a_\ell(t) - a_{\ell-1}(t)], \tag{6.2}$$

and hence, for all $t > 0$ and $j, \ell \in \mathbb{N},\ 1 \leqslant j \leqslant \ell \leqslant N$,

$$|a_\ell^{(j)}(t)| = |t^{-j}\sum_{k=0}^{j} C_{j,k}\, a_{\ell-k}(t)| \lesssim t^{-j}\, b^{-1}(t) \tag{6.3}$$

(with some constants $C_{j,k}$).

(b) Analogously, given $N \in \mathbb{N}$, we define

$$c_0(t) := b(t)\,, \qquad c_\ell(t) = t\int_t^\infty \frac{c_{\ell-1}(u)}{u^2}\,du\,, \qquad t > 0,\ \ \ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant N, \tag{6.4}$$

to obtain, for all $t > 0$ and $\ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant N$,

$$c_\ell(t) \approx b(t) \quad \text{and} \quad c'_\ell(t) = \frac{1}{t}[c_\ell(t) - c_{\ell-1}(t)], \tag{6.5}$$

and hence, for all $t > 0$ and $j, \ell \in \mathbb{N},\ 1 \leqslant j \leqslant \ell \leqslant N$,

$$|c_\ell^{(j)}(t)| = |t^{-j}\sum_{k=0}^{j} D_{j,k}\, c_{\ell-k}(t)| \lesssim t^{-j}\, b(t) \tag{6.6}$$

(with some constants $D_{j,k}$).

Now we introduce the subclass $SV_\uparrow$ of non-decreasing slowly varying functions by

$$SV_\uparrow := \{b \in SV : b \text{ is non-decreasing, } \lim_{t\to\infty} b(t) = \infty,\ \lim_{t\to 0_+} b(t) > 0\}, \tag{6.7}$$

and extend the *classical Riesz potential*

$$I^\sigma f := k_\sigma * f, \quad \text{where} \quad k_\sigma(x) := \mathcal{F}^{-1}[|\xi|^{-\sigma}](x), \quad 0 < \sigma < n,$$

to a *fractional integration with slowly varying component* b^{-1}, where $b \in SV_\uparrow$. To this end, if the slowly varying function a_N, $N \in \mathbb{N}$, is given by (6.1), set

$$I_N^{\sigma,b^{-1}} f := k_{\sigma,b^{-1};N} * f, \quad \text{where} \quad k_{\sigma,b^{-1};N} := \mathcal{F}^{-1}[|\xi|^{-\sigma} a_N(|\xi|)](x), \quad 0 < \sigma < n.$$

When we choose $N > (n+1)/2$, we can apply the formula

$$|\mathcal{F}^{-1}[m(|\xi|^2)](x)| \lesssim \int_0^{|x|^{-2}} t^{N-1+n/2}|m^{(N)}(t)|\,dt + |x|^{-N-(n-1)/2}\int_{|x|^{-2}}^\infty t^{N/2+(n-3)/4}|m^{(N)}(t)|\,dt,$$

contained in [62], with $m(t) = t^{-\sigma/2} a_N(\sqrt{t})$. To this end, observe that

$$m^{(N)}(t) = \sum_{\ell=0}^{N} t^{-N+\ell-\sigma/2}\sum_{k=0}^{\ell} c_{k,\ell,N}\, a_N^{(k)}(\sqrt{t})/(t^{\ell/2} t^{(\ell-k)/2})$$

(with some constants $c_{k,\ell,N}$) which, by (6.3), implies that $|m^{(N)}(t)| \lesssim t^{-N-\sigma/2} b^{-1}(\sqrt{t})$, and hence for all $x \neq 0$,

$$|k_{\sigma,b^{-1};N}(x)| \lesssim \int_0^{|x|^{-2}} t^{N-1+n/2} t^{-N-\sigma/2} b^{-1}(\sqrt{t})\,dt$$
$$+|x|^{-N-(n-1)/2} \int_{|x|^{-2}}^{\infty} t^{N/2+(n-3)/4} t^{-N-\sigma/2} b^{-1}(\sqrt{t})\,dt \lesssim |x|^{\sigma-n} b^{-1}(|x|^{-1}).$$

Consequently,

$$k^*_{\sigma,b^{-1};N}(t) \lesssim k^{**}_{\sigma,b^{-1};N}(t) \lesssim t^{\sigma/n-1} b^{-1}(t^{-1/n}) \quad \text{for all } t > 0.$$

Therefore, the proof of [26, Theorem 4.6] can be taken over to get the following analog of a *fractional integration* theorem.

Lemma 6.2 *Let* $1 < p < \infty$, $0 < \sigma < n/p$, $1/p^* = 1/p - \sigma/n$, $1 \leqslant r \leqslant s \leqslant \infty$, *and* $B \in SV$, $b \in SV_{\uparrow}$. *If* $N \in \mathbb{N}$, $N > (n+1)/2$, *and*

$$b_n(t) := b^{-1}(t^{-1/n}) \quad \textit{for all } t > 0, \tag{6.8}$$

then

$$\|I_N^{\sigma,b^{-1}} f\|_{p^*,s;B} \lesssim \|f\|_{p,r;b_n B} \quad \textit{for all } f \in L_{p,r;b_n B}(\mathbb{R}^n)\,.$$

The next lemma deals with a *Bernstein inequality* for slowly varying derivatives, based on the regularization of b. Throughout this section, given $R > 0$, we put

$$B_R(0) := \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$$

and denote by χ a $C^\infty[0,\infty)$—function such that

$$\chi(u) = 1 \quad \text{if} \quad 0 \leqslant u \leqslant 1 \quad \text{and} \quad \chi(u) = 0 \quad \text{if} \quad u \geqslant 2. \tag{6.9}$$

Lemma 6.3 *Let* $1 < p < \infty$, $1 \leqslant r \leqslant \infty$, *and* $g \in L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ *with* $\operatorname{supp} \widehat{g} \subset B_R(0)$, $R > 0$. *If* $B \in SV$, $b \in SV_{\uparrow}$, *and* $N \in \mathbb{N}$, $N > n/2$, *then*

$$\|\mathcal{F}^{-1}[c_N(|\xi|)\widehat{g}]\|_{p,r;B} \lesssim b(R)\,\|g\|_{p,r;B} \qquad \textit{for all } R > 0,$$

where the slowly varying function c_N *is given by* (6.4).

Proof Take $R > 0$, $N \in \mathbb{N}$, $N > n/2$, and define

$$m_{R;N}(t) := \chi(t/R)\, c_N(t)/b(R) \quad \text{for all } t > 0.$$

Then $m_{R;N}$ satisfies (cf. (6.5) and (6.6)) the condition

$$\sup_{t>0} |m_{R;N}(t)| + \sup_{\ell \in \mathbb{Z}} \int_{2^\ell}^{2^{\ell+1}} t^{N-1} |m_{R;N}^{(N)}(t)|\,dt \leqslant C, \qquad N > n/2, \tag{6.10}$$

which, by e.g. [10, Theorem 0.2], implies that $m_{R;N}(|\xi|)$ generates a uniformly bounded operator family on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, i.e., $\|\mathcal{F}^{-1}[m_{R;N}(|\xi|)\widehat{g}]\|_p \lesssim \|g\|_p$ if $1 < p < \infty$. Hence, cf. [25, Corollary 3.15], this is also true for the interpolation space $L_{p,r;B}(\mathbb{R}^n)$. Since $b(R) m_{R;N}(|\xi|)\widehat{g} = c_N(|\xi|)\widehat{g}$ if $g \in L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{g} \subset B_R(0)$, the assertion follows. □

A combination of these two lemmas gives the following embedding.

Lemma 6.4 *Let* $1 < p < \infty$, $0 < \sigma < n/p$, $1/p^* = 1/p - \sigma/n$, $1 \leqslant r \leqslant s \leqslant \infty$, $B \in SV$, $b \in SV_\uparrow$. *If* b_n *is defined by* (6.8), *then*

$$\|I^\sigma g\|_{p^*,s;B} \lesssim b(R)\, \|g\|_{p,r;b_nB}$$

for all $R > 0$ *and for all entire functions* $g \in L_{p,r;b_nB}(\mathbb{R}^n)$ *with supp* $\widehat{g} \subset B_R(0)$.

Proof Let $N \in \mathbb{N}$, $N > (n+1)/2$ and let the slowly varying functions a_N, c_N be given by (6.1) and (6.4). Then $1 = a_N c_N/(a_N c_N)$ on the interval $(0,\infty)$. Therefore, supposing that the Fourier symbol $1/(a_N(|\xi|)c_N(|\xi|))$ generates a bounded operator on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, then, by Lemma 6.2 and by Lemma 6.3, we obtain that

$$\begin{aligned}\|I^\sigma g\|_{p^*,s;B} &\lesssim \|\mathcal{F}^{-1}[|\xi|^{-\sigma} a_N(|\xi|)c_N(|\xi|)\widehat{g}(\xi)]\,\|_{p^*,s;B}\\ &\lesssim \|\mathcal{F}^{-1}[c_N(|\xi|)\widehat{g}(\xi)]\,\|_{p,r;b_nB} \lesssim b(R)\|g\|_{p,r;b_nB}\end{aligned}$$

for all $R > 0$ and for all entire functions $g \in L_{p,r;b_nB}(\mathbb{R}^n)$ with supp $\widehat{g} \subset B_R(0)$.

Thus, by [10, Theorem 0.2], it remains to show that $1/(a_N c_N)$ satisfies the condition (6.10) (with the function $m_{R;N}$ replaced by $1/(a_N c_N)$). Introduce the differential operator $D = t(d/dt)$, define D^0 to be the identity operator and $D^j = D\,D^{j-1}$, $j \in \mathbb{N}$. Now note that $t^N (d/dt)^N$ can be expressed as a linear combination of D^j, $1 \leqslant j \leqslant N$, that $D\,[a_N(t)c_N(t)] = a_{N-1}(t)c_N(t) - a_N(t)c_{N-1}(t)$ and, by induction, that

$$D^j\,(a_N(t)\,c_N(t)) = \sum_{k=0}^{j} (-1)^{k+1}\binom{j}{k} a_{N-k}(t)\,c_{N-j+k}(t)\,, \qquad 1 \leqslant j \leqslant N. \quad (6.11)$$

Therefore,

$$\left|D^j \frac{1}{a_N(t)\,c_N(t)}\right| \lesssim \sum_{k=1}^{j} \left|\frac{M_{j,k}(t)}{(a_N(t)\,c_N(t))^{k+1}}\right|, \quad 1 \leqslant j \leqslant N, \quad \text{for all } t > 0, \quad (6.12)$$

where the numerators $M_{j,k}(t)$ are appropriate linear combinations of terms of the type

$$\prod_{i=1}^{j} \left\{D^{\alpha_i^{k,j}}\,(a_k(t)\,c_\ell(t))\right\}^{\beta_i^{k,j}}, \qquad \alpha^{k,j}, \beta^{k,j} \in \mathbb{N}_0^j, \quad \sum_{i=1}^{j} \alpha_i^{k,j}\beta_i^{k,j} = j.$$

In view of (6.1) – (6.6), it is clear that the denominators on the right-hand side of (6.12) satisfy $(a_N(t)\,c_N(t))^{k+1} \approx 1$ for all $t > 0$, and that, on account of (6.11), $|M_{j,k}(t)| \lesssim 1$ for all $t > 0$ if $1 \leqslant j \leqslant N$ and $1 \leqslant k \leqslant j$. Therefore, $1/(a_N c_N)$ satisfies (6.10) and the proof is complete. □

The following variant of a *Nikol'skiĭ inequality* will turn out to be useful.

Lemma 6.5 *Let* $1 < p < \infty$, $0 < \sigma < n/p$, $1/p^* = 1/p - \sigma/n$, $1 \leqslant r \leqslant s \leqslant \infty$, $B \in SV$, $b \in SV_\uparrow$. *If* b_n *is defined by* (6.8), *then*

$$\|g\|_{p^*,s;B} \lesssim R^{n(1/p-1/p^*)} b(R)\, \|g\|_{p,r;b_nB}$$

for all $R > 0$ *and for all* $g \in L_{p,r;b_nB}(\mathbb{R}^n)$ *with supp* $\widehat{g} \subset B_R(0)$.

Proof Take χ defined by (6.9) and set $v_R(x) := \mathcal{F}^{-1}[\chi(|\xi|/R)](x)$, $x \in \mathbb{R}^n$, $R > 0$. Then, for all $x \in \mathbb{R}^n$, $t \in (0,\infty)$ and $R > 0$,

$$|v_R(x)| \lesssim \frac{R^n}{(1+R|x|)^n}, \qquad v_R^*(t) \lesssim \frac{R^n}{(1+Rt^{1/n})^n}, \qquad v_R^{**}(t) \lesssim \min\left\{R^n, \frac{1}{t}\right\}.$$

By the assumption on the support of the Fourier transform of g, we have $v_R * g = g$. Therefore, by O'Neil's inequality,

$$g^*(t) = (v_R * g)^*(t) \lesssim t\, v_R^{**}(t) g^{**}(t) + \int_t^\infty v_R^*(u) g^*(u)\, du.$$

Hence, for all $R > 0$,[4]

$$\begin{aligned} \|g\|_{p^*,s;B} \lesssim & \left(\int_0^\infty \left[t^{1/p^*} B(t) \min\left\{ R^n, \frac{1}{t} \right\} \int_0^t g^*(u)\, du \right]^s \frac{dt}{t} \right)^{1/s} \\ & + R^n \left(\int_0^\infty \left[t^{1/p^*} B(t) \int_t^\infty \frac{g^*(u)}{(1+Ru^{1/n})^n}\, du \right]^s \frac{dt}{t} \right)^{1/s} =: N_1 + N_2. \end{aligned}$$

Since $t^\varepsilon b_n(t)$, $\varepsilon > 0$, is almost non-decreasing and $t^{-\varepsilon} b_n(t)$ is almost non-increasing, elementary estimates lead to

$$\begin{aligned} N_1 \leqslant & R^n \left(\int_0^{R^{-n}} \left[\{ t^{1/p^*+1-1/p} b_n^{-1}(t) \}\, t^{1/p-1} b_n(t) B(t) \int_0^t g^*(u)\, du \right]^s \frac{dt}{t} \right)^{1/s} \\ & + \left(\int_{R^{-n}}^\infty \left[\{ t^{1/p^*-1/p} b_n^{-1}(t) \}\, t^{1/p-1} b_n(t) B(t) \int_0^t g^*(u)\, du \right]^s \frac{dt}{t} \right)^{1/s} \\ \lesssim & R^{n(1/p-1/p^*)} b(R) \left(\int_0^\infty \left[t^{1/p-1} b_n(t) B(t) \int_0^t g^*(u)\, du \right]^s \frac{dt}{t} \right)^{1/s} \quad \text{for all } R > 0. \end{aligned}$$

Now apply a Hardy-type inequality [29, Lemma 4.1] to obtain

$$N_1 \lesssim R^{n(1/p-1/p^*)} b(R)\, \|g\|_{p,r;b_n B}.$$

Similarly, handle the term N_2, use [29, Lemma 4.1] to arrive at

$$N_2 \lesssim R^n \left(\int_0^\infty \left[t^{1/p^*+1-1/r} B(t) \frac{g^*(t)}{(1+Rt^{1/n})^n} \right]^r dt \right)^{1/r} = R^n \left(\int_0^{R^{-n}} \cdots + \int_{R^{-n}}^\infty \cdots \right)^{1/r}.$$

Apply Minkowski's inequality, observe that

$$(1+Rt^{1/n})^n \approx \begin{cases} 1, & 0 < t < R^{-n}, \\ R^n t, & t \geqslant R^{-n}, \end{cases}$$

and use again almost monotonicity properties of $t^{\pm\varepsilon} b_n(t)$ to get

$$N_2 \lesssim R^{n(1/p-1/p^*)} b(R)\, \|g\|_{p,r;b_n B}.$$

□

We will need the Besov-type space $B^{\sigma,b}_{(p,r;B),s}(\mathbb{R}^n)$, modelled upon the Lorentz-Karamata space $L_{p,r;B}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \leqslant r \leqslant s \leqslant \infty$, $B \in SV$, whose smoothness order $\sigma > 0$ is perturbed by a slowly varying function $b \in SV_\uparrow$. To this end, we introduce the modulus of smoothness of fractional order $\kappa > 0$ on the Lorentz-Karamata space $L_{p,r;B}(\mathbb{R}^n)$ by (cf. (1.1))

$$\omega_\kappa(f,\delta)_{L_{p,r;B}} := \sup_{|h| \leqslant \delta} \left\| \Delta_h^\kappa f(x) \right\|_{L_{p,r;B}(\mathbb{R}^n)}$$

[4] We assume that $r, s < \infty$. If $s = \infty$ or $r = \infty$, then the proof is going along the same lines.

and then we set

$$B^{\sigma,b}_{(p,r;B),s}(\mathbb{R}^n) := \Big\{ f \in L_{p,r;B}(\mathbb{R}^n) : |f|^*_{B^{\sigma,b}_{(p,r;B),s}} < \infty \Big\}, \tag{6.13}$$

where

$$|f|^*_{B^{\sigma,b}_{(p,r;B),s}} := \|u^{-\sigma-1/s} b(u^{-1})\, \omega_{\kappa+\sigma}(f,u)_{L_{p,r;B}}\|_s .$$

This definition does not depend upon $\kappa > 0$, which follows from the Marchaud inequality (cf. [64, (1.12)]).

The following lemma is the key result to prove Theorem 6.7 mentioned below.

Lemma 6.6 *If* $1 < p < \infty$, $0 < \sigma < n/p$, $1/p^* = 1/p - \sigma/n$, $1 \leqslant r \leqslant s \leqslant \infty$, *and* $B \in SV$, $b \in SV_{\uparrow}$, *then*

$$\|f\|_{p^*,s;B} \lesssim |f|^*_{B^{\sigma,b}_{(p,r;b_nB),s}} \quad \textit{for all} \;\; f \in B^{\sigma,b}_{(p,r;b_nB),s}(\mathbb{R}^n).$$

The proof follows the same lines as the one of [30, Lemma 2.6]. Indeed, we use the Nikol'skiǐ inequality from Lemma 6.5, and the sequence space $\ell^\sigma_q(X)$, X a normed space, as the space of X-valued sequences $(F_j)_{j\in\mathbb{Z}}$ with

$$\|(F_j)_j\|_{\ell^\sigma_q} := \Big(\sum_{j\in\mathbb{Z}} [2^{j\sigma}\|b(2^j)F_j\|_X]^q \Big)^{1/q} < \infty.$$

Since, by [36] (see also [31, Lemma 5.5]),

$$(L_{p_0^*,s;B}, L_{p_1^*,s;B})_{\theta,q} = L_{p^*,q;B}, \quad 1/p^* = (1-\theta)/p_0^* + \theta/p_1^*, \quad 0 < \theta < 1,$$

the rest of the proof of [30, Lemma 2.6] carries over. □

We will also need the Riesz potential space $H^\lambda_{p,r;\,B} := H^\lambda L_{p,r;\,B}(\mathbb{R}^n)$ modelled upon the Lorentz-Karamata space $L_{p,r;B}$, $B \in SV$, and defined analogously to the space $H^\sigma_{p,r}$ introduced in Sect. 1.2. If $1 < p < \infty$, then the estimate

$$\omega_\lambda(f,t)_{L_{p,r;B}} \approx K_0(f, t^\lambda; L_{p,r;B}, H^\lambda_{p,r;\,B}) \quad \text{for all } f \in L_{p,r;B} \text{ and } t > 0 \tag{6.14}$$

can be verified analogously to estimate (1.13) in [30, Lemma 1.4].

Now we are in a position to prove the sharp Ul'yanov inequality between Lorentz–Karamata spaces.

Theorem 6.7 *Let* $\kappa > 0$, $1 < p < \infty$, $0 < \sigma < n/p$, $1/p^* = 1/p - \sigma/n$, $1 \leqslant r \leqslant s \leqslant \infty$, *and* $B \in SV$, $b \in SV_{\uparrow}$. *If* b_n *is defined by* (6.8), *then*

$$\omega_\kappa(f,\delta)_{L_{p^*,s;B}} \lesssim \Big(\int_0^\delta [t^{-\sigma} b(t^{-1})\, \omega_{\kappa+\sigma}(f,t)_{L_{p,r;b_nB}}]^s \frac{dt}{t} \Big)^{1/s}, \qquad \delta \to 0+, \tag{6.15}$$

for all $f \in B^{\sigma,b}_{(p,r;b_nB),s}(\mathbb{R}^n)$.

As an example, recalling that b is non-decreasing, we consider in Theorem 6.7

$$b(t) = \begin{cases} 1, & t \in (0,1] \\ (1+|\ln t|)^\gamma, \;\; \gamma \geqslant 0, & t \in (1,\infty) \end{cases}$$

and

$$B(t) = (1+|\ln t|)^\alpha, \;\; \alpha \in \mathbb{R}, \;\; t \in (0,\infty);$$

cf. [30].

Remark 6.8 Let all the assumptions of Theorem 6.7 be satisfied. Note that if $f \in L_{p,r;b_n B}(\mathbb{R}^n)$ and RHS(6.15) $< \infty$ for some $\delta > 0$, then $f \in B^{\sigma,b}_{(p,r;b_n B),s}(\mathbb{R}^n)$. Indeed, if $\delta > 0$, then, by Lemma 6.1 (iii), for all $f \in L_{p,r;b_n B}(\mathbb{R}^n)$,

$$\begin{aligned}\|u^{-\sigma-1/s}\, b(u^{-1})\, \omega_{\kappa+\sigma}(f,u)_{p,r;b_n B}\|_{s,(\delta,\infty)} &\lesssim \|f\|_{p,r;b_n B}\, \|u^{-\sigma-1/s} b(u^{-1})\|_{s,(\delta,\infty)}\\ &\approx \|f\|_{p,r;b_n B}\, \delta^{-\sigma} b(\delta^{-1}).\end{aligned}$$

Consequently, for all $f \in L_{p,r;b_n B}(\mathbb{R}^n)$,

$$|f|^*_{B^{\sigma,b}_{(p,r;b_n B),s}} \lesssim \text{RHS(6.15)} + \|f\|_{p,r;b_n B}\, \delta^{-\sigma} b(\delta^{-1}) < \infty,$$

and the result follows.

Since also

$$\text{RHS(6.15)} \leqslant |f|^*_{B^{\sigma,b}_{(p,r;b_n B),s}} \quad \text{for all } f \in B^{\sigma,b}_{(p,r;b_n B),s}(\mathbb{R}^n),$$

we see that

$$B^{\sigma,b}_{(p,r;b_n B),s}(\mathbb{R}^n) = \{f \in L_{p,r;b_n B}(\mathbb{R}^n) : \text{ RHS(6.15)} < \infty \text{ for some } \delta > 0\}.$$

Proof of Theorem 6.7 By (6.14), for all $f \in B^{\sigma,b}_{(p,r;b_n B),s}$, $g \in H^{\kappa}_{p^*,s;B}$ and $t > 0$,

$$\begin{aligned}\omega_\kappa(f,t)_{L_{p^*,s;B}} \approx& K_0(f, t^\kappa; L_{p^*,s;B}, H^\kappa_{p^*,s;B})\\ \leqslant& \|f-g\|_{p^*,s;B} + t^\kappa \|(-\Delta)^{\kappa/2} g\|_{p^*,s;B}.\end{aligned} \tag{6.16}$$

Take $g \in H^{\kappa+\sigma}_{p,r;b_n B}$ and consider its de la Vallée-Poussin means defined by

$$g_t := \mathcal{F}^{-1}[\chi(t|\xi|)] * g, \quad t > 0,$$

where χ is the cut-off function from (6.9). Then $\operatorname{supp} \widehat{g_t} \subset B_{2/t}(0)$. Note also that

$$\|g_t\|_{H^{\kappa+\sigma}_{p,r;b_n B}} \lesssim \|g\|_{H^{\kappa+\sigma}_{p,r;b_n B}} \quad \text{for all } t > 0 \text{ and } g \in H^{\kappa+\sigma}_{p,r;b_n B}$$

since $\|\mathcal{F}^{-1}[\chi(t|\xi|)]\|_1 \lesssim 1$ for all $t > 0$ by [61, Corollary 2.3]. Thus, using Lemma 6.4, we obtain

$$\|(-\Delta)^{\kappa/2} g_t\|_{p^*,s;B} \lesssim b(1/t)\, \|(-\Delta)^{(\kappa+\sigma)/2} g_t\|_{p,r;b_n B} \tag{6.17}$$

for all $t > 0$ and $g \in H^{\kappa+\sigma}_{p,r;b_n B}$.
Moreover, by Lemma 6.6,

$$\|f - g_t\|_{p^*,s;B} \lesssim |f-g_t|^*_{B^{\sigma,b}_{(p,r;b_n B),s}} \quad \text{for all } t > 0 \text{ and } g \in H^{\kappa+\sigma}_{p,r;b_n B}. \tag{6.18}$$

Combining estimates (6.16)–(6.18), we arrive at

$$\omega_\kappa(f,t)_{L_{p^*,s;B}} \lesssim |f-g_t|^*_{B^{\sigma,b}_{(p,r;b_n B),s}} + t^\kappa b(1/t) \|(-\Delta)^{(\kappa+\sigma)/2} g_t\|_{p,r;b_n B}$$

for all $f \in B^{\sigma,b}_{(p,r;b_n B),s}$, $g \in H^{\kappa+\sigma}_{p^*,s;B}$ and $t > 0$.

One gets rid of g_t estimating g_t by g in a way analogous to the proof of [30, Theorem 1.1 (i)]. Thus,

$$\omega_\kappa(f,t)_{L_{p^*,s;B}} \lesssim |f-g|^*_{B^{\sigma,b}_{(p,r;b_n B),s}} + t^\kappa b(1/t) \|(-\Delta)^{(\kappa+\sigma)/2} g\|_{p,r;b_n B}$$

for all $f \in B^{\sigma,b}_{(p,r;b_n B),s}$, $g \in H^{\kappa+\sigma}_{p^*,s;B}$ and $t > 0$. Hence, for all $f \in B^{\sigma,b}_{(p,r;b_n B),s}$ and $t > 0$,

$$\omega_\kappa(f,t)_{L_{p^*,s;B}} \lesssim K_0(f, t^\kappa b(1/t); B^{\sigma,b}_{(p,r;b_n B),s}, H^{\kappa+\sigma}_{p,r;b_n B}). \tag{6.19}$$

If we change the variable t^κ to $t^{1-\theta}$, with $\theta = \sigma/(\kappa+\sigma)$, set $b_0(t) := b(t^{-(1-\theta)/\kappa})$, and observe that $B^{\sigma,b}_{(p,r;b_n B),s} = (L_{p,r;b_n B}, H^{\kappa+\sigma}_{p,r;b_n B})_{\theta,s;b_0}$, then we can use the Holmstedt formula

$$K_0(f, t^{1-\theta} b_0(t); (X,Y)_{\theta,s;b_0}, Y) \approx \Big(\int_0^t [u^{-\theta} b_0(u) K_0(f,u;X,Y)]^s \frac{du}{u} \Big)^{1/s},$$

with $X = L_{p,r;b_n B}$ and $Y = H^{\kappa+\sigma}_{p,r;b_n B}$, which is proved in [31, Theorem 3.1 c)]. Thus,

$$\begin{aligned} &K_0(f, t^{1-\theta} b_0(t); B^{\sigma,b}_{(p,r;\,b_n B),s}, H^{\kappa+\sigma}_{p,r;\,b_n B}) \\ &\approx \Big(\int_0^t [u^{-\theta} b_0(u) K_0(f,u; L_{p,r;\,b_n B}, H^{\kappa+\sigma}_{p,r;\,b_n B})\,]^s \frac{du}{u} \Big)^{1/s} \end{aligned}$$

or, after the substitution $u = v^{\kappa+\sigma}$ under the integral sign and after cancelling the change of the variable t, one obtains

$$\begin{aligned} &K_0(f, t^\kappa b(1/t); B^{\sigma,b}_{(p,r;\,b_n B),s}, H^{\kappa+\sigma}_{p,r;\,b_n B}) \\ &\approx \Big(\int_0^t [v^{-\sigma} b(1/v) K_0(f, v^{\kappa+\sigma}; L_{p,r;\,b_n B}, H^{\kappa+\sigma}_{p,r;\,b_n B})\,]^s \frac{dv}{v} \Big)^{1/s}, \end{aligned}$$

which together with (6.19) and (6.14), implies the assertion of the theorem. □

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