



# Locally free Caldero–Chapoton functions via reflections

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## Abstract

We study the reflections of locally free Caldero–Chapoton functions associated to representations of Geiß–Leclerc–Schröer’s quivers with relations for symmetrizable Cartan matrices. We prove that for rank 2 cluster algebras, non-initial cluster variables are expressed as locally free Caldero–Chapoton functions of locally free indecomposable rigid representations. Our method gives rise to a new proof of the locally free Caldero–Chapoton formulas obtained by Geiß–Leclerc–Schröer in Dynkin cases. For general acyclic skew-symmetrizable cluster algebras, we prove the formula for any non-initial cluster variable obtained by almost sink and source mutations.

**Keywords** Cluster algebras · Caldero–Chapoton functions · Locally free representations

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## 1 Introduction

Cluster algebras are invented by Fomin and Zelevinsky [12] in connection with dual canonical bases and total positivity. A cluster algebra  $\mathcal{A}(B)$  associated to a skew-symmetrizable matrix  $B$  is a subalgebra of  $\mathbb{Q}(x_1, \dots, x_n)$  generated by a distinguished set of generators called *cluster variables* obtained by certain iterations called *mutations*. A first remarkable feature is that they turn out to be Laurent polynomials with integer coefficients. Much effort has been taken to give formulas or interpretations of these Laurent polynomials since the invention of cluster algebras.

The classification of finite type cluster algebras is identical to the Cartan–Killing classification of finite root systems [11]. In particular, non-initial cluster variables are naturally in bijection with positive roots of the corresponding root system. Meanwhile, Gabriel’s theorem [15] states that the indecomposable representations of a Dynkin quiver are in bijection with positive roots, thus further in bijection with non-initial cluster variables. Caldero and Chapoton [5] showed that any non-initial cluster variable can be obtained directly from its corresponding quiver representation as the generating function of Euler characteristics of quiver Grassmannians of subrepresentations, which we now call the Caldero–Chapoton function.

Caldero and Keller [4] have extended the above correspondence to cluster algebras associated to acyclic quivers, that is, non-initial cluster variables of  $\mathcal{A}(Q)$  are in bijection with real Schur roots in the root system associated to  $Q$ , and are again equal to the Caldero–Chapoton functions of the corresponding indecomposable rigid representations.

Geiß, Leclerc and Schröer [20] have defined a class of Iwanaga–Gorenstein algebras  $H$  associated to acyclic skew-symmetrizable matrices, generalizing the path algebras of acyclic quivers. These algebras are defined over arbitrary fields so certain geometric constructions valid for quivers carry over to them. The authors introduced *locally free Caldero–Chapoton functions* for locally free  $H$ -modules and showed that in Dynkin cases those of locally free indecomposable rigid modules are exactly non-initial cluster variables [16]. Their proof however does not explicitly interpret mutations of cluster variables in terms of representations but actually relies on [19] a realization of the positive part of the enveloping algebra of a simple Lie algebra using locally free  $H$ -modules and a known connection between cluster algebras of Dynkin types and (dual) enveloping algebras [32].

In this paper, we study the recursion of locally free Caldero–Chapoton functions of modules under reflection functors. These functors, introduced in [20] for  $H$ -modules, generalize the classical Bernstein–Gelfand–Ponomarev reflection functors [2] for representations of Dynkin quivers. We show that this recursion coincides with cluster mutations that happen at a sink or source, leading to our main results:

- (1) Non-initial cluster variables of a rank 2 cluster algebra are exactly locally free Caldero–Chapoton functions of locally free indecomposable rigid  $H$ -modules.
- (2) In Dynkin cases, we obtain a new proof of the aforementioned correspondence in [16] which does not rely on results in [19] and [32].
- (3) In general, any non-initial cluster variable obtained from almost sink and source mutations is expressed as the locally free Caldero–Chapoton function of a unique locally free indecomposable rigid  $H$ -module.

We next provide a more detailed summary of this paper.

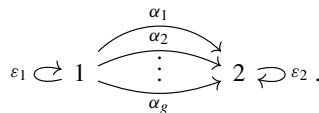
### 1.1 Rank 2 cluster algebras

Let  $b$  and  $c$  be two non-negative integers. The *cluster algebra*  $\mathcal{A}(b, c)$  is defined to be the subalgebra of  $\mathbb{Q}(x_1, x_2)$  generated by *cluster variables*  $\{x_n \mid n \in \mathbb{Z}\}$  satisfying relations

$$x_{n-1}x_{n+1} = \begin{cases} 1 + x_n^b & n \text{ is odd} \\ 1 + x_n^c & n \text{ is even.} \end{cases}$$

Every cluster variable  $x_n$  is viewed as a rational function of  $x_1$  and  $x_2$ . The cluster algebras  $\mathcal{A}(b, c)$  are said to be of rank 2 because the cardinality of each *cluster*  $\{x_n, x_{n+1}\}$  is 2.

Let  $c_1$  and  $c_2$  be two positive integers such that  $c_1b = c_2c$ . Let  $g := \text{gcd}(b, c)$ . Let  $Q$  be the quiver



Following [20], we define  $H = H(b, c, c_1, c_2)$  to be the path algebra  $\mathbb{C}\mathbb{Q}$  modulo the ideal

$$I := \langle \varepsilon_1^{c_1}, \varepsilon_2^{c_2}, \varepsilon_2^{b/g} \alpha_k - \alpha_k \varepsilon_1^{c/g} \mid k = 1, 2, \dots, g \rangle.$$

Denote by  $\text{rep } H$  the category of finitely generated left  $H$ -modules. For any  $M \in \text{rep } H$  and  $i \in \{1, 2\}$ , the subspace  $M_i := e_i M$  is a finitely generated module over the algebra  $H_i := e_i H e_i \cong \mathbb{C}[\varepsilon]/(\varepsilon^{c_i})$ . We say that  $M \in \text{rep } H$  is *locally free* (l.f. for short) if  $M_i$  is a free  $H_i$ -module for  $i = 1, 2$ . For such  $M$ , we define its *rank vector*

$$\text{rank } M := (m_1, m_2) \in \mathbb{N}^2$$

where  $m_i$  denotes the rank of  $M_i$  as a finitely generated free  $H_i$ -module. Let  $E_1$  (resp.  $E_2$ ) be the locally free module with rank vector  $(1, 0)$  (resp.  $(0, 1)$ ).

To any locally free  $M \in \text{rep } H$  with  $\text{rank } M = (m_1, m_2)$ , we associated a Laurent polynomial

$$X_M(x_1, x_2) = x_1^{-m_1} x_2^{-m_2} \sum_{\mathbf{r}=(r_1, r_2) \in \mathbb{N}^2} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) x_1^{b(m_2-r_2)} x_2^{c r_1} \in \mathbb{Z}[x_1^\pm, x_2^\pm], \quad (1.1)$$

where  $\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)$  is the *locally free quiver Grassmannian* (see Definition 4.1) which is a quasi-projective complex variety parametrizing locally free submodules of  $M$  with rank vector  $\mathbf{r}$ , and  $\chi(\cdot)$  denotes the Euler characteristic in complex analytic topology. The Laurent polynomial  $X_M$  is the *locally free Caldero–Chapoton function* associated to  $M$ .

Our first main result is

**Theorem 1.1** (Theorem 5.7) *For  $bc \geq 4$ , there is a class of locally free indecomposable rigid  $H$ -modules  $M(n)$  parametrized by  $\{n \in \mathbb{Z} \mid n \leq 0 \text{ or } n \geq 3\}$  such that*

$$X_{M(n)}(x_1, x_2) = x_n.$$

*In fact, this equality gives a bijection between all locally free indecomposable rigid  $H$ -modules (up to isomorphism) and non-initial cluster variables of  $\mathcal{A}(b, c)$ .*

**Remark 1.2** When  $bc < 4$ , the cluster variables  $x_n$  are periodic, that is, there are only finitely many distinguished  $x_n$ . These cases actually fall into another class of cluster algebras of *finite types* (or *Dynkin types*), which will be discussed in Sect. 1.3.

**Example 1.3** Let  $(b, c) = (2, 3)$  and  $(c_1, c_2) = (3, 2)$ . Then the algebra  $H = H(b, c, c_1, c_2)$  is the path algebra of the quiver  $Q$

$$\varepsilon_1 \curvearrowright 1 \xrightarrow{\alpha} 2 \curvearrowleft \varepsilon_2$$

modulo the relations  $\varepsilon_1^3 = 0$  and  $\varepsilon_2^2 = 0$ . According to the construction in Remark 5.2, we list first a few  $M(n) \in \text{rep}_{1.f.} H$  for  $n \geq 3$ .

- (1)  $M(3) = E_1$ . One calculates the only non-empty quiver Grassmannians' Euler characteristics  $\chi(\text{Gr}_{1.f.}((0, 0), E_1)) = \chi(\text{Gr}_{1.f.}((1, 0), E_1)) = 1$ . Thus  $X_{M(3)} = x_1^{-1}(1 + x_2^3) = x_3$ .
- (2)  $M(4) = I_2$ , the injective hull of  $E_2$ . It is obtained in a similar way as the module  $N$  in Example 5.5. It is easy to see that  $\chi(\text{Gr}_{1.f.}((0, 0), I_2)) = \chi(\text{Gr}_{1.f.}((2, 1), I_2)) = \chi(\text{Gr}_{1.f.}((0, 1), I_2)) = 1$  as in each case, the quiver Grassmannian consists of a single subrepresentation. We have  $\chi(\text{Gr}_{1.f.}((1, 1), I_2)) = \chi(\mathbb{P}^1) = 2$  according to Corollary 4.10. Thus

$$X_{M(4)} = x_1^{-2}x_2^{-1}(x_1^2 + x_2^6 + 1 + 2x_2^3) = x_4.$$

- (3)  $M(5)$  is calculated in Example 5.5. Computing  $X_{M(5)}$  is not so easy, but via (the proof of) Lemma 4.7, we have

$$X_{M(5)} = x_1^{-5}x_2^{-3} \left( x_1^6 + 3x_1^4(1 + x_2^3) + 3x_1^2(1 + x_2^3)^3 + (1 + x_2^3)^5 \right) = x_5.$$

### 1.2 Higher rank cluster algebras

Extending the construction of  $\mathcal{A}(b, c)$  to any  $n \times n$  integral skew-symmetrizable matrix  $B$ , there is an associated (coefficient-free) cluster algebra  $\mathcal{A}(B) \subset \mathbb{Q}(x_1, \dots, x_n)$  with the initial seed

$$\Sigma = (B, (x_1, \dots, x_n)).$$

Here we briefly review the construction by Fomin and Zelevinsky [12]. The previous definition of  $\mathcal{A}(b, c)$  in rank 2 corresponds to  $B = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$ .

Let  $\mathbb{T}_n$  be the infinite simple  $n$ -regular tree emanating from a given root  $t_0$  such that the  $n$  edges incident to any vertex are numbered by  $\{1, \dots, n\}$ . We associate  $\Sigma$  to  $t_0$ , and inductively if  $\Sigma_t = (B^t = (b_{ij}^t), (x_{1;t}, \dots, x_{n;t}))$  is associated to some vertex  $t \in \mathbb{T}_n$ , then

$$\Sigma_{t'} := \mu_k(\Sigma_t) := (\mu_k(B^t), (x_{1;t'}, \dots, x_{n;t'})) \tag{1.2}$$

is associated to  $t'$  for  $t \xrightarrow{k} t'$  in  $\mathbb{T}_n$ , where  $\mu_k(B^t)$  is Fomin–Zelevinsky's matrix mutation of  $B^t$  in direction  $k$  and

$$x_{i;t'} := x_{i;t} \text{ for } i \neq k \quad \text{and} \quad x_{k;t'} := x_{k;t}^{-1} \left( \prod_{i=1}^n x_{i;t}^{[b_{ik}^t]_+} + \prod_{i=1}^n x_{i;t}^{[-b_{ik}^t]_+} \right).$$

In this way, each  $t \in \mathbb{T}_n$  is associated with a well-defined seed  $(B^t, (x_{1;t}, \dots, x_{n;t}))$  where  $B^t$  is an  $n \times n$  integral skew-symmetrizable matrix and each rational function  $x_{i;t} \in \mathbb{Q}(x_1, \dots, x_n)$  is called a *cluster variable*. The cluster algebra  $\mathcal{A}(B)$  is then defined to be the subalgebra of  $\mathbb{Q}(x_1, \dots, x_n)$  generated by all cluster variables. The exchange between  $\Sigma_t$  and  $\Sigma_{t'}$  for  $t \xrightarrow{k} t'$  is usually called a *cluster mutation*.

### 1.3 Locally free Caldero–Chapoton formulas

Let  $(C, D, \Omega)$  be an  $n \times n$  symmetrizable Cartan matrix  $C = (c_{ij})$ , a symmetrizer  $D = \text{diag}(c_i)$ , and an acyclic orientation  $\Omega$  (see Sect. 2 for precise definitions). Geiß, Leclerc and Schröer [20] have associated a finite dimensional  $K$ -algebra  $H = H_K(C, D, \Omega)$  to the triple (where  $K$  is a field), generalizing the path algebra of an acyclic quiver. Similar to the rank 2 case, there are locally free  $H$ -modules, forming the subcategory  $\text{rep}_{\text{l.f.}} H \subset \text{rep } H$ . Analogously, each  $M \in \text{rep}_{\text{l.f.}} H$  has its rank vector  $\underline{\text{rank}} M \in \mathbb{N}^n$ . Let  $E_i$  be the locally free  $H$ -module with rank vector  $\alpha_i := (\delta_{ij} \mid j = 1, \dots, n)$ .

We define the bilinear form  $\langle -, - \rangle_H : \mathbb{Z}^n \otimes \mathbb{Z}^n \rightarrow \mathbb{Z}$  such that on the standard basis  $(\alpha_i)_{i=1}^n$ ,

$$\langle \alpha_i, \alpha_j \rangle_H = \begin{cases} c_i & \text{if } i = j, \\ c_i c_{ij} & \text{if } (j, i) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The skew-symmetrization of  $\langle -, - \rangle_H$  (on the basis  $(\alpha_i)_i$ ) defines a skew-symmetric matrix  $DB$  (thus defining a skew-symmetrizable matrix  $B = B(C, \Omega) = (b_{ij})$  actually having integer entries), i.e. explicitly, we have

$$b_{ij} = \begin{cases} c_{ij} & \text{if } (j, i) \in \Omega, \\ -c_{ij} & \text{if } (i, j) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.4** [16, Definition 1.1] For a locally free  $H_{\mathbb{C}}(C, D, \Omega)$ -module  $M$ , the associated locally free Caldero–Chapoton function is

$$X_M := \sum_{\mathbf{r} \in \mathbb{N}^n} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) \prod_{i=1}^n v_i^{-\langle \mathbf{r}, \alpha_i \rangle_H - \langle \alpha_i, \underline{\text{rank}} M - \mathbf{r} \rangle_H} \in \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}],$$

where  $v_i := x_i^{1/c_i}$ .

Suppose that  $k \in \{1, \dots, n\}$  is a sink of  $\Omega$  and let  $s_k(H) := H(C, D, s_k(\Omega))$  be the reflection of  $H$  at  $k$ . There is the (sink) reflection functor (see Sect. 3)

$$F_k^+ : \text{rep } H \rightarrow \text{rep } s_k(H),$$

which generalizes the classical BGP reflection functor on quiver representations.

The following proposition gives an algebraic identity on Caldero–Chapoton functions under reflections. It is the key recursion that makes connection with cluster mutations.

**Proposition 1.5** (Lemma 4.7 and Corollary 4.8) *Let  $M$  be a locally free  $H$ -module such that the map  $M_{k, \text{in}}$  is surjective. Then the reflection  $M' := F_k^+(M) \in \text{rep } s_k(H)$  is also locally free, and*

$$X_M(x_1, \dots, x_n) = X_{M'}(x'_1, \dots, x'_n),$$

where

$$x'_i = x_i \text{ for } i \neq k \text{ and } x'_k = x_k^{-1} \left( \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+} \right).$$

For  $B = B(C, \Omega)$ , an easy calculation shows that when  $k$  is a sink or source,  $\mu_k(B) = B(C, s_k(\Omega))$ . This hints that the recursion in Proposition 1.5 is closely related to cluster mutations as in (1.2) at sink or source, which actually leads to our next main result

**Theorem 1.6** (Theorem 7.4) *For any non-initial cluster variable  $x$  in  $\mathcal{A}(B)$  obtained by almost sink and source mutations, there is a unique locally free indecomposable rigid  $H$ -module  $M$  such that  $x = X_M$ .*

If  $C$  is of Dynkin type, it is known that any non-initial cluster variable can be obtained by almost sink and source mutations. Therefore our method provides a new proof of the following theorem of Geiß–Leclerc–Schröer [16, Theorem 1.2 (c) and (d)].

**Theorem 1.7** (Theorem 6.3) *If  $C$  is of Dynkin type, then the map  $M \mapsto X_M$  induces a bijection between isomorphism classes of locally free indecomposable rigid  $H$ -modules and the non-initial cluster variables of  $\mathcal{A}(B)$ .*

## 1.4 Other related work

Caldero and Zelevinsky [6] studied how the Caldero–Chapoton functions of representations of generalized Kronecker quivers behave under reflection functors and used them to express cluster variables of skew-symmetric rank 2 cluster algebras. Our result in rank 2 can thus be seen as a generalization to the skew-symmetrizable case.

We remark that the recursion in Proposition 1.5 has already been achieved in the skew-symmetric case for any reflection, not necessarily at sink or source, of any quiver by Derksen–Weyman–Zelevinsky [9, 10]. Extending their theory, especially obtaining Caldero–Chapoton type formulas, to the skew-symmetrizable case in full generality remains an open problem; see for example [1, 7, 22, 23, 27, 28].

There are several earlier works generalizing Caldero–Chapoton type formulas (or in the name of *cluster characters*) to the skew-symmetrizable case. Demonet [8] has obtained cluster characters for acyclic skew-symmetrizable cluster algebras by extending [18] to an equivariant version. Rupel [29, 30] has used representations of valued quivers over finite fields to obtain a quantum analogue of Caldero–Chapoton formula for quantum acyclic symmetrizable cluster algebras. The representation theories used in those work are however different from the one initiated in [20] which we follow in this paper.

Fu, Geng and Liu [14] have obtained locally free Caldero–Chapoton formulas for finite type  $C_n$  cluster algebras with respect to not necessarily acyclic clusters. In [26] with Labardini-Fragoso, we prove locally free Caldero–Chapoton formulas with respect to any cluster for cluster algebras associated to surfaces with boundary marks and orbifold points, which include finite type  $C_n$  and affine type  $\tilde{C}_n$ .

## 1.5 Organization

The paper is organized as follows. In Sect. 2, we recall the algebras  $H(C, D, \Omega)$  defined by Geiß–Leclerc–Schröer and review some necessary notions including locally free  $H$ -modules. In Sect. 3, we review the definition of reflection functors for  $H$ -modules and their properties. In Sect. 4 we study the reflections of  $F$ -polynomials of locally free modules, leading to a cluster type recursion of locally free Caldero–Chapoton functions. In Sects. 5, 6 and 7, we apply the results obtained in Sect. 4 to rank 2, Dynkin, and general cases respectively to obtain locally free Caldero–Chapoton formulas of cluster variables for skew-symmetrizable cluster algebras.

## 2 The algebras $H(C, D, \Omega)$

In this section, we review the algebras  $H(C, D, \Omega)$  defined in [20] and some relevant notions.

**2.1** Let  $(C, D, \Omega)$  be a *symmetrizable Cartan matrix*  $C$ , a *symmetrizer*  $D$  of  $C$ , and an *acyclic orientation*  $\Omega$  of  $C$ . Let  $I = \{1, \dots, n\}$ . Precisely, the matrix  $C = (c_{ij}) \in \mathbb{Z}^{I \times I}$  satisfies that

- $c_{ii} = 2$  for any  $i \in I$  and  $c_{ij} \leq 0$  for  $i \neq j$ , and
- there is some *symmetrizer*  $D = \text{diag}(c_1, \dots, c_n)$  where  $c_i \in \mathbb{Z}_{>0}$  for  $i \in I$  such that  $DC$  is symmetric.

An *orientation* of  $C$  is a subset  $\Omega \subset I \times I$  such that

- $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$  if and only if  $c_{ij} < 0$ , and
- if  $(i, j) \in \Omega$ , then  $(j, i) \notin \Omega$ .

Define  $Q^\circ = Q^\circ(C, \Omega) := (Q_0^\circ, Q_1^\circ, s, t)$  to be the quiver with

- the set of vertices  $Q_0^\circ := I$ , and
- the set of arrows  $Q_1^\circ := \{\alpha_{ij}^{(k)} : j \rightarrow i \mid (i, j) \in \Omega, k = 1, \dots, g_{ij}\}$  where  $g_{ij} := \text{gcd}(c_{ij}, c_{ji})$ .

We say  $\Omega$  *acyclic* if the quiver  $Q^\circ$  is acyclic, i.e., has no oriented cycles. Define  $Q = Q(C, \Omega)$  to be the quiver obtained from  $Q^\circ$  by adding one loop  $\varepsilon_i : i \rightarrow i$  to each vertex  $i \in I$ .

Following [20], we define (over some ground field  $K$ ) the algebra  $H := H_K(C, D, \Omega)$  to be the path algebra  $KQ$  modulo the ideal generated by the set of relations  $\mathcal{I}$  consisting of

- $\varepsilon_i^{c_i} = 0$  for  $i \in I$ ,
- $\varepsilon_i^{f_{ji}} \alpha_{ij}^{(k)} = \alpha_{ij}^{(k)} \varepsilon_j^{f_{ij}}$  for  $(i, j) \in \Omega$  and  $k = 1, \dots, g_{ij}$  where  $f_{ij} := -c_{ij}/g_{ij}$ .

The *opposite orientation* of  $\Omega$  is

$$\Omega^* := \{(i, j) \mid (j, i) \in \Omega\},$$

which clearly is an orientation of  $C$ . We denote  $H^* := H(C, D, \Omega^*)$ .

**2.2** From now on, we will always assume that  $\Omega$  is acyclic. For  $i \in I$ , let

$$H_i := e_i H e_i \cong K[\varepsilon_i]/(\varepsilon_i^{c_i}),$$

which is a subalgebra of  $H$ . For  $(i, j) \in \Omega$ , define the  $H_i$ - $H_j$ -sub-bimodule

$${}_i H_j := \langle \alpha_{ij}^{(k)} \mid k = 1, \dots, g_{ij} \rangle \subset H.$$

If  $(i, j) \in \Omega$ , then  $(j, i) \in \Omega^*$ . Consider the algebra  $H^* = H(C, D, \Omega^*)$ . We define  ${}_j H_i := {}_j (H^*)_i$ , which is an  $H_j$ - $H_i$ -bimodule.

As a right  $H_j$ -module,  ${}_i H_j$  is free of rank  $-c_{ji}$  with the basis given by

$$\{\varepsilon_i^{f_{ji}-1-f} \alpha_{ij}^{(k)} \mid 0 \leq f \leq f_{ji} - 1, 1 \leq k \leq g_{ij}\}.$$

While as a left  $H_i$ -module,  ${}_i H_j$  is free of rank  $-c_{ij}$  with the basis

$$\{\alpha_{ij}^{(k)} \varepsilon_j^f \mid 1 \leq k \leq g_{ij}, 0 \leq f \leq f_{ij} - 1\}.$$

There is then an  $H_i$ - $H_j$ -bimodule isomorphism

$$\rho : {}_i H_j \rightarrow \text{Hom}_{H_j}({}_j H_i, H_j), \quad \varepsilon_i^{f_{ji}-1-f} \alpha_{ij}^{(k)} \mapsto (\alpha_{ji}^{(k)} \varepsilon_i^f)^* \tag{2.1}$$

for  $0 \leq f \leq f_{ji} - 1$  and  $1 \leq k \leq g_{ij}$ . For more details, we refer to [20, Section 5.1].

**2.3** Let  $\text{rep } H$  denote the category of finitely generated left  $H$ -modules. We will often treat  $\text{rep } H$  as the equivalent category of quiver representations of  $Q$  satisfying relations in  $\mathcal{I}$ . For  $M \in \text{rep } H$  and  $i \in I$ , the subspace  $M_i := e_i M$  is a finitely generated module over  $H_i$ .

**Definition 2.1** We say that  $M \in \text{rep } H$  is *locally free* if for each  $i \in I$ , the  $H_i$ -module  $M_i$  is free, i.e. is isomorphic to  $H_i^{\oplus r_i}$  for some  $r_i \in \mathbb{N}$ .

Denote the full subcategory of locally free  $H$ -modules by  $\text{rep}_{\text{l.f.}} H$ . For  $M \in \text{rep}_{\text{l.f.}} H$ , define its *rank vector*

$$\text{rank } M := (r_1, \dots, r_n) \in \mathbb{Z}^I$$

where  $r_i$  stands for the rank of  $e_i M$  as a free  $H_i$ -module. Let  $E_i$  be the locally free  $H$ -module such that  $\text{rank } E_i = \alpha_i := (\delta_{ij} \mid j \in I) \in \mathbb{Z}^I$ .

We remark that  $H_i$  itself is the only indecomposable projective (also injective)  $H_i$ -module. Any indecomposable  $H_i$ -module is isomorphic to the submodule  $H_i \varepsilon_i^k \subset H_i$  for some  $k \in \{0, \dots, c_i - 1\}$ .

**2.4** Any  $M \in \text{rep } H$  is determined by the  $H_i$ -modules  $M_i$  for  $i \in I$  and the  $H_i$ -module homomorphisms

$$M_{ij} : {}_i H_j \otimes_{H_j} M_j \rightarrow M_i, \quad (\alpha_{ij}^{(k)}, m) \mapsto M_{\alpha_{ij}^{(k)}}(m)$$

for any  $(i, j) \in \Omega$ . We will later describe an  $H$ -module  $M$  by specifying the data  $(M_i, M_{ij})$ .

When there is no ambiguity, the subscript  $H_j$  under the tensor product will be omitted, hence the simplified notation  ${}_i H_j \otimes M_j$ .

### 3 Reflection functors

The Bernstein–Gelfand–Ponomarev reflection functors [2] are firstly defined to relate representations of an acyclic quiver  $Q$  with that of the reflection of  $Q$  at a sink or source vertex. These functors have been generalized to act on representations of  $H(C, D, \Omega)$  in [20]. In this section, we recall their definitions and review some useful properties.

For an orientation  $\Omega$  of  $C$  and  $i \in I$ , the *reflection of  $\Omega$  at  $i$*  is the following orientation of  $C$

$$s_i(\Omega) := \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cup \{(s, r) \in \Omega^* \mid i \in \{r, s\}\}.$$

We denote  $s_i(H) := H(C, D, s_i(\Omega))$ . Denote

$$\Omega(i, -) := \{j \in I \mid (i, j) \in \Omega\} \quad \text{and} \quad \Omega(-, i) := \{j \in I \mid (j, i) \in \Omega\}.$$

A vertex  $i \in I$  is called a sink (resp. source) of  $\Omega$  if it is a sink (resp. source) of the quiver  $Q^\circ$ , i.e.  $\Omega(-, i) = \emptyset$  (resp.  $\Omega(i, -) = \emptyset$ ). The only cases we will need are reflections at a sink or source.



### 3.1 Sink reflection

Let  $k$  be a sink of  $\Omega$ . In this subsection we define the *sink reflection functor*

$$F_k^+ : \text{rep } H \rightarrow \text{rep } s_k(H).$$

Denote  $M_{(k,-)} := \bigoplus_{j \in \Omega(k,-)} {}_k H_j \otimes M_j$ . Consider the  $H_k$ -module morphism

$$M_{k,\text{in}} := (M_{kj})_j : M_{(k,-)} \longrightarrow M_k.$$

Let  $N_k$  be  $\ker M_{k,\text{in}}$  as an  $H_k$ -module. Denote the inclusion  $N_k \subset M_{(k,-)}$  by

$$(\beta_{jk})_j : N_k \rightarrow M_{(k,-)}, \quad \beta_{jk} : N_k \rightarrow {}_k H_j \otimes M_j.$$

The isomorphism  $\rho : {}_k H_j \rightarrow \text{Hom}_{H_j}({}_j H_k, H_j)$  (2.1) induces an isomorphism

$$\rho : {}_k H_j \otimes M_j \rightarrow \text{Hom}_{H_j}({}_j H_k, M_j).$$

Then further by the tensor-hom adjunction, we have

$$\text{Hom}_{H_k}(N_k, {}_k H_j \otimes M_j) \cong \text{Hom}_{H_j}({}_j H_k \otimes N_k, M_j),$$

under which  $\beta_{jk}$  corresponds explicitly to the map

$$N_{jk} : {}_j H_k \otimes N_k \rightarrow M_j, \quad (\alpha, n) \mapsto \langle \alpha, \rho(\beta_{jk}(n)) \rangle \quad \text{for } n \in N_k \text{ and } \alpha \in {}_j H_k. \quad (3.1)$$

Now we define  $F_k^+(M) = (N_r, N_{rs})$  with  $(r, s) \in s_k(\Omega)$ , where

$$N_r := \begin{cases} M_r, & \text{if } r \neq k, \\ N_k, & \text{if } r = k \end{cases} \quad \text{and}$$

$$N_{rs} := \begin{cases} M_{rs} & \text{if } (r, s) \in \Omega \text{ and } r \neq k, \\ N_{rk} & \text{if } (r, s) \in \Omega^* \text{ and } s = k. \end{cases}$$

For a morphism  $f = (f_i)_{i \in I} : M \rightarrow M'$  in  $\text{rep } H$ , the morphism

$$F_k^+(f) = (f'_i)_{i \in I} : F_k^+(M) \rightarrow F_k^+(M')$$

is defined by setting  $f'_i = f_i$  for  $i \neq k$  and  $f'_k$  to be naturally induced between kernels. Thus  $F_k^+$  is functorial.

### 3.2 Source reflection

For  $k$  a source of  $\Omega$ , we define the *source reflection functor*

$$F_k^- : \text{rep } H \rightarrow \text{rep } s_k(H).$$

Denote  $M_{(-,k)} := \bigoplus_{j \in \Omega(-,k)} {}_k H_j \otimes M_j$ . Consider the  $H_k$ -module morphism

$$M_{k,\text{out}} := (\overline{M}_{jk})_j : M_k \rightarrow M_{(-,k)}$$

where each component  $\overline{M}_{jk} : M_k \rightarrow {}_k H_j \otimes M_j$  for  $(k, j) \in \Omega$  is defined through the structure morphism  $M_{jk}$  as follows. In fact, by the tensor-hom adjunction, we have the canonical isomorphism

$$\text{Hom}_{H_j}({}_j H_k \otimes M_k, M_j) \cong \text{Hom}_{H_k}(M_k, \text{Hom}_{H_j}({}_j H_k, M_j)),$$

where the later is further identified with  $\text{Hom}_{H_k}(M_k, {}_kH_j \otimes M_j)$  induced by the isomorphism

$$\rho: {}_kH_j \rightarrow \text{Hom}_{H_j}({}_jH_k, H_j).$$

Hence there is some  $\overline{M}_{jk} \in \text{Hom}_{H_k}(M_k, {}_kH_j \otimes M_j)$  that  $M_{jk} \in \text{Hom}_{H_j}({}_jH_k \otimes M_k, M_j)$  corresponds to.

Let  $N_k$  be coker  $M_{k,\text{out}}$  as an  $H_k$ -module. Denote the quotient  $M_{(-,k)} \rightarrow N_k$  by

$$(N_{kj})_j: M_{(-,k)} \rightarrow N_k, \quad N_{kj}: {}_kH_j \otimes M_j \rightarrow N_k.$$

We define  $F_k^-(M) = (N_r, N_{rs})$  with  $(r, s) \in s_k(\Omega)$ , where

$$N_r := \begin{cases} M_r, & \text{if } r \neq k, \\ N_k, & \text{if } r = k \end{cases} \quad \text{and}$$

$$N_{rs} := \begin{cases} M_{rs} & \text{if } (r, s) \in \Omega \text{ and } s \neq k, \\ N_{ks} & \text{if } (r, s) \in \Omega^* \text{ and } r = k. \end{cases}$$

Analogously to  $F_k^+$ , it is clear that  $F_k^-$  is also functorial.

### 3.3 Some properties of reflection functors

For  $i \in I$ , let  $S_i$  be the simple  $H$ -module supported at the vertex  $i$ . Note that  $S_i$  is at the same time the socle and the top (or head) of  $E_i$ . The following lemma is straightforward.

**Lemma 3.1** *For any  $M \in \text{rep } H$ , we have*

$$\begin{aligned} \text{Hom}_H(M, S_k) = 0 &\Leftrightarrow \text{Hom}_H(M, E_k) = 0 \Leftrightarrow M_{k,\text{in}} \text{ is surjective, and} \\ \text{Hom}_H(S_k, M) = 0 &\Leftrightarrow \text{Hom}_H(E_k, M) = 0 \Leftrightarrow M_{k,\text{out}} \text{ is injective.} \end{aligned}$$

**Proposition 3.2** ([20, Proposition 9.1 and Corollary 9.2]) *Let  $H = H(C, D, \Omega)$  and  $k$  be a sink of  $\Omega$ . The pair of reflection functors*

$$F_k^+ : \text{rep } H \rightarrow \text{rep } s_k(H) \quad \text{and} \quad F_k^- : \text{rep } s_k(H) \rightarrow \text{rep } H$$

*are (left and right) adjoint (additive) functors. They define inverse equivalences on subcategories*

$$\begin{aligned} \mathcal{T}_k^H &:= \{M \in \text{rep } H \mid \text{Hom}_H(M, S_k) = 0\} \subset \text{rep } H \quad \text{and} \\ \mathcal{S}_k^{s_k(H)} &:= \{M \in \text{rep } s_k(H) \mid \text{Hom}_{s_k(H)}(S_k, M) = 0\} \subset \text{rep } s_k(H). \end{aligned}$$

We now focus on the actions of reflection functors on locally free modules.

**Lemma 3.3** ([17, Lemma 3.6]) *Suppose that  $k$  is a sink (resp. source) of  $\Omega$  and  $M$  a locally free rigid  $H$ -module, with no direct summand isomorphic to  $E_k$ . Then we have  $\text{Hom}_H(M, E_k) = 0$  (resp.  $\text{Hom}_H(E_k, M) = 0$ ). In particular, the map  $M_{k,\text{in}}$  (resp.  $M_{k,\text{out}}$ ) is surjective (resp. injective).*

**Proof** The case where  $k$  is a sink is [17, Lemma 3.6]. The other case is simply a dual version. □

Let  $L = \mathbb{Z}^n$ . We think of rank vectors of locally free  $H$ -modules as living in  $L$  via  $\mathbb{N}^n \subset \mathbb{Z}^n$ . For  $i \in I$ , define the reflection

$$s_k : L \rightarrow L, \quad s_k(\alpha_i) := \alpha_i - c_{ki}\alpha_k \quad \text{for any } i \in I.$$

**Proposition 3.4** ([20, Proposition 9.6] and [17, Lemma 3.5]) *If  $k$  is a sink (resp. source) of  $\Omega$  and  $M$  is a locally free rigid  $H$ -module, then  $F_k^+(M)$  (resp.  $F_k^-(M)$ ) is locally free and rigid. If furthermore  $M_{k,\text{in}}$  is surjective (resp.  $M_{k,\text{out}}$  is injective), then*

$$\text{rank} F_k^\pm(M) = s_k(\text{rank} M).$$

**Remark 3.5** We remark that the functor  $F_k^+$  (resp.  $F_k^-$ ) preserves indecomposability if  $M_{k,\text{in}}$  is surjective (resp.  $M_{k,\text{out}}$  is injective) by Proposition 3.2 and Lemma 3.1.

## 4 Locally free Caldero–Chapoton functions

### 4.1 Locally free Caldero–Chapoton functions

Let  $H = H_{\mathbb{C}}(C, D, \Omega)$ .

**Definition 4.1** For  $M \in \text{rep}_{\text{l.f.}} H$  and a rank vector  $\mathbf{r} = (r_i)_{i \in I}$ , the *locally free quiver Grassmannian* is

$$\text{Gr}_{\text{l.f.}}(\mathbf{r}, M) := \{N \mid N \text{ is a locally free submodule of } M \text{ and } \underline{\text{rank}} N = \mathbf{r}\}.$$

**Remark 4.2** It is clear that the set  $\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)$  can be realized as a locally closed subvariety of the product of ordinary Grassmannians  $\prod_{i \in I} \text{Gr}(c_i r_i, M_i)$ . We take its analytic topology and denote by  $\chi(\cdot)$  the Euler characteristic.

**Definition 4.3** For  $M \in \text{rep}_{\text{l.f.}} H$ , we define its *locally free  $F$ -polynomial* as

$$F_M(y_1, \dots, y_n) := \sum_{\mathbf{r} \in \mathbb{N}^n} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) \prod_{i=1}^n y_i^{r_i} \in \mathbb{Z}[y_1, \dots, y_n].$$

Recall that we have defined in Sect. 1.3 the bilinear form  $\langle -, - \rangle_H : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$  and the skew-symmetrizable matrix  $B = (b_{ij})$  associated to  $(C, \Omega)$ .

**Definition 4.4** For  $M \in \text{rep}_{\text{l.f.}} H$  with  $\underline{\text{rank}} M = (m_i)_{i \in I}$ , the associated *locally free Caldero–Chapoton function* is the Laurent polynomial

$$X_M := \sum_{\mathbf{r} \in \mathbb{N}^n} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) \prod_{i=1}^n v_i^{-\langle \mathbf{r}, \alpha_i \rangle_H - \langle \alpha_i, \underline{\text{rank}} M - \mathbf{r} \rangle_H} \in \mathbb{Z}[x_1^\pm, \dots, x_n^\pm],$$

where  $v_i := x_i^{1/c_i}$ .

**Remark 4.5** Using the  $F$ -polynomial  $F_M$ , the Caldero–Chapoton function  $X_M$  can be rewritten as

$$\begin{aligned} X_M &= \prod_{i=1}^n x_i^{-m_i + \sum_{j=1}^n [-b_{ij}]_+ m_j} \cdot \sum_{\mathbf{r} \in \mathbb{N}^n} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) \prod_{i=1}^n x_i^{\sum_{j=1}^n b_{ij} r_j} \\ &= \prod_{i=1}^n x_i^{-m_i + \sum_{j=1}^n [-b_{ij}]_+ m_j} \cdot F_M(\hat{y}_1, \dots, \hat{y}_n), \end{aligned}$$

where  $\hat{y}_j = \prod_{i=1}^n x_i^{b_{ij}}$ . Another way to write  $X_M$  is

$$X_M = \prod_{i=1}^n x_i^{-m_i} \cdot \sum_{\mathbf{r} \in \mathbb{N}^n} \chi(\text{Gr}_{\text{l.f.}}(\mathbf{r}, M)) \prod_{i=1}^n x_i^{\sum_{j=1}^n [-b_{ij}]_+ m_j + b_{ij} r_j}.$$

We note that every term in the summation is an actual monomial since  $[-b_{ij}]_+ m_j + b_{ij} r_j \geq 0$  because we need  $r_j \leq m_j$  for the quiver Grassmannian to be non-empty. Moreover, for  $\mathbf{r} = 0$  and  $\mathbf{r} = \text{rank} M$ ,  $\chi(\text{Gr}_{1.f.}(\mathbf{r}, M)) = 1$  and the two corresponding monomials are coprime. Therefore  $\prod_{i=1}^n x_i^{m_i}$  can be characterized as the minimal denominator when expressing  $X_M = f/g$  as a quotient of a polynomial  $f$  and a monomial  $g$ . We thus call  $\text{rank} M$  the **d-vector** of the Laurent polynomial  $X_M$ .

**Example 4.6** For  $k \in I$  and  $E_k \in \text{rep}_{1.f.} H$ , the only non-empty locally free quiver Grassmannians are  $\text{Gr}_{1.f.}(0, E_k) = \{0\}$  and  $\text{Gr}_{1.f.}(\alpha_k, E_k) = \{E_k\}$ . Thus we have

$$X_{E_k} = \prod_{i=1}^n v_i^{-(\alpha_i, \alpha_k)_H} + \prod_{i=1}^n v_i^{-(\alpha_k, \alpha_i)_H} = x_k^{-1} \left( \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+} \right).$$

### 4.2 The key recursion

The following is the key proposition on the recursion of  $F$ -polynomials under reflections.

**Proposition 4.7** *Let  $M \in \text{rep}_{1.f.} H$  be of rank  $(m_i)_{i \in I}$  and  $k$  be a sink of  $H$ . Suppose that the map*

$$M_{k,\text{in}}: \bigoplus_{j \in \Omega(k, -)} {}_k H_j \otimes M_j \rightarrow M_k$$

*is surjective. Then the reflection  $M' := F_k^+(M)$  is locally free over  $s_k(H)$  with the rank vector  $(m'_i)_{i \in I}$  such that  $m'_i = m_i$  for  $i \neq k$  and*

$$m'_k = -m_k + \sum_{j \in \Omega(k, -)} b_{kj} m_j.$$

*Their  $F$ -polynomials satisfy the equation*

$$(1 + y_k)^{-m_k} F_M(y_1, \dots, y_n) = F_{M'}(y'_1, \dots, y'_n).$$

*where*

$$y'_i = y_i y_k^{b_{ki}} / (y_k + 1)^{b_{ki}} \text{ for } i \neq k, \quad \text{and } y'_k = y_k^{-1}.$$

**Proof** The first half of the statement is simply a recast of Proposition 3.4 without the hypothesis and conclusion on the rigidity. Explicitly, we observe that  $M'_k$  and  $M_k$  naturally fit into the following exact sequence (of  $H_k$ -modules)

$$0 \longrightarrow M'_k \xrightarrow{M'_{k,\text{out}}} \bigoplus_{j \in \Omega(k, -)} {}_k H_j \otimes M_j \xrightarrow{M_{k,\text{in}}} M_k \longrightarrow 0. \tag{4.1}$$

Since  $M_j$  is free over  $H_j$  of rank  $m_j$ , we have that for any  $j \in \Omega(k, -)$ , the bimodule  ${}_k H_j \otimes M_j$  is isomorphic to  ${}_k H_j^{\oplus m_j}$ , thus a free left  $H_k$ -module of rank  $b_{kj} m_j$ . The exact sequence splits since  $M_k$  is free. Then  $M'_k$  is also free. The calculation on its rank  $m'_k$  follows from the exact sequence.

Next we prove the recursion on  $F$ -polynomials.

**Step I.** Let  $\mathbf{e} = (e_i)_i \in \mathbb{N}^I$  be a rank vector. Decompose  $\text{Gr}_{1.f.}(\mathbf{e}, M)$  into constructible subsets  $\tilde{Z}_{\mathbf{e},r}(M)$  as follows. Let  $N \subset M$  be a locally free submodule. Denote

$$N_{(k, -)} := \bigoplus_{j \in \Omega(k, -)} {}_k H_j \otimes N_j.$$

Then  $M_{k,\text{in}}(N_{(k,-)})$  is an  $H_k$ -submodule of  $M_k$ . For  $r \in \mathbb{N}$ , let  $\tilde{Z}_{\mathbf{e};r}(M)$  be the subset of  $\text{Gr}_{1,\text{f.}}(\mathbf{e}, M)$  consisting of  $N \subset M$  such that

$$\text{rank } E(M_{k,\text{in}}(N_{(k,-)})) = r.$$

where  $E(\cdot)$  denotes the injective hull (of an  $H_k$ -module). Then  $\text{Gr}_{1,\text{f.}}(\mathbf{e}, M)$  is a disjoint union of (finitely many)  $\tilde{Z}_{\mathbf{e};r}(M)$  when  $r$  runs over  $\mathbb{N}$  and thus

$$\chi(\text{Gr}_{1,\text{f.}}(\mathbf{e}, M)) = \sum_{r \in \mathbb{N}} \chi(\tilde{Z}_{\mathbf{e};r}(M)).$$

**Step II.** Meanwhile for  $M' \in \text{rep}_{1,\text{f.}} s_k(H)$ , a rank vector  $\mathbf{e}$  and  $s \in \mathbb{N}$ , let  $\tilde{X}_{\mathbf{e};s}(M')$  be the most constructible subset of  $\text{Gr}_{1,\text{f.}}(\mathbf{e}, M')$  consisting of locally free submodules  $N \subset M'$  such that

$$\text{rank } F((M'_{k,\text{out}})^{-1}(N_{(-,k)})) = s,$$

where we denote

$$N_{(-,k)} := \bigoplus_{j \in s_k(\Omega)(-,k)} {}_k H_j \otimes N_j,$$

and  $F(\cdot)$  stands for the (isomorphism class of) maximal free submodule of an  $H_k$ -module. Decomposing  $\text{Gr}_{1,\text{f.}}(\mathbf{e}, M')$  into subsets  $\tilde{X}_{\mathbf{e};s}(M')$  where  $s$  runs over  $\mathbb{N}$ , we have

$$\chi(\text{Gr}_{1,\text{f.}}(\mathbf{e}, M')) = \sum_{s \in \mathbb{N}} \chi(\tilde{X}_{\mathbf{e};s}(M')).$$

**Step III.** Let  $\mathbf{e}'$  denote the rank vector (with  $n - 1$  entries) obtained from  $\mathbf{e}$  by forgetting the  $k$ -th component. Define for  $r \in \mathbb{N}$  the subset

$$Z_{\mathbf{e}';r}(M) \subset \prod_{i \neq k} \text{Gr}_{1,\text{f.}}(e_i, M_i) := \prod_{i \neq k} \{(N_i)_{i \neq k} \mid N_i \subset M_i, \ N_i \text{ is free of rank } e_i\}$$

such that

- (1)  $(N_i)_{i \neq k}$  is closed under the actions of arrows in (the quiver of)  $H$  that are not incident to  $k$ ;
- (2) The injective hull of  $M_{k,\text{in}}(N_{(k,-)})$  is of rank  $r$ .

There is the natural forgetful map

$$\pi : \tilde{Z}_{\mathbf{e};r}(M) \rightarrow Z_{\mathbf{e}';r}(M), \quad N \mapsto (N_i)_{i \neq k}.$$

The fiber over a point  $(N_i)_{i \neq k} \in Z_{\mathbf{e}';r}(M)$  is

$$\mathcal{V} = \mathcal{V}(e_k, M_{k,\text{in}}(N_{(k,-)}), M_k) := \{N_k \mid M_{k,\text{in}}(N_{(k,-)}) \subset N_k \subset M_k, \ N_k \text{ is free of rank } e_k\}.$$

Notice that  $\mathcal{V}$  is determined (up to isomorphism) by the isomorphism class of  $M_{k,\text{in}}(N_{(k,-)})$  as  $H_k$ -module. This means we can if necessary decompose  $Z_{\mathbf{e}';r}(M)$  further into (finitely many) locally closed subsets  $\bigsqcup_{j \in J} Z_j$  so that each  $\pi^{-1}(Z_j) \xrightarrow{\pi} Z_j$  is a fiber bundle. It is shown in Proposition 4.9 that the Euler characteristic of a fiber  $\mathcal{V}$  is  $\binom{m_k - r}{e_k - r}$ . Therefore we have

$$\chi(\tilde{Z}_{\mathbf{e};r}(M)) = \sum_{j \in J} \chi(\pi^{-1}(Z_j)) = \sum_{j \in J} \binom{m_k - r}{e_k - r} \chi(Z_j) = \binom{m_k - r}{e_k - r} \chi(Z_{\mathbf{e}';r}(M)). \tag{4.2}$$

For  $M'$  and  $s \in \mathbb{N}$ , we define the subset

$$X_{\mathbf{e}',s}(M') \subset \prod_{i \neq k} \text{Gr}_{\text{l.f.}}(e_i, M'_i) := \prod_{i \neq k} \{(N_i)_{i \neq k} \mid N_i \subset M'_i, \ N_i \text{ is free of rank } e_i\}$$

such that

- (1)  $(N_i)_{i \neq k}$  is closed under the actions of arrows in (the quiver of)  $s_k(H)$  not incident to  $k$ ;
- (2) The rank of a maximal free  $H_k$ -submodule of  $(M'_{k,\text{out}})^{-1}(N_{(-,k)})$  is  $s$ .

Then we have the forgetful map

$$\rho: \tilde{X}_{\mathbf{e};s}(M') \rightarrow X_{\mathbf{e}',s}(M'), \quad N \mapsto (N_i)_{i \neq k},$$

whose fiber at  $(N_i)_{i \neq k} \in X_{\mathbf{e}',s}(M')$  is

$$\begin{aligned} \mathcal{W} &= \mathcal{W}(e_k, (M'_{k,\text{out}})^{-1}(N_{(-,k)})) \\ &:= \{N_k \mid N_k \subset (M'_{k,\text{out}})^{-1}(N_{(-,k)}) \subset M'_k, \ N_k \text{ is free of rank } e_k\}, \end{aligned}$$

having Euler characteristic, according to Corollary 4.10,  $\chi(\mathcal{W}) = \binom{s}{e_k}$ . Analogous to (4.2), we have

$$\chi(\tilde{X}_{\mathbf{e};s}(M')) = \binom{s}{e_k} \chi(X_{\mathbf{e}',s}(M')). \tag{4.3}$$

**Step IV.** Now recall that  $M$  and  $M'$  are reflections of each other, i.e.  $M' = F_k^+(M)$ , with rank vectors  $(m_i)_i$  and  $(m'_i)_i$  respectively such that

$$m_i = m'_i \text{ for } i \neq k \quad \text{and} \quad m_k + m'_k = \sum_{j \in \Omega(k,-)} b_{kj} m_j.$$

We claim that for any  $\mathbf{e}' = (e_i)_{i \neq k}$

$$X_{\mathbf{e}',s}(M') = Z_{\mathbf{e}',r}(M) \quad \text{for } r + s = \sum_{j \in \Omega(k,-)} b_{kj} e_j. \tag{4.4}$$

In fact, let  $N_i$  be a free submodule (of rank  $e_i$ ) of  $M_i$  for any  $i \neq k$  and then we have from (4.1) the following short exact sequence of  $H_k$ -modules

$$0 \longrightarrow (M'_{k,\text{out}})^{-1}(N_{(-,k)}) \longrightarrow N_{(k,-)} \longrightarrow M_{k,\text{in}}(N_{(k,-)}) \longrightarrow 0,$$

where by our abuse of notation, the  $H_k$ -module  $N_{(-,k)}$  (which is for the orientation  $s_k(\Omega)$ ) is actually the same as  $N_{(k,-)}$  for  $\Omega$ . Denote  $A = (M'_{k,\text{out}})^{-1}(N_{(-,k)})$ ,  $B = N_{(k,-)}$ , and  $C = M_{k,\text{in}}(N_{(k,-)})$ . The injection  $A \rightarrow B$  then factors through the injective hull  $E(A)$  of  $A$  as  $B$  is free and thus injective. Thus  $C \cong E(A)/A \oplus B/E(A)$  where  $B/E(A)$  is free. The number of indecomposable summands of  $E(A)/A$  is easily seen to be  $\text{rank } E(A) - \text{rank } F(A)$ . The number of indecomposable summands of  $C$ , which equals  $\text{rank } E(C)$ , is just  $\text{rank } B - \text{rank } F(A)$ . We now have the equality

$$\text{rank } E(M_{k,\text{in}}(N_{(k,-)})) + \text{rank } F((M'_{k,\text{out}})^{-1}(N_{(-,k)})) = \text{rank } N_{(k,-)} = \sum_{j \in \Omega(k,-)} b_{kj} e_j.$$

Then one sees from their definitions that  $X_{\mathbf{e}',s}(M')$  and  $Z_{\mathbf{e}',r}(M)$  for any  $r$  and  $s$  such that  $r + s = \text{rank } N_{(k,-)}$  define the exact same tuples  $(N_i)_{i \neq k}$  in  $\prod_{i \neq k} \text{Gr}_{\text{l.f.}}(e_i, M_i)$ , hence (4.4).

Now we rewrite the  $F$ -polynomials as

$$\begin{aligned}
 F_M(y_1, \dots, y_n) &= \sum_{\mathbf{e} \in \mathbb{N}^n} \sum_{r \in \mathbb{N}} \chi(\tilde{Z}_{\mathbf{e};r}(M)) y^{\mathbf{e}} \\
 &\stackrel{(4.2)}{=} \sum_{\mathbf{e} \in \mathbb{N}^n} \sum_{r \in \mathbb{N}} \binom{m_k - r}{e_k - r} \chi(Z_{\mathbf{e}';r}(M)) y^{\mathbf{e}} \\
 &= \sum_{\mathbf{e}' \in \mathbb{N}^{n-1}} \sum_{r \in \mathbb{N}} \chi(Z_{\mathbf{e}';r}(M)) y_k^r (1 + y_k)^{m_k - r} \prod_{i \neq k} y_i^{e_i}, \\
 F_{M'}(z_1, \dots, z_n) &= \sum_{\mathbf{e} \in \mathbb{N}^n} \sum_{s \in \mathbb{N}} \chi(\tilde{X}_{\mathbf{e};s}(M')) z^{\mathbf{e}} \\
 &\stackrel{(4.3)}{=} \sum_{\mathbf{e} \in \mathbb{N}^n} \sum_{s \in \mathbb{N}} \binom{s}{e_k} \chi(X_{\mathbf{e}';s}(M')) z^{\mathbf{e}} \\
 &= \sum_{\mathbf{e}' \in \mathbb{N}^{n-1}} \sum_{s \in \mathbb{N}} \chi(X_{\mathbf{e}';s}(M')) (1 + z_k)^s \prod_{i \neq k} z_i^{e_i}.
 \end{aligned}$$

Let  $z_i = y'_i := y_i y_k^{b_{ki}} / (y_k + 1)^{b_{ki}}$  for  $i \neq k$  and  $z_k = y'_k := y_k^{-1}$ . Finally we have

$$\begin{aligned}
 F_{M'}(y'_1, \dots, y'_n) &= \sum_{\mathbf{e}' \in \mathbb{N}^{n-1}} \sum_{s \in \mathbb{N}} \chi(X_{\mathbf{e}';s}(M')) \left( \frac{y_k}{1 + y_k} \right)^{-s + \sum_{j \neq k} b_{kj} e_j} \prod_{i \neq k} y_i^{e_i} \\
 &\stackrel{(4.4)}{=} \sum_{\mathbf{e}' \in \mathbb{N}^{n-1}} \sum_{r \in \mathbb{N}} \chi(Z_{\mathbf{e}';r}(M)) \left( \frac{y_k}{1 + y_k} \right)^r \prod_{i \neq k} y_i^{e_i} \\
 &= (1 + y_k)^{-m_k} F_M(y_1, \dots, y_n).
 \end{aligned}$$

□

**Corollary 4.8** *In the setting of the above Lemma 4.7, we have*

$$X_M(x_1, \dots, x_n) = X_{M'}(x'_1, \dots, x'_n),$$

where  $x'_i = x_i$  for  $i \neq k$  and

$$x'_k = x_k^{-1} \left( \prod_{i \in I} x_i^{[b_{ik}]_+} + \prod_{i \in I} x_i^{[-b_{ik}]_+} \right) = x_k^{-1} \left( 1 + \prod_{i \in \Omega(k, -)} x_i^{-b_{ik}} \right).$$

**Proof** We first derive from Lemma 4.7 that

$$(1 + \hat{y}_k)^{-m_k} F_M(\hat{y}_1, \dots, \hat{y}_n) = F_{M'}(\hat{y}'_1, \dots, \hat{y}'_n),$$

where  $\hat{y}_j := \prod_{i=1}^n x_i^{b_{ij}}$  and  $\hat{y}'_j := \prod_{i=1}^n (x'_i)^{b'_{ij}}$  where  $(b'_{ij}) := \mu_k(B)$  or explicitly  $b'_{ij} = -b_{ij}$  if  $i = k$  or  $j = k$  and  $b'_{ij} = b_{ij}$  otherwise. In fact, this equality directly follows from the algebraic equations

$$y'_j |_{y_i \leftarrow \hat{y}_i, i=1, \dots, n} = \hat{y}'_j, \quad j \in I.$$

Then we spell out the two sides of the desired equation in the form of Remark 4.5. Now it amounts to show

$$\prod_{i=1}^n x_i^{-m_i + \sum_{j=1}^n [-b_{ij}]_+ m_j} = (1 + \hat{y}_k)^{-m_k} \prod_{i=1}^n (x'_i)^{-m'_i + \sum_{j=1}^n [-b'_{ij}]_+ m'_j},$$

which is straightforward to check. □

We finish this section by proving the following proposition (and Corollary 4.10) which has been used in the proof of Lemma 4.7.

**Proposition 4.9** *Let  $M$  be a (finitely generated)  $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ -module whose (any) maximal free submodule  $F(M)$  is of rank  $m$ . Let  $L \subset M$  be a submodule such that  $E(L)$  the injective hull of  $L$  is of rank  $\ell$ . Assume further that  $L$  is contained in a free submodule of  $M$ . Then for any integer  $e$  with  $\ell \leq e \leq m$ , the variety*

$$\mathcal{V} = \mathcal{V}(e, L, M) := \{N \mid L \subset N \subset M, \quad N \text{ is free of rank } e\}$$

has Euler characteristic  $\binom{m-\ell}{e-\ell}$ .

**Proof** By the assumption that  $L$  is contained in some free submodule of  $M$ , we can realize the injective hull  $E(L)$  such that  $L \subset E(L) \subset M$ . Then one can decompose  $M$  into indecomposable direct summands  $M = \bigoplus_{a \in A} M_a$  such that  $E(L) = \bigoplus_{a \in C} M_a$  where  $C$  is a subset of the maximal subset  $B \subset A$  such that for any  $a \in B$ , the summand  $M_a$  is free (of rank 1). Hence we have  $|C| = \ell$  and  $|B| = m$ . The decomposition can be particularly arranged so that  $L = \bigoplus_{a \in C} (L \cap M_a)$  (each summand being non-zero) since the injective hull is additive.

To compute the Euler characteristic of  $\mathcal{V} = \mathcal{V}(e, L, M)$ , we define an algebraic  $\mathbb{C}^*$ -action on  $\mathcal{V}$  and use the well-known fact [3, Corollary 2] that the Euler characteristic  $\chi(\mathcal{V})$  equals that of the fixed point subset. We first let the multiplicative group  $\mathbb{C}^*$  act on  $M = \bigoplus_{a \in A} M_a$  with distinct weights on different summands  $M_a$ , i.e. for any  $z \in \mathbb{C}^*$  and  $(m_a)_{a \in A} \in M$ ,

$$z \cdot (m_a)_{a \in A} = (z^{\lambda_a} m_a)_{a \in A} \in M$$

where  $(\lambda_a)_{a \in A} \in \mathbb{Z}^A$  are pairwise different. Since the  $\mathbb{C}^*$ -action commutes with the action of  $\varepsilon$ , it induces an action of  $\mathbb{C}^*$  on the set of all submodules of  $M$  preserving their isomorphism types. Also notice that a submodule  $N \subset M$  is fixed if and only if  $N = \bigoplus_{a \in A} (N \cap M_a)$ . Therefore the submodule  $L$  is fixed by the action and the  $\mathbb{C}^*$  acts on  $\mathcal{V}$ . Now the fixed points in  $\mathcal{V}$  are exactly those  $N = \bigoplus_{a \in D} M_a$  such that  $C \subset D \subset B$  with  $|D| = e$ , and there are  $\binom{m-\ell}{e-\ell}$  of them. Consequently  $\chi(\mathcal{V})$  equals  $\binom{m-\ell}{e-\ell}$  the number of fixed points. □

**Corollary 4.10** *Let  $M$  be a  $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ -module whose (any) maximal free submodule is of rank  $m$ . Then for any  $0 \leq e \leq m$ , the variety*

$$\mathcal{W}(e, M) := \{N \mid N \subset M, \quad N \text{ is free of rank } e\}$$

has Euler characteristic  $\binom{m}{e}$ .

**Proof** In the setting of Proposition 4.9, letting  $L = 0$ , the result follows. □

### 5 The rank 2 case

The purpose of this section is to prove Theorem 1.1 (Theorem 5.7). In fact, the construction of the algebra  $H$  in Sect. 1.1 can be seen as within the general framework introduced in Sect. 2, which we explain in below.

Let  $I = \{1, 2\}$ . Let  $C$  be the Cartan matrix

$$\begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix} \quad \text{for } b, c \in \mathbb{Z}_{\geq 0}.$$



There are only two possible orientations, i.e.  $\Omega = \{(1, 2)\}$  or  $\{(2, 1)\}$ . We take  $\Omega = \{(2, 1)\}$ . The associated matrix  $B$  is  $\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ . Assume in the rest of this section that  $bc \geq 4$ . The rest cases are of Dynkin types which will be covered in the next Sect. 6.

Let  $D = \text{diag}(c_1, c_2)$  be a symmetrizer of  $C$ . One easily sees that  $H := H(C, D, \Omega)$  is the same as  $H(b, c, c_1, c_2)$  defined in Sect. 1.1, where we denote the arrow  $\alpha_k$  there by  $\alpha_{21}^{(k)}$ . It is clear that

$$s_1(\Omega) = s_2(\Omega) = \Omega^* = \{(1, 2)\}.$$

Denote  $H^* := H(C, D, \Omega^*)$ . Then there are reflection functors

$$F^\pm: \text{rep } H \rightarrow \text{rep } H^*, \quad F^\pm: \text{rep } H^* \rightarrow \text{rep } H.$$

We omit the subscripts in the reflection functors since the sign  $\pm$  already specifies which vertex the reflection is performed at. Next we define a class of modules obtained from iterative reflections.

**Definition 5.1** We define for  $n \geq 0$  the following  $H$ -modules

$$M(n + 3) := \begin{cases} (F^+)^n E_1 & \text{if } n \text{ is even,} \\ (F^+)^n E_2 & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad M(-n) := \begin{cases} (F^-)^n E_2 & \text{if } n \text{ is even,} \\ (F^-)^n E_1 & \text{if } n \text{ is odd.} \end{cases}$$

**Remark 5.2** Let us clarify the above construction of  $M(n + 3)$ . For any  $n \geq 0$  and  $0 \leq k \leq n$ , let

$$H^{(k)} := \begin{cases} H & \text{if } k \text{ is even,} \\ H^* & \text{if } k \text{ is odd.} \end{cases}$$

Now we have a sequence of functors  $F^{(k)} := F^+ : \text{rep } H^{(k+1)} \rightarrow \text{rep } H^{(k)}$  for  $0 \leq k \leq n - 1$ . Then  $M(n + 3)$  is obtained by iteratively applying  $F^{(k)}$ , i.e.

$$M(n + 3) := \begin{cases} F^{(0)} \circ F^{(1)} \circ \dots \circ F^{(n-1)}(E_1) & \text{if } n \text{ is even,} \\ F^{(0)} \circ F^{(1)} \circ \dots \circ F^{(n-1)}(E_2) & \text{if } n \text{ is odd.} \end{cases}$$

The modules  $M(-n)$  are defined using  $F^-$  in a similar way.

**Lemma 5.3** For any  $n \geq 0$ , the  $H$ -module  $M = M(n + 3)$  (resp.  $M(-n)$ ) is locally free, indecomposable and rigid. The map  $M_{2,\text{in}}$  (resp.  $M_{1,\text{out}}$ ) is surjective (resp. injective).

**Proof** It follows from Proposition 3.4 that any  $M(n + 3)$  or  $M(-n)$  is locally free and rigid because so is  $E_1$  or  $E_2$ .

Now assume that for any  $0 \leq k \leq n$  the modules  $M(k + 3)$  and  $M(-k)$  are all indecomposable and that the map  $M_{1,\text{out}}$  is surjective for  $M(k + 3)$  and  $M_{2,\text{in}}$  is injective for any  $M(-k)$ . Denote the rank vectors by  $\alpha(n) := \text{rank } M(n)$ . Now by the construction of  $M(n + 4)$  and  $M(-(n + 1))$  and Proposition 3.4, we have that these two modules are locally free and rigid, and

$$\alpha(n + 4) = s_1 s_2 s_1 \cdots (\alpha_{(n+4)}), \quad \alpha(-(n + 1)) = s_2 s_1 s_2 \cdots (\alpha_{(-(n+1))}),$$

where  $\langle n \rangle \in \{1, 2\}$  is congruent to  $n$  modulo 2. It is then known that (in the case  $bc \geq 4$ ) both  $\alpha(n + 4)$  and  $\alpha(-(n + 1))$  are real positive roots of  $C$  (other than the simple roots  $\alpha_1$  and  $\alpha_2$ ) and in particular are strictly positive linear combinations of  $\alpha_1$  and  $\alpha_2$ ; see for example

[31, Section 3.1]. By Remark 3.5,  $M(n + 4)$  and  $M(-n + 1)$  are also indecomposable. So they cannot have any summand isomorphic to  $E_1$  or  $E_2$ . Now by Lemma 3.3, the induction is completed.  $\square$

**Remark 5.4** In fact, by [21], locally free indecomposable rigid  $H$ -modules are parametrized by their rank vectors as real Schur roots of  $C$  (depending on  $\Omega$ ). Since the rank vectors  $\alpha(n) = \text{rank}M(n)$  for  $n \leq 0$  and  $n \geq 3$  are exactly the real Schur roots (see for example [31]), we know that  $\{M(n) \mid n \leq 0 \text{ or } n \geq 3\}$  fully lists locally free indecomposable rigid  $H$ -modules.

**Example 5.5** Recall the algebra  $H$  considered in Example 1.3 where  $b = 2$  and  $c = 3$ . We calculate  $N := F_2^+(E_1) \in \text{rep } s_2(H)$  as follows. First, we have

$$N_1 = (E_1)_1 = H_1 \quad \text{and} \quad N_2 = {}_2H_1 \otimes H_1 = (H_2 \cdot \alpha_{21}) \oplus (H_2 \cdot \alpha_{21}\varepsilon_1) \oplus (H_2 \cdot \alpha_{21}\varepsilon_1^2).$$

The structure map  $N_{12}: {}_1H_2 \otimes N_2 \rightarrow N_1$  is given by

$$\alpha_{12} \otimes \alpha_{21}h \mapsto 0 \quad \text{and} \quad \alpha_{12} \otimes \varepsilon_2\alpha_{21}h \mapsto h$$

for any  $h \in H_1$ . Then one sees  $N \cong I_1 \in \text{rep } s_2(H)$ , which is locally free, indecomposable and rigid, and  $\text{rank}N = (1, 3) = s_2(\text{rank}E_1)$ .

We next calculate  $M = M(5) := F_1^+F_2^+(E_1) = F_1^+(N)$ . By definition,  $M_2 = N_2$  and

$$M_1 = \ker({}_1H_2 \otimes N_2 \xrightarrow{N_{12}} N_1),$$

which is a free  $H_1$ -module of rank 5 having the basis

$$\begin{aligned} e_1 &:= \alpha_{12} \otimes \alpha_{21}, & e_2 &:= \alpha_{12} \otimes \alpha_{21}\varepsilon_1, & e_3 &:= \alpha_{12} \otimes \alpha_{21}\varepsilon_1^2, \\ e_4 &:= \alpha_{12} \otimes \varepsilon_2\alpha_{21}\varepsilon_1 - \varepsilon_1\alpha_{12} \otimes \varepsilon_2\alpha_{21}, & e_5 &:= \alpha_{12} \otimes \varepsilon_2\alpha_{21}\varepsilon_1^2 - \varepsilon_1\alpha_{12} \otimes \varepsilon_2\alpha_{21}\varepsilon_1. \end{aligned}$$

Thus  $M_1 \cong \mathbb{C}^{15}$  as a vector space with the basis  $\{\varepsilon_1^k e_j \mid 1 \leq j \leq 5, 0 \leq k \leq 2\}$ . The action of  $\alpha_{21}$  on this basis is calculated in the table below (only non-zero terms shown).

|                          |                       |                            |                              |                             |   |  |   |
|--------------------------|-----------------------|----------------------------|------------------------------|-----------------------------|---|--|---|
| $\varepsilon_1^k e_j$    | $\varepsilon_1^2 e_1$ | $\varepsilon_1^2 e_2$      | $\varepsilon_1^2 e_3$        | $\varepsilon_1 e_4$         | $\varepsilon_1^2 e_4$                   | $\varepsilon_1 e_5$                      | $\varepsilon_1^2 e_5$                     |
| $M_{\alpha_{21}}(\cdot)$ | $\alpha_{21}$         | $\alpha_{21}\varepsilon_1$ | $\alpha_{21}\varepsilon_1^2$ | $-\varepsilon_2\alpha_{21}$ | $\varepsilon_2\alpha_{21}\varepsilon_1$ | $-\varepsilon_2\alpha_{21}\varepsilon_1$ | $\varepsilon_2\alpha_{21}\varepsilon_1^2$ |

For example, we have by (3.1) that

$$M_{\alpha_{21}}(\varepsilon_1 e_5) = \langle \alpha_{21}, \rho(\varepsilon_1\alpha_{12}) \rangle \otimes \varepsilon_2\alpha_{21}\varepsilon_1^2 - \langle \alpha_{21}, \rho(\varepsilon_1^2\alpha_{12}) \rangle \otimes \varepsilon_2\alpha_{21}\varepsilon_1 = -\varepsilon_2\alpha_{21}\varepsilon_1.$$

Recall the definition of the cluster variables  $x_n$  of rank 2 cluster algebras given in Sect. 1.1. We rewrite Corollary 4.8 in the current rank 2 situation.

**Corollary 5.6** *Let  $M \in \text{rep}_{l.f.} H$  such that  $M_{2,\text{in}}$  is surjective. Then  $M' := F_2^+(M) \in \text{rep } H^*$  is locally free and  $M'_{2,\text{out}}$  is injective such that  $\text{rank}M' = s_2(\text{rank}M)$ . We further have*

$$X_M(x_1, x_2) = X_{M'}(x'_1, x'_2)$$

where  $x'_1 = x_1$  and  $x'_2 = x_2^{-1}(1 + x_1^b)$ .

The following is the main result of this section.

**Theorem 5.7** *The map*

$$M \mapsto X_M \in \mathbb{Z}[x_1^\pm, x_2^\pm]$$

*induces a bijection*

$$\{M(n) \mid n \leq 0, n \geq 3\} \longleftrightarrow \{x_n \mid n \leq 0, n \geq 3\}$$

*such that  $X_{M(n)} = x_n$ . In particular, each  $x_n$  is distinct for  $n \in \mathbb{Z}$ .*

**Proof** We prove by induction on  $n$  that

$$X_{M(n+3)} = x_{n+3} \text{ and } X_{M(-n)} = x_{-n} \text{ for } n \geq 0 \tag{5.1}$$

using the recursion Corollary 5.6. For  $n = 0$ , we have  $M(0) = E_2$  and  $M(3) = E_1$ , thus by direct computation (as in Example 4.6)

$$X_{M(0)} = x_2^{-1}(1 + x_1^b) = x_0 \quad \text{and} \quad X_{M(3)} = x_1^{-1}(1 + x_2^c) = x_3.$$

Assume that the statement (5.1) is true for some  $n \geq 0$ . By the obvious symmetry between  $H$  and  $H^*$  by switching the orientation, we have by the induction hypothesis that

$$x_{n+4} = X_{M(n+3, H^*)}(x_3, x_2),$$

where the notation  $M(n + 3, H^*)$  stresses that the module  $M(n + 3, H^*)$  is constructed for  $H^*$  instead of  $H$ . We would like to apply Corollary 5.6 to  $M := M(n + 3, H^*)$  and the sink reflection functor

$$F_1^+ : \text{rep } H^* \rightarrow \text{rep } H.$$

The condition that  $M_{1, \text{in}}$  is surjective is guaranteed by Lemma 5.3 (applied to the algebra  $H^*$ ). Then by Corollary 5.6, we have

$$X_{M(n+3, H^*)}(x_3, x_2) = X_{M(n+4, H)}(x_1, x_2)$$

where  $M(n + 4, H) = F_1^+(M(n + 3, H^*))$  and  $x_1 = x_3^{-1}(1 + x_2^c)$ . Immediately we obtain

$$x_{n+4} = X_{M(n+4, H)}(x_1, x_2).$$

The proof for  $M(-n)$  for  $n \geq 0$  uses a similar induction.

Now that  $x_n = X_{M(n)}$  is a Laurent polynomial in  $x_1$  and  $x_2$ , the unique minimal common denominator (up to a scalar) is easily seen to be  $x^{\alpha(n)}$ . Since the positive roots  $\alpha(n)$  are distinct, so are the cluster variables  $x_n$ . □

**Remark 5.8** To a pair  $(b, c) \in \mathbb{Z}_{>0}^2$ , one can also associate an algebra  $H(b, c)$  defined as the path algebra  $\mathbb{C}\mathbb{Q}$  of the quiver  $Q = \varepsilon_1 \curvearrowright 1 \xrightarrow{\alpha} 2 \curvearrowleft \varepsilon_2$  modulo the relations  $\varepsilon_1^c = 0$  and  $\varepsilon_2^b = 0$ . When  $b$  and  $c$  are coprime, the algebra  $H(b, c)$  is the same as  $H(C, D, \Omega)$  for  $D = \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix}$ . However, when  $b$  and  $c$  are not coprime, the algebra  $H$  is not included in the construction of [20]. We note that Theorem 1.1 can be easily adapted to using the algebras  $H(b, c)$ . In the case  $b = c = 2$ , the algebra  $H(b, c)$  coincides with a construction in [25] where the ordinary Caldero–Chapoton functions are shown to give cluster variables of a generalized cluster algebra.

### 6 Dynkin cases

The purpose of this section is to give a new proof of Theorem 1.7 (Theorem 6.3). Let  $C$  be of Dynkin type and  $B = B(C, \Omega)$  the associated skew-symmetrizable matrix. Denote by  $\Delta_C^+$  the set of positive roots associated to  $C$ .

**Definition 6.1** A sequence  $\mathbf{i} = (i_1, \dots, i_{k+1}) \in I^{k+1}$  is called *adapted to an orientation  $\Omega$  of  $C$*  (or  $\Omega$ -adapted) if

- $i_1$  is a sink for  $\Omega$ ,
- $i_2$  is a sink for  $s_{i_1}(\Omega)$ ,
- $\vdots$
- $i_k$  is a sink for  $s_{i_{k-1}} \cdots s_{i_2} s_{i_1}(\Omega)$ .

The following lemma is well-known; see for example [24, Chapter 3].

**Lemma 6.2** *Let  $\beta$  be a positive root for  $C$  and  $\Omega$  be an orientation. Then there always exists a sequence  $\mathbf{i} = (i_1, \dots, i_{k+1})$  adapted to  $\Omega$  for  $\beta$  such that*

$$\begin{aligned} s_{i_k}(\alpha_{i_{k+1}}) &\in \Delta_C^+, \\ s_{i_{k-1}} s_{i_k}(\alpha_{i_{k+1}}) &\in \Delta_C^+, \\ &\vdots \\ \beta = s_{i_1} \cdots s_{i_{k-1}} s_{i_k}(\alpha_{i_{k+1}}) &\in \Delta_C^+. \end{aligned}$$

It is clear that for such a sequence  $\mathbf{i}$  in the above lemma,  $i_{k+1}$  must not be equal to  $i_k$ . To any sequence  $\mathbf{i} = (i_1, \dots, i_{k+1})$ , consider the following path in  $\mathbb{T}_n$

$$t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} \cdots \xrightarrow{i_k} t_k \xrightarrow{i_{k+1}} t_{k+1}.$$

Recall the cluster mutations as introduced in Sect. 1.3 which generate cluster variables. Recursively performing cluster mutations from  $t_0$  to  $t_{k+1}$  we obtain an  $n$ -tuple  $(x_{1;t_{k+1}}, \dots, x_{n;t_{k+1}})$  of cluster variables associated to  $t_{k+1}$ . We denote  $x_{\mathbf{i}} := x_{i_{k+1};t_{k+1}}$ .

Suppose that  $\mathbf{i}$  is adapted to  $\Omega$  for a positive root  $\beta$  as in Lemma 6.2. Note that  $i_\ell$  is a source for the orientation  $\Omega_\ell := s_{i_\ell} \cdots s_{i_1}(\Omega)$  for  $\ell = 1, \dots, k$ . Let  $H_\ell := H(C, D, \Omega_\ell)$  for  $\ell = 1, \dots, k$  and  $H_0 := H$ . We have source reflection functors

$$F_{i_\ell}^- : \text{rep } H_\ell \rightarrow \text{rep } H_{\ell-1}.$$

Define the module

$$M_{\mathbf{i}} := F_{i_1}^- \cdots F_{i_k}^- (E_{i_{k+1}}) \in \text{rep } H.$$

**Theorem 6.3** ([16, Theorem 1.2]) *The map  $M \mapsto X_M$  induces a bijection between isomorphism classes of locally free indecomposable rigid  $H(C, D, \Omega)$ -modules and the non-initial cluster variables of the cluster algebra  $\mathcal{A}(B)$ .*

**Proof** For  $\mathbf{i}$  adapted to  $\Omega$  for a positive root  $\beta$ , we show by induction on the length of  $\mathbf{i}$  that

$$M_{\mathbf{i}} \text{ is locally free, indecomposable and rigid with rank vector } \beta, \text{ and that } X_{M_{\mathbf{i}}} = x_{\mathbf{i}}. \quad (\star)$$

If  $\mathbf{i} = (i)$  is of length one, then  $M_{\mathbf{i}} = E_i$ . Notice that  $i$  is not necessarily a sink or source. As in Example 4.6, we have

$$X_{E_i} = x_i^{-1} \left( \prod_{j=1}^n x_j^{[b_{ji}]_+} + \prod_{j=1}^n x_j^{[-b_{ji}]_+} \right) = x(i).$$

Assume that  $(\star)$  is true for any  $\mathbf{i}$  of length no greater than  $k \in \mathbb{N}$ . Let  $\mathbf{i} = (i_\ell)_{\ell=1}^{k+1}$  and  $\mathbf{i}'$  be the sequence

$$(i_2, i_3, \dots, i_{k+1}) \in I^k,$$

which is adapted to the orientation  $s_{i_1}(\Omega)$ . By assumption, the module

$$M_{\mathbf{i}'} := F_{i_2}^- \cdots F_{i_k}^-(E_{i_{k+1}}) \in \text{rep } s_{i_1}(H)$$

is locally free, rigid and indecomposable with rank vector  $\beta' := s_{i_2} \dots s_{i_k}(\alpha_{i_{k+1}})$  and that  $X_{M_{\mathbf{i}'}} = x_{\mathbf{i}'} \in \mathcal{A}(\mu_{i_1}(B))$ . Since  $s_{i_1}(\beta') = \beta$  and  $\beta' \in \Delta^+(C)$ , the positive root  $\beta'$  cannot be a positive multiple of  $\alpha_{i_1}$ . Thus the indecomposable module  $M_{\mathbf{i}'}$  does not have any direct summand isomorphic to  $E_{i_1}$ . By Lemma 3.3, the map  $(M_{\mathbf{i}'})_{i_1, \text{out}}$  is injective. By Proposition 3.4, we have that  $M_{\mathbf{i}} = F_{i_1}^-(M_{\mathbf{i}'})$  is locally free, rigid and indecomposable with rank vector  $\beta = s_{i_1}(\beta')$ .

Applying Corollary 4.8 to  $M_{\mathbf{i}} \in \text{rep } H$  and  $M_{\mathbf{i}'} \cong F_{i_1}^+(M_{\mathbf{i}}) \in \text{rep } s_{i_1}(H)$ , we have

$$X_{M_{\mathbf{i}}}(x_1, \dots, x_{i_1}, \dots, x_n) = X_{M_{\mathbf{i}'}}(x_1, \dots, x'_{i_1}, \dots, x_n),$$

where

$$x'_{i_1} = x_{i_1}^{-1} \left( 1 + \prod_{j \in \Omega(i_1, -)} x_j^{-b_{ji_1}} \right).$$

Notice that

$$X_{M_{\mathbf{i}'}}(x_1, \dots, x'_{i_1}, \dots, x_n) = x_{\mathbf{i}'}(x_1, \dots, x'_{i_1}, \dots, x_n) = x_{\mathbf{i}}(x_1, \dots, x_{i_1}, \dots, x_n) \in \mathcal{A}(B).$$

Hence  $X_{M_{\mathbf{i}}} = x_{\mathbf{i}} \in \mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$ , which completes the induction and proves  $(\star)$ .

By [20, Theorem 1.3], the module  $M_{\mathbf{i}}$  constructed from  $\mathbf{i}$  only depends on the positive root  $\beta$  and the (thus well-defined) map  $\beta \mapsto M_{\mathbf{i}}$  induces a bijection from  $\Delta_C^+$  to locally free indecomposable rigid  $H$ -modules (up to isomorphism). Thus the formula  $x_{\mathbf{i}} = X_{M_{\mathbf{i}}}$  implies that the cluster variable  $x(\beta) := x_{\mathbf{i}}$  also only depends on  $\beta$ . In view of Remark 4.5, each  $x(\beta)$  has  $\mathbf{d}$ -vector  $\beta$ . By [11], these  $x(\beta)$  are exactly the non-initial cluster variables of  $\mathcal{A}(B)$ , hence the desired bijection.  $\square$

### 7 Beyond Dynkin and rank 2 cases

For  $(C, D, \Omega)$  which is neither of Dynkin type nor in rank 2, in general we will not be able to reach all locally free indecomposable rigid modules by reflections. In this section, we prove locally free Caldero–Chapoton formulas for cluster variables that can be obtained by *almost* sink and source mutations. In particular, any cluster variable on the bipartite belt [13] can be obtained this way.

**Definition 7.1** (cf. [29]) A sequence  $\mathbf{i} = (i_1, \dots, i_{k+1}) \in I^{k+1}$  is called *admissible* to an orientation  $\Omega$  of  $C$  if

- $i_1$  is a sink or source for  $\Omega$ ,
- $i_2$  is a sink or source for  $s_{i_1}(\Omega)$ ,
- $\vdots$
- $i_k$  is a sink or source for  $s_{i_{k-1}} \cdots s_{i_2} s_{i_1}(\Omega)$ .

Let  $B = B(C, \Omega)$  and  $\mathcal{A}(B)$  be the (coefficient-free) cluster algebra associated to  $B$ . As defined in Sect. 6, for an arbitrary sequence  $\mathbf{i}$ , there is the cluster variable  $x_{\mathbf{i}} \in \mathcal{A}(B)$  by successive cluster mutations.

**Definition 7.2** We say that the cluster variable  $x_{\mathbf{i}} \in \mathcal{A}(B)$  corresponding to a sequence  $\mathbf{i} = (i_{\ell})_{\ell}$  is obtained by *almost sink or source mutations* if  $\mathbf{i}$  is admissible to  $\Omega$ .

**Remark 7.3** We note that by definition the last index  $i_{k+1}$  can be arbitrary in  $I$ . It is the only step in the mutation sequence  $(\mu_{i_1}, \dots, \mu_{i_{k+1}})$  that may not be at a sink or source, thus the term *almost*.

The following is our main result in this section.

**Theorem 7.4** For any admissible sequence  $\mathbf{i}$ , either the cluster variable  $x_{\mathbf{i}}$  is an initial one or there is a locally free indecomposable rigid  $H(C, D, \Omega)$ -module  $M_{\mathbf{i}}$  such that

$$X_{M_{\mathbf{i}}}(x_1, \dots, x_n) = x_{\mathbf{i}}.$$

Moreover, the module  $M_{\mathbf{i}}$  is uniquely determined (up to isomorphism) by  $x_{\mathbf{i}}$ .

**Proof** We slightly modify the functors  $F_i^{\pm}$  to define the operations

$$F_i^{\pm}: \text{rep } H \cup \{x_1, \dots, x_n\} \rightarrow \text{rep } s_i(H) \cup \{x_1, \dots, x_n\}$$

such that

$$F_i^{\pm}(M) := \begin{cases} F_i^{\pm}(M) & \text{if } M \in \text{rep } H \text{ not isomorphic to } E_i, \\ x_i & \text{if } M \cong E_i, \\ E_i & \text{if } M = x_i, \\ x_j & \text{if } M = x_j \text{ for } j \neq i. \end{cases}$$

For an admissible sequence  $\mathbf{i}$ , let

$$M_{\mathbf{i}} := F_{i_1}^{\pm} F_{i_2}^{\pm} \cdots F_{i_k}^{\pm}(E_{i_{k+1}}) \in \text{rep } H \cup \{x_1, \dots, x_n\}$$

where each sign is chosen on whether  $i_{\ell}$  is a sink or source of  $s_{i_{\ell-1}} \cdots s_{i_1}(\Omega)$ . We define  $X_{M_{\mathbf{i}}}$  as in Definition 1.4 if  $M_{\mathbf{i}}$  is a locally free  $H$ -module or  $X_{M_{\mathbf{i}}} := x_{\mathbf{i}}$  if  $M_{\mathbf{i}} = x_i$  for some  $i \in \{1, \dots, n\}$ . We next show by induction on the length of  $\mathbf{i}$  that

- if  $M_{\mathbf{i}}$  is indeed a module, then it must be locally free, indecomposable and rigid, and
- $X_{M_{\mathbf{i}}} = x_{\mathbf{i}}$ .

The induction is a slight modification of the proof of Theorem 6.3. For  $\mathbf{i} = (i)$  of length one,  $M_{(i)} = E_i$  and the statement is clearly true. Assume that the statement is true for  $\mathbf{i}$  of length no greater than  $k$ . Let  $\mathbf{i} = (i_{\ell})_{\ell}$  be of length  $k + 1$  admissible to  $\Omega$  and  $\mathbf{i}' = (i_2, \dots, i_{k+1})$ , which is admissible to  $s_{i_1}(\Omega)$ . By assumption, either

- (1)  $M_{\mathbf{i}'}$   $\in$   $\text{rep } s_{i_1}(H)$  is locally free indecomposable rigid or
- (2)  $M_{\mathbf{i}'} = x_i$  for some  $i \in I$ .

In both cases, we have by assumption that  $X_{M_{i'}} = x_{i'} \in \mathcal{A}(\mu_{i_1}(B))$ .

In case (1), there are three sub-cases

- (1.1)  $M_{i'} \cong E_{i_1}$ ;
- (1.2)  $i_1$  is a sink of  $s_{i_1}(\Omega)$  and  $(M_{i'})_{i_1, \text{in}}$  is surjective by Lemma 3.3;
- (1.3)  $i_1$  is a source of  $s_{i_1}(\Omega)$  and  $(M_{i'})_{i_1, \text{out}}$  is injective by Lemma 3.3.

In case (2), there are two sub-cases

- (2.1)  $i = i_1$  and thus  $F_{i_1}^{\pm}(M_i) = E_i$ ;
- (2.2)  $i \neq i_1$  and thus  $F_{i_1}^{\pm}(M_i) = x_i$ .

By Corollary 4.8, it is easy to check that in each of the sub-cases we have

$$X_{M_i}(x_1, \dots, x_{i_1}, \dots, x_n) = X_{M_{i'}}(x'_1, \dots, x'_{i_1}, \dots, x'_n)$$

where  $x'_i = x_i$  if  $i \neq i_1$  and  $x'_{i_1} = x_{i_1}^{-1} \left( \prod_j x_j^{[b_{ji_1}]_+} + \prod_j x_j^{[-b_{ji_1}]_+} \right)$  and by Proposition 3.4 if  $M_i = F_{i_1}^{\pm}(M_{i'})$  is a module, it is locally free, indecomposable and rigid. In all cases, we have

$$X_{M_i}(x_1, \dots, x_{i_1}, \dots, x_n) = x_{i'}(x'_1, \dots, x'_{i_1}, \dots, x'_n) = x_i \in \mathcal{A}(B),$$

which finishes the induction.

By the Caldero–Chapoton formula, the rank vector  $\text{rank} M_i$  is just the  $\mathbf{d}$ -vector of  $x_i$ , thus only depending on  $x_i$ ; see Remark 4.5. It is shown in [21] that any locally free indecomposable rigid module is determined by its rank vector (which in particular is a real Schur root of  $(C, \Omega)$ ). Thus  $M_i$  is uniquely determined by  $x_i$ . □

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