Simples in a cotilting heart

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Abstract

Every cotilting module over a ring R induces a t-structure with a Grothendieck heart in the derived category D(Mod-R). We determine the simple objects in this heart and their injective envelopes, combining torsion-theoretic aspects with methods from the model theory of modules and Auslander-Reiten theory.

1 Introduction

The notion of a *t-structure* τ on a triangulated category $\mathcal{T}$ appears in the work of Beilinson, Bernstein, and Deligne [10] as a means to associate to $\mathcal{T}$ an abelian category, which then arises as the *heart* $\mathcal{H}_\tau$ of the *t*-structure. For example, if the triangulated category is the derived category D($\mathcal{A}$) of an abelian category $\mathcal{A}$, an appropriately chosen *t*-structure τ will recover, according to this process, the given abelian category $\mathcal{H}_\tau \cong \mathcal{A}$. Other choices of *t*-structure on D($\mathcal{A}$) will give rise to hearts that may be derived equivalent to $\mathcal{A}$.

The primary aim of [10] was to introduce the abelian category of *perverse sheaves* (see [25, Ch 8]) on a topological pseudomanifold X of even dimension, equipped with a stratification with no odd-dimensional strata. The triangulated category was the bounded derived category of sheaves on X and the *t*-structure was chosen to yield the perverse sheaves on X as the objects of the heart $\mathcal{H}_\tau$. As these perverse sheaves were seen to have finite length, attention naturally turned to the *simple* ones, which were determined [10, Theorem 4.3.1] by the intersection homology of the connected strata, appropriately shifted (see also [25, Theorem 8.1.8]).

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In the same spirit, our interest in this paper are the simple objects of the heart of the *HRS tilt* of a torsion pair $(\mathcal{T}, \mathcal{F})$ in the category Mod-R of modules over a ring. The HRS tilt of $(\mathcal{T}, \mathcal{F})$ is a *t*-structure τ introduced by Happel, Reiten, and Smalø [21] on the derived category D(Mod-R). The objects of the heart $\mathcal{H}_\tau$ need not all be of finite length, but Happel, Reiten and Smalø showed that $\mathcal{H}_\tau$ contains a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ whose constituent classes are equivalent to those of $(\mathcal{T}, \mathcal{F})$, with the roles reversed. When viewed with regard to the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$, every simple object of $\mathcal{H}_\tau$ is evidently either torsion or torsionfree.

We provide a *torsion theoretic* description of the simple objects of $\mathcal{H}_\tau$ using the notion of an **almost torsionfree** module (Definition 3.1) and its dual, that of an **almost torsion** module. Every torsionfree module is almost torsionfree, but there may be others, which are necessarily torsion. The dual statement also holds and we characterise in Theorem 3.6 the torsionfree simple objects of $\mathcal{H}_\tau$ to be those of the form $T[-1]$ where $T_R \in \mathcal{T} \subseteq$ Mod-R is an almost torsionfree torsion module, and the torsion simples of the heart to be the objects that correspond to almost torsion torsionfree modules.

If $C \in$ Mod-R is a 1-cotilting module, then the cotilting class $\mathcal{C} = {}^{\perp_1}C = \text{Cogen}(C)$ is a torsionfree class in Mod-R and we call the heart of the HRS tilt of the torsion pair $(\mathcal{Q}, \mathcal{C})$ a **cotilting heart.** Every cotilting heart is a Grothendieck category [14] whose subcategory $\text{Inj}(\mathcal{H}_\tau)$ of injective objects is known to be equivalent to $\text{Prod}(C) \subseteq \mathcal{C}$. As such, the injective objects of $\mathcal{H}_\tau$ are torsion, but when $\text{Prod}(C) \subseteq \mathcal{C} \subseteq$ Mod-R is regarded as a subcategory of R-modules, it consists of torsionfree modules. As cotilting modules are pure-injective, so are all the objects of $\text{Prod}(C)$. Corollary 5.12 makes use of the notion of a **neg-isolated** indecomposable pure-injective module (§5.5) from the *model theory of modules* to characterise the injective envelopes of simple objects of the heart, when they are considered as modules in the definable subcategory $\mathcal{C} \subseteq$ Mod-R. It states that they are precisely the neg-isolated indecomposable pure-injectives of $\mathcal{C}$ that belong $\text{Prod}(C)$.

Among the neg-isolated indecomposable pure-injective modules of a definable subcategory of Mod-R such as $\mathcal{C}$, there is the distinguished class of **critical** neg-isolated indecomposable pure-injectives U, determined by the property that every monomorphism $U \to V$ in Mod-R with $V \in \mathcal{C}$ is a split monomorphism (Proposition 5.16). These are the torsionfree modules that correspond to the injective envelopes of torsion simple objects of the heart. It is a general fact about definable subcategories that there exist enough critical neg-isolated indecomposables, in the sense that every torsionfree module $F \in \mathcal{C}$ may be embedded—not necessarily purely—into a direct product of critical neg-isolated indecomposables in $\mathcal{C}$. It follows that no 1-cotilting module is superdecomposable.

The characterisations of the torsionfree and torsion simple objects of a cotilting heart in terms of almost torsionfree and almost torsion modules are categorically dual and seem to give the two kinds of simple object equal status. The question of existence however does not. We call the neg-isolated indecomposable pure-injectives of $\mathcal{C}$ that correspond to injective envelopes of torsionfree simple objects of the heart **special.** In stark contrast to the critical neg-isolated indecomposables, there is a 1-cotilting module C_Λ over the Kronecker algebra Λ (Example 6.2) whose cotilting class contains no special neg-isolated indecomposable pure-injectives. In other words, every simple object of the cotilting heart of C_Λ is torsion.

All of our characterisations of the simple objects of a cotilting heart may be regarded as part of *Auslander-Reiten theory,* but only the last makes direct appeal to almost split morphisms in the module category. This final description relies on the *approximation theory* of the complete cotorsion pair $(\mathcal{C}, \mathcal{C}^\perp)$. The almost split morphisms that appear are left almost split morphisms that enjoy the **strong** uniqueness property (Definition 2.6). It is included in the following summary of all our results on the torsion simple objects of a cotilting heart.

Theorem A (Theorems 3.6 and 4.2, Proposition 4.1, Corollary 5.18)
  The following statements are equivalent for a module N.

(1) *N is isomorphic to the injective envelope of a torsion simple S in $\mathcal{H}_\tau$.*
(2) *N is a critical neg-isolated module in $\mathcal{C}$.*
(3) *There exists a short exact sequence*

$$0 \longrightarrow S \xrightarrow{\ a\ } N \xrightarrow{\ b\ } \bar{N} \longrightarrow 0$$

  in Mod-R, *where S is torsionfree, almost torsion, a is a $\mathcal{C}^{\perp_1}$-envelope, and b is a strong left almost split morphism in $\mathcal{C}$.*

A strong left almost split morphism is either a monomorphism or epimorphism (Lemma 4.3). Theorem A includes a characterisation of the torsion simple objects of the heart as the torsionfree modules that appear as *kernels* of strong left almost split morphisms in $\mathcal{C}$, while its dual, the next Theorem B, characterises the torsionfree simple objects as shifts of torsion modules that arise as *cokernels* of strong left almost split morphisms in $\mathcal{C}$.

Theorem B (Theorems 3.6 and 4.2, Proposition 4.1, Proposition 5.19)
  The following statements are equivalent for a module N.

(1) *N is isomorphic to the injective envelope of a torsionfree simple $S[-1]$ in $\mathcal{H}_\tau$.*
(2) *N is a special neg-isolated module in $\mathcal{C}$.*
(3) *There exists a short exact sequence*

$$0 \longrightarrow N \xrightarrow{\ a\ } \bar{N} \xrightarrow{\ b\ } S \longrightarrow 0$$

  in Mod-R, *where S is torsion, almost torsionfree, a is a strong left almost split morphism in $\mathcal{C}$, and b is a $\mathcal{C}$-cover.*

  The simple objects in cotilting hearts are crucial to understanding the phenomenon of mutation and to describe the lattice tors-R of torsion classes in the category mod-R of finite dimensional modules over a finite dimensional algebra R. Indeed, the simple objects in the heart $\mathcal{H}_\tau$ correspond to the arrows in the Hasse quiver of tors-R which are incident to the torsion class $\mathcal{Q} \cap$ mod-R, or equivalently, to the irreducible mutations of the cotilting module C, cf. [7, 16, 24]. In a forthcoming paper [5], we will employ Theorems A and B to obtain an explicit description of mutation of cotilting (or more generally, cosilting) modules. This will allow us to interpret mutation as an operation on the Ziegler spectrum of R which will amount to replacing critical neg-isolated summands by special ones, or viceversa.

2 Background

2.1 Notation

In this section we fix our basic notations and conventions.
  Let R be a unital associative ring. We denote the category of right R-modules by Mod-R and the category of left R-modules by R-Mod. The full subcategories of finitely presented modules are denoted mod-R and R-mod respectively. The derived category of Mod-R is denoted D(Mod-R). We abbreviate the Hom-spaces in D(Mod-R) in the following way:

$$\mathrm{Hom}_{\mathrm{D}(R)}(X, Y) := \mathrm{Hom}_{\mathrm{D(Mod}\text{-}R)}(X, Y)$$

for all complexes X, Y.

All subcategories will be strict (i.e. closed under isomorphisms) and, for a full subcategory $\mathcal{B}$, we will use the notation $B \in \mathcal{B}$ to indicate that B is an object of $\mathcal{B}$.

Let $\mathcal{X}$ be a set of objects in an additive category $\mathcal{A}$ with products. Then we use the notation $\text{Prod}(\mathcal{X})$ for the set of direct summands of products of copies of objects contained in $\mathcal{X}$. In the case where $\mathcal{A}$ is Grothendieck abelian, we will use $\text{Cogen}(\mathcal{X})$ to denote the set of subobjects of objects contained in $\text{Prod}(\mathcal{X})$. We will write $\text{Inj}(\mathcal{X})$ for the class of injective objects in the category $\mathcal{A}$ that are contained in $\mathcal{X}$. We will consider the following full perpendicular subcategories determined by a subset $I \subseteq \{0, 1\}$:

$$\mathcal{X}^{\perp_I} := \{M \in \mathcal{A} \mid \text{Ext}^i_{\mathcal{A}}(X, M) = 0 \text{ for all } X \in \mathcal{X} \text{ and } i \in I\}$$
$$^{\perp_I}\mathcal{X} := \{M \in \mathcal{A} \mid \text{Ext}^i_{\mathcal{A}}(M, X) = 0 \text{ for all } X \in \mathcal{X} \text{ and } i \in I\}.$$

In the case where $\mathcal{X} = \{X\}$, we will use the notation $X^{\perp_I}$ for $\mathcal{X}^{\perp_I}$ and $\text{Prod}(X)$ for $\text{Prod}(\mathcal{X})$ etc. Furthermore, we will often just write $\mathcal{X}^{\perp_0}$ instead of $\mathcal{X}^{\perp_{\{0\}}}$ etc.

2.2 Torsion pairs and HRS-tilts

In this subsection we introduce the notion of an HRS-tilt, due to Happel, Reiten and Smalø. The idea of their work is to produce a t-structure in the derived category $\text{D}(\text{Mod-}R)$ from a given torsion pair in Mod-R. More details about the construction and properties of this t-structure can be found in [21].

Torsion pairs, first introduced by Dickson [17], will be a central object of study in the latter sections of this article. The following is the definition of a torsion pair in an abelian category $\mathcal{A}$.

Definition 2.1 A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called a **torsion pair** if the following conditions hold.

(1) For every $T \in \mathcal{T}$ and $F \in \mathcal{F}$, we have that $\text{Hom}_{\mathcal{A}}(T, F) = 0$.
(2) For every X in $\mathcal{A}$, there exists a short exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0$$

where $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

We call $\mathcal{T}$ the **torsion class** and $\mathcal{F}$ the **torsionfree class**. If, in addition, the class $\mathcal{T}$ is closed under subobjects, then the torsion pair is called **hereditary**.

We extend the above terminology to objects: the objects T in $\mathcal{T}$ are called **torsion** and the objects F in $\mathcal{F}$ are called **torsionfree**.

The next result shows that such a torsion pair in Mod-R yields a t-structure in $\text{D}(\text{Mod-}R)$, in the sense of [10]. Note that we define our t-structure to consist of two Hom-orthogonal classes; this differs from the original definition by a shift.

Proposition 2.2 ([21, Prop. I.2.1]) *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\text{D}(\text{Mod-}R)$. *The two full subcategories*

$$\mathcal{U}_\tau = \{X \in \text{D}(\text{Mod-}R) \mid H^0(X) \in \mathcal{T}, H^i(X) = 0 \text{ for } i > 0\}$$
$$\mathcal{V}_\tau = \{X \in \text{D}(\text{Mod-}R) \mid H^0(X) \in \mathcal{F}, H^i(X) = 0 \text{ for } i < 0\}$$

of D(Mod-R) form a t-structure.

We will refer to this t-structure as the **HRS-tilt of** $(\mathcal{T}, \mathcal{F})$. It is shown in [10] that the
heart $\mathcal{H}_\tau := \mathcal{U}_\tau[-1] \cap \mathcal{V}_\tau$ of the t-structure $(\mathcal{U}_\tau, \mathcal{V}_\tau)$ is an abelian category whose short
exact sequences $0 \to X \to Y \to Z \to 0$ are given by the triangles $X \to Y \to Z \to X[1]$
of D(Mod-R) such that X, Y and Z are contained in $\mathcal{H}_\tau$. For any two objects X and Y in
$\mathcal{H}_\tau$, there are functorial isomorphisms

$$\mathrm{Hom}_{D(R)}(X, Y[i]) \cong \mathrm{Ext}^i_{\mathcal{H}_\tau}(X, Y) \text{ for } i = 0, 1.$$

Moreover, $(\mathcal{F}, \mathcal{T}[-1])$ is a torsion pair in $\mathcal{H}_\tau$ by [21, Cor. I.2.2].

We will make use of the following lemma in Sect. 3.

Lemma 2.3 *Let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* Mod-R.

(1) Let $f : X \to Y$ *be a morphism in* $\mathcal{H}_\tau$*, and let* Z *be the cone of* f *in* D(Mod-R)*. Consider
the canonical triangle*

$$K \to Z \to W \to K[1]$$

where $K \in \mathcal{U}_\tau$ *and* $W \in \mathcal{V}_\tau$*. Then*

$$\mathrm{Ker}_{\mathcal{H}_\tau}(f) = K[-1], \quad \mathrm{Coker}_{\mathcal{H}_\tau}(f) = W.$$

(2) Let $h : Y \to X$ *be an* R*-homomorphism with* $Y, X \in \mathcal{F}$*. The morphism* h *is a monomor-
phism in* $\mathcal{H}_\tau$ *if and only if* $\mathrm{Ker}(h) = 0$ *and* $\mathrm{Coker}(h) \in \mathcal{F}$*, and* h *is an epimorphism in*
$\mathcal{H}_\tau$ *if and only if* $\mathrm{Coker}(h) \in \mathcal{T}$*.*
(3) Let $h : Y \to X$ *be a* R*-homomorphism with* $Y, X \in \mathcal{T}$*. The morphism* $h[-1]$ *is a
monomorphism in* $\mathcal{H}_\tau$ *if and only if* $\mathrm{Ker}(h) \in \mathcal{F}$*, and* $h[-1]$ *is an epimorphism in* $\mathcal{H}_\tau$ *if
and only if* $\mathrm{Coker}(h) = 0$ *and* $\mathrm{Ker}(h) \in \mathcal{T}$*.*

Proof Recall that the cone of a morphism h in Mod-R has homologies $\mathrm{Ker}(h)$ in degree -1,
$\mathrm{Coker}(h)$ in degree 0, and zero elsewhere.

(1) This is a standard property of t-structures. See, for example, [18, pp.281].
(2) We know from (1) that $\mathrm{Ker}_{\mathcal{H}_\tau}(h) = 0$ if and only if the cone of h belongs to $\mathcal{V}_\tau$. This
means $\mathrm{Ker}(h) = 0$ and $\mathrm{Coker}(h) \in \mathcal{F}$. Similarly, $\mathrm{Coker}_{\mathcal{H}_\tau}(h) = 0$ if and only if the
cone of h belongs to $\mathcal{U}_\tau$, which means that $\mathrm{Coker}(h) \in \mathcal{T}$.
(3) The cone of $h[-1]$ belongs to $\mathcal{V}_\tau$ if and only if $\mathrm{Ker}(h) \in \mathcal{F}$, and it belongs to $\mathcal{U}_\tau$ if and
only if $\mathrm{Coker}(h) = 0$ and $\mathrm{Ker}(h) \in \mathcal{T}$.

$\square$

2.3 Cotilting modules and cotorsion pairs

In this paper we will focus on HRS-tilts of torsion pairs induced by cotilting modules. We
now introduce these modules and collect together some of their important properties. The
definition of a (possibly infinitely generated) cotilting module first appeared in [13], dualising
the definition of [12].

Definition 2.4 A right R-module C is called a **cotilting module** if the following three state-
ments hold.

(1) $\mathrm{Injdim}_R C \leq 1$.
(2) $\mathrm{Ext}^1_R(C^\kappa, C) = 0$ for all cardinals κ.

(3) There exists a short exact sequence $0 \to C_1 \to C_0 \to I \to 0$ where $C_i \in \mathrm{Prod}(C)$ for $i = 0, 1$ and I is an injective cogenerator of Mod-R.

We say that cotilting modules C and C' are **equivalent** if $\mathrm{Prod}(C) = \mathrm{Prod}(C')$.

In [13, Prop. 1.7], the authors show that $\mathrm{Cogen}(C) = {}^{\perp_1}C$ and, moreover, that this equality characterises cotilting modules. We call this class $\mathcal{C} := \mathrm{Cogen}(C) = {}^{\perp_1}C$ the **cotilting class** associated to C and it follows that $\tau = (\mathcal{Q}, \mathcal{C}) := ({}^{\perp_0}C, \mathrm{Cogen}(C))$ is a (faithful) torsion pair. We call the heart of the HRS-tilt of τ the associated **cotilting heart**.

We know from [14] that a cotilting heart $\mathcal{H}_\tau$ is a Grothendieck category with injective cogenerator C so, in particular, we have $\mathrm{Inj}(\mathcal{H}_\tau) = \mathrm{Prod}(C)$.

Remark 2.5 Often the term cotilting module is used for the more general notion of an n-cotilting module, which was first defined in [1]. In that context, the modules specified in Definition 2.4 are called 1-cotilting modules. Since we will not be considering n-cotilting modules for $n > 1$, we will use the term cotilting module to refer to a 1-cotilting module.

It was shown in [8] that every cotilting module is pure-injective and every cotilting class is definable (see Sects. 5.3 and 5.4 for definitions of these terms). As a consequence, the class $\mathcal{C}$ is closed under direct limits, and the cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1}) = ({}^{\perp_1}C, ({}^{\perp_1}C)^{\perp_1})$ cogenerated by C is a perfect cotorsion pair. In particular, for every module M in Mod-R, there exist special approximation sequences

$$0 \longrightarrow X \longrightarrow Y \overset{a}{\longrightarrow} M \longrightarrow 0$$
$$0 \longrightarrow M \overset{b}{\longrightarrow} X' \longrightarrow Y' \longrightarrow 0$$

such that $X, X' \in \mathcal{C}^{\perp_1}$ and $Y, Y' \in \mathcal{C}$. In particular, a is a $\mathcal{C}$-cover and b is a $\mathcal{C}^{\perp_1}$-envelope. Moreover, we have that $\mathcal{C} \cap \mathcal{C}^{\perp_1} = \mathrm{Prod}(C)$. For more details on covers, envelopes and cotorsion pairs, we refer the reader to [19].

2.4 Injective envelopes of simples and left almost split morphisms

In this section we will prove some preliminary results connecting simple objects in a cotilting heart to left almost split morphisms. Our considerations are inspired by [15].

Definition 2.6 Let $\mathcal{X}$ be an additive category. A morphism $f : X \to Y$ in $\mathcal{X}$ is called a **left almost split morphism** if it is not a split monomorphism and, for any $g : X \to Z$ that is not a split monomorphism, there exists a morphism $h : Y \to Z$ such that $g = hf$. If the morphism h is unique for every such g, then we call f a **strong** left almost split morphism.

We begin with the following general result about Grothendieck abelian categories.

Proposition 2.7 *Let $\mathcal{G}$ be a Grothendieck abelian category and, for any object M in $\mathcal{G}$, let $E(M)$ denote the injective envelope of M.*

(1) If S is a simple object, then the canonical morphism $E(S) \to E(E(S)/S)$ is a left almost split morphism in $\mathrm{Inj}(\mathcal{G})$.

(2) If $f : E \to E^+$ is a left almost split morphism in $\mathrm{Inj}(\mathcal{G})$, then the kernel $\mathrm{Ker}(f)$ is simple and the canonical embedding $\mathrm{Ker}(f) \to E$ is the injective envelope of $\mathrm{Ker}(f)$.

(3) If $f : E \to E^+$ is a left almost split morphism in $\mathrm{Inj}(\mathcal{G})$, then the canonical epimorphism $g : E \to \mathrm{Im}(f)$ is a strong left almost split morphism in $\mathcal{G}$.

(4) If $g\colon E \to \tilde{E}$ is a left almost split morphism in $\mathcal{G}$ with E in $\mathrm{Inj}(\mathcal{G})$, and $e\colon \tilde{E} \to E(\tilde{E})$
is the injective envelope of $\tilde{E}$, then $f := eg$ is a left almost split morphism in $\mathrm{Inj}(\mathcal{G})$.

Proof (1) Consider the morphism f given by the composition of the quotient $E(S) \to E(S)/S$ with the injective envelope $E(S)/S \to E(E(S)/S)$. This morphism is not a split monomorphism. Any other morphism $g\colon E(S) \to F$ in $\mathrm{Inj}(\mathcal{G})$ that is not a split monomorphism must have a non-trivial kernel K and so K necessarily contains S because S is essential in $E(S)$. It follows that g factors through f as required.

(2) Consider the kernel $0 \to K \xrightarrow{k} E \xrightarrow{f} E^+$ of f in $\mathcal{G}$. We will show that $K = \mathrm{Ker}\,(f)$ is simple. Clearly $K \neq 0$ because f is not a split monomorphism. Moreover, every non-zero subobject $G \subset K$ coincides with K, because the composition of the quotient $E \to E/G$ with the injective envelope $E/G \to E(E/G)$ of E/G is not a split monomorphism and thus factors through f.

Let $e\colon K \to E(K)$ be the injective envelope of K. Since $k\colon K \to E$ is a monomorphism and E is injective, there exists a split epimorphism $m\colon E \to E(K)$ such that $e = mk$. If m is not a monomorphism, then there exists a morphism $g\colon E^+ \to E(K)$ such that $gf = m$. This implies that $0 = gfk = mk = e$, which is a contradiction. Therefore m is an isomorphism.

(3) By (2), we have an exact sequence $0 \to S \xrightarrow{i} E \xrightarrow{f} E^+$ where i is the injective envelope of S and S is simple. Consider the short exact sequence

$$0 \to S \xrightarrow{i} E \xrightarrow{g} E/S \to 0.$$

We will show that g is a strong left almost split morphism in $\mathcal{G}$. Note that g is not a monomorphism and so cannot be a split monomorphism. Consider a morphism $a\colon E \to M$ that is not a split monomorphism. If $ai \neq 0$, then ai must be a monomorphism because S is simple. Then a is a monomorphism because i is an essential monomorphism. This implies that a splits because E is injective, but this is a contradiction. Thus $ai = 0$ and therefore a factors uniquely through the cokernel $E/S \cong \mathrm{Im}(f)$ of i, as required.

(4) Consider $f := eg$ where $e\colon \tilde{E} \to E(\tilde{E})$ is the injective envelope of $\tilde{E}$. We will show that f is a left almost split morphism in $\mathrm{Inj}(\mathcal{G})$. Firstly, f is not a split monomorphism because otherwise g is a monomorphism and therefore split (since E is injective). Let $a\colon E \to E'$ be a morphism in $\mathrm{Inj}(\mathcal{G})$ that is not a split monomorphism. As g is a left almost split morphism in $\mathcal{G}$, we have that there exists a morphism $b\colon \tilde{E} \to E'$ such that $a = bg$. Moreover, since E' is injective and e is a monomorphism, we have that there exists a morphism $c\colon E(\tilde{E}) \to E'$ such that $a = c(eg) = cf$, as required. $\square$

Remark 2.8 Following all the notation of Proposition 2.7, assume that the Grothendieck category $\mathcal{G} = \mathcal{H}_\tau$ is a cotilting heart with respect to the cotilting torsion pair $\tau = (\mathcal{Q}, \mathcal{C})$. Then, in the argument for Proposition 2.7(3), the object $\mathrm{Im}_{\mathcal{H}_\tau}(f)$ is in $\mathcal{C}$ because $\mathcal{C}$ is a torsion class. Hence g is a strong left almost split morphism in the subcategory $\mathcal{C}$. Moreover, the argument for Proposition 2.7(4) only requires that g is a left almost split morphism in $\mathcal{C}$ since the injective objects in $\mathrm{Inj}(\mathcal{H}_\tau) = \mathrm{Prod}(C)$ are contained in $\mathcal{C}$. So, every left almost split morphism $E \to \tilde{E}$ in $\mathcal{C}$ with $E \in \mathrm{Prod}(C)$ induces a left almost split morphism in $\mathrm{Inj}(\mathcal{H}_\tau)$.

Corollary 2.9 *Let $\mathcal{G}$ be a Grothendieck abelian category and let $\mathrm{Inj}(\mathcal{G})$ denote the full subcategory of injective objects in $\mathcal{G}$. The following statements are equivalent for an object E of $\mathrm{Inj}(\mathcal{G})$.*

(1) E is isomorphic to the injective envelope $E(S)$ of a simple object S in $\mathcal{G}$.
(2) There exists a left almost split morphism $f\colon E \to E^+$ in $\mathrm{Inj}(\mathcal{G})$.

(3) There exists a strong left almost split morphism $g\colon E \to \tilde{E}$ in $\mathcal{G}$.
(4) There exists a left almost split morphism $g\colon E \to \tilde{E}$ in $\mathcal{G}$.

Proposition 2.7 and Remark 2.8 yield the following corollary in the special case where $\mathcal{G}$ is a cotilting heart.

Corollary 2.10 *Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair in* Mod-R *and let* $\mathrm{Inj}(\mathcal{H}_\tau)$ *denote the full subcategory of injective objects in $\mathcal{H}_\tau$. The following statements are equivalent for an object E of* $\mathrm{Inj}(\mathcal{H}_\tau)$.

(1) E is isomorphic to the injective envelope $E(S)$ of a simple object S in $\mathcal{H}_\tau$.
(2) There exists a left almost split morphism $f\colon E \to E^+$ in $\mathrm{Inj}(\mathcal{H}_\tau)$.
(3) There exists a (strong) left almost split morphism $g\colon E \to \tilde{E}$ in $\mathcal{H}_\tau$.
(4) There exists a (strong) left almost split morphism $g\colon E \to \tilde{E}$ in the torsion class $\mathcal{C}$.

2.5 Localisation in abelian Grothendieck categories

A torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category is **hereditary** if the torsion class $\mathcal{T}$ is closed under subobjects. If the abelian category is Grothendieck, this is equivalent to the torsionfree class being closed under injective envelopes. In that case, the torsion pair $(\mathcal{T}, \mathcal{F})$ is cogenerated by its torsionfree injective objects. Define the **localisation** $\mathcal{G}/\mathcal{T}$ of $\mathcal{G}$ at $\mathcal{T}$ to be the category whose objects are the same as the objects X of $\mathcal{G}$, but denoted by $X_\mathcal{T}$. The morphisms between two objects are given by the set $\mathrm{Hom}_{\mathcal{G}/\mathcal{T}}(X_\mathcal{T}, Y_\mathcal{T}) := \varinjlim \mathrm{Hom}_\mathcal{G}(X', Y/Y')$ where X' ranges over the subobjects of X such that $X/X' \in \mathcal{T}$ and Y' ranges over the subobjects of Y such that $Y' \in \mathcal{T}$. The work of Gabriel shows that the **localisation functor** $L_\mathcal{T}\colon \mathcal{G} \to \mathcal{G}/\mathcal{T}$, $X \mapsto X_\mathcal{T}$, is the left adjoint of an adjunction

$$\mathcal{G} \underset{R_\mathcal{T}}{\overset{L_\mathcal{T}}{\rightleftarrows}} \perp \ \mathcal{G}/\mathcal{T}. \tag{2.1}$$

The adjoint property allows us to calculate hom groups in the localisation: if $X \in \mathcal{G}$ and $Y_\mathcal{T} \in \mathcal{G}/\mathcal{T}$, then $\mathrm{Hom}_{\mathcal{G}/\mathcal{T}}(X_\mathcal{T}, Y_\mathcal{T}) \cong \mathrm{Hom}_\mathcal{G}(X, R_\mathcal{T}(Y_\mathcal{T}))$.

The *left* adjoint $L_\mathcal{T}$ is exact and the *right* adjoint $R_\mathcal{T}\colon \mathcal{G}/\mathcal{T} \to \mathcal{G}$ is fully faithful. We may therefore identify the localisation category $\mathcal{G}/\mathcal{T}$ with the full subcategory of $\mathcal{G}$ given by the image of $R_\mathcal{T}$. We note that $\mathcal{G}/\mathcal{T}$ is contained in $\mathcal{F}$. Because the right adjoint of an exact functor preserves injective objects, we may regard $\mathrm{Inj}(\mathcal{G}/\mathcal{T})$ under this identification as a subcategory of $\mathrm{Inj}(\mathcal{G})$; it is precisely the subcategory $\mathrm{Inj}(\mathcal{G}/\mathcal{T}) = \mathrm{Inj}(\mathcal{G}) \cap \mathcal{F}$ of torsionfree injective objects.

As the right adjoint $R_\mathcal{T}$ is left exact, we may identify the entire localisation $\mathcal{G}/\mathcal{T}$ with the equivalent subcategory $\mathrm{Cogen}^2(\mathrm{Inj}(\mathcal{G}/\mathcal{T})) \subseteq \mathcal{G}$ consisting of the objects in $\mathcal{G}$ with a copresentation by torsionfree injectives. For more details on localisation in Grothendieck categories, we refer the reader to [29, Ch. 4].

3 Simple objects in the heart

In this section we consider the simple objects in the heart $\mathcal{H}_\tau$ of the HRS-tilt of a torsion pair $\tau = (\mathcal{T}, \mathcal{F})$ in Mod-R. Since $(\mathcal{F}, \mathcal{T}[-1])$ is a torsion pair in $\mathcal{H}_\tau$, it follows that any simple

object S in $\mathcal{H}_\tau$ is either of the form $S = F$ for some F in $\mathcal{F}$ or $S = T[-1]$ for some T in $\mathcal{T}$. In other words, the simple objects in $\mathcal{H}_\tau$ correspond to certain *modules* in Mod-R. The aim of this section is to identify these modules. We remark that our results remain valid when replacing Mod-R by an arbitrary abelian category.

Definition 3.1 Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion pair in Mod-R.
A non-zero module T is called **almost torsionfree** if the following conditions are satisfied.

(ATF1) Every proper submodule of T is contained in $\mathcal{F}$.
(ATF2) For each short exact sequence $0 \to A \to B \to T \to 0$, if B is in $\mathcal{T}$, then A is in $\mathcal{T}$.

A non-zero module F is called **almost torsion** if the following conditions are satisfied.

(AT1) Every proper quotient of F is contained in $\mathcal{T}$.
(AT2) For each short exact sequence $0 \to F \to A \to B \to 0$, if A is in $\mathcal{F}$, then B is in $\mathcal{F}$.
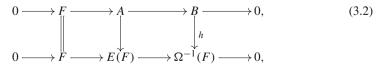
Remark 3.2 Any torsionfree module is trivially almost torsionfree and any torsion module is trivially almost torsion. The condition (ATF1) implies that if an almost torsionfree module is not torsionfree, then it must be torsion. Similarly, any almost torsion module is either torsion or torsionfree. We will consider the non-trivial cases: the module contained in $\mathcal{T}$ that are almost torsionfree and the module contained in $\mathcal{F}$ that are almost torsion. These module are also known as **torsion, almost torsionfree** and **torsionfree, almost torsion** respectively.

Example 3.3 Suppose that the torsion pair $\tau = (\mathcal{T}, \mathcal{F})$ in Mod-R is hereditary. If T is a torsion, almost torsionfree module, then (ATF1) implies that T is simple. Conversely, if $T \in \mathcal{T}$ is simple, then (ATF1) is clearly satisfied, and (ATF2) follows from the hereditary property of $\mathcal{T}$.

Next, we show that the torsionfree, almost torsion modules are precisely the modules in $\mathrm{Cogen}^2(\mathrm{Inj}(\mathcal{F})) \cong \mathrm{Mod}\text{-}R/\mathcal{T}$ which become simple in the localisation. To see that a torsionfree, almost torsion module F belongs to $\mathrm{Cogen}^2(\mathrm{Inj}(\mathcal{F}))$, take the injective envelope of F,

$$0 \longrightarrow F \xrightarrow{\ a\ } E(F) \longrightarrow \Omega^{-1}(F) \longrightarrow 0, \tag{3.1}$$

which is torsionfree by the hereditary property. Condition (AT2) implies that $\Omega^{-1}(F)$ too is torsionfree; if we take its injective envelope, we get a copresentation of $F = F_\mathcal{T}$ by torsionfree injective modules. Conversely, any module F in $\mathrm{Cogen}^2(\mathrm{Inj}(\mathcal{F}))$ satisfies condition (AT2). For, suppose w.l.o.g. that there is a short exact sequence $0 \to F \to A \to B \to 0$ with A in $\mathcal{F}$ and $B \neq 0$ in $\mathcal{T}$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0, \\
& & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & F & \longrightarrow & E(F) & \longrightarrow & \Omega^{-1}(F) & \longrightarrow & 0,
\end{array} \tag{3.2}$$

where $\Omega^{-1}(F) \in \mathcal{F}$ and thus $h = 0$. But then the upper row is split exact, a contradiction. Now it is easy to see that a module in $\mathrm{Cogen}^2(\mathrm{Inj}(\mathcal{F}))$, regarded as an object of the localisation, contains no proper subobjects, and must therefore be simple, if and only if it satisfies condition (AT1).

Finally, observe that the torsionfree, almost torsion module F is uniform, for if $M_1 \cap M_2 = 0$ are two nonzero submodules of F, then, by (AT1), the direct sum $F/M_1 \oplus F/M_2$ is a torsion module. As F embeds in a canonical way into this direct sum, the hereditary property would give the contradiction that F was also torsion. We conclude that $E(F)$ is indecomposable

and, because $\Omega^{-1}(F)$ also belongs to $\mathcal{F}$, the injective envelope $a \colon F \to E(F)$ in the short exact sequence (3.1) is a special $\mathcal{F}^{\perp_1}$-envelope of F in Mod-R (cf. Theorem 4.2(2)).

The following proposition is essentially a rephrasing of [38, Lem. 2.3].

Proposition 3.4 *Let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* D(Mod-R).

(1) The following statements are equivalent for a non-zero module T.

    (a) T is almost torsionfree.
    (b) For every exact sequence $0 \to X \to Y \overset{g}{\to} T$ with X in $\mathcal{F}$, either
        (i) Y is in $\mathcal{F}$; or
        (ii) g is a split epimorphism.

(2) The following statements are equivalent for a non-zero module F.

    (a) F is almost torsion.
    (b) For every exact sequence $F \overset{g}{\to} X \to Y \to 0$ with Y in $\mathcal{T}$, either
        (i) X is in $\mathcal{T}$; or
        (ii) g is a split monomorphism.

Proof We will prove (1), the argument for (2) is completely dual.

(1) [(a)⇒(b)] Assume T is almost torsionfree and consider an arbitrary exact sequence $0 \to X \to Y \overset{g}{\to} T$ with X in $\mathcal{F}$. The case where g is an epimorphism is covered by the dual of [38, Lem. 2.3] (noting that the argument does not require T to be in $\mathcal{T}$). It remains to consider the case where g is not an epimorphism. Then $\mathrm{Im}(g)$ is in $\mathcal{F}$ by (ATF1), so Y is in $\mathcal{F}$ because $\mathcal{F}$ is closed under extensions.

[(b)⇒(a)] Suppose T satisfies (1)(b). By the dual of [38, Lem. 2.3] (noting again that the argument does not require T to be in $\mathcal{T}$), it suffices to show that every proper subobject of T is in $\mathcal{F}$. But this follows immediately if we consider the exact sequence $0 \to 0 \to Y \to T$. $\qquad\square$

Remark 3.5 Almost torsionfree and almost torsion modules are closely related to the minimal (co)extending modules over finite-dimensional algebras introduced in [7], and also the brick labelling given in [6] for functorially finite torsion pairs and in [16] for general torsion pairs. The precise connections between these concepts are made clear in [38].

If $\tau = (\mathcal{Q}, \mathcal{C})$ is a cotilting torsion pair, then $\mathcal{C}$ is closed under direct limits, hence all torsion, almost torsionfree modules are finitely generated. On the other hand, there may be torsionfree, almost torsion modules which are not finitely generated, as Example 6.1 will show.

Theorem 3.6 *Let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* Mod-R. *The simple objects S in the heart* $\mathcal{H}_\tau$ *of the HRS-tilt of* $(\mathcal{T}, \mathcal{F})$ *are precisely those of the form $S = T[-1]$ with T torsion, almost torsionfree and $S = F$ with F torsionfree, almost torsion.*

Proof Using the canonical exact sequence $0 \to F \to S \to T[-1] \to 0$ in $\mathcal{H}_\tau$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$, we see that a simple object S is either of the form $S = F$ or $S = T[-1]$. Let us show that an object of the form $S = F$ with $F \in \mathcal{F}$ is simple if and only if F is almost torsion. The other case is proven dually.

For the only-if part, we start by considering a proper submodule U of F. Then, since $F = S$ is simple in $\mathcal{H}_\tau$, the map $h : U \to F$ gives rise to an epimorphism $h : U \to F = S$ in $\mathcal{H}_\tau$, hence the module $F/U = \mathrm{Coker}(h)$ is contained in $\mathcal{T}$ by Lemma 2.3. So, (AT1) is

verified. To prove (AT2), we consider a short exact sequence $0 \to F \xrightarrow{h} A \to B \to 0$ in Mod-R with $A \in \mathcal{F}$. Here $h : F = S \to A$ is a monomorphism in $\mathcal{H}_\tau$, and so the module $B = \mathrm{Coker}(h)$ is contained in $\mathcal{F}$ again by Lemma 2.3.

Conversely, we show that (AT1) and (AT2) imply that $S = F$ is simple. To this end, we claim that every morphism $0 \neq f : S \to A$ in $\mathcal{H}_\tau$ is a monomorphism. Since f factors through the torsion part of A with respect to the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$, we can assume that $A = C$ for some $C \in \mathcal{F}$. Then f is a morphism in Mod-R, and f is a monomorphism (by (AT1)) with cokernel in $\mathcal{F}$ (by (AT2)). But then it follows from Lemma 2.3 that $\mathrm{Ker}_{\mathcal{H}_\tau}(f) = 0$, and the claim is proven.　　　　　　　　　　　　　　　　　　　□

Corollary 3.7 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion pair in Mod-R.*

(1) Assume that T, T' are both torsion, almost torsionfree. If $g : T \to T'$ is non-zero, then g is an isomorphism.
(2) Assume that F, F' are both torsionfree, almost torsion. If $f : F \to F'$ is non-zero, then f is an isomorphism.

In particular, we have shown that the torsion, almost torsionfree modules and the torsionfree, almost torsion modules are **bricks** (i.e., their endomorphism rings are division rings).

4 Injective envelopes in a cotilting heart

In this section we will consider the case where our torsion pair $\tau = (\mathcal{Q}, \mathcal{C})$ is a cotilting torsion pair. We know from [14] that the associated cotilting heart $\mathcal{H}_\tau$ is a Grothendieck category and so, in particular, has enough injectives. Next we relate the injective envelopes of simple objects in $\mathcal{H}_\tau$ to the special approximation sequences induced by the perfect cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1}) = (^{\perp_1}\mathcal{C}, (^{\perp_1}\mathcal{C})^{\perp_1})$.

Proposition 4.1 *Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair with associated cotilting module C and cotilting heart $\mathcal{H}_\tau$.*

(1) Let $M \in \mathcal{Q}$ and consider a short exact sequence $0 \to X \xrightarrow{a} Y \xrightarrow{b} M \to 0$ in Mod-R. Let

$$X \xrightarrow{a} Y \xrightarrow{b} M \xrightarrow{c} X[1]$$

be the corresponding triangle in $\mathrm{D}(\mathrm{Mod}\text{-}R)$. The following statements are equivalent.

(a) The morphism $b : Y \to M$ is a special $\mathcal{C}$-cover of M in Mod-R.
(b) The morphism $c[-1] : M[-1] \to X$ is an injective envelope of $M[-1]$ in $\mathcal{H}_\tau$.

(2) Let $M \in \mathcal{C}$ and consider a short exact sequence $0 \to M \xrightarrow{a} X \xrightarrow{b} Y \to 0$ in Mod-R. The following statements are equivalent.

(a) The morphism $a : M \to X$ is a special $\mathcal{C}^{\perp_1}$-envelope of M in Mod-R.
(b) The morphism $a : M \to X$ is an injective envelope of M in $\mathcal{H}_\tau$.

Proof **(1)** **[(a)$\Rightarrow$(b)]** Since $\mathcal{C}$ is closed under submodules and b is a special $\mathcal{C}$-cover, $X \in \mathcal{C} \cap \mathcal{C}^{\perp_1} = \mathrm{Prod}(C)$, so X is injective in $\mathcal{H}_\tau$. Moreover, it follows from Lemma 2.3 that there is an exact sequence

$$0 \to M[-1] \xrightarrow{c[-1]} X \xrightarrow{a} Y \to 0$$

in $\mathcal{H}_\tau$.

It remains to check that $c[-1]$ is left minimal. Consider an endomorphism $h \in \mathrm{End}_{\mathcal{H}_\tau}(X)$ with $h \circ c[-1] = c[-1]$. Then there is $g \in \mathrm{End}_{\mathcal{H}_\tau}(Y)$ yielding a commutative diagram whose rows are given by triangles

$$
\begin{array}{ccccccc}
M[-1] & \xrightarrow{c[-1]} & X & \xrightarrow{a} & Y & \xrightarrow{b} & M \\
\| & & \downarrow{h} & & \downarrow{g} & & \| \\
M[-1] & \xrightarrow{c[-1]} & X & \xrightarrow{a} & Y & \xrightarrow{b} & M
\end{array}
$$

It follows that $b = b \circ g$ and hence g is an isomorphism by the minimality of b. As g is an isomorphism, we conclude that h is an isomorphism as desired.

[(b)$\Rightarrow$(a)] Let $X' \xrightarrow{a'} Y' \xrightarrow{b'} M \xrightarrow{c'} X'[1]$ be the triangle in $\mathrm{D}(\mathrm{Mod}\text{-}R)$ induced by a special $\mathcal{C}$-cover b' of M. We have already seen that $c'[-1]$ is an injective envelope of $M[-1]$ in $\mathcal{H}_\tau$. Since injective envelopes are unique up to isomorphism, there exists an isomorphism $h\colon X \to X'$ and a commutative diagram:

$$
\begin{array}{ccccccc}
X & \xrightarrow{a} & Y & \xrightarrow{b} & M & \xrightarrow{c} & X[1] \\
\cong \downarrow{h} & & \downarrow{f} & & \| & & \cong \downarrow{h[1]} \\
X' & \xrightarrow{a'} & Y' & \xrightarrow{b'} & M & \xrightarrow{c'} & X'[1]
\end{array}
$$

where the induced morphism f must also be an isomosrphism. It follows that b is a $\mathcal{C}$-cover of M in $\mathrm{Mod}\text{-}R$.

(2) [(a)$\Rightarrow$(b)] Since M and Y are in $\mathcal{C}$, we have $X \in \mathcal{C} \cap \mathcal{C}^{\perp_1} = \mathrm{Prod}(C)$, so X is injective in $\mathcal{H}_\tau$. Moreover, it follows from Lemma 2.3 that there is an exact sequence $0 \to M \xrightarrow{a} X \to Y \to 0$ in $\mathcal{H}_\tau$. Finally, a is left minimal in $\mathcal{H}_\tau$ since so is a in $\mathrm{Mod}\text{-}R$.

[(b)$\Rightarrow$(a)] Let $a'\colon M \to X'$ be a special $\mathcal{C}^{\perp_1}$-envelope of M. We have seen that a' is an injective envelope of M in $\mathcal{H}_\tau$. Since injective envelopes are unique up to isomorphism, there exists an isomorphism $h\colon X \to X'$ such that $ha = a'$. It follows that a is a special $\mathcal{C}^{\perp_1}$-envelope of M. $\qquad\square$

Theorem 4.2 *Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair with associated cotilting module C. Consider a short exact sequence $0 \to L \xrightarrow{a} M \xrightarrow{b} N \to 0$ in $\mathrm{Mod}\text{-}R$.*

(1) The following statements are equivalent.

(a) The module N is torsion, almost torsionfree and the morphism b is a special $\mathcal{C}$-cover of N in $\mathrm{Mod}\text{-}R$.

(b) The module L is in $\mathrm{Prod}(C)$ and the morphism a is a strong left almost split morphism in $\mathcal{C}$.

(2) The following statements are equivalent.

(a) The module L is torsionfree, almost torsion and the morphism a is a special $\mathcal{C}^{\perp_1}$-envelope of L in $\mathrm{Mod}\text{-}R$.

(b) The module M is in $\mathrm{Prod}(C)$ and the morphism b is a strong left almost split morphism in $\mathcal{C}$.

Proof **(1)[(a)$\Rightarrow$(b)]** By Theorem 3.6, the object N is simple in the heart $\mathcal{H}_\tau$ and by Proposition 4.1, we have that $c[-1]\colon N[-1] \to L$ is an injective envelope, where

$L! \to !M! \to !N \overset{c}{\to} L[1]$ is the completion of the exact sequence to a triangle in $D(\text{Mod-}R)$. In particular, this means that L is injective in $\mathcal{H}_\tau$ and hence $L \in \text{Prod}(C)$. It follows from Lemma 2.3, that $0 \to N[-1] \overset{c[-1]}{\to} L \overset{a}{\to} M \to 0$ is a short exact sequence in $\mathcal{H}_\tau$. Then Remark 2.8 tells us that a is a strong left almost split morphism in $\mathcal{C}$.

[**(b)**$\Rightarrow$**(a)**] It follows from our assumptions that L is injective in $\mathcal{H}_\tau$. By Proposition 2.7 and Remark 2.8, the kernel $S := \text{Ker}_{\mathcal{H}_\tau}(a)$ is simple and the inclusion $0 \to S \overset{c}{\to} L$ is an injective envelope. Moreover, since strong left almost split morphisms starting at an object are unique up to isomorphism, a is an epimorphism in $\mathcal{H}_\tau$. By Lemma 2.3, we have that N is in $\mathcal{Q}$ and also that $N[-1] \cong S$. By Theorem 3.6 we have that N is almost torsionfree. Finally, since c is an injective envelope, it follows from Proposition 4.1 that b is a $\mathcal{C}$-cover of N.

(2)[**(a)**$\Rightarrow$**(b)**] By Theorem 3.6, we have that L is simple in $\mathcal{H}_\tau$ and by Proposition 4.1 the morphism a is an injective envelope in $\mathcal{H}_\tau$. In particular, we have that M is contained in $\text{Prod}(C)$ and the sequence $0 \to L \overset{a}{\to} M \overset{b}{\to} N \to 0$ is exact in $\mathcal{H}_\tau$. By Remark 2.8 and Proposition 2.7(1) we have that b is a strong left almost split morphism in $\mathcal{C}$.

[**(b)**$\Rightarrow$**(a)**] Since M is in $\mathcal{C}$ and $\mathcal{C}$ is closed under submodules, we have that L is also in $\mathcal{C}$. Therefore $0 \to L \overset{a}{\to} M \overset{b}{\to} N \to 0$ is a short exact sequence in $\mathcal{H}_\tau$. By our assumption, we have that b is a strong left almost split morphism in $\mathcal{C}$ and so, by Remark 2.8 and Proposition 2.7(2), we have that L is simple and a is an injective envelope. By Theorem 3.6 and Proposition 4.1, we have shown that condition (a) holds. $\qquad\square$

Next we show that the strong left almost split morphisms arising in Theorem 4.2 are the only strong left almost split morphisms in $\mathcal{C}$ with domain contained in $\text{Prod}(C)$.

Lemma 4.3 *Let $\mathcal{M}$ be a full subcategory of $\text{Mod-}R$ that is closed under subobjects. Then the following are equivalent for a module M in $\mathcal{M}$.*

(1) There is a left almost split morphism $f \colon M \to \tilde{M}$ in $\mathcal{M}$ that is not a monomorphism.
(2) There is a left almost split morphism $g \colon M \to \bar{M}$ in $\mathcal{M}$ that is an epimorphism.

Moreover, a strong left almost split morphism is either a monomorphism or an epimorphism.

Proof [**(2)** $\Rightarrow$ **(1)**] is trivially true. To prove [**(1)** $\Rightarrow$ **(2)**], observe that, if $f \colon M \to \tilde{M}$ is a left almost split morphism in $\mathcal{M}$ that is not a monomorphism, then $g \colon M \to \text{Im}(f)$ is a left almost split morphism in $\mathcal{M}$ that is an epimorphism because $\text{Im}(f) \in \mathcal{M}$ by assumption.

The final statements follows immediately because strong left almost split morphisms are unique up to isomorphism. $\qquad\square$

Define

$$\mathcal{N}_C := \{N \in \text{Prod}(C) \mid \exists N \to \bar{N} \text{ a (strong) left almost split morphism in } \mathcal{C}\}.$$

Note that, by Proposition 2.7 and the subsequent corollaries, the set $\mathcal{N}_C$ does not depend on whether we choose to include the word strong or not. The previous lemma shows that $\mathcal{N}_C$ is a disjoint union $\mathcal{N}_C = \mathcal{M}_C \sqcup \mathcal{E}_C$ where

$$\mathcal{M}_C := \{L \in \text{Prod}(C) \mid \exists L \to \bar{L} \text{ a strong left almost split monomorphism in } \mathcal{C}\} \quad \text{and}$$
$$\mathcal{E}_C := \{M \in \text{Prod}(C) \mid \exists M \to \bar{M} \text{ a strong left almost split epimorphism in } \mathcal{C}\}.$$

Note that $\mathcal{M}_C$ consists of the modules $L \in \text{Prod}(C)$ arising in Theorem 4.2(1) and $\mathcal{E}_C$ consists of the modules $M \in \text{Prod}(C)$ arising in Theorem 4.2(2).

Corollary 4.4 *The following statements hold for a module N in* $\mathrm{Prod}(C)$.

(1) $N \in \mathcal{N}_C$ if and only if N is the injective envelope of a simple object in $\mathcal{H}_\tau$. In this case N is isomorphic to an indecomposable direct summand of any cotilting module that is equivalent to C.

(2) $N \in \mathcal{M}_C$ if and only if N is the injective envelope of $T[-1]$ in $\mathcal{H}_\tau$ where T is a torsion, almost torsionfree module with respect to τ. In this case T is the cokernel of the strong left almost split monomorphism $N \to \bar{N}$ in $\mathcal{C}$.

(3) $N \in \mathcal{E}_C$ if and only if N is the injective envelope of F in $\mathcal{H}_\tau$ where F is a torsionfree, almost torsion module with respect to τ. In this case F is the kernel of the strong left almost split epimorphism $N \to \bar{N}$ in $\mathcal{C}$.

Proof (1) The first statement follows immediately from Corollary 2.10. The latter statement follows from the fact that the injective envelope of a simple object in a Grothendieck category is indecomposable and the fact that the injective envelopes of simple objects arise as direct summands of any injective cogenerator, up to isomorphism.

The statements (2) and (3) follow directly from Theorem 4.2 and Proposition 4.1.   □

In some cases, the heart $\mathcal{H}_\tau$ turns out to be locally finitely generated and in this case we have the converse of the second part of Corollary 4.4.

Corollary 4.5 *Suppose $\mathcal{H}_\tau$ is locally finitely generated.*

(1) Let $D \in \mathrm{Prod}(C)$. Every $N \in \mathcal{N}_C$ is isomorphic to a direct summand of D if and only if D is a cotilting module that is equivalent to C.

(2) Let $\tilde{C}$ be a special $\mathcal{C}^{\perp_1}$-envelope of $\bigoplus_{N \in \mathcal{N}_C} N$. Then $\tilde{C}$ is a cotilting module that is equivalent to C and, moreover, $\tilde{C}$ is isomorphic to a direct summand of every other cotilting module that is equivalent to C.

Proof We have already seen that every $N \in \mathcal{N}_C$ arises as a direct summand of a cotilting module D that is equivalent to C. The rest of the corollary follows from the corresponding statements for locally finitely generated Grothendieck categories (see, for example, [27, Prop. 3.17 and Cor. 3.18]). Note that the special $\mathcal{C}^{\perp_1}$-envelope of $\bigoplus_{N \in \mathcal{N}_C} N$ becomes the injective envelope of $\bigoplus_{N \in \mathcal{N}_C} N$ in the heart by Proposition 4.1.   □

Example 4.6 There are some important cases where we know that the heart $\mathcal{H}_\tau$ is locally finitely generated and so we may apply Corollary 4.5.

(1) If C is a cotilting module of cofinite type, then $\mathcal{H}_\tau$ is the heart of a compactly generated t-structure by [3, Lemma 3.7] and [11, Thm. 2.3]. It follows from [37, Thm. 8.31] that $\mathcal{H}_\tau$ is locally finitely presented and hence locally finitely generated.

(2) A cotilting module C is an elementary cogenerator if and only if $\mathcal{H}_\tau$ is locally coherent. One implication follows from the description of the heart as a localisation of the functor category given in [42], the other is shown in [28, Thm. 5.12].

(3) [19, Thm. 15.31], [36, Thm. 5.2] If R is a right noetherian ring, then (1) and (2) apply, and the finitely presented objects in $\mathcal{H}_\tau$ are precisely the objects which belong to the bounded derived category $\mathcal{D}^b(\mathrm{mod}\text{-}R)$.

5 Neg-isolated modules

Let R be a ring and consider the category $(R\text{-mod}, \mathrm{Ab})$ of additive functors from the category R-mod of finitely presented left R-modules to the category Ab of abelian groups. This functor

category is a locally coherent Grothendieck category. We will use the notation $(M, -) :=$ $\mathrm{Hom}_R(M, -)$ for the representable objects in (mod-R, Ab), $M \in$ mod-R, and we will write $[F, G]$ to denote the set $\mathrm{Hom}_{(R\text{-mod}, \mathrm{Ab})}(F, G)$ of natural transformations from F to G.

5.1 Finitely generated subfunctors of $\mathcal{I}$

In this section, we study the subfunctors of the most important object of (mod-R, Ab), the forgetful functor $\mathcal{I}\colon$ mod-$R \to$ Ab. There is a natural isomorphism $\mathcal{I} \xrightarrow{\iota} (R, -)$ between $\mathcal{I}$ and the functor represented by R_R. For $M \in$ mod-R, the M-component is given by the morphism $M \xrightarrow{\iota_M} (R, M)$, $m \mapsto (f_m\colon 1 \mapsto m)$ in Ab. This is actually a morphism of R-modules which induces an isomorphism of pointed R-modules $(M, m) \longrightarrow ((R, M), f_m)$ for each $m \in M$. So if $\phi \subseteq \mathcal{I}$ is a subfunctor of the forgetful functor, then $m \in \phi(M)$ if and only if $f_m \in \phi(R, M)$. In this way we observe that the rule $\phi \mapsto \phi(R, -)$ is a bijective correspondence between the subfunctors of the forgetful functor and those of $(R, -)$.

Associated to a pointed finitely presented module (M, m), is the finitely generated subfunctor $\mathrm{Im}\,(f_m, -) \subseteq (R, -)$ induced by the natural transformation $(f_m, -)\colon$ $(M, -) \longrightarrow (R, -)$. It corresponds to the subfunctor $H_{M,m}$ of the forgetful functor $\mathcal{I}$ which takes $N \in$ mod-R to the finite matrix subgroup (or pp-definable subgroup) $H_{M,m}(N) = \{h(m) \mid h \in \mathrm{Hom}_R(M, N)\}$. On the other hand, if $\phi \subseteq (R, -)$ is a finitely generated subfunctor, then there is a natural transformation $\eta\colon (M, -) \to (R, -)$ with image $\mathrm{Im}\,\eta = \phi$. By Yoneda's Lemma, there exists an R-linear morphism $f_m : R \longrightarrow M$ such that $\eta = (f_m, -)$. Thus, every finitely generated subfunctor of $\mathcal{I}$ arises from a pointed finitely presented module (M, m) in this way.

The finitely presented objects of (mod-R, Ab) admit a canonical extension to Mod-R that respects direct limits. As all representable functors and their finitely generated subfunctors are finitely presented, this pertains to the finitely generated subfunctors $\phi \subseteq \mathcal{I}$ of the forgetful functor.

Proposition 5.1 *Let (M, m) be a pointed finitely presented module and $\phi = H_{M,m} \subseteq \mathcal{I}$. Then (M, m) is a **free realisation** of ϕ, in the sense that $m \in \phi(M)$, and whenever (N, n) is a pointed module with $n \in \phi(N)$, then there exists a morphism $h\colon (M, m) \to (N, n)$ of pointed modules.*

Proof The case when (N, n) is finitely presented is clear by definition of $H_{M,m}$. For the general case, use the fact that ϕ respects direct limits, so there exists a pointed finitely presented module (N', n') and a morphism $(N', n') \to (N, n)$ of pointed modules with the property that $n' \in \phi(N')$. Now use again the definition of ϕ. $\qquad\square$

5.2 The pp-type of a pointed module

The finitely generated subobjects of $\mathcal{I}$ form a modular lattice. We will now use techniques from the model theory of modules [30, 31, 44] to investigate this lattice. The model theoretic approach allows us to represent the finitely generated subfunctors of $\mathcal{I}$ by formulas $\phi(x)$ in a certain first-order language. The formula endows the corresponding subfunctor $\phi \subseteq \mathcal{I}$ with semantic content, so that if (M, m) is a pointed right R-module, we may evaluate the statement $\phi(m)$ as true in M, denoted by $M \models \phi(m)$, or not. In other words, the M-component of the inclusion $\phi \subseteq \mathcal{I}$ consists of the "solutions" in M to the formula $\phi(x)$. We refer to the formulas $\phi(x)$ as **pp-formulas**, see [31, §12.2] for details.

The **pp-type,** denoted by $\mathrm{pp}(N, n)$, of a pointed module (N, n) is the collection of pp-formulas $\phi(x)$, for which $N \models \phi(n)$. Equivalently, we can think of $\mathrm{pp}(N, n)$ as the collection of finitely generated subfunctors $\phi \subseteq \mathcal{I}$ for which $n \in \phi(N)$. As such, the pp-type $\mathrm{pp}(N, n)$ may be regarded as a filter Φ in the lattice of finitely generated subfunctors of the forgetful functor $\mathcal{I} \in (\mathrm{mod}\text{-}R, \mathrm{Ab})$; we say that (N, n) is a **realisation** of Φ. The Completeness Theorem of first-order logic ensures that every filter Φ arises as the pp-type $\Phi = \mathrm{pp}(N, n)$ of some pointed module. The functorial property of pp-formulas ensures that if $f \colon (M, n) \to (N, n)$ is a morphism of pointed R-modules, then $\mathrm{pp}(M, m) \subseteq \mathrm{pp}(N, n)$.

The general question thus arises of when an inclusion $\mathrm{pp}(M, m) \subseteq \mathrm{pp}(N, n)$ of pp-types is induced by a morphism of pointed modules. If M is finitely presented, it is easy to see that $\mathrm{pp}(M, m)$ is the principal filter generated by the subfunctor $H_{M, m} \subseteq \mathcal{I}$. Proposition 5.1 therefore implies that every inclusion $\mathrm{pp}(M, m) \subseteq \mathrm{pp}(N, n)$ is induced by a morphism $h \colon (M, m) \to (N, n)$. Next we consider a condition on the module (N, n) that ensures the existence of a morphism of pointed modules.

5.3 Pure-injective modules

The assignment $M \mapsto (M \otimes_R -)$ defines a fully faithful right exact functor

$$c\mathcal{Y} \colon \mathrm{Mod}\text{-}R \to (R\text{-mod}, \mathrm{Ab}) \tag{5.1}$$

which is called coYoneda embedding. It allows us to consider the exact structure of the functor category inside the module category. This is known as the pure exact structure in Mod-R.

Definition 5.2 A short exact sequence $0 \longrightarrow N \xrightarrow{\ f\ } M \xrightarrow{\ g\ } L \longrightarrow 0$ in Mod-R is called **pure** if

$$0 \longrightarrow N \otimes_R - \xrightarrow{\ f \otimes -\ } M \otimes_R - \xrightarrow{\ g \otimes -\ } L \otimes_R - \longrightarrow 0 \tag{5.2}$$

is exact in $(R\text{-mod}, \mathrm{Ab})$. In this case, we refer to f as a **pure monomorphism** and to g as a **pure epimorphism**.

It is natural to consider the modules that are injective with respect to the pure exact structure.

Definition 5.3 A module N is called **pure-injective** if every pure exact sequence of the form (5.2) is a split exact sequence.

Clearly any module that becomes an injective object under the coYoneda embedding is pure-injective and, in fact, all injective objects in $(R\text{-mod}, \mathrm{Ab})$ arise in this way. That is, the coYoneda embedding restricts to an equivalence

$$c\mathcal{Y} \colon \mathrm{Pinj}(R) \xrightarrow{\ \sim\ } \mathrm{Inj}(R\text{-mod}, \mathrm{Ab})$$

where $\mathrm{Pinj}(R)$ denotes the full subcategory of pure-injective objects in Mod-R and $\mathrm{Inj}(R\text{-mod}, \mathrm{Ab})$ denotes the full subcategory of injective objects in $(R\text{-mod}, \mathrm{Ab})$. Furthermore, if M_R is an R-module with pure-injective envelope $\iota \colon M \to \mathrm{PE}(M)$, then the corresponding monomorphism $\iota \otimes - \colon M \otimes_R - \to \mathrm{PE}(M) \otimes_R -$ is the injective envelope of $M \otimes_R -$ in $(R\text{-mod}, \mathrm{Ab})$.

A pure monomorphism $f : M \to N$ may also be characterised [31, Proposition 2.1.6] in terms of pp-formulas, which is equivalent to the condition that $\mathrm{pp}(M, m) = \mathrm{pp}(N, f(m))$ for every $m \in M$. So if Φ is a filter of finitely generated subfunctors of the forgetful functor with realisation $\Phi = \mathrm{pp}(N, n)$, then the pure-injective envelope $\Phi = \mathrm{pp}(\mathrm{PE}(N), n)$ too is a realisation of Φ.

Remark 5.4 [44, Corollary 3.3 (1)] If N is pure-injective, then every inclusion $\mathrm{pp}(M, m) \subseteq \mathrm{pp}(N, n)$ is induced by a morphism $f : (M, m) \to (N, n)$.

5.4 Definable subcategories of modules

We are interested in localising the functor category $(R\text{-mod}, \mathrm{Ab})$ at hereditary torsion classes associated to a particular kind of category of modules.

Definition 5.5 A full subcategory $\mathcal{D}$ of Mod-R is called **definable** if it is closed under products, pure submodules and directed colimits.

A definable subcategory $\mathcal{D} \subseteq \mathrm{Mod}\text{-}R$ is closed under pure-injective envelopes, so that its image under the coYoneda embedding $\mathrm{c}\mathcal{Y}(\mathcal{D}) = \mathcal{D} \otimes - \subseteq (R\text{-mod}, \mathrm{Ab})$ is closed under injective envelopes in $(R\text{-mod}, \mathrm{Ab})$. It follows that the torsion pair $(\mathcal{T}_{\mathcal{D}}, \mathrm{Cogen}(\mathcal{D} \otimes -))$ in $(R\text{-mod}, \mathrm{Ab})$ cogenerated by $\mathcal{D} \otimes -$, is hereditary.

Notation 5.6 We will denote the localisation $(R\text{-mod}, \mathrm{Ab})/\mathcal{T}_{\mathcal{D}}$ by $(R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}$ and the corresponding localisation functor by

$$(-)_{\mathcal{D}} : (R\text{-mod}, \mathrm{Ab}) \to (R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}.$$

We will denote the Hom-spaces between two objects F, G in $(R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}$ by $[F, G]_{\mathcal{D}}$.

If $\mathcal{D} \subseteq \mathrm{Mod}\text{-}R$ is a definable subcategory, then the hereditary torsion pair $(\mathcal{T}_{\mathcal{D}}, \mathrm{Cogen}(\mathcal{D} \otimes -))$ in $(R\text{-mod}, \mathrm{Ab})$ is of finite type in the sense of the following definition. We refer the reader to [22] for more details on the theory surrounding hereditary torsion pairs of finite type in locally coherent Grothendieck categories.

Definition 5.7 A torsion pair $(\mathcal{T}, \mathcal{F})$ (not necessarily hereditary) in a Grothendieck category $\mathcal{G}$ is said to be of **finite type** if the torsionfree class $\mathcal{F}$ is closed under directed limits in $\mathcal{G}$.

The following theorem will be very important in what follows. For details, we refer to [31, §12.3].

Theorem 5.8 *The rule $\mathcal{D} \mapsto (\mathcal{T}_{\mathcal{D}}, \mathcal{F}_{\mathcal{D}})$ is a bijective correspondence between the collection of definable subcategories $\mathcal{D} \subseteq \mathrm{Mod}\text{-}R$ and hereditary torsion pairs in $(R\text{-mod}, \mathrm{Ab})$ of finite type, with inverse given by $(\mathcal{T}, \mathcal{F}) \mapsto \mathcal{D} = \{M \in \mathrm{Mod}\text{-}R \mid M \otimes_R - \in \mathcal{F}\}$.*

Consider the functor Mod-$R \to (R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}$ given by the composition of the functor (5.1) with the localisation functor $(-)_{\mathcal{D}}$. Since the injective objects in $(R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}$ coincide with the injective objects in $(R\text{-mod}, \mathrm{Ab})$ that are contained in the torsionfree class, this functor restricts to an equivalence of categories

$$\mathrm{Pinj}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Inj}((R\text{-mod}, \mathrm{Ab})_{\mathcal{D}})$$

where $\mathrm{Pinj}(\mathcal{D})$ denotes the full subcategory of pure-injective objects in $\mathcal{D}$ and $\mathrm{Inj}((R\text{-mod}, \mathrm{Ab})_{\mathcal{D}})$ denotes the full subcategory of injective objects in $(R\text{-mod}, \mathrm{Ab})_{\mathcal{D}}$.

Remark 5.9 In the proofs below we will use the following observation several times. If $\mathcal{D}$ is a definable subcategory of $(R\text{-mod, Ab})$, then, for any $M \in \text{Pinj}(\mathcal{D})$ and any R-module L, we have that

$$[\,(L \otimes -)_{\mathcal{D}}, (M \otimes -)_{\mathcal{D}}\,]_{\mathcal{D}} \cong [(L \otimes -), (M \otimes -)] \cong \text{Hom}_R(L, M). \qquad (5.3)$$

This follows directly from the fact that $(M \otimes -)$ is injective and torsionfree with respect to the torsion pair $(\mathcal{T}_{\mathcal{D}}, \mathcal{F}_{\mathcal{D}})$ induced by $\mathcal{D}$ in $(R\text{-mod, Ab})$. In particular, we have

$$[\,(R \otimes -)_{\mathcal{D}}, (M \otimes -)_{\mathcal{D}}\,]_{\mathcal{D}} \cong [(R \otimes -), (M \otimes -)] \cong \text{Hom}_R(R, M) \cong M. \qquad (5.4)$$

5.5 Neg-isolated pure-injective modules

Next we consider the pure-injective objects in a given definable subcategory that correspond to injective envelopes of simple objects in the Grothendieck category $(R\text{-mod, Ab})_{\mathcal{D}}$. They will be characterised by a condition on pp-types.

A filter Φ in the lattice of finitely generated subfunctors of the forgetful functor in $(\text{mod-}R, \text{Ab})$ will be called a $\mathcal{D}$-**filter** if it admits a realisation $\text{pp}(D, d) = \Phi$ with D in $\mathcal{D}$. Recall that we can always choose D to belong to the full subcategory $\text{Pinj}(\mathcal{D})$ of pure-injective objects in $\mathcal{D}$.

Theorem 5.10 *Let $\mathcal{D}$ be a definable subcategory in* Mod-*R. The following statements are equivalent for an indecomposable pure-injective module N in $\mathcal{D}$.*
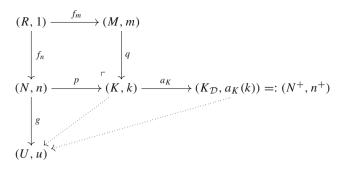
(1) $(N \otimes_R -)_{\mathcal{D}}$ is the injective envelope of a simple object in $(R\text{-mod, Ab})_{\mathcal{D}}$.
(2) There exists a left almost split morphism $N \to N^+$ in $\text{Pinj}(\mathcal{D})$.
(3) If $n \in N$ is a nonzero element and $\Phi = \text{pp}(N, n)$ is the associated pp-type, then there exists a $\mathcal{D}$-filter $\Phi^+ \supset \Phi$ which properly contains Φ such that whenever a $\mathcal{D}$-filter $\Psi \supset \Phi$ properly contains Φ, then $\Psi \supseteq \Phi^+$.
(4) N is the source of an almost split morphism in $\mathcal{D}$.

Proof Notice that, since N is an indecomposable pure-injective module, the endomorphism ring of N is then local. This implies that every endomorphism of pointed modules $(N, n) \to (N, n)$ with $n \in N$ being nonzero is an automorphism. The equivalence

[(1) $\Leftrightarrow$ (2)] follows directly from Corollary 2.9 and the discussion following Notation 5.6.

[(2) $\Rightarrow$ (3)] Let $n \in N$ be nonzero and let $f: N \to N^+$ be a left almost split morphism in $\text{Pinj}(\mathcal{D})$. Set $n^+ = f(n)$ and consider the $\mathcal{D}$-filters $\Phi = \text{pp}(N, n)$ and $\Phi^+ = \text{pp}(N^+, n^+)$. Clearly $\Phi^+ \supseteq \Phi$, and equality would imply by Remark 5.4 that there is a map of pointed modules $(N^+, n^+) \to (N, n)$. Because f is not a split monomorphism, we get that $\Phi^+ \supset \Phi$ is strictly larger. On the other hand, if $\Psi \supset \Phi$ is any strictly larger $\mathcal{D}$-filter with realisation $\Psi = (U, u)$ where U in $\text{Pinj}(\mathcal{D})$, then there is a morphism $g: (N, n) \to (U, u)$ which is not a split monomorphism. Then g factors through f, and $\text{pp}(U, u) \supseteq \text{pp}(N^+, n^+)$, that is, $\Psi \supseteq \Phi^+$.

[(3) $\Rightarrow$ (4)] Given a nonzero element $n \in N$, set $\Phi = \text{pp}(N, n)$ and pick $\psi \in \Phi^+ \setminus \Phi$ with a free realisation (M, m), i.e. a finitely presented pointed module (M, m) such that $\psi = H_{M,m}$. Consider the pushout of (N, n) with (M, m), postcomposed with a $\mathcal{D}$-approximation (which exists e.g. by [31, Proposition 3.4.39]) as in the middle row of

$$(R, 1) \xrightarrow{\;f_m\;} (M, m)$$

$$(N, n) \xrightarrow{\;p\;} (K, k) \xrightarrow{\;a_K\;} (K_{\mathcal{D}}, a_K(k)) =: (N^+, n^+)$$

$$(U, u)$$

with vertical maps f_n, q, g.

Clearly, $\mathrm{pp}(N^+, n^+)$ contains both Φ and ψ, so it must contain Φ^+. Then the composition $a_K \circ p \colon (N, n) \to (N^+, n^+)$ cannot be a split monomorphism. In fact, it must be a left almost split morphism in $\mathcal{D}$. For, suppose that we are given a morphism $g \colon N \to U$ in $\mathcal{D}$ that is not a split monomorphism. Then there can't be a morphism of pointed modules $(U, u) \to (N, n)$, and we infer from Remark 5.4 that $\mathrm{pp}(U, u) \supset \Phi$ is strictly larger. Thus $\psi \in \mathrm{pp}(U, u)$. By Proposition 5.1 there exist morphism $(M, m) \to (U, u)$ from the free realisation of ψ and a factorisation through the pushout. As $U \in \mathcal{D}$, this map from the pushout then factorises through its $\mathcal{D}$-approximation, as required.

[(4) $\Rightarrow$ (2)] Suppose there exists a left almost split morphism $h \colon N \to \bar{N}$ in $\mathcal{D}$ and let $e \colon \bar{N} \to \mathrm{PE}(\bar{N})$ be the pure-injective envelope of $\bar{N}$. We will show that $g := eh$ is a left almost split morphism in $\mathrm{Pinj}(\mathcal{D})$. Let $u \colon N \to U$ be a morphism in $\mathrm{Pinj}(\mathcal{D})$ that is not a split monomorphism. Then there exists some $v \colon \bar{N} \to U$ such that $u = vh$. Using that e is a pure monomorphism and that U is pure-injective, we have that there exists $f \colon \mathrm{PE}(\bar{N}) \to U$ such that $fg = feh = vh = u$ as desired. $\qquad\square$

The theorem above is a relative version of results in [31, §5.3.5] for the case $\mathcal{D} = \mathrm{Mod}\text{-}R$. Indeed, in that case condition (3) means precisely that $\Phi = \mathrm{pp}(N, n)$ is a **neg-isolated pp-type**. This suggests the following terminology.

Definition 5.11 Let $\mathcal{D}$ be a definable subcategory of $\mathrm{Mod}\text{-}R$. A pure-injective module N in $\mathcal{D}$ is called **neg-isolated in** $\mathcal{D}$ if it satisfies the equivalent conditions of Theorem 5.10.

We have seen that left almost split morphisms in a cotilting class are intimately related to the injective envelopes of simple objects in the cotilting heart. Since cotilting classes $\mathcal{C}$ are always definable subcategories, it is natural to ask how the results of Sect. 4 are related to the neg-isolated modules in $\mathcal{C}$.

Corollary 5.12 *Let C be a cotilting module with torsion pair $\tau = (\mathcal{Q}, \mathcal{C})$. The R-modules that become injective envelopes of simple objects in $\mathcal{H}_\tau$ are precisely the neg-isolated modules in $\mathcal{C}$ which lie in* $\mathrm{Prod}(C)$.

Proof This follows immediately from Theorem 5.10 and Corollary 4.4. $\qquad\square$

5.6 Critical modules

Let $(\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair in $\mathrm{Mod}\text{-}R$. The aim of this section is to investigate the neg-isolated modules in $\mathcal{C}$ that are domains of left almost split morphisms in $\mathcal{C}$ that are epimorphisms. In Lemma 4.3 we saw that these coincide with the domains of left almost split morphisms in $\mathcal{C}$ that are not monomorphisms; these are the critical neg-isolated modules.

Definition 5.13 Let $\mathcal{D}$ be a definable subcategory of Mod-R. We call a neg-isolated module N in $\mathcal{D}$ **critical** in $\mathcal{D}$ if there exists a morphism $h \colon N \rightarrow N^+$ that is a left almost split morphism in $\mathrm{Pinj}(\mathcal{D})$ such that h is not a monomorphism.

When the definable subcategory is a torsionfree class, we have the following alternative characterisation of critical modules showing that they are exactly the neg-isolated modules such that the associated strong left almost split morphism is an epimorphism.

Proposition 5.14 *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* Mod-R *such that $\mathcal{F}$ is a definable subcategory. The following statements are equivalent for a module N in $\mathcal{F}$.*

(1) N is a critical neg-isolated module in $\mathcal{F}$.
(2) There exists a left almost split morphism $f \colon N \rightarrow \bar{N}$ in $\mathcal{F}$ that is an epimorphism. In particular, f is a strong left almost split morphism.

Proof **[(1) $\Rightarrow$ (2)]** Let $h \colon N \rightarrow N^+$ be a left almost split morphism in $\mathrm{Pinj}(\mathcal{F})$ that is not a monomorphism. Note that $\mathrm{Im}(h)$ is contained in $\mathcal{F}$ because $\mathcal{F}$ is closed under submodules. We will show that $\bar{h} \colon N \rightarrow \mathrm{Im}(h)$ is a left almost split morphism in $\mathcal{F}$. Since h is not a monomorphism, it follows that $\bar{h}$ is not a split monomorphism. Suppose $u \colon N \rightarrow U$ is a morphism in $\mathcal{F}$ that is not a split monomorphism and consider $eu \colon N \rightarrow \mathrm{PE}(U)$ where e is the pure-injective envelope of U. Then eu can't be a split monomorphism. Since h is left almost split in $\mathrm{Pinj}(\mathcal{F})$ and $\mathrm{PE}(U)$ lies in $\mathcal{F}$ by [31, Thm. 3.4.8], there exists a morphism $f \colon N^+ \rightarrow \mathrm{PE}(U)$ such that $eu = fh$. Let $k \colon \mathrm{Ker}(h) \rightarrow N$ be the kernel of h. Then $euk = fhk = 0$ and, moreover, $uk = 0$ because e is a monomorphism. Using that $\bar{h}$ is the cokernel of k, we conclude that there exists a unique morphism $g \colon \mathrm{Im}(h) \rightarrow U$ such that $u = g\bar{h}$.

[(2) $\Rightarrow$ (1)] Let $h \colon N \rightarrow \bar{N}$ be a left almost split morphism in $\mathcal{F}$ that is an epimorphism. Then N must belong to $\mathrm{Pinj}(\mathcal{F})$, since otherwise the pure-injective envelope $e \colon N \rightarrow \mathrm{PE}(N)$ would factor through h and h would be an isomorphism. In the proof of Theorem 5.10, we showed that $eh \colon N \rightarrow \bar{N} \rightarrow \mathrm{PE}(\bar{N})$ is a left almost split morphism in $\mathrm{Pinj}(\mathcal{F})$. Since h is an epimorphism, it is clear that eh is not a monomorphism. $\square$

We saw in Theorem 5.10 that neg-isolated modules in definable subcategories are in bijection with injective envelopes of simple objects in the corresponding localisation of the functor category. In the next lemma we identify which injective envelopes of simple objects give rise to critical modules.

Lemma 5.15 *Let $\mathcal{D}$ be a definable subcategory of* Mod-R *and suppose N is neg-isolated in $\mathcal{D}$. The following statements are equivalent.*

(1) N is critical in $\mathcal{D}$.
(2) $(N \otimes_R -)_{\mathcal{D}}$ is the injective envelope of a simple object S in $(R$-mod, Ab$)_{\mathcal{D}}$ such that $[(R \otimes_R -)_{\mathcal{D}}, S]_{\mathcal{D}} \neq 0$.

Proof Since N is neg-isolated in $\mathcal{D}$, we have the following set up according to Sect. 5.3. There exists a left almost split morphism $h \colon N \rightarrow N^+$ in $\mathrm{Pinj}(\mathcal{D})$ and, moreover, the morphism $(h \otimes -)_{\mathcal{D}} \colon (N \otimes_R -)_{\mathcal{D}} \rightarrow (N^+ \otimes -)_{\mathcal{D}}$ is a left almost split morphism in $\mathrm{Inj}(R$-mod, Ab$)_{\mathcal{D}}$. By Proposition 2.7, we see that the kernel of $(h \otimes -)_{\mathcal{D}}$ is isomorphic to the monomorphism $i \colon S \rightarrow E(S) = (N \otimes_R -)_{\mathcal{D}}$.

[(1)$\Rightarrow$(2)] Suppose that $\mathrm{Ker}(h) \neq 0$. Consider an element $k \in \mathrm{Ker}(h)$ and consider it as the morphism $k \colon R \rightarrow N$ that takes $1 \mapsto k$. This yields a non-zero morphism

$(k \otimes -) \in [(R \otimes_R -), (N \otimes_R -)]$, and it follows from Remark 5.9 that $(k \otimes -)_{\mathcal{D}} \neq 0$. Moreover, we have that $(h \otimes -) \circ (k \otimes -) = (h(k) \otimes -) = 0$ and so $(h \otimes -)_{\mathcal{D}} \circ (k \otimes -)_{\mathcal{D}} = 0$. Therefore, the morphism $(k \otimes -)_{\mathcal{D}}$ factors through the kernel of $(h \otimes -)_{\mathcal{D}}$. Thus we have a factorisation $i \circ g = (k \otimes -)_{\mathcal{D}}$ and, in particular, we have a non-zero morphism $g \colon (R \otimes_R -)_{\mathcal{D}} \to S$.

[(2)⇒(1)] Now suppose that there exists a non-zero morphism $f \colon (R \otimes_R -)_{\mathcal{D}} \to S$ and consider $i \circ f \colon (R \otimes_R -)_{\mathcal{D}} \to (N \otimes_R -)_{\mathcal{D}}$. Then, by Remark 5.9, there exists a non-zero morphism $g \colon (R \otimes_R -) \to (N \otimes_R -)$ such that $g_{\mathcal{D}} = i \circ f$. Since the coYoneda embedding given in Sect. 5.3 is fully faithful, we can find a non-zero element $n \in N$ that determines a morphism $n \colon R \to N$ such that $1 \mapsto n$ and $(n \otimes -) = g$. By definition, we have that $(n \otimes -)_{\mathcal{D}} = i \circ f$ and so we have

$$(h(n) \otimes -)_{\mathcal{D}} = ((h \otimes -) \circ (n \otimes -))_{\mathcal{D}} = (h \otimes -)_{\mathcal{D}} \circ (n \otimes -)_{\mathcal{D}} = (h \otimes -)_{\mathcal{D}} \circ i \circ f = 0.$$

By Remark 5.9, this implies that $(h(n) \otimes -) = 0$ and hence $0 \neq n \in \mathrm{Ker}\,(h)$.   □

This characterisation of critical modules in a definable subcategory $\mathcal{D}$ allows us to show in the next proposition that they cogenerate $\mathcal{D}$. Moreover, it follows from this that the critical modules in a definable subcategory $\mathcal{D}$ are related to the split injective modules in $\mathcal{D}$. A module M in a full subcategory $\mathcal{M}$ of Mod-R is called **split injective in** $\mathcal{M}$ if every monomorphism $M \to M'$ with M' in $\mathcal{M}$ is a split monomorphism.

Proposition 5.16 *Let $\mathcal{D}$ be a definable subcategory of* Mod-R *and let $\mathcal{D}_0$ denote the set of critical modules in $\mathcal{D}$.*
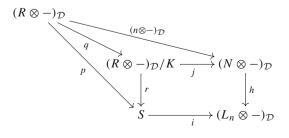
(1) The subcategory $\mathcal{D}$ is contained in $\mathrm{Cogen}(\mathcal{D}_0)$.
(2) The split injective modules in $\mathcal{D}$ are contained in the set $\mathrm{Prod}(\mathcal{D}_0)$.
(3) A module L belongs to $\mathcal{D}_0$ if and only if it is neg-isolated and split injective in $\mathcal{D}$.

Proof **(1)** The definable subcategory $\mathcal{D}$ is closed under pure-injective envelopes and so it suffices to show that $\mathrm{Pinj}(\mathcal{D}) \subseteq \mathrm{Cogen}(\mathcal{D}_0)$. Let $N \in \mathrm{Pinj}(\mathcal{D})$ and let $n \in N$ and consider

$$0 \to K \to (R \otimes -)_{\mathcal{D}} \overset{(n \otimes -)_{\mathcal{D}}}{\longrightarrow} (N \otimes -)_{\mathcal{D}}$$

Write $(n \otimes -)_{\mathcal{D}} = jq$ where $q \colon (R \otimes -)_{\mathcal{D}} \to (R \otimes -)_{\mathcal{D}}/K$ is the canonical quotient morphism and j is a monomorphism. Observe that the functor $(R \otimes -) \cong \mathrm{Hom}_R(R, -)$ is finitely presented, and so is its localisation $(R \otimes -)_{\mathcal{D}}$ by [22, Prop. 2.15]. Then K must be contained in a maximal subobject M, which induces an epimorphism $r \colon (R \otimes -)_{\mathcal{D}}/K \to (R \otimes -)_{\mathcal{D}}/M$ such that $p = rq$ where p is the quotient morphism $(R \otimes -)_{\mathcal{D}} \to (R \otimes -)_{\mathcal{D}}/M$. Since $S := (R \otimes -)_{\mathcal{D}}/M$ is simple and there is a non-zero morphism $(R \otimes -)_{\mathcal{D}} \to S$, it follows that there is a critical module L_n in $\mathcal{D}$ such that $i \colon S \to (L_n \otimes -)_{\mathcal{D}}$ is an injective envelope.

As $(L_n \otimes -)_{\mathcal{D}}$ is injective, there exists a morphism $h \colon (N \otimes -)_{\mathcal{D}} \to (L_n \otimes -)_{\mathcal{D}}$ such that the following diagram is commutative.

The isomorphisms in Remark 5.9 yield that $h \cong (h_n \otimes -)_{\mathcal{D}}$ for some $h_n \colon N \to L_n$. The commutativity of the above diagram yields that $0 \neq (h_n \otimes -)_{\mathcal{D}} \circ (n \otimes -)_{\mathcal{D}} \cong (h_n(n) \otimes -)_{\mathcal{D}}$ and it follows that $h_n(n) \neq 0$.

Applying this argument to every element of N, yields a monomorphism $N \to \prod_{n \in N} L_n$ with components equal to h_n.

(2) By (1), every split injective in $\mathcal{D}$ is contained in $\text{Cogen}(\mathcal{D}_0)$ and hence in $\text{Prod}(\mathcal{D}_0)$.

(3) Let $m : L \to N$ be a monomorphism in Mod-R with $N \in \mathcal{D}$ and $L \in \mathcal{D}_0$. If m is not split then, the composition em is not split, where $e \colon N \to \text{PE}(N)$ is the pure-injective envelope of N. But then em must factor through the left almost split morphism $h \colon L \to L^+$ in $\text{Pinj}(\mathcal{D})$. But this is not possible because h is not a monomorphism.

Conversely, if L is neg-isolated and split injective in $\mathcal{D}$, then there exists a left almost split morphism $h : L \to L^+$ in $\text{Pinj}(\mathcal{D})$, and h cannot be a monomorphism.   □

Proposition 5.17 *Now suppose $(\mathcal{Q}, \mathcal{C})$ is a cotilting torsion pair and let $\mathcal{C}_0$ denote the set of critical modules in $\mathcal{C}$. We fix an injective cogenerator I of* Mod-R *with a special $\mathcal{C}$-cover*

$$0 \to C_1 \to C_0 \overset{g}{\to} I \to 0$$

Then C_0 is split injective in $\mathcal{C}$ and $\text{Prod}(C_0) = \text{Prod}(\mathcal{C}_0)$.

Proof First we show that C_0 is split injective. Consider a monomorphism $h : C_0 \to C \in \mathcal{C}$. Since C is cogenerated by C_0 and I is injective, there are a cardinal α and maps $e : C \hookrightarrow C_0^{\alpha}$ and $f : C_0^{\alpha} \to I$ such that $feh = g$. As g is a $\mathcal{C}$-cover, there is also $f' : C_0^{\alpha} \to C_0$ such that $f = gf'$. Now the right minimality of g yields that h is a split monomorphism.

Next we show that $\text{Prod}(C_0) = \text{Prod}(\mathcal{C}_0)$. By the first part of the proof and Proposition 5.16(2), we have that $C_0 \in \text{Prod}(\mathcal{C}_0)$ and hence $\text{Prod}(C_0) \subseteq \text{Prod}(\mathcal{C}_0)$. As in [2, Lem. 1.1], we see that C_0 is a cogenerator of $\mathcal{C}$. Therefore, by Proposition 5.16(3), we have that $\mathcal{C}_0 \subseteq \text{Prod}(C_0)$. Thus we have that $\text{Prod}(C_0) = \text{Prod}(\mathcal{C}_0)$.   □

The proposition above shows in particular that every cotilting class contains critical neg-isolated modules and torsionfree, almost torsion modules. In contrast, we will see in Example 6.2 that torsion, almost torsionfree modules need not exist. Notice moreover that in general the class of split injectives in $\mathcal{C}$ is properly contained in $\text{Prod}(\mathcal{C}_0)$, cf. Example 6.3.

Corollary 5.18 *The set of critical neg-isolated modules in $\mathcal{C}$ coincides with the set $\mathcal{E}_C$ of modules $M \in \text{Prod}(C)$ admitting a strong left almost split epimorphism $M \to \bar{M}$ in $\mathcal{C}$.*

Proof The statement follows immediately from Propositions 5.14 and 5.17 since $\text{Prod}(C_0) \subseteq \text{Prod}(C)$.   □

5.7 Special modules

Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair with cotilting module C. We saw in Corollary 4.4 that the injective envelopes of simple objects in the heart $\mathcal{H}_\tau$ are exactly the objects in the set

$$\mathcal{N}_C := \{N \in \text{Prod}(C) \mid \exists N \to \bar{N} \text{ a (strong) left almost split morphism in } \mathcal{C}\}.$$

By Corollary 5.12 the elements of $\mathcal{N}_C$ are the neg-isolated modules in $\mathcal{C}$ which belong to $\text{Prod}(C)$. Moreover, we showed in Lemma 4.3 that $\mathcal{N}_C = \mathcal{E}_C \sqcup \mathcal{M}_C$ where

$$\mathcal{E}_C := \{M \in \text{Prod}(C) \mid \exists M \to \bar{M} \text{ a strong left almost split epimorphism in } \mathcal{C}\} \text{ and}$$

$$\mathcal{M}_C := \{L \in \text{Prod}(C) \mid \exists L \to \bar{L} \text{ a strong left almost split monomorphism in } \mathcal{C}\}.$$

By Corollary 5.18 we have that the elements of $\mathcal{E}_C$ are the critical neg-isolated modules in $\mathcal{C}$. In this section we will identify which of the non-critical neg-isolated modules in $\mathcal{C}$ are contained in $\mathcal{M}_C$. In other words, we wish to determine the non-critical neg-isolated modules in $\mathcal{C}$ which are contained in $\mathrm{Prod}(C)$.
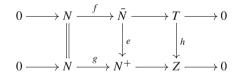
Proposition 5.19 *Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair. The following statements are equivalent for a module N in $\mathcal{C}$.*

(1) N is contained in $\mathrm{Prod}(C)$ and is a non-critical neg-isolated module in $\mathcal{C}$.
(2) N is contained in the set $\mathcal{M}_C$ of modules in $\mathrm{Prod}(C)$ that admit a strong left almost split monomorphism $f \colon N \to \bar{N}$ in $\mathcal{C}$.
(3) There exists a strong left almost split monomorphism $f \colon N \to \bar{N}$ in $\mathcal{C}$ such that the cokernel of f is torsion, almost torsionfree.
(4) There exists a left almost split morphism $g \colon N \to N^+$ in $\mathrm{Pinj}(\mathcal{C})$ such that the cokernel of g is not contained in $\mathcal{C}$.
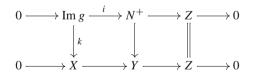
Proof **[(1)⇒(2)]** By Theorem 5.10 and Proposition 5.18, if $N \in \mathrm{Prod}(C)$ is non-critical neg-isolated, then N is contained in $\mathcal{N}_C \backslash \mathcal{E}_C = \mathcal{M}_C$.
[(2)⇒(3)] This follows from Theorem 4.2(1).
[(3)⇒(4)] Let $f \colon N \to \bar{N}$ be as in (3) and let $T := \mathrm{Coker} f$. By the proof of Theorem 5.10, we have that the composition $g := ef$ is a left almost split morphism in $\mathrm{Pinj}(\mathcal{C})$ where $e \colon \bar{N} \to N^+ := \mathrm{PE}(\bar{N})$ is the pure-injective envelope of $\bar{N}$. We show that $Z := \mathrm{Coker} g$ is not contained in $\mathcal{C}$. Thus we have a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & \bar{N} & \longrightarrow & T & \longrightarrow & 0 \\
  &   & \| &   & \downarrow{\scriptstyle e} &   & \downarrow{\scriptstyle h} &   &   \\
0 & \longrightarrow & N & \xrightarrow{g} & N^+ & \longrightarrow & Z & \longrightarrow & 0
\end{array}
$$

where h must be non-zero. Indeed, if $h = 0$, then e factors through g and f is a split monomorphism, a contradiction. Since $T \in \mathcal{Q}$, we conclude that Z is not in $\mathcal{C} = \mathcal{Q}^{\perp_0}$.
[(4)⇒(1)] Let $g \colon N \to N^+$ be as in (4). Note that, by definition, the module N is neg-isolated in $\mathcal{C}$. Set $Z := \mathrm{Coker} g$ and consider the special $\mathcal{C}$-cover $0 \to X \to Y \to Z \to 0$ of Z. Note that $X \in \mathcal{C} \cap \mathcal{C}^{\perp_1} = \mathrm{Prod}(C) \subseteq \mathrm{Pinj}(\mathcal{C})$ because X is a subobject of $Y \in \mathcal{C}$. Since N^+ is in $\mathcal{C}$, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}\, g & \xrightarrow{i} & N^+ & \longrightarrow & Z & \longrightarrow & 0 \\
  &   & \downarrow{\scriptstyle k} &   & \downarrow &   & \| &   &   \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}
$$

where $g = ip$ is the canonical factorisation of g through $\mathrm{Im}\, g$. Then kp must be a split monomorphism. Indeed, if kp is not a split monomorphism, then there exists a morphism $l \colon N^+ \to X$ such that $kp = lg = lip$. Since p is an epimorphism, it follows that $k = li$. A standard argument shows that the bottom sequence splits, which is not possible because Z would then be isomorphic to a summand of $Y \in \mathcal{C}$. We have therefore shown that N is contained in $\mathrm{Prod}(C)$ because it is isomorphic to a direct summand of $X \in \mathrm{Prod}(C)$. Moreover, since $kp = lg$ is a monomorphism, so is g and hence N is not critical. □

Definition 5.20 Let $\tau = (\mathcal{Q}, \mathcal{C})$ be a cotilting torsion pair. A neg-isolated module N in $\mathcal{C}$ is called **special** if it satisfies the equivalent conditions of Proposition 5.19.

Corollary 5.21 *Let I be an injective cogenerator of Mod-R with a special C-cover*

$$0 \to C_1 \to C_0 \overset{g}{\to} I \to 0. \tag{5.5}$$

(1) Every critical neg-isolated module in $\mathcal{C}$ is a direct summand of C_0.
(2) Every special neg-isolated module in $\mathcal{C}$ is a direct summand of C_1.

Proof To prove the corollary we will use the following general property of neg-isolated modules (see [30, Prop. 9.29]): if a neg-isolated module N in a definable subcategory $\mathcal{D}$ is a direct summand of $\prod_{i \in I} M_i$ where $M_i \in \mathcal{D}$ for all $i \in I$, then N is a direct summand of M_i for some $i \in I$. Since $C_0 \oplus C_1$ is a cotilting module equivalent to C, it follows from Corollary 4.4(1) that every $N \in \mathcal{N}_C$ is a direct summand of $C_0 \oplus C_1$. By Proposition 5.16(3), the neg-isolated summands of C_0 are exactly the critical ones and so the special neg-isolated modules are all direct summands of C_1 by [30, Prop. 9.29].   □

Corollary 5.22 *If $\tau = (\mathcal{Q}, \mathcal{C})$ is a cotilting torsion pair with cotilting module C, then* $\mathrm{PE}(\oplus_{\mathcal{N}_C} N)$ *is isomorphic to a direct summand of C.*

Proof The argument used in the first part of the proof of Corollary 5.21 implies that every $N \in \mathcal{N}_C$ arises as a direct summand of C. Equivalently, the functor $N \otimes_R -$, regarded as an object in the localised functor category $(R\text{-mod}, \mathrm{Ab})_{\mathcal{C}}$, is a coproduct factor of the injective object $C \otimes_R -$. Since N is neg-isolated, the **socle** $\mathrm{soc}(N \otimes_R -)$ is a simple object in $(R\text{-mod}, \mathrm{Ab})_{\mathcal{C}}$. It follows that $\coprod_{N \in \mathcal{N}_C} \mathrm{soc}(N \otimes_R -) \subseteq \mathrm{soc}(C \otimes_R -)$ and therefore that the injective envelope

$$E(\coprod_{N \in \mathcal{N}_C} \mathrm{soc}(N \otimes_R -)) = E(\coprod_{\mathcal{N}_C} (N \otimes_R -)) = E((\oplus_{\mathcal{N}_C} N) \otimes_R -) = \mathrm{PE}(\oplus_{\mathcal{N}_C} N) \otimes_R -$$

is a coproduct factor of $C \otimes_R -$, as required.   □

6 Examples

In this section, we discuss some examples that illustrate our results. We also study the special case of cotilting modules induced by ring epimorphisms, where we establish some interesting properties of special and critical neg-isolated modules.

6.1 The Kronecker algebra

Let Λ be the Kronecker algebra, i.e. the path algebra of the quiver $\bullet \overset{\longrightarrow}{\longrightarrow} \bullet$ over an algebraically closed field k. It is well known that the category of finite dimensional indecomposable modules admits a canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ denote the families given by the preprojective and preinjective modules, respectively, and $\mathbf{t} = \bigcup_{x \in \mathbb{X}} \mathbf{t}_x$ is the tubular family formed by the regular modules and indexed over the projective line $\mathbb{X} = \mathbb{P}^1(k)$. Given a simple regular module S, we denote by S_∞ and $S_{-\infty}$ the Prüfer and the adic module on the corresponding ray and coray, respectively. Further, we denote by G the generic module. Recall from [34] that $\mathrm{End}_R G$ is a division ring, and G is the unique (up to isomorphism) indecomposable module which has infinite length over Λ, but finite length over its endomorphism ring. Moreover, G cogenerates the class $\mathcal{F} = \mathbf{t}^{\perp_0}$ of all torsionfree modules, it generates the class $\mathcal{D} = {}^{\perp_0}\mathbf{t}$ of all divisible modules, and the intersection $\mathcal{F} \cap \mathcal{D} = \mathrm{Add}\,G$ is

equivalent to the category of all modules over a simple artinian ring Q which is obtained as universal localization of Λ at $\mathbf{t}$ and is Morita equivalent to $\mathrm{End}_R\, G$.

A complete description of the hearts of all cotilting Λ-modules is given in [32]. Let us focus on two special cases.

Example 6.1 Consider the cotilting torsion pair $\tau = (\mathcal{T}, \mathcal{F})$ generated by $\mathbf{t}$. It is cogenerated by the cotilting module

$$C = G \oplus \prod \{\text{all adic modules } S_{-\infty}\}.$$

The torsion, almost torsionfree modules coincide with the simple regular modules, while G is the only torsionfree, almost torsion module. We refer to [32] for the first statement and prove the second for the reader's convenience.

(AT1) Let $g : G \to B$ be a proper epimorphism, and $0 \to B' \to B \to \overline{B} \to 0$ the canonical exact sequence with $B' \in \mathcal{T}$ and $\overline{B} \in \mathcal{F}$. Then $\overline{B}$ lies in $\mathcal{F} \cap \mathcal{D} = \mathrm{Add} G$, so $G \xrightarrow{g} B \to \overline{B}$ is a morphism in $\mathrm{Add} G$ and is therefore zero or a split monomorphism. It follows that $\overline{B} = 0$ and $B \in \mathcal{T}$.

(AT2) Let $0 \to G \to B \to C \to 0$ be an exact sequence with $B \in \mathcal{F}$. By applying the functor $\mathrm{Hom}_\Lambda(S, -)$ given by a simple regular module S, we obtain an exact sequence

$$\mathrm{Hom}_\Lambda(S, B) \to \mathrm{Hom}_\Lambda(S, C) \to \mathrm{Ext}^1_\Lambda(S, G) \cong D\mathrm{Hom}_\Lambda(G, S)$$

where the first and third term are zero. Hence $C \in \mathcal{F}$.

We have shown that G is almost torsion. For the uniqueness, observe that any other torsionfree, almost torsion module $X \in \mathcal{F}$ is cogenerated by G, hence $\mathrm{Hom}_\Lambda(X, G) \neq 0$, and $X \cong G$ by Corollary 3.7.

It follows that G is simple injective in the heart $\mathcal{H}_\tau$. Moreover, every simple regular module S gives rise to a short exact sequence

$$0 \to S_{-\infty} \xrightarrow{a} S_{-\infty} \xrightarrow{b} S \to 0$$

as in Theorem B, and to a minimal injective coresolution

$$0 \to S[-1] \longrightarrow S_{-\infty} \longrightarrow S_{-\infty} \to 0$$

of the simple torsionfree object $S[-1]$ in $\mathcal{H}_\tau$. We infer that G is the only critical neg-isolated module in $\mathcal{F}$, and the special neg-isolated modules coincide with the adic modules. These are the neg-isolated modules in $\mathcal{F}$ which belong to $\mathrm{Prod}(C)$. Observe that also the modules in $\mathbf{p}$ are neg-isolated in $\mathcal{F}$, see Theorem 5.10.

Finally, let us remark that $\mathcal{H}_\tau$ is not hereditary. Indeed, it is shown in [41, 5.2] that the heart of a torsion pair is hereditary only if the torsion pair splits. But if S_x denotes the simple regular in the tube $\mathbf{t}_x$, then by [35, Prop. 5] there is a non-split exact sequence

$$0 \to \bigoplus_{x \in \mathbb{X}} S_x \to \prod_{x \in \mathbb{X}} S_x \to G^{(\alpha)} \to 0$$

with $\bigoplus_{x \in \mathbb{X}} S_x \in \mathcal{T}$ and $G^{(\alpha)} \in \mathcal{F}$. This shows that $(\mathcal{T}, \mathcal{F})$ is not a split torsion pair.

We have just seen that torsionfree, almost torsion modules may be infinite dimensional, while this is not possible for torsion, almost torsionfree modules according to Remark 3.5. The next example, however, exhibits a cotilting module without torsion, almost torsionfree modules.

Example 6.2 Consider now the torsion pair $\tau = (\mathcal{Q}, \mathcal{C})$ in Mod-Λ generated by the set **q**. It is cogenerated by the cotilting module

$$C = G \oplus \bigoplus \{\text{all Prüfer modules } S_\infty\}.$$

The heart $\mathcal{H}_\tau$ is locally coherent and hereditary, and it is equivalent to the category of quasi-coherent sheaves over $\mathbb{X}$. In particular, all simple objects in $\mathcal{H}_\tau$ are torsion. In other words, there are no torsion, almost torsionfree modules, and the torsionfree, almost torsion modules coincide with the simple regular modules. For details, we refer again to [32].

Every simple regular module S gives rise to a short exact sequence

$$0 \to S \xrightarrow{a} S_\infty \xrightarrow{b} S_\infty \to 0$$

as in Theorem A, which can also be regarded as the minimal injective coresolution of the simple torsion object S in $\mathcal{H}_\tau$. The critical neg-isolated modules in $\mathcal{C}$ thus coincide with the Prüfer modules, and there are no special neg-isolated modules.

The last example also shows that in general the class $\mathrm{Prod}C_0$ in Proposition 5.16 does not coincide with the class of split injectives in $\mathcal{C}$.

Example 6.3 Let $\tau = (\mathcal{Q}, \mathcal{C})$ be as in Example 6.2. Given an injective cogenerator I of Mod-Λ, there is a short exact sequence

$$0 \to C_1 \to C_0 \xrightarrow{g} I \to 0 \tag{6.1}$$

where g is a $\mathcal{C}$-cover, C_0 is a direct sum of Prüfer modules and C_1 is a direct sum of copies of G, see [33, Thm. 7.1]. Observe that C_1 is a direct summand of a product of copies of C_0 by [35, Prop. 3], but it is not split injective in $\mathcal{C}$ because the sequence (6.1) is not split.

6.2 Hereditary torsion pairs

In this section, we assume that $\tau = (\mathcal{Q}, \mathcal{C})$ is a **hereditary** cotilting torsion pair with cotilting module C. Equivalently, the torsionfree class $\mathcal{C}$ is closed under injective envelopes. If $V \in \mathcal{C}$ is split injective, then its injective envelope $V \to E(V)$ is a monomorphism in $\mathcal{C}$, and must therefore be an isomorphism. We conclude that the subcategory of split injective objects of $\mathcal{C}$ is given by the category $\mathrm{Inj}(\mathrm{Mod}\text{-}R) \cap \mathcal{C}$ of torsionfree injective modules, and that if $\mathcal{C}_0 = \mathcal{E}_\mathcal{C}$ denotes the set of critical modules in $\mathcal{C}$ as in Proposition 5.16, then

$$\mathrm{Prod}(\mathcal{C}_0) = \mathcal{C} \cap \mathrm{Inj}(\mathrm{Mod}\text{-}R).$$

Given a module M, we denote by $\mathcal{C}(M)$ a $\mathcal{C}$-cover of M and by $\mathcal{K}(M)$ its kernel,

$$0 \longrightarrow \mathcal{K}(M) \longrightarrow \mathcal{C}(M) \longrightarrow M \longrightarrow 0.$$

If C is a cotilting module cogenerating $\mathcal{C}$, then $\mathcal{K}(M)$ is an object of $\mathrm{Prod}(C)$ which is uniquely determined by M, up to isomorphism. If $F \in \mathcal{C}$ is a simple torsionfree module, then its injective envelope $E(F)$ is also torsionfree and so serves as its own $\mathcal{C}$-cover, but if $Q \in \mathcal{Q}$ is a *torsion* simple module, then its injective envelope $E(Q)$ is not torsionfree so its $\mathcal{C}$-cover is given by the epimorphism in

$$0 \longrightarrow \mathcal{K}(E(Q)) \longrightarrow \mathcal{C}(E(Q)) \xrightarrow{c} E(Q) \longrightarrow 0, \tag{6.2}$$

where $\mathcal{K}(E(Q)) \neq 0$.

In Example 3.3, we elaborated on the equivalence of Theorem 4.2(2) for a hereditary torsion pair, showing that the torsionfree, almost torsion modules correspond to the simple objects of the localisation Mod-$R/\mathcal{Q}$, and that their injective envelopes are the critical neg-isolated indecomposable pure-injectives. The following elaborates on the equivalence given by Theorem 4.2(1).

Theorem 6.4 *Suppose that $(\mathcal{Q}, \mathcal{C})$ is a hereditary cotilting torsion pair and $Q_R \in \mathcal{Q}$ is a torsion simple module. The $\mathcal{C}$-cover of Q is given by the pullback of (6.2) along its injective envelope*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}(E(Q)) & \overset{a}{\longrightarrow} & F & \overset{b}{\longrightarrow} & Q & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{K}(E(Q)) & \longrightarrow & \mathcal{C}(E(Q)) & \overset{c}{\longrightarrow} & E(Q) & \longrightarrow & 0.
\end{array}
$$

Consequently, $F = \mathcal{C}(Q)$, $\mathcal{K}(Q) = \mathcal{K}(E(Q))$ and $\mathcal{C}(E(Q)) = E(\mathcal{C}(Q))$. Furthermore, $a\colon \mathcal{K}(Q) \to \mathcal{C}(Q)$ is a left almost morphism in $\mathcal{C}$ and $\mathcal{K}(Q)$ is a special neg-isolated indecomposable pure-injective.

Proof As in the argument used in the proof of Proposition 5.17, the module $\mathcal{C}(E(Q))$ must be split injective and therefore, by the hereditary property, injective. Moreover, any indecomposable summand of $\mathcal{C}(E(Q))$ which does not intersect $\mathcal{K}(E(Q))$ must be isomorphic to $E(Q)$, which is impossible as $Q \in \mathcal{Q}$. We conclude that the kernel morphism of c is an injective envelope. This also implies that the embedding of F into $\mathcal{C}(E(Q))$ is an injective envelope.

Furthermore, it follows that $\mathcal{K}(E(Q))$ is indecomposable. For, suppose that $\mathcal{K}(E(Q)) = K_1 \oplus K_2$ were a proper decomposition. Neither of the summands can be injective, since they are contained in the kernel of a $\mathcal{C}$-cover. The injective envelope of $\mathcal{K}(E(Q))$ as well as its cosyzygy would then be decomposable, contradicting the fact that $E(Q)$ is not.

Because F is torsionfree and $\mathcal{K}(E(Q))$ belongs to $\mathcal{C}^{\perp_1}$, $b\colon F \to Q$ is a special $\mathcal{C}$-precover. As such, it contains the $\mathcal{C}$-cover $\mathcal{C}(Q) \to Q$ as a direct summand, whose kernel would be a direct summand of $\mathcal{K}(E(Q))$. But $\mathcal{K}(E(Q))$ is indecomposable, and Q is not torsionfree, so the $\mathcal{C}$-cover of Q must contain $\mathcal{K}(E(Q))$, and properly so. As Q is simple, so we see that $b\colon F = \mathcal{C}(Q) \to Q$ is the $\mathcal{C}$-cover of Q and $\mathcal{K}(Q) = \mathcal{K}(E(Q))$.

The last statement follows from Theorem 4.2(1). □

Theorem 6.4 allows us to infer that the module $\mathrm{PE}(\oplus_{\mathcal{N}_C} N)$ of Corollary 5.22 that arises as a summand of any cotilting module C for $\mathcal{C}$ is itself cotilting. Because it is a summand of a cotilting module, it certainly satisfies the first two conditions of being one. To verify the third, consider the injective cogenerator $I = \prod_{\mathrm{Sim}(\mathrm{Mod}\text{-}R)} E(S)$ of Mod-R, where the index set runs over the set of all simple modules. It may be decomposed as

$$
I = \prod_{Q \in \mathrm{Sim}(\mathcal{Q})} E(Q) \quad \oplus \quad \prod_{F \in \mathrm{Sim}(\mathcal{C})} E(F),
$$

where the index set has been partitioned into the simple torsion and simple torsionfree modules, respectively. Take the special $\mathcal{C}$-precover of I given by the product of the respective $\mathcal{C}$-covers,

$$
0 \longrightarrow \prod_{Q \in \mathrm{Sim}(\mathcal{Q})} \mathcal{K}(Q) \longrightarrow \prod_{Q \in \mathrm{Sim}(\mathcal{Q})} \mathcal{C}(E(Q)) \oplus \prod_{F \in \mathrm{Sim}(\mathcal{C})} E(F) \longrightarrow I \longrightarrow 0.
$$

Because every $\mathcal{K}(Q)$ is neg-isolated, the kernel belongs to $\mathrm{Prod}(\mathrm{PE}(\oplus_{\mathcal{N}_C} N))$. On the other hand, the middle term is injective and therefore belongs to $\mathcal{C} \cap \mathrm{Inj}(\mathrm{Mod}\text{-}R) = \mathrm{Prod}(\mathcal{C}_0)$.

Corollary 6.5 *If* $(\mathcal{Q}, \mathcal{C})$ *is a hereditary cotilting torsion pair with cotilting module* C, *then the module* $\mathrm{PE}(\oplus_{\mathcal{N}_C} N) = \mathrm{PE}((\oplus_{\mathcal{E}_C} M) \oplus (\oplus_{\mathcal{M}_C} L))$, *given more explicitly by*

$$\mathrm{PE}(\bigoplus_{Q \in \mathrm{Sim}(\mathcal{Q})} \mathcal{K}(Q) \oplus \bigoplus_{F \in \mathrm{Sim}(\mathit{Mod}\text{-}R/\mathcal{Q})} E(F)),$$

is a 1-*cotilting module for* $\mathcal{C}$ *that is isomorphic to a direct summand of* C.

6.3 Commutative noetherian rings

Let now R be a commutative noetherian ring. It is shown in [23] that the cotilting torsion pairs are precisely the hereditary torsion pairs in Mod-R with R being torsionfree. They are parametrized by the subsets $P \subset \mathrm{Spec}(R)$ that are closed under specialization and satisfy $\mathrm{Ass}\,R \cap P = \emptyset$. More precisely, P corresponds to the hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ given by

$$\mathcal{T} = \{M \in \mathrm{Mod}\text{-}R \mid \mathrm{Supp}\,M \subset P\}$$
$$\mathcal{F} = \{M \in \mathrm{Mod}\text{-}R \mid \mathrm{Ass}\,M \cap P = \emptyset\}$$

where $\mathcal{T}$ contains all $E(R/\mathfrak{p})$ with $\mathfrak{p} \in P$, and $\mathcal{F}$ contains all $E(R/\mathfrak{q})$ with $\mathfrak{q} \in Q = \mathrm{Spec}(R) \backslash P$.

We denote by $\mathrm{Max}(P)$ the set of all maximal ideals of R which lie in P and by $\mathrm{Max}(Q) = \{\mathfrak{q} \in Q \mid V(\mathfrak{q}) \backslash \{\mathfrak{q}\} \subset P\}$ the set of all prime ideals which are maximal in Q.

Proposition 6.6 *The module*

$$PE(\bigoplus_{\mathfrak{m} \in \mathit{Max}(P)} \mathcal{K}(R/\mathfrak{m})) \oplus \bigoplus_{\mathfrak{q} \in \mathit{Max}(Q)} E(R/\mathfrak{q})$$

is a cotilting module cogenerating $\mathcal{F}$ *which is isomorphic to a direct summand of any other cotilting module cogenerating* $\mathcal{F}$.

Proof The torsion simple modules are precisely the modules of the form $R/\mathfrak{m}$ with $\mathfrak{m} \in \mathrm{Max}(P)$. Further, since the ring is noetherian, the direct sum $\bigoplus_{F \in \mathrm{Sim}(\mathrm{Mod}\text{-}R/\mathcal{T})} E(F)$ is (pure-)injective. By Corollary 6.5, it remains to show that the modules $E(F)$ with $F \in \mathrm{Sim}(\mathrm{Mod}\text{-}R/\mathcal{T})$ are precisely the indecomposable injectives $E(R/\mathfrak{q})$ with $\mathfrak{q} \in \mathrm{Max}(Q)$, up to isomorphism.

In order to verify this, recall first that any such $E(R/\mathfrak{q})$ is torsionfree and therefore cogenerated by critical modules. In particular $\mathrm{Hom}_R(E(R/\mathfrak{q}), E(F)) \neq 0$ for some $F \in \mathrm{Sim}(\mathrm{Mod}\text{-}R/\mathcal{T})$. But $E(F)$ is indecomposable injective and thus of the form $E(R/\mathfrak{p})$ for some $\mathfrak{p} \in V(\mathfrak{q})$. Moreover $\mathfrak{p} \in Q$ as $E(F) \in \mathcal{F}$. It follows by assumption that $\mathfrak{p} = \mathfrak{q}$, hence $E(R/\mathfrak{q}) \cong E(F)$.

For the converse implication, we employ [43, Theorem 5.2] which states that

$$C = \mathcal{K}(\bigoplus_{\mathfrak{m} \in \mathrm{Max}(P)} E(R/\mathfrak{m})) \oplus \bigoplus_{\mathfrak{q} \in \mathrm{Max}(Q)} E(R/\mathfrak{q})$$

is a cotilting module with the stated properties. It follows that every $E(F)$ is isomorphic to a direct summand of C. Now observe that $\mathcal{K}(\bigoplus_{\mathfrak{m} \in \mathrm{Max}(P)} E(R/\mathfrak{m}))$ cannot have injective summands by minimality, so $E(F)$ must be isomorphic to some $E(R/\mathfrak{q})$ with $\mathfrak{q} \in \mathrm{Max}(Q)$. $\square$

We remark that the modules in $\mathrm{Sim}(\mathrm{Mod}\text{-}R/\mathcal{T})$ need not have the form $R/\mathfrak{q}$, as we are going to see next.

Example 6.7 Let $P = \mathrm{Max}(\mathbb{Z})$ be the (specialization closed) set of all maximal ideals of $\mathbb{Z}$. Then $Q = \mathrm{Max}(Q) = \{0\}$, and $(\mathcal{T}, \mathcal{F})$ is the torsion pair formed by the torsion and torsionfree abelian groups, respectively. So the only simple object in $\mathrm{Mod}\text{-}\mathbb{Z}/\mathcal{T}$ is $\mathbb{Q}$, which coincides with its injective envelope and is also the only module which is critical neg-isolated in $\mathcal{F}$.

6.4 Minimal cotilting modules

Throughout this section, we assume that C is a cotilting R-module with torsion pair $\tau = (\mathcal{Q}, \mathcal{C})$ and that there is an injective cogenerator I of $\mathrm{Mod}\text{-}R$ with a special $\mathcal{C}$-cover

$$0 \to C_1 \to C_0 \overset{g}{\to} I \to 0 \tag{6.3}$$

such that

(M1) The left perpendicular category $\mathcal{Y} = {}^{\perp_{0,1}}C_1$ is closed under direct products;
(M2) $\mathrm{Hom}_R(C_0, C_1) = 0$.

Cotilting modules with this property are called **minimal** and we call the associated class $\mathcal{C}$ a **minimal cotilting class**.

Observe that condition (M1) amounts to the fact that the inclusion functor $\mathcal{Y} \to \mathrm{Mod}\text{-}R$ has a left adjoint and a right adjoint. This is equivalent to the existence of a ring epimorphism $\lambda : R \to S$ which is **pseudoflat**, i.e. $\mathrm{Tor}_1^R(S, S) = 0$, such that $\mathcal{Y}$ is the essential image of the functor $\lambda_* : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ given by restriction of scalars.

In fact, minimal cotilting modules are closely related to pseudoflat ring epimorphisms. Let k be a commutative ring such that R is a k-algebra, and let $D = \mathrm{Hom}_k(-, E)$ be the duality induced by an injective cogenerator of $\mathrm{Mod}\text{-}k$. Any injective ring epimorphism $\lambda : R \to S$ with the property that $\mathrm{Cogen}\, D({}_R S) \subset {}^{\perp_1} D({}_R S)$ is pseudoflat and induces a minimal cotilting right R-module $C = D({}_R S) \oplus D({}_R S/R)$, for which the sequence (6.3) can be chosen as

$$0 \to D({}_R S/R) \to D({}_R S) \overset{g}{\to} D({}_R R) \to 0 \tag{6.4}$$

We refer to [4, Section 4] for details. In particular, we have

Theorem 6.8 *[4, Theorem 4.16, Corollaries 4.18 and 4.19] The map assigning to a ring epimorphism $\lambda : R \to S$ the class $\mathrm{Cogen}\, D({}_R S)$ yields a bijection between*

(i) epiclasses of injective ring epimorphisms $\lambda : R \to S$ with $\mathrm{Cogen}\, D({}_R S) \subset {}^{\perp_1} D({}_R S)$,
(ii) minimal cotilting classes.

Moreover, the set in (i) equals the set of

(i') epiclasses of injective pseudoflat ring epimorphisms,

provided that R has weak global dimension at most one or is a commutative noetherian ring.

Example 6.9 (**1**) Let Λ be the Kronecker algebra over a field k. Consider the pseudoflat ring epimorphism $\lambda : \Lambda \to \Lambda_{\mathbf{t}}$ given by universal localization at $\mathbf{t}$. We have a short exact sequence

$$0 \to \Lambda \to \Lambda_{\mathbf{t}} \cong G \oplus G \to \bigoplus \{\text{all Prüfer modules } S_\infty\} \to 0$$

in Λ-Mod. Applying the duality $D = \mathrm{Hom}_k(-, k)$ we obtain a short exact sequence in Mod-Λ

$$0 \rightarrow \prod \{\text{all adic modules } S_{-\infty}\} \rightarrow D(G \oplus G) \rightarrow D(\Lambda) \rightarrow 0$$

and a minimal cotilting module equivalent to the cotilting module in Example 6.1.

(2) [4, Example 4.10] The cotilting module in Example 6.2 is not minimal. In fact, it is the only non-minimal cotilting Λ-module up to equivalence.

(3) [4, Example 4.22 (3)] Over a commutative noetherian ring of Krull dimension at most one, all cotilting modules are minimal.

Let us collect some properties of minimal cotilting modules. In what follows we will make use of the notation introduced before Corollary 4.4, noting that the set of neg-isolated modules in $\mathcal{C}$ that are contained in $\mathrm{Prod}(C)$ coincides with $\mathcal{N}_C$, the set of critical neg-isolated modules coincides with $\mathcal{E}_C$, and the set of special neg-isolated modules coincides with $\mathcal{M}_C$. See Sect. 5.7 for more details.

Lemma 6.10 *Let assumptions and notation be as above, and suppose that the heart $\mathcal{H}_\tau$ is locally finitely generated.*

(1) $\mathrm{Prod}(C_0) = \mathrm{Prod}(\mathcal{E}_C)$ *and* $\mathrm{Prod}(C_1) = \mathrm{Prod}(\mathcal{M}_C)$.
(2) *An indecomposable module X lies in $\mathrm{Prod}(C)$ if and only if it lies in $\mathrm{Prod}(C_0)$ or $\mathrm{Prod}(C_1)$, and not in both.*
(3) *If $\mathcal{C}$ is a proper subcategory of Mod-R, then it contains special neg-isolated modules, and $\mathcal{Q}$ contains torsion, almost torsionfree modules.*

Proof The first statement in (1) and the inclusion $\mathrm{Prod}(C_1) \supseteq \mathrm{Prod}(\mathcal{M}_C)$ are true in general, see Proposition 5.17 and Corollary 5.21(2).

When $\mathcal{H}_\tau$ is locally finitely generated, then the direct product of all injective envelopes of simple objects is an injective cogenerator of $\mathcal{H}_\tau$. Hence $\mathrm{Prod}(C) = \mathrm{Prod}(\mathcal{N}_C)$ where $\mathcal{N}_C = \mathcal{E}_C \cup \mathcal{M}_C$ is the set of all neg-isolated modules in $\mathcal{C}$ which belong to $\mathrm{Prod}(C)$. So, every module $X \in \mathrm{Prod}(C)$ admits a split monomorphism $\iota : X \hookrightarrow M_0 \oplus M_1$ where M_i is a direct product of modules in $\mathcal{C}_i$ for $i = 0, 1$. Then we can write $\mathrm{id}_X = \pi_0 \iota_0 + \pi_1 \iota_1$ where ι_i and π_i are the components of ι and of its left inverse $\pi : M_0 \oplus M_1 \rightarrow X$.

Now, if $X \in \mathrm{Prod}(C_1)$, then it follows from condition (M2) that $C_0 \in {}^{\perp_0} X$, hence by condition (M1) we have $\mathrm{Prod}(C_0) \subseteq {}^{\perp_0} X$, which implies that $\pi_0 : M_0 \rightarrow X$ vanishes. Thus $\mathrm{id}_X = \pi_1 \iota_1$ and X lies in $\mathrm{Prod}(\mathcal{M}_C)$. This concludes the proof of statement (1).

Moreover, if X is an indecomposable module in $\mathrm{Prod}(C)$, then it is pure-injective and therefore has a local endomorphism ring. This shows that $\pi_i \iota_i$ must be an isomorphism for $i = 0$ or $i = 1$, that is, X lies in $\mathrm{Prod}(C_0)$ or $\mathrm{Prod}(C_1)$. Finally, X can't lie in both, because $\mathrm{Prod}(C_0) \subset {}^{\perp_0} C_1$ by conditions (M1) and (M2). This proves statement (2).

For statement (3), observe that the $\mathcal{C}$-cover in (6.3) can't be an isomorphism, so $C_1 \neq 0$ and $\mathcal{C}_1 \neq \emptyset$. The existence of torsion, almost torsionfree modules then follows from Theorem B. $\square$

Proposition 6.11 [39] *Let assumptions and notation be as above, and suppose that R is right artinian. Assume further that $\mathcal{C}$ is a proper subcategory of Mod-R, and let $\mathcal{W}$ be the class of all modules with a finite filtration by torsion, almost torsionfree modules. Then the class of all direct limits of modules in $\mathcal{W}$ coincides with the left perpendicular category ${}^{\perp_{0,1}} C_0$ of the module C_0 in (6.3).*

Theorem 6.12 *Let R be a right artinian ring and let C be a minimal cotilting module with associated ring epimorphism $\lambda : R \to S$.*

(1) Every indecomposable summand of a product of special neg-isolated modules in $\mathcal{C}$ is special neg-isolated in $\mathcal{C}$.

(2) S is a right coherent ring.

(3) C_0 is an elementary cogenerator.

Proof (1) Suppose M is an indecomposable module in $\mathrm{Prod}(\mathcal{M}_C)$ which is not special. Recall that $\mathcal{Y} = {}^{\perp_{0,1}}C_1$ is the essential image of λ_*. By [4, Proposition 4.15] there is an exact sequence

$$0 \to M \xrightarrow{\eta} M \otimes_R S \to M'' \to 0$$

where M'' belongs to $\mathcal{Q}$ and $\mathrm{Hom}_R(\eta, Y)$ is an isomorphism for every module $Y \in \mathcal{Y}$. In particular, $\mathrm{Hom}_R(\eta, C_0)$ is an isomorphism. Furthermore, $\mathrm{Ext}^1_R(M \otimes_R S, C_0) = 0$ because $M \otimes_R S \in \mathcal{Y} \subseteq {}^{\perp_1}C_1 = {}^{\perp_1}C$ and $C_0 \in \mathrm{Prod}(C)$. It follows that $\mathrm{Ext}^1_R(M'', C_0) = 0$, and of course also $\mathrm{Hom}_R(M'', C_0) = 0$. But then M'' lies in the left perpendicular category of C_0, and by Proposition 6.11 it is a direct limit of modules in $\mathcal{W}$, the class of all modules with a finite filtration by torsion, almost torsionfree modules.

Consider now $W \in \mathcal{W}$. We have $\mathrm{Hom}_R(W, M \otimes_R S) = 0$ because W is torsion and $M \otimes_R S$ is torsionfree. Moreover, $\mathrm{Ext}^1_R(W, M) = 0$, because M is an indecomposable, non-special module in $\mathrm{Prod}(C)$, thus becomes indecomposable injective in $\mathcal{H}_\tau$ and satisfies $\mathrm{Ext}^1_R(S, M) \cong \mathrm{Hom}_{\mathcal{H}_\tau}(S[-1], M) = 0$ for all torsion, almost torsionfree modules S. We conclude that $\mathrm{Hom}_R(W, M'') = 0$.

In conclusion, we have shown that $M'' = 0$ and $M \cong M \otimes_R S$ belongs to $\mathcal{Y}$. But then $M \in {}^{\perp_0}C_1$, which contradicts the assumption $M \in \mathrm{Prod}(\mathcal{M}_C) = \mathrm{Prod}(C_1)$.

(2) By Example 4.6, the heart $\mathcal{H}_\tau$ is a locally coherent Grothendieck category. Then we know from [22], [26] that the (isoclasses of) indecomposable injective objects form a topological space $\mathrm{Spec}(\mathcal{H}_\tau)$, where a basis of open subsets is given by the collection

$$\mathcal{O}(M) = \{E \in \mathrm{Spec}(\mathcal{H}_\tau) \mid \mathrm{Hom}_{\mathcal{H}_\tau}(M, E) \neq 0\}, \ M \in \mathcal{H}_\tau \cap \mathcal{D}^b(\mathrm{mod}\text{-}R).$$

Moreover, there is a one-one-correspondence between the open subsets of $\mathrm{Spec}(\mathcal{H}_\tau)$ and the hereditary torsion pairs of finite type in $\mathcal{H}_\tau$, which maps a torsion pair $(\mathcal{S}, \mathcal{R})$ to the set of indecomposable injectives which are not contained in $\mathcal{R}$.

Let us now consider the set $\Omega = \{S[-1] \mid S \text{ torsion, almost torsionfree}\}$ of all simple torsionfree objects in $\mathcal{H}_\tau$. It generates a hereditary torsion pair $(\mathcal{S}, \mathcal{R})$ of finite type in $\mathcal{H}_\tau$, which is associated to the open set

$$\mathcal{O} = \{E \in \mathrm{Spec}(\mathcal{H}) \mid \mathrm{Hom}_{\mathcal{H}}(M, E) \neq 0 \text{ for some } M \in \Omega\} = \mathcal{M}_C$$

in $\mathrm{Spec}(\mathcal{H}_\tau)$. Moreover, $\mathcal{S}$ is a localizing subcategory of $\mathcal{H}_\tau$, and the corresponding quotient category $\mathcal{H}_\tau/\mathcal{S}$ is again a locally coherent Grothendieck category whose spectrum is formed by the indecomposable injective objects in $\mathcal{R}$, that is, by the complement $\mathcal{O}^c$ of $\mathcal{O}$, cf. [22, Thm. 2.16 and Prop. 3.6].

We have seen in (1) that an indecomposable module in $\mathrm{Prod}(C)$ that is not in $\mathcal{M}_C$ can't belong to $\mathrm{Prod}(\mathcal{M}_C)$ and thus must lie in $\mathrm{Prod}(C_0)$ by Proposition 6.10. That is, every indecomposable injective in $\mathcal{R}$ is contained in $\mathrm{Prod}(C_0)$. Since every injective in $\mathcal{R}$ is a direct summand of a product of indecomposables, it follows that the injectives in $\mathcal{R}$ coincide with $\mathrm{Prod}(C_0)$. In other words, the class of injective objects of $\mathcal{H}_\tau/\mathcal{S}$ is $\mathrm{Prod}(C_0) = \mathrm{Prod}(D(S))$. Observe that these are precisely the indecomposable injective

right S-modules. Since Grothendieck categories are determined up to equivalence by their injective objects (cf. [42, p.81]), it follows that $\mathcal{H}_\tau/\mathcal{S}$ and Mod-S are equivalent, and in particular, S must be right coherent.

(3) Recall that a module M is said to be fp-injective if every short exact sequence starting at M and ending at a finitely presented module is split exact, or equivalently, if M is a pure submodule of an injective module. Since S is right coherent, we know from [40] that the class of fp-injective right S-modules is closed under direct limits. In other words, the class $\mathrm{Cogen}_*(D(S))$ of all pure S-submodules of products of copies of the injective cogenerator $D(S)$ is a definable subcategory of Mod-S. Observe that the functor $\lambda_* : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ takes $\mathrm{Cogen}_*(D(S))$ to the class $\mathrm{Cogen}_*(C_0)$ of all pure R-submodules of products of copies of C_0. Indeed, this is due to the fact that λ_* commutes with direct products and direct limits, and that pure-exact sequences are direct limits of split exact sequences. Conversely, any module X in $\mathrm{Cogen}_*(C_0)$ is an S-module, and any pure embedding of X in a product of copies of C_0 is also a pure embedding of S-modules by [31, Cor. 6.1.11], so that X lies in $\mathrm{Cogen}_*(D(S))$. Now we can conclude by [31, Thm. 6.1.11] that $\mathrm{Cogen}_*(C_0)$ is a definable subcategory of Mod-R, that is, C_0 is an elementary cogenerator. □

Remark 6.13 Let C be a minimal cotilting module with associated ring epimorphism $\lambda : R \to S$. Then S is right noetherian if and only if C_0 is Σ-pure-injective. Indeed, the only-if-part follows from the fact that C_0 is an injective cogenerator of Mod-S and $\mathrm{Prod}(C_0)$ is thus closed under coproducts. Assume conversely that C_0 is Σ-pure-injective. By [20] there is a cardinal κ such that $\mathrm{Prod}(C_0) \subseteq \mathrm{Add}(M)$ where M is the direct sum of a set of representatives of modules in $\mathrm{Prod}(C_0)$ of cardinality at most κ. It is not difficult to see that $\mathrm{Prod}(C_0)$ then even equals $\mathrm{Add}(M)$ and is therefore closed under coproducts, which shows that S is right noetherian.

Example 6.14 **(1)** If R is a commutative noetherian ring, then we know from Proposition 6.6 that C_0 is injective and thus Σ-(pure-)injective.

(2) Consider the hereditary noetherian ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. It is shown in [9, Example 4] that the non-noetherian ring $S = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$, where p, q are two prime numbers, is a universal localization of R at a set of matrices Σ. Hence for the minimal cotilting module C given by the pseudoflat injective ring epimorphism $R \to S$ we have that C_0 is not Σ-pure-injective.

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