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Ill-posedness for the Half wave Schrödinger equation

Isao Kato[1]

Abstract

We study the Cauchy problem for the half wave Schrödinger equation introduced by Xu [9]. There are some well-posedness results for the equation, however there is no ill-posedness result. We focus on the scale critical space and obtain the ill-posedness in the super-critical or at the critical space under certain condition. The proofs in the super-critical space are based on the argument established by Christ, Colliander and Tao [4]. More precisely, we analyze dispersionless equation with smooth initial data, namely the Schwartz function and it is locally well-posed in some weighted Sobolev space. We construct the solution for the half wave Schrödinger equation by using the solution for the dispersionless equation and we can exploit the norm inflation or the decoherence properties. For the critical space, we use the standing wave solution, which was proved the existence by Bahri, Ibrahim and Kikuchi [1].

Keywords Cauchy problem · Ill-posedness · Schrödinger equation · Critical space · Standing wave

Mathematics Subject Classification 35Q55 · 35A01

1 Introduction

We consider the Cauchy problem for the following equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u - |D_y|u = \mu |u|^{p-1}u, & (t, x, y) \in \mathbb{R}^3, \\ u(0, x, y) = u_0(x, y), \end{cases} \tag{1.1}$$

where $|D_y| = (-\partial_y^2)^{\frac{1}{2}}$, $\mu = \pm 1$, $p > 1$. (1.1) with $\mu = 1$, $p = 3$ is firstly considered by Xu [9] on the cylinder $(x, y) \in \mathbb{R} \times \mathbb{T}$. In [9], large time behavior for (1.1), namely the modified scattering for small regular solution is derived. For the case $\mu = -1$ and $1 < p < 5$, Bahri, Ibrahim and Kikuchi [1] proved existence and the stability of the ground states $Q_\omega(x, y)$ with $\omega > 0$ which satisfy

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$$-\partial_x^2 Q + |D_y|Q + \omega Q - |Q|^{p-1}Q = 0.$$

They also found the traveling wave solutions $e^{i\omega t} Q_{\omega,v}(x, y - vt)$, $\omega > 0$, $v \in \mathbb{R}$ exist and derived the behavior of $Q_{\omega,v}$ in $H^{1,0}(\mathbb{R}^2)$ or $L^2(\mathbb{R}^2)$ as the velocity v tends to 1. See the definition of $H^{s_1,s_2}(\mathbb{R}^2)$ in the following section. Here $Q_{\omega,v}$ is a solution to the following equation:

$$-\partial_x^2 Q + |D_y|Q + iv\partial_y Q + \omega Q - |Q|^{p-1}Q = 0.$$

Concerning the Cauchy problem, [1] showed the local well-posedness in the scale sub-critical space, namely $H^{0,s}(\mathbb{R}^2)$ with $\frac{1}{2} < s < 1$ and $1 < p \leq 5$ by the standard fixed point argument (see section 2 for the definition of scale sub-critical space)[1]. The following Strichartz estimate is a key role to show local well-posedness: For $S(t) := e^{it(\partial_x^2 - |D_y|)}$ and $s > \frac{1}{2}$, there exists $C > 0$ such that

$$\|S(t)f\|_{L_T^4 L_{x,y}^\infty} \leq C\|f\|_{H^{0,s}}, \qquad f \in H^{0,s}(\mathbb{R}^2),$$

$$\left\| \int_0^t S(t - s)F(s)\,\mathrm{d}s \right\|_{L_T^4 L_{x,y}^\infty} \leq C\|F\|_{L_T^1 H^{0,s}}, \qquad F \in L_T^1 H^{0,s}(\mathbb{R}^2).$$

So far there is no result for lower regularity space, for instance the energy space $E := H^{1,0}(\mathbb{R}^2) \cap H^{0,\frac{1}{2}}(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_E := \left(\|\partial_x u\|_{L^2}^2 + \||D_y|^{\frac{1}{2}}u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

In Theorem 1.1 below, we find that for $p > 5$ the ill-posedness (norm inflation) holds in E since E is the super-critical space in this case. However for $1 < p < 5$ (resp. $p = 5$), E is the sub-critical (resp. scale-critical) space and we do not obtain any result yet. We explain some difficulties to show well-posedness in E for $1 < p \leq 5$. The first one is the restriction of regularity with respect to y variable, namely $s > \frac{1}{2}$ for the above Strichartz estimate. We do not know whether the Strichartz estimate hold or not for $s = \frac{1}{2}$. If we neglect the term $\partial_x^2 u$ and x variable, (1.1) is nothing but the half wave equation and it is ill-posed in $H^s(\mathbb{R})$ for $s < \frac{1}{2}$ and $p = 3$ (see e.g. Theorem 1.3 [3]). For the delicate case $s = \frac{1}{2}$, one may apply the argument in the cubic Szegő equation [6] or the half wave equation [8]. However, for instance (1.1) with $p = 3$, the Yudovich argument, which was applied to show uniqueness of the solution for Szegő equation [6] or the half wave equation [8], is difficult to apply. In fact, we cannot replace the right-hand side of the following inequality X by the energy space E. For $q \in [2, \infty)$, $u \in X = H^{\frac{1}{2},\frac{1}{2}}(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$, there exists $C > 0$ such that the following holds:

$$\|u\|_{L^q(\mathbb{R}^2)} \leq C\sqrt{q}\,\|u\|_X.$$

This is the second difficulty. Moreover, the Gagliardo-Nirenberg inequality (Lemma 2.1 [1]):

$$\|u\|_{L^{q+1}(\mathbb{R}^2)}^{q+1} \leq C_{\mathrm{GN}}\|\partial_x u\|_{L^2(\mathbb{R}^2)}^{\frac{q-1}{2}}\||D_y|^{\frac{1}{2}}u\|_{L^2(\mathbb{R}^2)}^{q-1}\|u\|_{L^2(\mathbb{R}^2)}^{\frac{5-q}{2}}, \qquad C_{\mathrm{GN}} > 0$$

holds only for $1 < q < 5$. However, we need $q \to \infty$ to show the uniqueness. These are the main reasons why we cannot apply the standard technique. Thus the Cauchy problem (1.1) in E for $1 < p \leq 5$ is an interesting open problem.

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[1] We can also obtain the local well-posedness in $H^{1,0}(\mathbb{R}^2) \cap H^{0,s}(\mathbb{R}^2)$, $s > \frac{1}{2}$ for (1.1), $\mu = \pm 1$, $p = 3$.

The aim of this paper is to investigate the ill-posedness for (1.1) in the super-critical (resp. at the critical) space H^{s_1,s_2} for $\mu = \pm 1$ (resp. $\mu = -1$) since there is no ill-posedness result. The first statement is the norm inflation in the super-critical space.

Theorem 1.1 *Let $p > 1$ be an odd integer and $\mu = \pm 1$. Suppose that either $s_1, s_2 \geq 0$ except for $s_1 = s_2 = 0$ and $s_1 + 2s_2 < \frac{3}{2} - \frac{2}{p-1}$ or $s_1, s_2 \leq -\frac{1}{2}$. Then for any $\varepsilon > 0$ there exist a solution u for (1.1) and $t > 0$ such that $u(0) \in \mathcal{S}(\mathbb{R}^2)$,*

$$\|u(0)\|_{H^{s_1,s_2}} < \varepsilon, \qquad 0 < t < \varepsilon, \qquad \|u(t)\|_{H^{s_1,s_2}} > \frac{1}{\varepsilon}.$$

If $p > 1$ is not an odd integer, then the same conclusion holds provided that there exists an integer $k > 1$ such that $p \geq k + 1$ and either $s_1, s_2 \geq 0$ except for $s_1 = s_2 = 0$ and $s_1 + 2s_2 < \frac{3}{2} - \frac{2}{p-1}$ or $s_1, s_2 \leq -\frac{1}{2}$.

The second result is the decoherence in H^{s_1,s_2} with non-positive s_1, s_2. Here, the decoherence means that the flow map is not uniformly continuous at time 0 in H^{s_1,s_2}.

Theorem 1.2 *Let $p > 1$ be an odd integer and $\mu = \pm 1$. For any $s_1, s_2 \leq 0$, there exist the constants $C > 0$ such that the following estimates (1.2)–(1.4) hold. For any $0 < \delta, \varepsilon < 1$ and $t > 0$ there exist solutions for (1.1) with initial data $u_1(0), u_2(0) \in \mathcal{S}(\mathbb{R}^2)$ such that*

$$\|u_1(0)\|_{H^{s_1,s_2}}, \|u_2(0)\|_{H^{s_1,s_2}} \leq C\varepsilon, \tag{1.2}$$

$$\|u_1(0) - u_2(0)\|_{H^{s_1,s_2}} \leq C\delta, \tag{1.3}$$

$$\|u_1(t) - u_2(t)\|_{H^{s_1,s_2}} \geq C\varepsilon. \tag{1.4}$$

If $p > 1$ is not an odd integer, then the same conclusion holds provided that there exists an integer $k > \frac{3}{2}$ such that $p \geq k + 1$ and $s_1, s_2 \leq 0$.

The proofs for Theorems 1.1 and 1.2 are based on the work of Christ, Colliander and Tao [4] (Theorems 2, 1). In [4], they proved the norm inflation and the decoherence for the Schrödinger equation and the wave equation with power type nonlinearity $\mu|u|^{p-1}u$, $\mu = \pm 1$, by regarding the nonlinearity as the main term and the linear dispersive part as the perturbation. To show the decoherence for the Schrödinger equation, the Galilean invariance parameter plays a crucial role. Using this idea with suitable modification (since (1.1) has different dispersion in x and y variable), we can obtain the ill-posedness for (1.1). We remark that we consider the ill-posedness in the super-critical space in Theorems 1.1 and 1.2 and in Theorem 1.2 more regularity (or weight) k is needed than in Theorem 1 [4].

The third main result is also the decoherence, and we consider at the scale critical space with non-negative s_1, s_2.

Theorem 1.3 *Let $1 < p < 5$ and $\mu = -1$. For any $s_1, s_2 \geq 0, s_1 + 2s_2 = \frac{3}{2} - \frac{2}{p-1}, 0 < \delta, \varepsilon < 1$ and $t > 0$ there exist solutions for (1.1) with initial data $u_1(0), u_2(0) \in H^{s_1,s_2}(\mathbb{R}^2)$ such that (1.3) and (1.4) hold in Theorem 1.2.*

We note that in Theorem 1.3, we only consider the focusing nonlinearity $\mu = -1$ and $1 < p < 5$ since we use the standing wave solutions for (1.1) with $\mu = -1$ obtained by [1].

Let us introduce an organization of this paper. In section 2, we give some notations and lemmas (Lemmas 2.1, 2.2). By the usual energy method, we obtain Lemma 2.1, namely local well-posedness for (2.2) (the equation with small dispersion parameter ν) with regular initial data. In section 3, we prove the norm inflation in the super-critical space. In section 4, we prove the decoherence. We apply Lemma 2.2 together with the scaling to show the decoherence occurs in a very short time. Finally in section 5, we prove the decoherence at the critical space for the focusing case.

2 Preliminaries

$\hat{f}$ denotes the Fourier transform of f with respect to the spatial variable x and y. We define the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2) = (1 - \partial_x^2)^{-\frac{s_1}{2}}(1 - \partial_y^2)^{-\frac{s_2}{2}} L^2(\mathbb{R}^2)$ and $\dot{H}^{s_1,s_2} = (-\partial_x^2)^{-\frac{s_1}{2}}(-\partial_y^2)^{-\frac{s_2}{2}} L^2(\mathbb{R}^2)$ with norms

$$\|f\|_{H^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{f}(\xi, \eta)|^2 \, d\xi \, d\eta \right)^{\frac{1}{2}}, \qquad \langle \cdot \rangle := \sqrt{1 + |\cdot|^2},$$

$$\|f\|_{\dot{H}^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} |\hat{f}(\xi, \eta)|^2 \, d\xi \, d\eta \right)^{\frac{1}{2}}.$$

(1.1) is scale-invariant under the transformation

$$u_\lambda(t, x, y) = \lambda^{-\frac{2}{p-1}} u \left(\frac{t}{\lambda^2}, \frac{x}{\lambda}, \frac{y}{\lambda^2} \right)$$

for all $\lambda > 0$. If s_1, s_2 satisfy

$$s_1 + 2s_2 = \frac{3}{2} - \frac{2}{p-1}, \tag{2.1}$$

then we have $\|u_\lambda(0)\|_{\dot{H}^{s_1,s_2}} = \|u_0\|_{\dot{H}^{s_1,s_2}}$. We call H^{s_1,s_2} scale critical[2] if s_1, s_2 satisfy (2.1) and super-critical if s_1, s_2 satisfy

$$s_1 + 2s_2 < \frac{3}{2} - \frac{2}{p-1}.$$

Similarly the scale sub-critical space H^{s_1,s_2} be such that s_1, s_2 satisfy

$$s_1 + 2s_2 > \frac{3}{2} - \frac{2}{p-1}.$$

$\mathcal{H}^{k,k}$ denotes the weighted Sobolev space endowed with the norm

$$\|u\|_{\mathcal{H}^{k,k}} := \sum_{i=0}^{k} \|(1 + |x| + |y|)^{k-i} \, \partial^i u\|_{L^2(\mathbb{R}^2)}.$$

Let $\phi_0 = aw, a \in [\frac{1}{2}, 1]$ and the Schwartz function $w \in \mathcal{S}(\mathbb{R}^2)$ be given and let $\phi = \phi^{(a,\nu)}(t, x, y)$ satisfies

$$\begin{cases} i\partial_t \phi + \nu^2 \partial_x^2 \phi - \nu^2 |D_y| \phi = \mu |\phi|^{p-1} \phi, \\ \phi(0, x, y) = \phi_0(x, y) \in \mathcal{S}(\mathbb{R}^2), \end{cases} \tag{2.2}$$

where $0 < \nu \ll 1$, $\mu = \pm 1$. By letting $\nu \to 0$, (2.2) approaches

$$\begin{cases} i\partial_t \phi = \mu |\phi|^{p-1} \phi, \\ \phi(0, x, y) = \phi_0(x, y) \in \mathcal{S}(\mathbb{R}^2). \end{cases} \tag{2.3}$$

The solution $\phi = \phi^{(0)}$ for (2.3) can be expressed as

$$\phi^{(0)} = \phi_0 e^{-i\mu t |\phi_0|^{p-1}}. \tag{2.4}$$

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[2] The energy space $H^{1,0}(\mathbb{R}^2) \cap H^{0,\frac{1}{2}}(\mathbb{R}^2)$ is scale critical when $p = 5$.

For $\frac{1}{2} \le a \le 1, 0 < \lambda \ll 1, v \in \mathbb{R}$ we set

$$u^{(a,v,\lambda,v)}(t, x, y) := \lambda^{-\frac{2}{p-1}} e^{-\frac{i}{2} v \cdot x} e^{-\frac{i}{4}|v|^2 t} \phi^{(a,v)} \left(\frac{t}{\lambda^2}, \frac{v}{\lambda}(x + vt), \frac{v^2}{\lambda^2} y \right). \qquad (2.5)$$

Then $u = u^{(a,v,\lambda,v)}$ satisfies

$$i \partial_t u + \partial_x^2 u - |D_y| u = \mu |u|^{p-1} u.$$

The following lemma will be applied in the proof of Theorems 1.1, 1.2.

Lemma 2.1 (Lemma 2.1 [4]) *Let $p \ge 1, k > 1$ and let $\phi^{(0)}(t)$ satisfies (2.4). If p is not an odd integer, then we also assume $p \ge k + 1$. Then there exist $C, c > 0$ depending on p, k such that the following holds: For sufficiently small $0 < v \le c$ and $T = c|\log v|^c$, there exists a solution $\phi(t) \in C^1([-T, T]; \mathcal{H}^{k,k})$ of (2.2) satisfying*

$$\|\phi(t) - \phi^{(0)}(t)\|_{\mathcal{H}^{k,k}} \le Cv \qquad (2.6)$$

for all $|t| \le c|\log v|^c$.

Proof Let $\phi = \phi^{(0)} + \psi$ where $\phi, \phi^{(0)}$ satisfy (2.2), (2.4) respectively. Then ψ satisfies

$$\begin{cases} i \partial_t \psi + v^2 \partial_x^2 \psi - v^2 |D_y| \psi = -v^2 \partial_x^2 \phi^{(0)} + v^2 |D_y| \phi^{(0)} + \mu |\phi^{(0)} \\ \quad + \psi|^{p-1}(\phi^{(0)} + \psi) - \mu |\phi^{(0)}|^{p-1} \phi^{(0)}, \\ \psi(0, x, y) = 0. \end{cases}$$

To show (2.6) we need to prove

$$\sup_{|t| \le T} \|\psi(t)\|_{\mathcal{H}^{k,k}} \le Cv \qquad (2.7)$$

for $0 \le T \le c|\log v|^c$. For simplicity, let $t \ge 0$. The energy inequality gives

$$\partial_t \|\psi(t)\|_{\mathcal{H}^{k,k}} \le C \| - v^2(\partial_x^2 - |D_y|)\phi^{(0)}(t) + \mu (|\phi^{(0)} + \psi|^{p-1}(\phi^{(0)} + \psi)$$

$$- |\phi^{(0)}|^{p-1} \phi^{(0)})(t)\|_{\mathcal{H}^{k,k}}$$

$$+ C \|\psi(t)\|_{\mathcal{H}^{k,k}}. \qquad (2.8)$$

From (2.4), $\phi_0 \in \mathcal{S}(\mathbb{R}^2)$ and

$$\|\phi^{(0)}(t)\|_{\mathcal{H}^{k,k}} + \|\phi^{(0)}(t)\|_{C^k} \le C(1 + t)^k, \qquad (2.9)$$

we obtain

$$\|(\partial_x^2 - |D_y|)\phi^{(0)}(t)\|_{\mathcal{H}^{k,k}} \le C(1 + t)^k. \qquad (2.10)$$

We show

$$\|F(\phi^{(0)} + \psi)(t) - F(\phi^{(0)})(t)\|_{\mathcal{H}^{k,k}} \le C(1 + t)^C \|\psi(t)\|_{\mathcal{H}^{k,k}} (1 + \|\psi(t)\|_{\mathcal{H}^{k,k}})^{p-1}, \qquad (2.11)$$

where $F : \mathbb{C} \to \mathbb{C}, F(z) := \mu |z|^{p-1} z$. For $j = 0, 1, ..., k$, we see

$$|F^{(j)}(z)| \le C(1 + |z|)^{p-j},$$

$$|F^{(j)}(z + \tilde{z}) - F^{(j)}(z)| \le C|\tilde{z}| (1 + |z| + |\tilde{z}|)^{p-j-1}.$$

Thus the chain rule, the Leibnitz rule and (2.9) imply

$$|F^{(j)}(\phi^{(0)} + \psi)(t, x, y) - F^{(j)}(\phi^{(0)})(t, x, y)|$$
$$\leq C(1 + t)^C \sum_{\alpha_1, \ldots, \alpha_r} |\psi^{(\alpha_1)}(t, x, y)| \cdots |\psi^{(\alpha_r)}(t, x, y)|(1 + |\psi(t, x, y)|^{p-r})$$

for each $0 \leq j \leq k$, where $\alpha_1, \ldots, \alpha_r$ range over all finite collections of non-negative integers with $1 \leq r \leq k + 1$ and $\alpha_1 + \ldots + \alpha_r \leq k$. Thus (2.11) follows from the Hölder inequality and the Sobolev inequality. Collecting (2.8), (2.10) and (2.11),

$$\partial_t \|\psi(t)\|_{\mathcal{H}^{k,k}} \leq C\nu^2 (1 + t)^C + C(1 + t)^C \|\psi(t)\|_{\mathcal{H}^{k,k}} + C(1 + t)^C \|\psi(t)\|_{\mathcal{H}^{k,k}}^p,$$

where $C > 0$ depends on k. Under the a priori assumption $\|\psi(t)\|_{\mathcal{H}^{k,k}} \leq 1$, the above inequality becomes

$$\partial_t \|\psi(t)\|_{\mathcal{H}^{k,k}} \leq C\nu^2 (1 + t)^C + C(1 + t)^C \|\psi(t)\|_{\mathcal{H}^{k,k}}.$$

By the Gronwall inequality and $\psi(0) = 0$, we have

$$\|\psi(t)\|_{\mathcal{H}^{k,k}} \leq C\nu^2 e^{C(1+t)^C},$$

when $0 \leq t \leq c|\log \nu|^c$ for suitably chosen c and sufficiently small $\nu > 0$. Then (2.7) follows and the a priori assumption $\|\psi(t)\|_{\mathcal{H}^{k,k}} \leq 1$ is attained. Hence we obtain the desired result. $\qquad\square$

We prepare the following lemma for the proof of Theorem 1.2.

Lemma 2.2 (Lemma 3.1 [4]) *Let* $0 \neq w \in \mathcal{S}(\mathbb{R}^2)$ *and* $s_1, s_2 < 0$. *Suppose that* $p, k, \phi = \phi^{(a,\nu)}, \phi^{(0)}$ *be given in Lemma 2.1. Also assume that* $|\nu| \geq 1, \frac{1}{2} \leq a, a' \leq 2$ *and* $0 < \nu \leq \lambda \ll 1$. *Then for* $u^{(a,\nu,\lambda,\nu)}$ *given by (2.5) there exist* $C, c > 0$ *such that the followings hold:*

$$\|u^{(a,\nu,\lambda,\nu)}(0)\|_{H^{s_1,s_2}} \leq C\lambda^{-\frac{2}{p-1}} |\nu|^{s_1} \left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}}, \tag{2.12}$$

$$\|u^{(a,\nu,\lambda,\nu)}(0) - u^{(a',\nu,\lambda,\nu)}(0)\|_{H^{s_1,s_2}} \leq C\lambda^{-\frac{2}{p-1}} |\nu|^{s_1} \left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}} |a - a'|, \tag{2.13}$$

$$\|u^{(a,\nu,\lambda,\nu)}(t) - u^{(a',\nu,\lambda,\nu)}(t)\|_{H^{s_1,s_2}}$$
$$\geq C\lambda^{-\frac{2}{p-1}} |\nu|^{s_1}$$
$$\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}} \left(\left\| \phi^{(a,\nu)} \left(\frac{t}{\lambda^2}\right) - \phi^{(a',\nu)} \left(\frac{t}{\lambda^2}\right) \right\|_{H^{0,s_2}} - C|\log \nu|^C \left(\frac{\lambda}{\nu}\right)^{-k} |\nu|^{-s_1-k} \right) \tag{2.14}$$

whenever $|t| \leq c|\log \nu|^c \lambda^2$.

Proof From (2.5), we have

$$[u^{(a,\nu,\lambda,\nu)}(0)]^\wedge(\xi, \eta) = \lambda^{-\frac{2}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 [\phi^{(a,\nu)}(0)]^\wedge \left(\frac{\lambda}{\nu}\left(\xi + \frac{\nu}{2}\right), \frac{\lambda^2}{\nu^2}\eta\right).$$

By Lemma 2.1, for $0 < \nu \leq c$ there exists the solution $\phi(t, x, y) = \phi^{(a,\nu)}(t, x, y)$ to (2.2) with initial data $\phi^{(a,\nu)}(0, x, y) = aw(x, y)$, $0 \neq w \in \mathcal{S}(\mathbb{R}^2)$. From $\frac{1}{2} \leq a \leq 2$ and $s_1, s_2 < 0$ we have

$$\|u^{(a,\nu,\lambda,\nu)}(0)\|^2_{H^{s_1,s_2}} = Ca^2 \lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3$$

$$\int_{\mathbb{R}^2} \left(1 + \left|\frac{\nu}{\lambda}\xi - \frac{\nu}{2}\right|\right)^{2s_1} \left(1 + \left|\frac{\nu^2}{\lambda^2}\eta\right|\right)^{2s_2} |\widehat{w}(\xi, \eta)|^2 \, d\xi \, d\eta$$

$$\leq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 \left(\int_{|\xi| \leq \frac{\lambda}{4\nu}|\nu|} |\nu|^{2s_1} \|\mathcal{F}_x[w](\xi, \cdot)\|^2_{L^2_y} \, d\xi + \int_{|\xi| \geq \frac{\lambda}{4\nu}|\nu|} \|\mathcal{F}_x[w](\xi, \cdot)\|^2_{L^2_y} \, d\xi\right)$$

$$\leq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 \left(|\nu|^{2s_1} + C_N \left(\frac{\lambda}{\nu}|\nu|\right)^{-N}\right) \leq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 |\nu|^{2s_1}$$

for all $N < \infty$ since $w \in \mathcal{S}(\mathbb{R}^2)$. In the last step we use $\nu \leq \lambda$. Hence (2.12) follows. (2.13) is clear from $u^{(a,\nu,\lambda,\nu)}(0) - u^{(a',\nu,\lambda,\nu)}(0) = (a - a')u^{(1,\nu,\lambda,\nu)}(0)$ and (2.12).

Finally we prove (2.14) below. From (2.5),

$$\|u^{(a,\nu,\nu)}(t) - u^{(a',\nu,\lambda,\nu)}(t)\|^2_{H^{s_1,s_2}}$$

$$= \lambda^{-\frac{4}{p-1}} \left\|e^{-\frac{i}{2}\nu \cdot x}\left(\phi^{(a,\nu)}\left(\frac{t}{\lambda^2}, \frac{\nu}{\lambda}(x + vt), \frac{\nu^2}{\lambda^2}y\right)\right.\right.$$

$$\left.\left. - \phi^{(a',\nu)}\left(\frac{t}{\lambda^2}, \frac{\nu}{\lambda}(x + vt), \frac{\nu^2}{\lambda^2}y\right)\right)\right\|^2_{H^{s_1,s_2}}.$$

Here

$$\left\|e^{-\frac{i}{2}\nu \cdot x}\left(\phi^{(a,\nu)}\left(\frac{t}{\lambda^2}, \frac{\nu}{\lambda}(x + vt), \frac{\nu^2}{\lambda^2}y\right) - \phi^{(a',\nu)}\left(\frac{t}{\lambda^2}, \frac{\nu}{\lambda}(x + vt), \frac{\nu^2}{\lambda^2}y\right)\right)\right\|^2_{H^{s_1,s_2}}$$

$$= C\left(\frac{\lambda}{\nu}\right)^3 \int_{\mathbb{R}^2}$$

$$\left(1 + \left|\frac{\nu}{\lambda}\xi - \frac{\nu}{2}\right|\right)^{2s_1} \left(1 + \left|\frac{\nu^2}{\lambda^2}\eta\right|\right)^{2s_2} \left|\widehat{\phi}^{(a,\nu)}\left(\frac{t}{\lambda^2}, \xi, \eta\right) - \widehat{\phi}^{(a',\nu)}\left(\frac{t}{\lambda^2}, \xi, \eta\right)\right|^2 \, d\xi \, d\eta.$$

Hence from $\nu \leq \lambda$ and $s_2 < 0$ we have

$$\|u^{(a,\nu,\nu)}(t) - u^{(a',\nu,\lambda,\nu)}(t)\|^2_{H^{s_1,s_2}}$$

$$\geq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 \int_{|\xi| \leq \frac{\lambda}{4\nu}|\nu|} |\nu|^{2s_1} \left\|\mathcal{F}_x[\phi^{(a,\nu)}]\left(\frac{t}{\lambda^2}, \xi, \cdot\right) - \mathcal{F}_x[\phi^{(a',\nu)}]\left(\frac{t}{\lambda^2}, \xi, \cdot\right)\right\|^2_{H^{s_2}} \, d\xi$$

$$\geq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 |\nu|^{2s_1} \left(\left\|\phi^{(a,\nu)}\left(\frac{t}{\lambda^2}\right) - \phi^{(a',\nu)}\left(\frac{t}{\lambda^2}\right)\right\|^2_{H^{0,s_2}}\right.$$

$$\left. - \int_{|\xi| \geq \frac{\lambda}{4\nu}|\nu|} \left(\left\|\mathcal{F}_x[\phi^{(a,\nu)}]\left(\frac{t}{\lambda^2}, \xi, \cdot\right)\right\|^2_{H^{s_2}} + \left\|\mathcal{F}_x[\phi^{(a',\nu)}]\left(\frac{t}{\lambda^2}, \xi, \cdot\right)\right\|^2_{H^{s_2}}\right) \, d\xi\right). \quad (2.15)$$

By Lemma 2.1, (2.9) and $s_2 < 0$, we have

$$\|\phi(t)\|_{H^{k,s_2}} \leq C\|\phi(t)\|_{\mathcal{H}^{k,k}} \leq C\|\phi(t) - \phi^{(0)}(t)\|_{\mathcal{H}^{k,k}} + C\|\phi^{(0)}(t)\|_{\mathcal{H}^{k,k}}$$

$$\leq C\nu + C(1 + |t|)^k \leq C|\log \nu|^C \quad (2.16)$$

for $|t| \leq c|\log v|^c$. On the other hand,

$$
\|\phi(t)\|_{H^{k,s_2}} \geq \left(\int_{|\xi| \geq \frac{\lambda}{4v}|v|} \langle \xi \rangle^{2k} \|\mathcal{F}_x[\phi](t, \xi, \cdot)\|^2_{H^{s_2}} \, d\xi \right)^{\frac{1}{2}}
$$

$$
\geq C \left(\frac{\lambda}{v}|v| \right)^k \left(\int_{|\xi| \geq \frac{\lambda}{4v}|v|} \|\mathcal{F}_x[\phi](t, \xi, \cdot)\|^2_{H^{s_2}} \, d\xi \right)^{\frac{1}{2}}. \tag{2.17}
$$

Collecting (2.16) and (2.17), we see

$$
\int_{|\xi| \geq \frac{\lambda}{4v}|v|} \left(\left\| \mathcal{F}_x[\phi^{(a,v)}] \left(\frac{t}{\lambda^2}, \xi, \cdot \right) \right\|^2_{H^{s_2}} + \left\| \mathcal{F}_x[\phi^{(a',v)}] \left(\frac{t}{\lambda^2}, \xi, \cdot \right) \right\|^2_{H^{s_2}} \right) d\xi
$$

$$
\leq C|\log v|^C \left(\frac{\lambda}{v}|v| \right)^{-2k}
$$

when $\left| \frac{t}{\lambda^2} \right| \leq c|\log v|^c$. Then

(RHS of (2.15))

$$
\geq C\lambda^{-\frac{4}{p-1}} \left(\frac{\lambda}{v} \right)^3 |v|^{2s_1} \left(\left\| \phi^{(a,v)} \left(\frac{t}{\lambda^2} \right) - \phi^{(a',v)} \left(\frac{t}{\lambda^2} \right) \right\|^2_{H^{0,s_2}} - C|\log v|^C \left(\frac{\lambda}{v}|v| \right)^{-2k} \right).
$$

Hence from $s_1 < 0$ and $|v| \geq 1$, we obtain (2.14).

$\square$

3 Norm inflation

In this section, we prove Theorem 1.1. We consider either (i) $s_1, s_2 \geq 0$ except for $s_1 = s_2 = 0$ and $s_1 + 2s_2 < \frac{3}{2} - \frac{2}{p-1}$ or (ii) $s_1, s_2 \leq -\frac{1}{2}$. In addition we assume that $0 < \lambda \leq v \ll 1$.

Proof of Theorem 1.1 We see from $\phi^{(a,v)}(0, x, y) = aw(x, y), 0 \neq w \in \mathcal{S}(\mathbb{R}^2)$ that

$$
u^{(a,v,\lambda,0)}(0, x, y) = \lambda^{-\frac{2}{p-1}} \phi^{(a,v)} \left(0, \frac{v}{\lambda}x, \frac{v^2}{\lambda^2}y \right) = a\lambda^{-\frac{2}{p-1}} w \left(\frac{v}{\lambda}x, \frac{v^2}{\lambda^2}y \right),
$$

$$
[u^{(a,v,\lambda,0)}(0)]^{\wedge}(\xi, \eta) = a\lambda^{-\frac{2}{p-1}} \left(\frac{\lambda}{v} \right)^3 \widehat{w} \left(\frac{\lambda}{v}\xi, \frac{\lambda^2}{v^2}\eta \right).
$$

Hence if $\lambda \leq \nu$, then

$$
\|u^{(a,\nu,\lambda,0)}(0)\|_{H^{s_1,s_2}}^2
$$

$$
= a^2\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^3 \int_{\mathbb{R}^2}\left(1+\left|\frac{\nu}{\lambda}\xi\right|^2\right)^{s_1}\left(1+\left|\frac{\nu^2}{\lambda^2}\eta\right|^2\right)^{s_2}|\widehat{w}(\xi,\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta
$$

$$
\leq C\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^{3-4s_2}\left(\int_{|\xi|\geq\frac{\lambda}{\nu}}\left(\frac{\nu}{\lambda}|\xi|\right)^{2s_1}\|\mathcal{F}_x[w](\xi,\cdot)\|_{H^{s_2}}^2\,\mathrm{d}\xi\right.
$$

$$
\left.+\int_{|\xi|\leq\frac{\lambda}{\nu}}\|\mathcal{F}_x[w](\xi,\cdot)\|_{H^{s_2}}^2\,\mathrm{d}\xi\right)
$$

$$
= C\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^{3-2s_1-4s_2}\left(\int_{\mathbb{R}}|\xi|^{2s_1}\|\mathcal{F}_x[w](\xi,\cdot)\|_{H^{s_2}}^2\,\mathrm{d}\xi\right.
$$

$$
\left.-\int_{|\xi|\leq\frac{\lambda}{\nu}}\left(|\xi|^{2s_1}-\left(\frac{\lambda}{\nu}\right)^{2s_1}\right)\|\mathcal{F}_x[w](\xi,\cdot)\|_{H^{s_2}}^2\,\mathrm{d}\xi\right). \tag{3.1}
$$

Thus if $\lambda \leq \nu$, then

$$
\|u^{(a,\nu,\lambda,0)}(0)\|_{H^{s_1,s_2}} \leq C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}-s_1-2s_2}\left(1+O\left(\left(\frac{\lambda}{\nu}\right)^{s_1+\frac{1}{2}}\right)\right)
$$

$$
\leq C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}-s_1-2s_2} = C\lambda^{\frac{3}{2}-\frac{2}{p-1}-s_1-2s_2}\nu^{s_1+2s_2-\frac{3}{2}}.
$$

Taking $\lambda = \nu^\sigma$, $\sigma := \frac{\frac{3}{2}-s_1-2s_2+\delta}{\frac{3}{2}-\frac{2}{p-1}-s_1-2s_2} > 1$ and $\delta > 0$ be such that $\nu^\delta = \varepsilon$, then we have

$$
\|u^{(a,\nu,\lambda,0)}(0)\|_{H^{s_1,s_2}} \leq C\varepsilon.
$$

Similarly from $[u^{(a,\nu,\lambda,0)}(\lambda^2 t)](\xi,\eta) = \lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^3\widehat{\phi}^{(a,\nu)}\left(t,\frac{\lambda}{\nu}\xi,\frac{\lambda^2}{\nu^2}\eta\right)$, $s_1, s_2 \geq 0$ except for $s_1 = s_2 = 0$ and $\lambda \leq \nu$, we see

$$
\|u^{(a,\nu,\lambda,0)}(\lambda^2 t)\|_{H^{s_1,s_2}}^2
$$

$$
= C\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^3\int_{\mathbb{R}^2}\left(1+\left|\frac{\nu}{\lambda}\xi\right|\right)^{2s_1}\left(1+\left|\frac{\nu^2}{\lambda^2}\eta\right|\right)^{2s_2}|\widehat{\phi}^{(a,\nu)}(t,\xi,\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta
$$

$$
\geq C\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^3\int_{|\xi|\geq1,|\eta|\geq1}\left(\frac{\nu}{\lambda}|\xi|\right)^{2s_1}\left(\frac{\nu^2}{\lambda^2}|\eta|\right)^{2s_2}|\widehat{\phi}^{(a,\nu)}(t,\xi,\cdot)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta
$$

$$
\geq C\lambda^{-\frac{4}{p-1}}\left(\frac{\lambda}{\nu}\right)^{3-2s_1-4s_2}(\|\phi^{(a,\nu)}(t)\|_{H^{s_1,s_2}}^2 - \|\phi^{(a,\nu)}(t)\|_{L^2}^2).
$$

We estimate $\|\phi^{(a,\nu)}(t)\|_{H^{s_1,s_2}}$ below. If we can show

$$
\|\phi^{(a,0)}(t)\|_{H^{s_1,s_2}} \sim t^{s_1+s_2}, \qquad t \gg 1 \tag{3.2}
$$

then we apply Lemma 2.1 and obtain $\|\phi^{(a,\nu)}(t)\|_{H^{s_1,s_2}} \sim t^{s_1+s_2}$. This leads to

$$
\|u^{(a,\nu,\lambda,0)}(\lambda^2 t)\|_{H^{s_1,s_2}} \geq C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}-s_1-2s_2}\|\phi^{(a,\nu)}(t)\|_{H^{s_1,s_2}} \sim \varepsilon t^{s_1+s_2},
$$

when $\nu \ll 1$ and $1 \ll t \leq c|\log \nu|^c$. (3.2) is confirmed by the same manner as in the proof of Theorem 2 (page 17 [4]). Indeed, from

$$\phi^{(a,0)}(t, x, y) = aw(x, y)e^{i\mu a^{p-1}t|w(x,y)|^{p-1}} \tag{3.3}$$

and

$$\partial_x^{j_1} \partial_y^{j_2} \phi^{(a,0)}(t, x, y) = aw(x, y)t^{j_1+j_2}(i\mu a^{p-1}\nabla_{x,y}|w(x, y)|^{p-1})^{j_1+j_2}e^{i\mu a^{p-1}t|w(x,y)|^{p-1}}$$
$$+ O(t^{j_1+j_2})$$

for all integers $j_1, j_2 \geq 0$ if p is an odd integer, and all j_1, j_2 satisfying $0 \leq j_1 + j_2 \leq p - 1$ otherwise, we have

$$\|\phi^{(a,0)}(t)\|_{H^{j_1,j_2}} \sim t^{j_1+j_2}.$$

Under the assumptions of s_1, s_2, k, p in Theorem 1.1, we have (3.2).

Next, we prove the case $s_1, s_2 \leq -\frac{1}{2}$. From (3.1) and if $\|\mathcal{F}_x[w](\xi, \cdot)\|_{H^{s_2}} \sim O(|\xi|^\kappa)$ as $\xi \to 0$ for some $\kappa > -s_1 - \frac{1}{2}$, then from $\lambda \leq \nu$,

$$\int_{|\xi| \leq \frac{\lambda}{\nu}} \|\mathcal{F}_x[w](\xi, \cdot)\|_{H^{s_2}}^2 \left(\left(\frac{\lambda}{\nu}\right)^{2s_1} - |\xi|^{2s_1}\right) d\xi \leq C \left(\frac{\lambda}{\nu}\right)^{2\kappa+2s_1+1} < \infty.$$

This yields $\|u^{(a,\nu,\lambda,0)}(0)\|_{H^{s_1,s_2}} \leq C\varepsilon$. We will show that for any $\varepsilon > 0$, it holds $\|u^{(a,\nu,\lambda,0)}(\lambda^2)\|_{H^{s_1,s_2}} > \frac{1}{\varepsilon}$ for sufficiently small $0 < \lambda \leq \nu$. We check the case $s_1, s_2 < -\frac{1}{2}$ at first.

$$\left|\int_{\mathbb{R}^2} \phi^{(a,0)}(1, x, y) \, dx \, dy\right| \geq C$$

is equivalent to $|\widehat{\phi}^{(a,0)}(1, 0, 0)| \geq C$. Here $\phi^{(a,0)}(1)$ is rapidly decreasing, hence by continuity we see for $|\xi|, |\eta| \leq c$ with $0 < c \ll 1$ that

$$|\widehat{\phi}^{(a,0)}(1, \xi, \eta)| \geq C. \tag{3.4}$$

By $|\xi|, |\eta| \leq c \ll 1$, the Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$|\widehat{\phi}^{(a,\nu)}(1, \xi, \eta) - \widehat{\phi}^{(a,0)}(1, \xi, \eta)| \leq \int_{|\xi|,|\eta| \leq c \ll 1} |\phi^{(a,\nu)}(1, \xi, \eta) - \phi^{(a,0)}(1, \xi, \eta)| \, d\xi \, d\eta$$
$$\leq C\|\phi^{(a,\nu)}(1) - \phi^{(a,0)}(1)\|_{\mathcal{H}^{k,k}} \leq C\nu. \tag{3.5}$$

Collecting (3.4) and (3.5), we have

$$|\widehat{\phi}^{(a,\nu)}(1, \xi, \eta)| \geq C \tag{3.6}$$

for $|\xi|, |\eta| \leq c$ and sufficiently small $\nu > 0$. From (2.5), we compute

$$\widehat{u}^{(a,\nu,\lambda,0)}(\lambda^2, \xi, \eta) = \lambda^{-\frac{2}{p-1}} \left(\frac{\lambda}{\nu}\right)^3 \widehat{\phi}^{(a,\nu)}\left(1, \frac{\lambda}{\nu}\xi, \frac{\lambda^2}{\nu^2}\eta\right).$$

If $|\xi| \le c\frac{\nu}{\lambda}$, $|\eta| \le c\frac{\nu^2}{\lambda^2}$, then by (3.6), we have $|\widehat{u}^{(a,\nu,\lambda,0)}(\lambda^2,\xi,\eta)| \ge C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^3$. From this and $s_1, s_2 < -\frac{1}{2}$, we see

$$\|u^{(a,\nu,\lambda,0)}(\lambda^2)\|_{H^{s_1,s_2}} \ge C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^3 \left(\int_{|\xi|\le c\frac{\nu}{\lambda},|\nu|\le c\frac{\nu^2}{\lambda^2}} (1+|\xi|)^{2s_1}(1+|\eta|)^{2s_2}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)^{\frac{1}{2}}$$

$$= C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}-s_1-2s_2}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}+s_1+2s_2} = C\varepsilon\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}+s_1+2s_2}. \quad (3.7)$$

If $\lambda \le \nu$, then $\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}+s_1+2s_2} \to \infty$ as $\nu \to 0$ since $s_1 + 2s_2 < -\frac{3}{2}$. Thus the claim follows by taking ν sufficiently small depending on ε. We consider the case $s_1 = s_2 = -\frac{1}{2}$ secondly. From (3.7), $\lambda \le \nu$ and

$$\int_{|\xi|\le c\frac{\nu}{\lambda},|\nu|\le c\frac{\nu^2}{\lambda^2}} (1+|\xi|)^{-1}(1+|\eta|)^{-1}\,\mathrm{d}\xi\,\mathrm{d}\eta \ge C\log\left(\frac{\nu}{\lambda}\right),$$

we obtain

$$\|u^{(a,\nu,\lambda,0)}(\lambda^2)\|_{H^{s_1,s_2}} \ge C\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^3 \log\left(\frac{\nu}{\lambda}\right) = C\varepsilon\log\left(\frac{\nu}{\lambda}\right).$$

Thus the desired result follows. It remains to show the claim when $s_1 = -\frac{1}{2}$, $s_2 < -\frac{1}{2}$ or $s_1 < -\frac{1}{2}$, $s_2 = -\frac{1}{2}$. These cases are handled in a similar way, hence we omit the detail.
$\square$

4 Decoherence

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2 Firstly, we consider the case $s_1, s_2 < 0$. Set $\lambda^{-\frac{2}{p-1}}\left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}}|\nu|^{s_1} = \varepsilon$, $\lambda = \nu^\sigma$ such that $0 < \varepsilon, \sigma \ll 1$ and σ will be fixed later. This condition is equivalent to $|\nu| = \nu^{\frac{1}{s_1}\left(\frac{3}{2}(1-\sigma)+\frac{2\sigma}{p-1}\right)} \cdot \varepsilon^{\frac{1}{s_1}}$, namely $|\nu|$ grows negative power of ν. From (2.12) and (2.13) we obtain

$$\|u^{(a,\nu,\lambda,\nu)}(0)\|_{H^{s_1,s_2}} + \|u^{(a',\nu,\lambda,\nu)}(0)\|_{H^{s_1,s_2}} \le C\varepsilon,$$

$$\|u^{(a,\nu,\lambda,\nu)}(0) - u^{(a',\nu,\lambda,\nu)}(0)\|_{H^{s_1,s_2}} \le C\varepsilon|a-a'|.$$

Taking $\delta = \varepsilon|a-a'|$ then we obtain (1.2) and (1.3). There exists some $T = T(a,a') > 0$ such that

$$\|\phi^{(a,0)}(T) - \phi^{(a',0)}(T)\|_{H^{0,s_2}} = \|awe^{i\mu Ta^{p-1}|w|^{p-1}} - a'we^{i\mu Ta'^{p-1}|w|^{p-1}}\|_{H^{0,s_2}} \ge C. \quad (4.1)$$

Here $C > 0$ is independent of a, a' and fix this T. By Lemma 2.1, for all $0 < T \le c|\log\nu|^c$

$$\|\phi^{(a,\nu)}(T) - \phi^{(a,0)}(T)\|_{H^{k,k}(\mathbb{R}^2)} \le C\nu, \quad (4.2)$$

where $\phi^{(a,0)}(t)$ is defined in (3.3). From (4.1) and (4.2), we have

$$\|\phi^{(a,v)}(T) - \phi^{(a',v)}(T)\|_{H^{0,s_2}}$$
$$\geq \|\phi^{(a,0)}(T) - \phi^{(a',0)}(T)\|_{H^{0,s_2}} - \|\phi^{(a,v)}(T) - \phi^{(a,0)}(T)\|_{H^{0,s_2}}$$
$$- \|\phi^{(a',0)}(T) - \phi^{(a',v)}(T)\|_{H^{0,s_2}}$$
$$\geq C - 2Cv \geq C$$

provided $s_2 < 0$ and $v > 0$ is sufficiently small. From (2.14), if $0 < T \leq c|\log v|^c$, then we have

$$\|u^{(a,v,\lambda,v)}(\lambda^2 T) - u^{(a',v,\lambda,v)}(\lambda^2 T)\|_{H^{s_1,s_2}}$$
$$\geq C\varepsilon \left(\|\phi^{(a,v)}(T) - \phi^{(a',v)}(T)\|_{H^{0,s_2}} - C|\log v|^C \left(\frac{\lambda}{v}\right)^{-k} |v|^{-s_1-k} \right)$$
$$\geq C\varepsilon - C\varepsilon \left(\frac{\lambda}{v}\right)^{-k} |v|^{-s_1-k}|\log v|^C. \tag{4.3}$$

We set $\lambda = v^\sigma$ with sufficiently small $\sigma > 0$ such that

$$\left(\frac{\lambda}{v}\right)^{-k} |v|^{-s_1-k}|\log v|^C = v^{k - \frac{3(s_1+k)}{2s_1} + \left(\frac{s_1+k}{s_1}\left(\frac{3}{2} - \frac{2}{p-1}\right) - k\right)\sigma} |\log v|^C \varepsilon^{-\frac{s_1+k}{s_1}} \to 0$$

as $v \to 0$. Indeed, if $s_1 < 0$ and $k > \frac{3}{2}$ we see $k - \frac{3(s_1+k)}{2s_1} > -\frac{9}{4}s_1 > 0$. Therefore

$$(\text{RHS of (4.3)}) \geq C\varepsilon.$$

As $v \to 0$, we have $\lambda = v^\sigma \to 0$, hence $\lambda^2 T \to 0$. Therefore we conclude (1.4). Secondly we check the case $s_1 = s_2 = 0$. We see

$$\|\phi^{(a,0)}(t) - \phi^{(a',0)}(t)\|_{L^2} \geq C > 0$$

when $t \geq C|a - a'|^{-1}$. If $v > 0$ is sufficiently small and $C|a - a'|^{-1} \leq t \leq c|\log v|^c$, then (2.6) gives

$$\|\phi^{(a,v)}(t) - \phi^{(a',v)}(t)\|_{L^2} \geq \|\phi^{(a,0)}(t) - \phi^{(a',0)}(t)\|_{L^2} - \|\phi^{(a,v)}(t) - \phi^{(a,0)}(t)\|_{L^2}$$
$$- \|\phi^{(a',0)}(t) - \phi^{(a',v)}(t)\|_{L^2}$$
$$\geq C - 2Cv \geq C.$$

We see

$$\|u^{(a,v,\lambda,0)}(\lambda^2 t) - u^{(a',v,\lambda,0)}(\lambda^2 t)\|_{L^2} = \lambda^{-\frac{2}{p-1}} \left(\frac{\lambda}{v}\right)^{\frac{3}{2}} \|\phi^{(a,v)}(t) - \phi^{(a',v)}(t)\|_{L^2} \geq C\varepsilon.$$

Of course it holds that

$$\|u^{(a,v,\lambda,0)}(0)\|_{L^2} \leq C\varepsilon, \qquad \|u^{(a,v\lambda,0)}(0) - u^{(a',v,\lambda,0)}(0)\|_{L^2} \leq C\varepsilon|a - a'|.$$

Therefore the claim follows for the case $s_1 = s_2 = 0$. The cases $s_1 = 0$ and $s_2 < 0$ or $s_1 < 0$ and $s_2 = 0$ would be similar, hence we omit the proof.

$\square$

5 Decoherence at the critical regularity for focusing case

In this section, we consider (1.1) with $\mu = -1$ (focusing case):

$$i\partial_t u + \partial_x^2 u - |D_y|u = -|u|^{p-1}u. \tag{5.1}$$

For $1 < p < 5$, there exists the standing wave for (5.1), namely $u(t, x, y) = e^{i\beta t} Q_\beta(x, y)$, $\beta > 0$. Here Q_β satisfies the following equation (see (1.5) [1]):

$$-\partial_x^2 Q + |D_y|Q + \beta Q - |Q|^{p-1}Q = 0.$$

We note that

$$Q_\beta(x, y) = \beta^{\frac{1}{p-1}} Q_1(\sqrt{\beta}x, \beta y). \tag{5.2}$$

Proof of Theorem 1.3 We set two solutions u_1, u_2 for (5.1) as

$$u_j(t, x, y) = e^{i\beta_j t} Q_{\beta_j}(x, y), \tag{5.3}$$

where $\beta_j > 0$, $j = 1, 2$. From (5.2), (5.3), change of variables and $s_1 + 2s_2 = \frac{3}{2} - \frac{2}{p-1}$ we see

$$
\begin{aligned}
\|u_1(0) - u_2(0)\|_{\dot{H}^{s_1,s_2}}^2 &= \|Q_{\beta_1} - Q_{\beta_2}\|_{\dot{H}^{s_1,s_2}}^2 \\
&= \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \left| \beta_1^{\frac{1}{p-1}-\frac{3}{2}} \widehat{Q}_1\left(\frac{\xi}{\sqrt{\beta_1}}, \frac{\eta}{\beta_1}\right) - \beta_2^{\frac{1}{p-1}-\frac{3}{2}} \widehat{Q}_1\left(\frac{\xi}{\sqrt{\beta_2}}, \frac{\eta}{\beta_2}\right) \right|^2 \, d\xi \, d\eta \\
&= \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} |\widehat{Q}_1(\xi, \eta)|^2 \, d\xi \, d\eta + \left(\frac{\beta_2}{\beta_1}\right)^{\frac{2}{p-1}-3} \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \left| \widehat{Q}_1\left(\sqrt{\frac{\beta_1}{\beta_2}}\xi, \frac{\beta_1}{\beta_2}\eta\right) \right|^2 \, d\xi \, d\eta \\
&\quad - 2 \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{p-1}-\frac{3}{2}} \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \, \widehat{Q}_1(\xi, \eta) \, \overline{\widehat{Q}}_1\left(\sqrt{\frac{\beta_1}{\beta_2}}\xi, \frac{\beta_1}{\beta_2}\eta\right) \, d\xi \, d\eta \\
&\to 0
\end{aligned}
$$

as $\frac{\beta_2}{\beta_1} \to 1$. Hence (1.3) follows. Finally, we check (1.4).

$$\|u_1(t) - u_2(t)\|_{\dot{H}^{s_1,s_2}}^2 = \|u_1(t)\|_{\dot{H}^{s_1,s_2}}^2 + \|u_2(t)\|_{\dot{H}^{s_1,s_2}}^2 - 2\langle u_1(t), u_2(t) \rangle_{s_1,s_2}.$$

Similar to the above argument, we see $\|u_i(t)\|_{\dot{H}^{s_1,s_2}} = \|Q_1\|_{\dot{H}^{s_1,s_2}}$ for $i = 1, 2$.

$$
\begin{aligned}
\langle u_1(t), u_2(t) \rangle_{s_1,s_2} &= \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} e^{i(\beta_1-\beta_2)t} \, \widehat{Q}_{\beta_1}(\xi, \eta) \, \overline{\widehat{Q}}_{\beta_2}(\xi, \eta) \, d\xi \, d\eta \\
&= e^{i(\beta_1-\beta_2)t} (\beta_1\beta_2)^{\frac{1}{p-1}-\frac{3}{2}} \\
&\quad \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \, \widehat{Q}_1\left(\frac{\xi}{\sqrt{\beta_1}}, \frac{\eta}{\beta_1}\right) \overline{\widehat{Q}}_1\left(\frac{\xi}{\sqrt{\beta_2}}, \frac{\eta}{\beta_2}\right) \, d\xi \, d\eta \\
&= e^{i(\beta_1-\beta_2)t} (\beta_1\beta_2)^{\frac{1}{p-1}-\frac{3}{2}} \beta_1^{s_1+2s_2+\frac{3}{2}} \\
&\quad \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \, \widehat{Q}_1(\xi, \eta) \, \overline{\widehat{Q}}_1\left(\sqrt{\frac{\beta_1}{\beta_2}}\xi, \frac{\beta_1}{\beta_2}\eta\right) \, d\xi \, d\eta.
\end{aligned}
$$

If we take $\beta_1 = n^2 + n$, $\beta_2 = n^2$, $t = \frac{\pi}{n}$ for $n \in \mathbb{N}$, then we obtain

$$\langle u_1(t), u_2(t) \rangle_{s_1, s_2} = -\left(\frac{n^2}{n^2 + n}\right)^{\frac{1}{p-1} - \frac{3}{2}} \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{Q}_1(\xi, \eta) \overline{\widehat{Q}}_1 \left(\sqrt{\frac{n^2 + n}{n^2}} \xi, \frac{n^2 + n}{n^2} \eta\right) \mathrm{d}\xi \mathrm{d}\eta$$

$$\to -\|Q_1\|^2_{\dot{H}^{s_1, s_2}}, \qquad n \to \infty$$

since $s_1 + 2s_2 = \frac{3}{2} - \frac{2}{p-1}$. Therefore

$$\|u_1(t) - u_2(t)\|^2_{\dot{H}^{s_1, s_2}} \to 4\|Q_1\|^2_{\dot{H}^{s_1, s_2}}, \qquad n \to \infty.$$

Hence (1.4) follows. $\qquad\square$

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