



Hybrid algebras

Karin Erdmann¹ · Andrzej Skowroński²

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Abstract

We introduce a new class of symmetric algebras, which we call hybrid algebras. This class contains on one extreme Brauer graph algebras, and on the other extreme general weighted surface algebras. We show that hybrid algebras are precisely the blocks of idempotent algebras of weighted surface algebras, up to socle deformations. More generally, for tame symmetric algebras whose Gabriel quiver is 2-regular, we show that the tree class of an arbitrary Auslander–Reiten component is Dynkin or Euclidean or one of the infinite trees A_∞ , A_∞^∞ or D_∞ .

Keywords Periodic algebra · Self-injective algebra · Symmetric algebra · Surface algebra · Tame algebra · Auslander–Reiten component

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1 Introduction

We are interested in the representation theory of tame self-injective algebras. In this paper, all algebras are finite-dimensional basic associative and indecomposable K -algebras over an algebraically closed field K of arbitrary characteristic.

In the modular representation theory of finite groups representation-infinite tame blocks occur only over fields of characteristic 2, and their defect groups are dihedral, semidihedral, or (generalized) quaternion 2-groups. Such blocks were studied in a more general setting: this led to algebras of dihedral, semidihedral and quaternion type, over algebraically closed fields

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✉ Karin Erdmann
erdmann@maths.ox.ac.uk

¹ Mathematical Institute, Oxford University, ROQ, Oxford OX2 6GG, UK

² Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

of arbitrary characteristic, which were introduced and investigated in [5]. These algebras are quite restrictive, for example the number of simple modules can be at most 3, and one would like to know how these fit into a wider context.

Recently cluster theory has led to new directions. Inspired by this, we study in [8], [10], [14] and [3] a class of symmetric algebras defined in terms of surface triangulations, which we call *weighted surface algebras*. They are periodic as algebras of period 4 (with a few exceptions). All but one of the algebras of quaternion type occur in this setting. Furthermore, most algebras of dihedral type, and of semidihedral type occur naturally as degenerations of these weighted surface algebras. As well, Brauer graph algebras, which includes blocks of finite type, appear. This places blocks of finite or tame representation type into a much wider context, which also connects with other parts of mathematics.

In this paper, we present a unified approach. We introduce a new class of algebras, which we call *hybrid algebras*. At one extreme it contains all Brauer graph algebras, and at the other extreme it contains all weighted surface algebras, which are almost all periodic as algebras, of period four (see [8] and [10]). Furthermore, the class contains many other symmetric algebras of tame or finite representation type. In particular it contains all blocks of group algebras, or type A Hecke algebras, of tame or finite type, up to Morita equivalence.

Consider tame symmetric algebras more generally. One observes that being tame is a strong restriction on the Gabriel quiver of the algebra. At any given vertex there are not too many arrows starting or ending, and also not too few, avoiding finite type. The situation when one can expect larger classes of algebras occurs when the Gabriel quiver is 2-regular. We ask whether all tame symmetric algebra with a 2-regular Gabriel quiver are hybrid algebras, up to some small list of exceptions, and up to derived equivalence. Our result on the tree class of stable AR components holds for any tame symmetric algebra with 2-regular Gabriel quiver, and could be thought of as some evidence.

A motivation is that various basic tame, or finite type, symmetric algebras studied in recent years have a unified description, of the form $\Lambda = KQ/I$ with (Q, I) satisfying certain combinatorial restrictions. Namely, the quiver Q is 2-regular, that is, there are two arrows starting and two arrows ending at each vertex. Here I may contain arrows of Q , so that the Gabriel quiver can be seen as a subset of Q . The fact that Q is 2-regular, gives rise to symmetry. There is a permutation f of the arrows such that $t(\alpha) = s(f(\alpha))$ for each arrow α . This determines uniquely a different permutation g where $t(\alpha) = s(g(\alpha))$ but $f(\alpha) \neq g(\alpha)$. Such permutations have been studied for Brauer graph algebras: the permutation g describes the cyclic order in the Brauer graph, and the permutation f has been called the 'Green walk'. Here we will see that these permutations f and g exist more generally.

The permutation f encodes minimal relations, and the permutation g describes, roughly speaking, a basis for the indecomposable projective modules. Consider $e_i \Lambda$, and let $\alpha, \bar{\alpha}$ be the arrows starting at i . Then $e_i \Lambda$ has a basis consisting of monomials along the g -cycles of α and of $\bar{\alpha}$, and the socle of $e_i \Lambda$ is spanned by B_α (or $B_{\bar{\alpha}}$), where B_α is the longest monomial starting with α which is non-zero in Λ . Let also A_α be the submonomial of B_α such that $B_\alpha = A_\alpha \gamma$ where γ is the arrow with $g(\gamma) = \alpha$.

For each arrow α there is a minimal relation determined by f , either 'biserial', or 'quaternion':

- (B) $\alpha f(\alpha) \in I$, or
- (Q) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}} \in I$

(where the $c_{\bar{\alpha}}$ are non-zero scalars constant on g -cycles). With these data, together with suitable zero relations, and up to socle deformations, the following hold.

The algebra Λ is a Brauer graph algebra if all minimal relations are biserial. If $f^3 = 1$ and all minimal relations are quaternion, then the algebra A is a weighted surface algebra (as in [8, 10, 14]). When $f^3 = 1$, and some but not all minimal relations are biserial, we get algebras generalizing algebras of semidihedral type, as in [5] (see also [16]). As well algebras of finite type can occur naturally (which we also call tame in this context).

The known structure of tame local symmetric algebras should be further motivation. As one can find in [5], section III, up to socle deformations, only relations of the form (B) and (Q) occur. This suggests that 'generally' it should be sufficient to incorporate these types of relations. Cycles of f of length 3 (or 1) play a special role in the algebras studied in [5]. A relation (Q) only occurs if α belongs such a cycle. Namely we have $A_{\bar{\alpha}} g^{-1}(\bar{\alpha}) = B_{\bar{\alpha}}$ and $g^{-1}(\bar{\alpha}) = f^{-1}(\alpha)$ therefore $\alpha f(\alpha) f^{-1}(\alpha)$ is a cyclic path, so the arrow α occurs in some triangle.

We call the set of arrows in an f -cycle of length 3 or 1 a *triangle*. Describing a *hybrid algebra* H in broad terms, we fix a set \mathcal{T} of triangles in Q . Then $H = H_{\mathcal{T}} = KQ/I$ where

- (i) an arrow $\alpha \in \mathcal{T}$ satisfies the quaternion relation, and
- (ii) an arrow $\alpha \notin \mathcal{T}$ satisfies the biserial relation.

In addition there are zero relations.

We start with a hybrid algebra where the quiver Q for the definition is the Gabriel quiver, this is introduced and studied in Sect. 2. We call the algebras *regular* hybrid algebras. This is extended in Sect. 3. Our first main result is the following.

Theorem 1.1 (i) Assume Λ is a weighted surface algebra and e is an idempotent of Λ , then every block component of $e\Lambda e$ is a hybrid algebra (up to socle equivalence).

(ii) Assume H is a hybrid algebra. Then there is a weighted surface algebra Λ and an idempotent e of Λ such that H is isomorphic to a block component of $e\Lambda e$.

The second part of this theorem generalises [15] where we prove that every Brauer graph algebra occurs as an idempotent algebra of a weighted surface algebra. For the second part, given a hybrid algebra H , to construct the weighted surface algebra Λ with H as a component of $e\Lambda e$, we use the $*$ construction introduced in [15].

Idempotent algebras of weighted surface algebras include many local algebra, therefore our definition of hybrid algebras must included these. In our general construction of weighted surface algebras [10], we have allowed virtual arrows, with the benefit of essentially enlarging the class of algebras. The price to pay is that zero relations have to be treated with care (see [14]), and naturally this is also the case for hybrid algebras. In particular we need to exclude a few small algebras (see Assumption 3.4).

All local symmetric algebras of tame or finite type, and almost all algebras of dihedral, semidihedral or quaternion type as in [5] are hybrid algebras. There is one family of algebras of quaternion type which are not hybrid algebras, but are derived equivalent to algebras of quaternion type (algebras $Q(3C)^{k,s}$, see [19]).

Hybrid algebras place blocks into a wider context; in [9] we define algebras of generalized quaternion type, as tame symmetric algebras with periodic module categories, that is, generalizing quaternion blocks, and show that the ones with 2-regular Gabriel quiver are almost all weighted surface algebras. As well in [13] we define algebras of generalized dihedral type, in terms of homological properties generalizing dihedral blocks, and show that almost all are the biserial weighted surface algebras as in [8]. One would like a similar homological description of the hybrid algebras which generalize semidihedral blocks.

In order to understand the representation theory for all these algebras, the structure of the stable Auslander–Reiten quiver is essential. Our second main result is more general, it

describes its graph structure for arbitrary tame symmetric algebras with 2-regular Gabriel quiver:

Theorem 1.2 *Assume Λ is a tame symmetric algebra with a 2-regular Gabriel quiver. Then the tree classes of stable Auslander–Reiten components of Λ are one of the infinite trees A_∞ , A_∞^∞ or D_∞ , or Euclidean or Dynkin.*

It would be interesting to know whether a component with tree class A_∞ of a tame symmetric algebra is necessarily a tube.

We describe the organisation of the paper. In Sect. 2, we present and study a simplified version of hybrid algebras, which we call regular. For such an algebra, Q is the Gabriel quiver. In this case we prove a weaker version of Theorem 1.1, which will show how virtual arrows occur.

In Sect. 3 we give the general definition, and discuss exceptions for the zero relations. The details for consistency and bases are refinements of results in Sect. 2 and are therefore only given in an “Appendix”. Originally we had incorporated socle deformations into the general definition of a hybrid algebras. This is not done here, as it has caused further technical work. Note however that socle deformations can occur but are easy to identify.

In Sect. 4 we discuss algebras with few simple modules and small multiplicities. In Sect. 5 we prove Theorem 1.1, extending the version in Sect. 2. Section 6 is valid more generally, for arbitrary tame symmetric algebras with 2-regular Gabriel quiver. The main result is Theorem 1.2 on stable Auslander–Reiten components. In the case of hybrid algebras, we identify components containing simple modules, and see in particular that the infinite trees in the list all occur.

For further background and motivation, we refer to [1, 2], and to the introductions of [8, 10], or [15].

2 Preliminaries and regular hybrid algebras

2.1 The setup

Recall that a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, and where s, t are maps $Q_1 \rightarrow Q_0$ associating to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its target $t(\alpha)$. We say that α starts at $s(\alpha)$ and ends at $t(\alpha)$. We assume throughout that any quiver is connected. The quiver Q is 2-regular if at each vertex, two arrows start and two arrows end.

Denote by KQ the path algebra of Q over K . The underlying space has basis given by the set of all paths in Q , in particular for each vertex i , let ε_i be the path of length zero at i in KQ . We will consider algebras of the form $\Lambda = KQ/I$ for some ideal I of KQ . Let $e_i = \varepsilon_i + I$, then the e_i are pairwise orthogonal idempotents, and their sum is the identity of Λ . We assume that the ideal I contains all paths of length $\geq N$ for some $N \geq 2$, so that the algebra is finite-dimensional and basic. The Gabriel quiver Q_Λ of Λ has by definition the same vertices as Q and its arrows are in bijection with a basis for J/J^2 where J is the radical of Λ . Usually, Q_Λ can be taken as a subquiver of Q .

2.2 Notation

Recall that a biserial quiver is a pair (Q, f) where Q is a 2-regular quiver, and f is a permutation of the arrows such that for each arrow α we have $s(f(\alpha)) = t(\alpha)$. This was defined in [15], but here we also allow the quiver Q with only one vertex. Moreover, we have an involution $(-)$ on the arrows, taking $\bar{\alpha}$ to be the arrow $\neq \alpha$ with the same starting vertex. Given f , this uniquely determines the permutation g on arrows, defined by $g(\alpha) = f(\bar{\alpha})$.

Let \mathcal{O} be the set of g -orbits on Q_1 . We fix a weight function (or multiplicity function), that is, a function $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}$, and we fix a parameter function, that is, a function $c_\bullet : \mathcal{O}(g) \rightarrow K^*$. Moreover, n_α is the size of the g -orbit of $\alpha \in Q_1$.

For an arrow α of Q , let B_α be the monomial along the g -cycle of α which starts with α , of length $m_\alpha n_\alpha$, and let A_α be the submonomial of B_α starting with α of length $m_\alpha n_\alpha - 1$, so that $B_\alpha = A_\alpha g^{-1}(\alpha)$.

For a path p in KQ we write $|p|$ for the length of p . We will sometimes write $p \equiv q$ if p and q are paths in KQ such that $p = \lambda q$ in some algebra KQ/I for $0 \neq \lambda \in K$.

2.3 Regular hybrid algebras

The arrows in f -orbits of length 3 or 1 play a special role, we refer to these as *triangles*. Note that any set of triangles is invariant under the permutation f . The regular hybrid algebra is defined so that it has Q as its Gabriel quiver, this is ensured by the following:

(*) We assume $m_\alpha n_\alpha \geq 2$ for any arrow α , and $m_\alpha n_\alpha \geq 3$ if $\bar{\alpha} \in \mathcal{T}$.

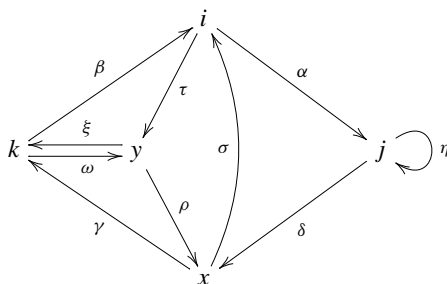
Definition 2.1 Let (Q, f) be a biserial quiver with the data m_\bullet, c_\bullet as in 2.2, and let \mathcal{T} be a set of distinguished triangles. The regular hybrid algebra $H = H_{\mathcal{T}} = H_{\mathcal{T}}(Q, f, m_\bullet, c_\bullet)$ associated to \mathcal{T} , with assumption (*), is the algebra $H = KQ/I$ where I is generated by the following elements:

- (1) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ for $\alpha \in \mathcal{T}$ and $\alpha f(\alpha)$ for $\alpha \notin \mathcal{T}$.
- (2) $\alpha f(\alpha)g(f(\alpha))$ and $\alpha g(\alpha)f(g(\alpha))$ for any arrow α of Q .
- (3) $c_\alpha B_\alpha - c_{\bar{\alpha}} B_{\bar{\alpha}}$ for any arrow α of Q .

Let i be a vertex and $\alpha, \bar{\alpha}$ the arrows starting at i . We say that i is *biserial* if α and $\bar{\alpha}$ are both not in \mathcal{T} . We call the vertex i a *quaternion* vertex if α and $\bar{\alpha}$ are both in \mathcal{T} . Otherwise, we say that i is *hybrid*.

The conditions (*) imply that arrows are not contained in I , so that Q is the Gabriel quiver of H . If $\mathcal{T} = \emptyset$, then the algebra H is special biserial and symmetric, that is, a Brauer graph algebra (BGA). At the other extreme, if $\mathcal{T} = Q_1$ then H is a weighted surface algebra (WSA), as defined in [8], if Q has at least three vertices, or it occurs amongst the algebras of quaternion type in [5].

Example 2.2 Consider the quiver



As the permutation f , we take

$$f = (\alpha \delta \sigma)(\rho \gamma \omega)(\xi \beta \tau)(\eta)$$

Then

$$g = (\alpha \eta \delta \gamma \beta)(\tau \rho \sigma)(\xi \omega)$$

We take $m_\alpha = 1 = m_\tau$ and $m_\xi = 2$ and $c_\alpha = c$, $c_{\bar{\alpha}} = d$ and $c_\xi = 1$.

The permutation f has four cycles, each of size 1 or 3, so there are several choices for the set \mathcal{T} of distinguished triangles.

- (a) If $\mathcal{T} = Q_1$ then the algebra $H_{\mathcal{T}}$ is a weighted surface algebra, as in [8].
- (b) If $\mathcal{T} = \emptyset$ then the algebra is special biserial and symmetric, hence a Brauer graph algebra.
- (c) An example for an intermediate choice of \mathcal{T} might be $\mathcal{T} = \{\alpha, \delta, \sigma, \eta\}$. Then the relations for the paths of length 2 between arrows of \mathcal{T} are

$$\alpha\delta = dA_\tau, \quad \delta\sigma = cA_\eta, \quad \sigma\alpha = cA_\gamma, \quad \eta^2 = cA_\delta;$$

and products of paths of length two along each other f -cycle are zero in $H_{\mathcal{T}}$. In this case, vertices i and x are hybrid, vertex j is quaternion, and vertices k, y are biserial.

Lemma 2.3 *The conditions (1) to (3) in Definition 2.1 are consistent. In particular B_α is non-zero on H .*

Proof We show that the condition for $\alpha f(\alpha)$ from (1) and the conditions for $g^{-1}(\alpha)\alpha f(\alpha)$ and $\alpha f(\alpha)g(f(\alpha))$ from (2) agree. This is clear when $\alpha \notin \mathcal{T}$ since then condition (1) requires $\alpha f(\alpha) = 0$ in H .

Assume now that $\alpha \notin \mathcal{T}$, then we substitute $\alpha f(\alpha) = c_{\bar{\alpha}}A_{\bar{\alpha}}$. We should have that $g^{-1}(\alpha)A_{\bar{\alpha}} = 0$ in H . By the definition of the permutations, we have $g^{-1}(\alpha) = f^{-1}(\bar{\alpha})$, and by the assumption (*), the monomial $A_{\bar{\alpha}}$ has length at least 2 and therefore $f^{-1}(\bar{\alpha})A_{\bar{\alpha}} = f^{-1}(\bar{\alpha})\bar{\alpha}g(\bar{\alpha})p$ for some monomial $p \in KQ$ of length ≥ 0 . Now condition (2) gives that this is zero in H . Similarly, $A_{\bar{\alpha}}g(f(\alpha)) = qg^{-1}(\beta)\beta f(\beta)$ where $\beta = g^{-2}(\bar{\alpha})$ is the last arrow of $A_{\bar{\alpha}}$ and $q \in KQ$ a monomial of length ≥ 0 , and this is zero in H by condition (2). Similarly one verifies that conditions (1) and (3) agree. \square

Lemma 2.4 *For each vertex i and arrow α starting at i , we have $B_\alpha J = 0$ and $JB_\alpha = 0$ where J is the radical of H . In particular $B_\alpha \neq 0$ belongs to the socle of $e_i H$.*

Proof We have $B_\alpha \alpha = \alpha B_{g(\alpha)} \equiv \alpha B_{g^{-1}(\bar{\alpha})} = \alpha B_{f(\alpha)} = \alpha f(\alpha)g(f(\alpha))p$ where p is some monomial of length ≥ 0 and this is zero by condition (2). Then we have as well that $B_{\bar{\alpha}} \bar{\alpha} \equiv B_{\bar{\alpha}} \bar{\alpha} = 0$. \square

We write $(B_\alpha)_j$ for the initial submonomial $\alpha g(\alpha) \dots g^{j-1}(\alpha)$ of B_α of length j .

Lemma 2.5 *Let $\alpha \in Q_1$, and let $\mathcal{B}_\alpha := \{(B_\alpha)_j \mid 1 \leq j \leq |B_\alpha|\}$ be the set of all initial submonomials of B_α .*

(a) *The set \mathcal{B}_α is linearly independent in H .*

(b) *Assume that $\alpha, \bar{\alpha}$ are both in \mathcal{T} , then $\mathcal{B}_\alpha \cup \mathcal{B}_{\bar{\alpha}}$ also is linearly independent.*

Proof (a) Let

$$\sum_{j=1}^{|B_\alpha|} a_j (B_\alpha)_j = 0 \quad (a_j \in K).$$

Premultiplying with $A_{g^{-1}(\alpha)}$ gives $0 = a_1 A_{g^{-1}(\alpha)} \alpha = a_1 B_{g^{-1}(\alpha)}$ and hence $a_1 = 0$. Suppose we have $a_1 = \dots = a_{r-1} = 0$. We premultiply with the submonomial q of B_α such that $q(B_\alpha)_r$ is equal to B_γ for the appropriate γ . This annihilates all terms except one, leaving only $a_r B_\gamma = 0$ and so $a_r = 0$.

(b) Let $\sum_{j=1}^{|B_\alpha|} a_j (B_\alpha)_j + b A_{\bar{\alpha}} = 0$ with a_j and b in K . We premultiply with $f^{-1}(\bar{\alpha}) = g^{-1}(\alpha)$. By condition (2) of Definition 2.1, using also that $|A_{\bar{\alpha}}| \geq 2$ we get $f^{-1}(\bar{\alpha}) A_{\bar{\alpha}} = 0$, and this leaves

$$\sum_{j=1}^{|B_\alpha|} a_j (B_{g^{-1}(\alpha)})_{j+1} = 0.$$

Hence $a_1 = \dots = a_{|B_\alpha|-1} = 0$, by (a), and we are left with $a_{|B_\alpha|} B_\alpha + b A_{\bar{\alpha}} = 0$. Using that $B_\alpha \equiv B_{\bar{\alpha}}$, we have linear combination of two initial submonomials of $B_{\bar{\alpha}}$, and by part (a) (applied to $\bar{\alpha}$), the coefficients are zero. \square

Lemma 2.6 *The module $e_i H$ has basis $\{e_i\} \cup \mathcal{B}_\alpha \cup \mathcal{B}_{\bar{\alpha}} \setminus \{B_{\bar{\alpha}}\}$. Hence $\dim e_i H = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$.*

Proof Suppose we have

$$\sum_{j=1}^{|B_\alpha|} a_j (B_\alpha)_j + \sum_{t=1}^{|B_{\bar{\alpha}}|-1} \bar{a}_t (B_{\bar{\alpha}})_t = 0. \quad (*)$$

(a) Assume first that (say) $\bar{\alpha}$ is not in \mathcal{T} . We premultiply (*) with $f^{-1}(\bar{\alpha})$, this annihilates the second sum. Recall $f^{-1}(\bar{\alpha}) = g^{-1}(\alpha)$, therefore the first sum becomes

$$0 = \sum a_j (B_{g^{-1}(\alpha)})_{j+1},$$

and by Lemma 2.5, $a_j = 0$ for all $j < |B_\alpha|$. Then (*) becomes

$$0 = a_{|B_\alpha|} B_\alpha + \sum_{t=1}^{|B_{\bar{\alpha}}|-1} \bar{a}_t (B_{\bar{\alpha}})_t = 0.$$

Since $B_\alpha \equiv B_{\bar{\alpha}}$ we can again apply Lemma 2.5 and deduce that all coefficients are zero.

(b) Assume $\alpha, \bar{\alpha}$ are both in \mathcal{T} . We premultiply with $\gamma = f^{-1}(\bar{\alpha})$. We have $\gamma \bar{\alpha} = c_{\bar{\gamma}} A_{\bar{\gamma}}$ but $\gamma \bar{\alpha} g(\bar{\alpha}) = 0$ and there is only one non-zero term from the second sum, namely a multiple of $A_{\bar{\gamma}}$. The first sum is a linear combination of elements $(B_\gamma)_j$ since $\gamma = g^{-1}(\alpha)$. We apply part (b) of Lemma 2.5 and deduce that all scalar coefficients are zero. \square

2.4 Idempotent algebras of WSA's

In [8] we have studied weighted surface algebras whose Gabriel quiver is 2-regular (with at least three vertices). One may ask whether an idempotent algebra of such a WSA is a regular hybrid algebra. We will investigate this, and determine when exactly this is the case, and at the same time it will illustrate why we should allow virtual arrows for general hybrid algebras. Examples can be found in 2.8 below.

Proposition 2.7 *Assume Λ is a WSA with a 2-regular Gabriel quiver. Let Γ be a subset of Q_0 and $e = \sum_{i \in \Gamma} e_i$, and let $R = e\Lambda e$.*

- (i) *The idempotent algebra R satisfies conditions (1) to (3) of Definition 2.1.*
- (ii) *R satisfies the multiplicity condition (*) unless for some $i \in \Gamma$ and α starting at i we have*

- (*1) *$m_\alpha = 1$ and the g -cycle of α intersects Γ only in i (with no repetition); or*
- (*2) *$m_\alpha = 1$ and $\tilde{n}_\alpha = 2$, and Γ contains both $s(g^{-1}(\alpha))$ and $t(\tilde{\alpha})$.*

Proof Let Λ be a WSA with 2-regular Gabriel quiver, that is it has a presentation $\Lambda = KQ/I$ of a (regular) hybrid algebra such that $\mathcal{T} = Q_1$. In particular we have then $m_\alpha n_\alpha \geq 3$ for all α . The only additional assumption in [8] is that the quiver has at least three vertices (see the text following [8, Theorem 1.4]). Take a subset Γ of Q_0 , and let $e = \sum_{i \in \Gamma} e_i$ and $R := e\Lambda e$.

(i) We compute the basic algebra for R . Let \tilde{Q} be the quiver with vertices corresponding to the primitive idempotents of R , that is the $e_i (= ee_i e)$ with $i \in \Gamma$. For $\alpha \in Q_1$ and $s(\alpha) = i \in \Gamma$, let $\tilde{\alpha}$ be the shortest path in Q along the g -cycle of α , starting with α , and ending at a vertex in Γ . We define \tilde{Q} by taking the set

$$\tilde{Q}_1 = \{\tilde{\alpha} \in KQ \mid \alpha \in Q_1, \alpha = e_i \alpha \text{ for } i \in \Gamma\}$$

as its set of arrows. The set \tilde{Q} is a generating set for the radical of R , and hence we have a surjective algebra map $\psi : K\tilde{Q} \rightarrow R$, and $R \cong K\tilde{Q}/\tilde{I}$ where \tilde{I} is the kernel of ψ .

(a) We observe that the quiver \tilde{Q} is 2-regular: We have two arrows starting at each vertex, and also two arrows ending at each vertex (write $B_\alpha, B_{\tilde{\alpha}}$ as a product of elements in \tilde{Q}_1 , then B_α and $B_{\tilde{\alpha}}$ end with distinct arrows of \tilde{Q}).

We define a permutation \tilde{f} . Let $\tilde{\alpha} = \alpha g(\alpha) \dots g^p(\alpha)$ and $\beta := f(g^p(\alpha))$, then

$$\tilde{f}(\tilde{\alpha}) := \tilde{\beta}.$$

With this, each connected component of (\tilde{Q}, \tilde{f}) is a biserial quiver. Furthermore, the permutation \tilde{g} is obtained from the cycles of g in Q , by factorizing them at each vertex in Γ . In particular if \tilde{n}_α is the length of the cycle of $\tilde{\alpha}$, then $1 \leq \tilde{n}_\alpha \leq n_\alpha$. The multiplicity function \tilde{m} for \tilde{Q} must be taken as $\tilde{m}_\alpha = m_\alpha$, and the parameter function \tilde{c} is taken as $\tilde{c}_\alpha = c_\alpha$ for each arrow $\tilde{\alpha}$. Note that we may view the path algebra $K\tilde{Q}$ as a subspace of KQ , and if so then $B_{\tilde{\alpha}}$ is equal to B_α .

(b) There is a canonical set $\tilde{\mathcal{T}}$ of distinguished triangles of \tilde{Q} . Let

$$\tilde{\mathcal{T}} := \{\tilde{\alpha} \mid \alpha = \tilde{\alpha} \text{ and } \widetilde{f(\alpha)} = f(\alpha)\}$$

Note that if $\alpha = \tilde{\alpha}$ and also $f(\alpha) = \widetilde{f(\alpha)}$ then both $s(\alpha)$ and $t(f(\alpha))$ are in Γ , and hence $f^2(\alpha) = \widetilde{f^2(\alpha)}$. Therefore $\tilde{\mathcal{T}}$ is closed under the permutation \tilde{f} . Furthermore, the arrows in $\tilde{\mathcal{T}}$ satisfy the relations (1) of Definition 2.1.

(c) We show now that for $\tilde{\alpha} \notin \tilde{\mathcal{T}}$ we have $\tilde{\alpha}\tilde{f}(\tilde{\alpha}) = 0$. With the notation as in (a) we have

$$\tilde{\alpha}\tilde{f}(\tilde{\alpha}) = \alpha g(\alpha) \dots g^p(\alpha) f(g^p(\alpha))q \quad (*)$$

for some monomial $q \in KQ$. If $p \geq 1$ this is zero in Λ , by condition (2) of Definition 2.1. Suppose now that $p = 0$, so that $\tilde{\alpha} = \alpha$, then $\tilde{f}(\tilde{\alpha}) \neq f(\alpha)$ since $\tilde{\alpha} \notin \tilde{\mathcal{T}}$. Therefore q has length ≥ 1 and $(*)$ has a factor $\alpha f(\alpha)g(f(\alpha))$ which is zero in Λ .

(d) So far we have verified that condition (1) of Definition 2.1 holds. Condition (3) is also satisfied, from analogous conditions in Λ . We can also see that condition (2) holds: For example consider

$$\tilde{\alpha}\tilde{f}(\tilde{\alpha})\tilde{g}(\tilde{f}(\tilde{\alpha})). \quad (**)$$

If $\tilde{\alpha}$ is not in $\tilde{\mathcal{T}}$ then already the product of the first two terms is zero. Suppose $\tilde{\alpha} \in \tilde{\mathcal{T}}$, then $(**)$ is equal to $\alpha f(\alpha)\tilde{g}(f(\alpha))$, which has a factor $\alpha f(\alpha)g(f(\alpha))$ and is zero in Λ . Similarly one obtains the other identity.

(ii) We investigate when R satisfies the condition $(*)$, that is

$$\tilde{m}_{\tilde{\alpha}}\tilde{n}_{\tilde{\alpha}} \geq 2 \text{ and } \tilde{m}_{\tilde{\alpha}}\tilde{n}_{\tilde{\alpha}} \geq 3 \text{ if } \tilde{\alpha} \in \tilde{\mathcal{T}}.$$

Recall $\tilde{m}_{\tilde{\alpha}} = m_{\alpha}$, hence if $m_{\alpha} \geq 3$ then this condition holds. Assume now that $m_{\alpha} = 2$, then the first part of $(*)$ holds. Suppose that we have $m_{\alpha}\tilde{n}_{\tilde{\alpha}} = 2$, then we need to show that then $\tilde{\alpha}$ is not in $\tilde{\mathcal{T}}$.

Write $\tilde{\alpha} = \alpha \dots g^p(\alpha)$, then $B_{\alpha} = \tilde{\alpha}^2$, of length ≥ 3 as an element of KQ (by the assumption on Λ), and hence $p \geq 1$. So we have $t(g^p(\alpha)) = i$ but $s(g^p(\alpha))$ is not in Γ . Assume for a contradiction that $\tilde{\alpha}$ is in $\tilde{\mathcal{T}}$, then $\tilde{\alpha} = \tilde{\alpha} = \bar{\alpha}$ and the vertices between $\bar{\alpha}$, $f(\bar{\alpha})$ and $f^2(\bar{\alpha})$ belong to Γ . Now, $f^2(\bar{\alpha}) = g^{-1}(\alpha) = g^p(\alpha)$ and therefore $s(g^p(\alpha))$ is in Γ , a contradiction. We have shown that when $m_{\alpha} = 2$ for an arrow α starting at i , the condition $(*)$ holds for α .

Assume now that $m_{\alpha} = 1$. It is possible that $\tilde{n}_{\tilde{\alpha}} = 1$ so that already the first part of $(*)$ fails. (For example, take $B_{\alpha} = \tilde{\alpha}$ of length ≥ 3 and $s(\alpha)$ is the only vertex along B_{α} which is in Γ . This is the exception $(*)1$.) We continue with $m_{\alpha} = 1$, and we assume now $\tilde{n}_{\tilde{\alpha}} = 2$, in this case the first part of $(*)$ holds.

We write $B_{\alpha} = (\alpha \dots g^p(\alpha))(g^{p+1}(\alpha) \dots g^r(\alpha))$ where $(\alpha \dots g^p(\alpha)) = \tilde{\alpha}$, so that we have $\tilde{g}(\tilde{\alpha}) = (g^{p+1}(\alpha) \dots g^r(\alpha))$. Then $i = s(\alpha)$ and $j = s(g^{p+1}(\alpha))$ are the only vertices along the g -cycle of α which belong to Γ . The condition $(*)$ fails in this case if and only if $\tilde{\alpha}$ belongs to $\tilde{\mathcal{T}}$.

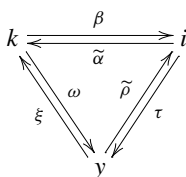
We observe that $\tilde{\alpha} = \tilde{\alpha}$, and this belongs to $\tilde{\mathcal{T}}$ if and only if all the vertices between $\bar{\alpha}$, $f(\bar{\alpha})$ and $f^2(\bar{\alpha})$ belong to Γ , that is, each of i and $t(\bar{\alpha})$ and $s(f^2(\bar{\alpha}))$ is in Γ .

We have $f^2(\bar{\alpha}) = g^{-1}(\alpha) = g^r(\alpha)$, and therefore by the construction $r = p + 1$ and the vertex $s(g^r(\alpha))$ is what we called j . In addition we have $t(\bar{\alpha})$ in Γ . We have arrived at condition $(*)2$. \square

Example 2.8 We take the quiver and the weighted surface algebra Λ as in Example 2.2, that is we take $\Lambda = H_{\mathcal{T}}$ with $\mathcal{T} = Q_1$. The following examples illustrate that the arrows of \tilde{Q} need not be a minimal generating set, that is, \tilde{Q} may not be the Gabriel quiver of the algebra $e\Lambda e$.

(a) Let $\Gamma = \{i\}$. The algebra $R = e\Lambda e$ has the quiver with vertex i and two loops, $\tilde{\alpha}$ and $\tilde{\tau}$. We have $m_{\alpha} = 1$ and $\tilde{n}_{\tilde{\alpha}} = 1$ since $\tilde{\alpha} = B_{\alpha}$. This is an example for the exception $(*)1$ of Proposition 2.7. In fact we also have that $m_{\tilde{\tau}} = 1$ and $\tilde{n}_{\tilde{\tau}} = 1$. Here \tilde{Q} is not the Gabriel quiver of R .

(b) Let $\Gamma = \{i, k, y\}$. Then again $\tilde{n}_{\tilde{\alpha}} = 2$. Now $\tilde{\mathcal{T}} = \{\tau, \xi, \beta\}$ and $\tau = \bar{\alpha}$. The quiver of R is triangular,

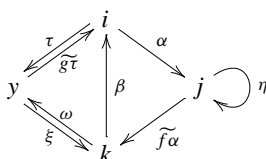


here $\tilde{\alpha} = \alpha\eta\delta\gamma$ and $\tilde{\rho} = \rho\sigma$. The permutation \tilde{g} is the product of three 2-cycles,

$$(\tilde{\alpha} \beta)(\xi \omega)(\tau \tilde{\rho}).$$

The arrow $\tilde{\alpha} = \tau$ is in $\tilde{\mathcal{T}}$ and we have an example for the exception (*) of Proposition 2.7. Note that $m_{\tilde{\rho}} = 1$ and $m_{\tilde{\alpha}} = 1$.

(c) Let $\Gamma = \{i, j, k, y\}$. The algebra R has quiver



and $\tilde{g} = (\xi \omega)(\tau \tilde{\rho})(\beta \alpha \eta \tilde{\delta})$ with multiplicities $m_{\xi} = 2$, $m_{\tau} = 1$ and $m_{\beta} = 1$. We have

$$\tilde{f} = (\omega \tilde{\rho} \alpha \tilde{\delta})(\tau \xi \beta)(\eta)$$

In this case the set of distinguished arrow is $\tilde{\mathcal{T}} = \{\tau, \xi, \beta, \eta\}$. We can see directly using identity (2) of Definition 2.1 that products of arrows in the 4-cycle of f are zero. We observe that $m_{\tilde{\rho}}\tilde{n}_{\tilde{\rho}} = 2$, and $\tilde{\rho} = \xi \in \tilde{\mathcal{T}}$, that is the multiplicity condition is not satisfied. Indeed, we have $s(g^{-1}(\rho)) = i \in \Gamma$ and $t(\xi) = k \in \Gamma$ and we have again an example for the exception (*) of Proposition 2.7.

3 General hybrid algebras

We present now the general definition. The multiplicity condition (*) in 2.3 is replaced by the weaker requirement (**). This has the effect that the quiver Q need not be the Gabriel quiver of the algebra, and therefore we get many more algebras. However now there are exceptions for the zero relations, and they are the main reason for much of the work.

We use the notation as in 2.2, in particular \mathcal{T} is a fixed set of triangles (see 2.3). The condition (*) in 2.3 is replaced by the following.

(**) We assume $m_{\alpha}n_{\alpha} \geq 2$ for all $\alpha \in Q_1$, except that $m_{\alpha}n_{\alpha} = 1$ is allowed when $\alpha, \bar{\alpha}$ are both not in \mathcal{T} .

Then sometimes an arrow may not be part of the Gabriel quiver, and this motivates our definition of virtual arrows:

Definition 3.1 Let i be a vertex, and let α be an arrow starting at i . Then α is a *virtual* arrow if one of the following holds:

- (a) $m_{\alpha}n_{\alpha} = 1$ and $\alpha, \bar{\alpha} \notin \mathcal{T}$; or
- (b) $m_{\alpha}n_{\alpha} = 2$ and $\bar{\alpha} \in \mathcal{T}$. That is, $|A_{\alpha}| = 1$ and $\bar{\alpha} \in \mathcal{T}$.

For the general definition of a hybrid algebra, there are exceptions for the zero relations. To spell these out explicitly, we will use the term 'critical' as in the following definition.

Definition 3.2 Let α be an arrow. We say that α is *critical* if $|A_\alpha| = 2$ and $\alpha \in \mathcal{T}$, and moreover $f(\alpha)$ is virtual (so that $|A_{f(\alpha)}| = 1$ and $g(\alpha) \in \mathcal{T}$).

In Sect. 3.1 we present diagrams showing the quiver near a virtual arrow, or near a critical arrow.

Definition 3.3 Let (Q, f) be a biserial quiver with the data m_\bullet, c_\bullet as in 2.2, and let \mathcal{T} be a set of distinguished triangles. The hybrid algebra $H = H_{\mathcal{T}} = H_{\mathcal{T}}(Q, f, m_\bullet, c_\bullet)$, with assumption (**), is the algebra $H = KQ/I$ where I is generated by the following elements:

- (1) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ for $\alpha \in \mathcal{T}$ and $\alpha f(\alpha)$ for $\alpha \notin \mathcal{T}$.
- (2) $\alpha f(\alpha) g(f(\alpha))$ unless $\alpha, \bar{\alpha} \in \mathcal{T}$, and $\bar{\alpha}$ is either virtual, or is critical.
- (2') $\alpha g(\alpha) f(g(\alpha))$ unless $\alpha, g(\alpha) \in \mathcal{T}$, and $f(\alpha)$ is either virtual, or is critical.
- (3) $c_\alpha B_\alpha - c_{\bar{\alpha}} B_{\bar{\alpha}}$ for any arrow α of Q .
- (4) If all arrows of Q are virtual, then we require $B_\alpha \alpha \in I$ and $\alpha B_{g(\alpha)} \in I$ for each arrow α .

If $\mathcal{T} = Q_1$ and $|Q_0| \geq 2$ this is the same as the definition of a weighted surface algebra in [10], but there we did not use the term 'critical'. If $\mathcal{T} = \emptyset$ then the algebra $H_{\mathcal{T}}$ is special biserial (by (1)), and identities (2) and (2') hold automatically. We will mainly discuss algebras where $\mathcal{T} \neq \emptyset$.

The details for the definition of a hybrid algebra are chosen to ensure that they are precisely the idempotent algebras of weighted surface algebras, up to socle equivalence. Furthermore, we require that hybrid algebras are symmetric, and finite-dimensional. Therefore a few small algebras need to be excluded, which actually are the same which were excluded for weighted surface algebras:

Assumption 3.4 We exclude four algebras, they are not symmetric.

- (1) $|Q_0| = 2$, $\mathcal{T} = Q_1$, with a virtual loop, and the 3-cycle of g has multiplicity $m = 1$ (see 4.2(2a)).
- (2) $|Q_0| = 3$, $\mathcal{T} = Q_1$, the singular algebra with a triangular quiver (see 4.3(3)), or the singular algebra with a linear quiver (see 4.4).
- (3) $|Q_0| = 3$ with a triangular quiver, $\mathcal{T} = Q_1$ and $m \equiv 1$ (see 4.3(1)).
- (4) $|Q_0| = 6$, $\mathcal{T} = Q_1$ when H is the singular spherical algebra as in [10, 3.6] (see 4.7).

In [10, 2.7], we had formulated a slightly different assumption, this is covered by the above (modulo minor changes). One would have liked to have that the Gabriel quiver of H is obtained from Q by removing the virtual arrows. There is however one exception of a local algebra, which is a hybrid algebra (it occurs as an idempotent algebra of a weighted surface algebra, see Example 2.8(a)).

Remark 3.5 In the following there will be computations using the permutations f and g , we describe some basic properties. We will use these freely.

- (1) We always have that $f^{-1}(\alpha) = g^{-1}(\bar{\alpha})$. If $\alpha \in \mathcal{T}$ then $f^{-1}(\alpha) = f^2(\alpha)$ (which may be α).
- (2) Assume i is a quaternion vertex. Then we have, exactly as in [8, 10],

$$\alpha f(\alpha) f^2(\alpha) = c_{\bar{\alpha}} A_{\bar{\alpha}} f^2(\alpha) = c_{\bar{\alpha}} B_{\bar{\alpha}} = c_\alpha B_\alpha = \bar{\alpha} f(\bar{\alpha}) f^2(\bar{\alpha}).$$

Lemma 3.6 *Assume $H = KQ/I$ is a hybrid algebra. Then the Gabriel quiver Q_H of H is obtained from Q by removing the virtual arrows, except when H is local with two virtual loops.*

Proof Suppose i is a vertex with arrows $\alpha, \bar{\alpha}$ starting at i . If they are not virtual then they are part of the Gabriel quiver. As well, if (say) α is virtual but $\bar{\alpha}$ is not virtual then $\bar{\alpha}$ is part of the Gabriel quiver but α is not. Suppose now that $\alpha, \bar{\alpha}$ are both virtual.

(1) Suppose (say) α is a virtual loop and $\bar{\alpha}$ is virtual but not a loop. Then $\bar{\alpha}$ must be virtual of type (b) as in Definition 3.1, and $m_{\bar{\alpha}}n_{\bar{\alpha}} = 2$ which shows $g(\bar{\alpha}) : t(\bar{\alpha}) \rightarrow i$, and $\alpha \in \mathcal{T}$. The arrow $f(\alpha)$ starts at i , so we have either $f(\alpha) = \alpha$, or $f(\alpha) = \bar{\alpha}$. In the first case we would have $g(\alpha) = \bar{\alpha} = g^2(\bar{\alpha})$ and $\alpha = g(\bar{\alpha})$, so that $t(\bar{\alpha}) = i$ and $\bar{\alpha}$ is a loop, which is not the case. Therefore we can only have $f(\alpha) = \bar{\alpha}$, and since $f^2(\alpha)$ must end at i we have $f^2(\alpha) = f(\bar{\alpha}) : t(\bar{\alpha}) \rightarrow i$ and it follows that $f(\bar{\alpha}) = g(\bar{\alpha})$, a contradiction. So this cannot happen.

(2) Suppose that α and $\bar{\alpha}$ are virtual but not loops, then they are both in \mathcal{T} (and they cannot be double arrows since then g would consist of two 2-cycles, and Q would have only two vertices, hence the arrows cannot be in 3-cycles of f). Then Q has a subquiver of the form

$$3 \begin{array}{c} \xrightarrow{g(\bar{\alpha})} \\ \xleftarrow{\bar{\alpha}} \end{array} i \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{g(\alpha)} \end{array} 2$$

with $m_{\alpha} = 1 = m_{\bar{\alpha}}$. By definition of virtual, α and $\bar{\alpha}$ are in \mathcal{T} , hence they must lie in 3-cycles of f . Then $f^2(\alpha)$ ends at vertex 1, so it is either $g(\alpha)$ or $g(\bar{\alpha})$. Since $f(f^2(\alpha)) = \alpha = g(g(\alpha))$ it follows that $f^2(\alpha) \neq g(\alpha)$, hence it is equal to $g(\bar{\alpha})$. Therefore, $f(\alpha)$ must be an arrow $2 \rightarrow 3$. Similarly $f(\bar{\alpha})$ is an arrow $3 \rightarrow 2$. That is, Q is the triangular quiver, with three vertices, and g is a product of 2-cycles. We have $m_{\alpha} = 1 = m_{\bar{\alpha}}$ and we have excluded in Assumption 3.4(3) that $m \equiv 1$. It follows that $m_{f(\alpha)} \geq 2$ and $f(\alpha), f(\bar{\alpha})$ are not virtual. We will see in Lemma 4.2 that such an algebra has finite type, and that the Gabriel quiver is obtained by removing the virtual arrows.

(3) Assume both $\alpha, \bar{\alpha}$ are virtual loops. First, suppose (say) that α is in \mathcal{T} , then both $\alpha, \bar{\alpha}$ are virtual of type (b). We have $f = (\alpha)(\bar{\alpha})$ and $g = (\alpha \bar{\alpha})$ with $m_{\alpha} = 1$. This algebra is dealt with in 4.1(2a), and we will see that $H \cong K$. Hence the Gabriel quiver of H is obtained by removing the virtual arrows.

If $\alpha, \bar{\alpha}$ are not in \mathcal{T} , that is they are virtual of type (a) in Definition 3.1, then $m_{\alpha} = m_{\bar{\alpha}} = 1$. We see that $H \cong K[x]/(x^2)$, and that its Gabriel quiver is not obtained from Q by removing the virtual arrows. \square

Corollary 3.7 *The only hybrid algebras for which all arrows are virtual are local algebras 4.1 (2a) and 4.1(1) with $m_{\bullet} \equiv 1$.*

Proof Assume α is virtual of type (a), then $\alpha, \bar{\alpha}$ are not in \mathcal{T} . Since we also assume $\bar{\alpha}$ is virtual it must also be of type (a). By (3) of the above proof, H is as stated. Suppose now all arrows are virtual of type (b). Then we can proceed as in part (2) of the proof of Lemma 3.6, and get H is the algebra with triangular quiver and $m \equiv 1$. But this is excluded (see Assumption 3.4(3)). \square

3.1 The exceptions in relations (2) and (2')

The exceptions in (2) and (2') of Definition 3.3 create special cases in various proofs to come.

First we show that there is a unique algebra with two vertices where a critical arrow occurs in a g -cycle with a loop (see Lemma 3.8 below). Otherwise the exceptions always arise in specific subquivers of the same kind, for which we will now fix notation, to be used later. We write $\zeta_\alpha = \alpha f(\alpha)g(f(\alpha))$ and $\xi_\alpha = \alpha g(\alpha)f(g(\alpha))$. We always have $\alpha, \bar{\alpha} \in \mathcal{T}$, hence all virtual arrows are of type (b).

We take care of critical arrows whose g -cycle contains a loop.

Lemma 3.8 *Assume τ is critical.*

(a) *The g -cycle of τ contains a loop if and only if $|Q_0| = 2$ and H is the algebra in 4.2(2c).*

(b) *Assume the g -cycle of τ does not contain a loop, then $f(\tau)$ cannot be a loop.*

Proof Assume τ is critical, then $|A_{g^i(\tau)}| \neq |A_{f(\tau)}|$ and hence $f(\tau)$ does not belong to the g -cycle of τ .

(a) For H as in 4.2(2c) one checks directly that the arrow $\tau := \gamma$ is critical and its g -cycle contains a loop. For the converse, assume τ is critical. If $g(\tau) = \tau$ then H cannot be local

(if so then τ would be in a 2-cycle of f). Hence Q contains $\tau \circlearrowleft i \xrightleftharpoons[f^2(\tau)]{f(\tau)} j$ but then since

$f(\tau)$ is virtual we have $g(f(\tau)) = f^2(\tau) = f(f(\tau))$ which is a contradiction. It follows that the g -cycle of τ has length 3 and is a subquiver of Q of the form $\circlearrowleft i \xrightleftharpoons{f(\tau)} j$.

Now, $f(\tau)$ is not part of this subquiver but τ is in \mathcal{T} . It follows that $f(\tau)$ is a loop at j and τ is the arrow $i \rightarrow j$. In particular Q has three vertices and H is the algebra in 4.2(2c) with γ as the critical arrow.

(b) Suppose $\tau : j \rightarrow y$, and assume $f(\tau)$ is a loop. Then since $\tau \in \mathcal{T}$ we must have that $f^2(\tau) : y \rightarrow j$. But as well the arrow $g(\tau) (\neq f(\tau))$ starts at y . Since Q is 2-regular, we deduce $g(\tau) = f^2(\tau)$ and since $g^3(\tau) = \tau$ it follows that $g^2(\tau)$ is a loop at j , a contradiction. \square

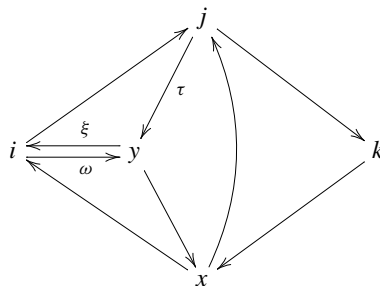
In the following, we exclude the algebra 4.2(2c). That is we assume that a critical arrow does not occur in a g -cycle with a loop, and that the g -cycle with a critical arrow has size 3.

3.1.1 The subquiver around a critical arrow

We will see that in the exceptional cases

$$\zeta_\alpha \equiv A_\alpha \text{ and } \xi_\alpha \equiv A_\alpha.$$

Let τ be a critical arrow, in a g -cycle of length three, then by definition τ and $f(\tau)$ belong to \mathcal{T} . In order to study the paths ζ_α and ξ_α near τ in the exceptional cases, we also assume that $g^2(\tau)$ belongs to \mathcal{T} . Then by Lemma 3.8 the quiver near τ has the following form



The permutation f has 3-cycles through vertices j, y, i and y, x, i and j, k, x . At vertex k the quiver there is at least one other arrow, to have a 2-regular quiver. We assume that τ is critical, so that $m_\tau = 1$ and moreover $\xi = f(\tau)$ is virtual. Since all f -cycles shown belong to \mathcal{T} , the arrow ω is also virtual.

(a) We study the path $\zeta_\alpha = \alpha f(\alpha)g(f(\alpha))$ when $\bar{\alpha}$ is critical, using the above diagram. That is we take for α the arrow $j \rightarrow k$, so that $\bar{\alpha} = \tau$. Then we have

$$\zeta_\alpha = c_{\bar{\alpha}} A_{\bar{\alpha}} g(f(\alpha)) = c_\tau \tau g(\tau) f(g(\tau)) = c_\tau c_\xi \tau \xi = c_\tau c_\xi c_\alpha A_\alpha.$$

This must be non-zero since we require that $A_\alpha \neq 0$. We note that $A_\alpha = \alpha \cdot C \cdot f(\alpha)g(f(\alpha))$ where C is a monomial of positive length.

(b) We study the path $\xi_\alpha = \alpha g(\alpha) f(g(\alpha))$ when $f(\alpha)$ is critical, using the above diagram. Here we take for α the arrow $i \rightarrow j$. Then

$$\xi_\alpha = \alpha \cdot c_\tau A_\tau = c_\tau \alpha \tau g(\tau) = c_\tau c_\omega \omega g(\tau) = c_\tau c_\omega c_\alpha A_\alpha$$

which again must be non-zero. We note that $A_\alpha = \alpha \cdot C \cdot f(g(\alpha))$ where again C is a monomial of length ≥ 1 .

Remark (a) It is not possible that α and $\bar{\alpha}$ are both critical. Suppose $\tau = \bar{\alpha}$ and $\alpha : j \rightarrow k$ is also critical, then $f(\alpha) : k \rightarrow x$ is virtual, so there must be an arrow $x \rightarrow k$ and three arrows start at x , a contradiction.

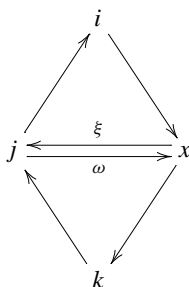
(b) If τ is critical in a g -cycle of length three then in general $g(\tau)$ need not be in \mathcal{T} .

3.1.2 Subquivers around a virtual arrow

We will see that in the exceptional cases

$$\zeta_\alpha \equiv A_\alpha \text{ and } \xi_\alpha \equiv A_{\bar{\alpha}}.$$

(1) Assume first that the virtual arrow is not a loop, then there is a pair ξ, ω of virtual arrows, and the quiver contains



The arrows shown form two 3-cycles of f , and belong to \mathcal{T} . First we assume $|Q_0| > 3$, that is $i \neq k$. We assume ξ, ω are virtual, then the other arrows in the diagram are not virtual.

(a) Consider $\zeta_\alpha = \alpha f(\alpha)g(f(\alpha))$ for $\bar{\alpha}$ virtual, then $\bar{\alpha}$ is one of ξ or ω .

Consider the case $\bar{\alpha} = \xi$, then we take for α the arrow $x \rightarrow k$. Then

$$\zeta_\alpha = c_\xi \xi g(f(\alpha)) = c_\xi c_\alpha A_\alpha$$

and this must be non-zero. One can write $A_\alpha = \alpha \cdot C$ where C is a monomial of length ≥ 1 . When $\bar{\alpha} = \omega$ then we take for α the arrow $x \rightarrow i$ and we get similarly

$$\zeta_\alpha = c_\omega c_\alpha A_\alpha$$

and we can write $A_\alpha = \alpha C$ with C a monomial of length ≥ 1 .

(b) Consider $\xi_\alpha = \alpha g(\alpha) f(g(\alpha))$ for $f(\alpha)$ virtual, that is $f(\alpha) = \xi$ or ω . If $f(\alpha) = \xi$ then we take for α the arrow $i \rightarrow j$, and

$$\xi_\alpha = \alpha c_\xi \xi = c_\xi c_{\bar{\alpha}} A_{\bar{\alpha}}$$

and this must be non-zero. We can write $A_{\bar{\alpha}} = C f(g(\alpha))$ where C is a monomial of positive length. Suppose $f(\alpha) = \omega$, then we take for α the arrow $k \rightarrow x$, and we get

$$\xi_\alpha = c_\omega c_{\bar{\alpha}} A_{\bar{\alpha}}$$

which must be non-zero. We can write $A_{\bar{\alpha}} = C f(g(\alpha))$ for a monomial C of positive length.

(2) Now assume $i = k$ so that $|Q_0| = 3$. By 4.3(2) we can assume the multiplicities are not $(m, 1, 1)$ (as this gives a Nakayama algebra), and in 4.3(3) we deal with multiplicities $(2, 2, 1)$. This leaves multiplicities $(m_1, m_2, 1)$ where $(m_1, m_2) \neq (2, 2)$ and $m_i \geq 2$. This case is similar to the above, we omit details.

(3) Now we consider a virtual loop, and analyze the exceptions. Here we can use the quiver

$$\omega \begin{array}{c} \curvearrowright \\ \end{array} i \rightleftarrows k$$

where ω is virtual. Consider $\zeta_\alpha = \alpha f(\alpha) g(f(\alpha))$ when $\bar{\alpha}$ is virtual using this diagram, that is $\omega = \bar{\alpha}$. We take for α the arrow $i \rightarrow k$. By assumption $\omega = g(\omega)$ and therefore f has cycle $(\omega \alpha f(\alpha))$. Moreover $g(f(\alpha)) = \alpha$. We have

$$\zeta_\alpha = c_\omega \omega \alpha = c_\omega c_\alpha A_\alpha.$$

Now consider $\xi_\alpha = \alpha g(\alpha) f(g(\alpha))$ when $f(\alpha)$ is virtual, using this diagram. That is $f(\alpha) = \omega$. We take for α the arrow $j \rightarrow i$. Then $g(\alpha) : i \rightarrow j$ and $g(\alpha) = f^2(\alpha)$ and $f(g(\alpha)) = \alpha$. We have

$$\xi_\alpha = \alpha f^2(\alpha) \alpha = \alpha c_\omega \omega = c_\omega c_{\bar{\alpha}} A_{\bar{\alpha}}$$

□

As in 3.1.1, we can deduce a general description of a path of type ζ or ξ in a subquiver of the above forms (allowing also for arrows at i or k): The following Corollary gives already Lemma 7.1

Corollary 3.9 *Consider any path of length three of the form ζ_σ or ξ_σ in the subquiver of 3.1.1 or 3.1.2 shown.*

- (a) *If the path does not contain ξ or ζ then it must be non-zero in H .*
- (b) *If the path contains ξ or ζ then it is zero in H .*

Part (a) is implicitly part of the discussion. Part (b) can be seen using the relations (2) and (2') of Definition 3.3.

3.2 Consistency, bases and dimensions

This is an update for the case done in Sect. 2, when virtual arrows are allowed. This may be found in the “Appendix”.

4 Some hybrid algebras with at most three simple modules

In [8] and [10] we have excluded small quivers, to avoid technical problems obscuring the general structure. However, here one of the main aims is characterize hybrid algebras as idempotent algebras of weighted surface algebras. This forces us to include small algebras as well.

In this section we consider some hybrid algebras whose quiver has at most four vertices. We will mainly discuss algebras where $\mathcal{T} \neq \emptyset$, and which can have virtual arrows of type (b), for small multiplicities. Note that given (Q, f) and \mathcal{T} , together with m_\bullet, c_\bullet , the algebra is completely determined, and we will usually not write down relations explicitly.

4.1 Local algebras

Here Q consists of one vertex and two loops, denoted by α and β . There are two possibilities for f and g , and if f is the identity permutation there are three possibilities, depending on \mathcal{T} .

(1) Consider an algebra where $f = (\alpha \beta)$ and $g = (\alpha)(\beta)$, then we must have $\mathcal{T} = \emptyset$. We may assume $m_\alpha \geq m_\beta$.

If $m_\beta = 1$ then $H \cong K[x]/(x^{m_\alpha+1})$. Otherwise it is an algebra of dihedral type as in [5, III.1(a)]:

The relations are:

$$\alpha\beta = 0 = \beta\alpha, \quad c_\alpha B_\alpha = c_\alpha \alpha^{m_\alpha} = c_\beta \beta^{m_\beta} = c_\beta B_\beta.$$

If $m_\beta = 1$ so that β is virtual (of type (a) of Definition 3.1), then $H \cong K[x]/(x^{m_\alpha+1})$. This also holds when $m_\alpha = 1$; in this case the Gabriel quiver of H is not obtained from Q by removing the virtual arrows (see also Lemma 3.6). If $m_\beta \geq 2$ then H is special biserial, of infinite type and is a (commutative) algebra of dihedral type, as defined in [5, III.1(a)].

(2) Consider hybrid algebras where $f = (\alpha)(\beta)$ and $g = (\alpha \beta)$, so $m_\alpha n_\alpha \geq 2$.

(2a) Assume first that $\mathcal{T} = Q_1$. If $m_\alpha = 1$ then $H \cong K$, and if $m_\alpha \geq 2$ then H is an algebra as in [5, III.1(e)] of quaternion type:

Assume that $m_\alpha = 1$, we may assume that $c_\alpha = 1$. The relations are

$$\alpha^2 = A_\beta = \beta \quad \text{and} \quad \beta^2 = A_\alpha = \alpha,$$

that is, both arrows are virtual. By condition (4) of Definition 3.3 we have that $B_\alpha \alpha = 0 = \alpha B_{g(\alpha)}$. Relation (3) gives $B_\alpha = \alpha\beta = B_\beta = \beta\alpha$. and hence $\alpha^2\beta = 0$ and therefore

$$0 = \alpha^4 = \beta^2 = \alpha$$

and similarly $\beta = 0$. We have shown that $H \cong K$. On the other hand, when $m_\alpha \geq 2$ then we see directly that we get an algebra of quaternion type, as in [5, III.1.(e)]. The algebras where $H \cong K$ cannot occur as an idempotent algebra of a WSA Λ , since $e_i \Lambda e_i$ has at least two independent elements: the idempotent e_i and the generator of the socle of $e_i \Lambda$.

(2b) Assume $\mathcal{T} = \{\beta\}$. If $m_\alpha = 1$ then $H \cong K[x]/(x^4)$. Otherwise H is an algebra as in [5, III.1(d)] (of semidihedral type):

We may assume $c_\alpha = 1$, and we have the relations

$$\beta^2 = A_\alpha = (\alpha\beta)^{m_\alpha-1}\alpha, \quad \alpha^2 = 0.$$

If $m_\alpha \geq 2$, this gives precisely the algebras in [5, III.1(d)]. Suppose $m_\alpha = 1$ so that the arrow α is virtual. Then we see $\beta^3 \equiv B_\alpha$ and $\beta^4 = 0$ which shows that H is isomorphic

to $K[x]/(x^4)$. In this case the Gabriel quiver is obtained from Q by removing the virtual arrows.

(2c) Assume $\mathcal{T} = \emptyset$. Then H is an algebra as in [5, III.1(b)]. For $m_\alpha = 1$ it is four-dimensional commutative: This is seen directly from the relations. \square

4.2 Hybrid algebras with two simple modules

Let H be a hybrid algebra with two simple modules, then $H = KQ/I$ where the quiver Q is of the form

$$\alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \sigma$$

We consider only the cycle structures of f, g for which \mathcal{T} can be non-empty and the algebra can have virtual arrows of type (b).

(1) Consider algebras with

$$f = (\alpha)(\beta\gamma)(\sigma) \text{ with } g = (\alpha \beta \sigma \gamma).$$

Suppose $\mathcal{T} \neq \emptyset$, then \mathcal{T} consists of one or two loops, and there are no virtual arrows. The algebras look similar to algebras of semidihedral type in [5], however they have always singular Cartan matrices, which was excluded for semidihedral type.

(2) Consider algebras where

$$f = (\alpha \beta \gamma)(\sigma) \text{ with } g = (\alpha)(\beta \sigma \gamma).$$

For hybrid algebras with $\mathcal{T} \neq \emptyset$, the possibilities for \mathcal{T} are either Q_1 , or $\mathcal{T} = \{\sigma\}$, or $\mathcal{T} = \{\alpha, \beta, \gamma\}$.

4.2(2a) **The case $\mathcal{T} = Q_1$ and $(t, m) = (2, 1)$** This is excluded in Assumption 3.4(1). In [10] it was excluded because the algebras appeared to be of finite type. However the argument was based on the incorrect relations. Here we review this algebra, with amended relations.

We may take $c_\bullet = (1, c)$. Note that α is virtual and γ is critical. The associated hybrid algebra is given by the relations

$$\begin{array}{llll} \beta\gamma = \alpha, & \gamma\alpha = c\sigma\gamma, & \alpha\beta = c\beta\sigma, & \sigma^2 = c\gamma\beta, \\ \alpha\beta\sigma = 0, & \gamma\alpha^2 = 0, & \sigma\gamma\alpha = 0, & \alpha^2\beta = 0 \end{array}$$

These imply that the algebra is not symmetric. Alternatively, there is a quick way to get a contradiction. Namely

$$0 = \beta\sigma^2 \equiv \beta\gamma\beta = \alpha\beta$$

and $\alpha \notin \mathcal{T}$.

4.2 (2b) **The case $\mathcal{T} = Q_1$ and $t = 3$ and $m = 1$.** This was dealt with in [10, Example 3.1(1)], the algebra is called *disc algebra*, and is denoted by $D(\lambda)$. Viewed in the context of periodicity, it has a singular version: when the scalar parameter $\lambda = 1$ it is not periodic, but it is a hybrid algebra. In that case $\text{rad}(e_1 H)/S_1 \cong \text{rad}(e_2 H)/S_2$ and is indecomposable, and the simple modules belong to an Auslander–Reiten component of type D . The algebra is of semidihedral type, part of the family $SD(2\mathcal{B})_3$ in [5] and it is a hybrid algebra.

4.2 (2c) **Algebras with $\mathcal{T} = \{\alpha, \beta, \gamma\}$ and $(t, m) = (2, 1)$.** This is the only algebra where the g -cycle of a critical arrow has a loop (see Lemma 3.8). However the algebra is seen below to be special biserial and we do not have to consider it further. The arrow α is virtual, and γ

is critical. We may take $c_\alpha = 1$, and we set $c_\beta = c$. Then the associated hybrid algebra is given by the relations:

$$\begin{aligned} \beta\gamma &= \alpha, & \gamma\alpha &= c\sigma\gamma = \gamma\beta\gamma, & \alpha\beta &= c\beta\sigma = \beta\gamma\beta, & \sigma^2 &= 0, \\ \beta\gamma\beta\sigma &= 0, & (\gamma\beta)^2\gamma &= 0, & (\beta\gamma)^2\beta &= 0, & \sigma^2\gamma &= 0, & \beta\sigma^2 &= 0, & \sigma\gamma\beta\gamma &= 0. \end{aligned}$$

Note that $\gamma\beta\sigma = \sigma\gamma\beta = c^{-1}(\gamma\beta)^2$ and $B_\beta J = 0 = B_\gamma J$.

Lemma 4.1 *The algebra H is special biserial. More precise, let $\sigma' := (c\sigma - \gamma\beta)$. Then $\sigma'\gamma = 0$ and $\beta\sigma' = 0$.*

Then H has presentation $k\tilde{Q}/\tilde{I}$ where \tilde{Q} is the quiver

$$1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 2 \begin{array}{c} \curvearrowright \sigma \\ \curvearrowleft \end{array}$$

and $\tilde{I} = \langle \sigma\gamma, \beta\sigma, \sigma^2 - (\gamma\beta)^2 \rangle$.

Proof Rewriting the relations gives that $\sigma'\gamma = 0$ and $\beta\sigma' = 0$. Note that σ' may be taken as an arrow. We have $\sigma'\sigma = c\sigma^2 - \gamma\beta\sigma$ and it is non-zero in the socle of e_2A . We have $\sigma'\gamma\beta = c(\sigma\gamma\beta - (\gamma\beta)^2) = 0$, hence

$$(\sigma')^2 = -(\gamma\beta)^2$$

We may rescale σ' and then obtain the presentation as stated. \square

One may introduce a virtual loop at 1, which gives a presentation of a hybrid algebra.

4.2 (2d) Algebras with $\mathcal{T} = \{\sigma\}$ and $t = m = 1$.

Here α is virtual of type (a) (note that α and $\bar{\alpha} = \beta$ are not in \mathcal{T}). We can take $c_\alpha = 1$ and we set $c_\beta = c$. Then the relations are

$$\begin{aligned} \alpha\beta &= 0, & \beta\gamma &= 0, & \gamma\alpha &= 0, & \sigma^2\gamma &= 0, \\ \beta\sigma^2 &= 0, & \sigma^2 &= c\gamma\beta, & \alpha &= c(\beta\sigma\gamma), & cB_\gamma &= cB_\sigma. \end{aligned}$$

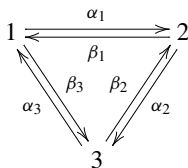
This algebra occurs in (3.6) of [22], with a slightly different presentation. It is an algebra of finite (Dynkin) type \mathbb{D} .

We consider now some algebras with three simple modules.

In total there are five possible quivers for which f has at least one 3-cycle. We will discuss algebras with 'triangular' and 'linear' quiver in some detail first, and will briefly consider the other three later.

4.3 Algebras with triangular quiver

Let Q be the quiver



The only cycle structure for which \mathcal{T} can be non-empty is given by $f = (\alpha_1 \alpha_2 \alpha_3)(\beta_1 \beta_3 \beta_2)$, so that $g = (\alpha_1 \beta_1)(\alpha_2 \beta_2)(\alpha_3 \beta_3)$. We write $m_i = m_{\alpha_i}$ and $c_i = c_{\alpha_i}$.

4.3 (1) **Algebras with $\mathcal{T} = Q_1$ and $m_\bullet = (1, 1, 1)$.** Such an algebra is excluded in Assumption 3.4(3). It was excluded in [10, 4.4], though the argument was not correct. The algebra is given by the relations

$$\alpha_i \alpha_{i+1} = c_{i+2} \beta_{i+2}, \quad \beta_i \beta_{i-1} = c_{i-2} \alpha_{i-2}$$

(indices modulo 3). As well $B_{\alpha_i} = \alpha_i \beta_i \equiv B_{\beta_{i-1}} = \beta_{i-1} \alpha_{i-1}$, and there are no zero relations of types (2) or (2'). We observe that

$$\alpha_1 \beta_1 \equiv \beta_3 \beta_2 \alpha_2 \alpha_3 \equiv \beta_3 \alpha_3 \beta_3 \alpha_3 \equiv (\alpha_1 \beta_1)^2 = 0$$

and this is zero by condition (4) of Definition 3.3. Similarly all paths $\alpha_i \beta_i$ and $\beta_i \alpha_i$ are zero, and then any cyclic path of positive length is zero in the algebra. Therefore the algebra is not symmetric.

4.3 (2) **Algebras with $\mathcal{T} = Q_1$ and $m_\bullet = (m, 1, 1)$ and $m \geq 2$.**

Such an algebra was excluded in [10, 4.4], as it was said to be not finite-dimensional. However this is not correct, it has even finite type, as we will now show. Note also that the Gabriel quiver is obtained by removing the virtual arrows.

Lemma 4.2 *With these conditions, H has finite type, it is isomorphic to the direct sum of a Nakayama algebra*

$$KQ / \langle (\alpha\beta)^{m-1} \alpha, (\beta\alpha)^{m-1} \beta \rangle$$

with a copy of K , where Q is the quiver $1 \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} 2$.

Proof The relations are as follows.

$$\begin{aligned} \alpha_1 \alpha_2 &= c_3 \beta_3, & \alpha_2 \alpha_3 &= c_1 A_{\beta_1}, & \alpha_3 \alpha_1 &= c_2 \beta_2 \\ \beta_1 \beta_3 &= c_2 \alpha_2, & \beta_3 \beta_2 &= c_1 A_{\alpha_1}, & \beta_2 \beta_1 &= c_3 \alpha_3, \\ \alpha_2 \alpha_3 \beta_3 &= 0, & \beta_3 \beta_2 \alpha_2 &= 0, & \alpha_3 \beta_3 \beta_2 &= 0, & \beta_2 \alpha_2 \alpha_3 &= 0. \end{aligned}$$

Moreover we have the consequences

$$c_1 B_{\alpha_1} = c_3 B_{\beta_3}, \quad c_1 B_{\beta_1} = c_2 B_{\alpha_2}, \quad c_2 B_{\beta_2} = c_3 B_{\alpha_3}$$

(1) Starting with the relation $0 = \alpha_2 \alpha_3 \beta_3 (= \alpha_2 B_{\alpha_3})$ we show that $\beta_1 \alpha_1 \alpha_2 = 0$: Namely

$$0 = \alpha_2 B_{\alpha_3} \equiv \alpha_2 B_{\beta_2} = B_{\alpha_2} \alpha_2 \equiv B_{\beta_1} \alpha_2 = (\beta_1 \alpha_1)^m \alpha_2.$$

Next we have

$$(\beta_1 \alpha_1)^m \alpha_2 = (\beta_1 \alpha_1)^{m-1} \beta_1 \alpha_1 \alpha_2 \equiv (\beta_1 \alpha_1)^{m-1} \beta_1 \beta_3 \equiv (\beta_1 \alpha_1)^{m-1} \alpha_2.$$

Repeating this reduction gives $\beta_1 \alpha_1 \alpha_2 = 0$ and then $\alpha_2 = 0$. Similarly we have $0 = \beta_3 = \beta_2 = \alpha_3$. Hence the algebra has a direct summand spanned by e_3 which is isomorphic to K . Furthermore, from the relations we have $A_{\beta_1} = 0$ and $A_{\alpha_1} = 0$, and there are no further restrictions. This shows that the subalgebra generated by e_1, e_2 and α_1, β_1 is the Nakayama algebra as stated. \square

4.3 (3) **Algebras with $\mathcal{T} = Q_1$ and $m_\bullet = (2, 2, 1)$.**

They are called *triangle algebras*, as discussed in [10, Example 3.3 (1)], and denoted by $T(\lambda)$, where $c_\bullet = (\lambda, 1, 1)$. The algebra with $\lambda = 1$ is not symmetric, as it was shown in [10, 3.3], and therefore it is excluded in Assumption 3.4(2).

4.3 (4) **Algebras with $\mathcal{T} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $m_\bullet = (1, 1, 1)$.** In this case, the arrows β_i are virtual, and the algebra $H_{\mathcal{T}}$ is a Nakayama algebra of finite type: The relations are

$$\alpha_i \alpha_{i+1} = c_{i+2} \beta_{i+2} \text{ and } \beta_i \beta_{i-1} = 0.$$

There are no exceptions to the zero relations in (2) and (2') since for any arrow α we have $\alpha \notin \mathcal{T}$ or $\bar{\alpha} \notin \mathcal{T}$, and $\alpha \notin \mathcal{T}$ or $g(\alpha) \notin \mathcal{T}$. It is straightforward to check that H is the Nakayama algebra where the quiver is cyclic with three vertices, and where all paths of length four are zero in the algebra.

4.4 Algebras with linear quiver

Consider algebras whose quiver is of the form

$$\alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 2 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\delta} \end{array} 3 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \eta$$

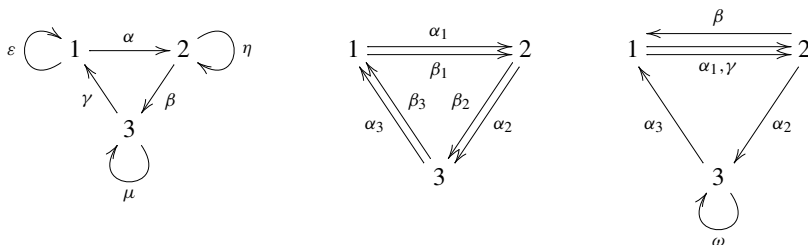
To have that $\mathcal{T} \neq \emptyset$ containing some virtual arrows of type (b), we have two possibilities for the permutations f and g :

$$\begin{aligned} f &= (\alpha \beta \gamma)(\sigma \eta \delta) \text{ and } g = (\alpha)(\beta \sigma \delta \gamma)(\eta), \text{ or} \\ f &= (\alpha \beta \gamma)(\sigma \delta)(\eta) \text{ and } g = (\alpha)(\beta \sigma \eta \delta \gamma). \end{aligned}$$

For most of the hybrid algebras with these cycle structures, virtual arrows do not lead to special cases. We only discuss algebras with the first cycle structure and where $m_\bullet = (2, 1, 2)$. This has been considered in Example 3.4 of [10]. It is shown that we may assume $c_\bullet = (1, \lambda, 1)$, the algebra is called $\Sigma(\lambda)$. Furthermore, it is proved (in Lemma 3.5 of [10]) that $\Sigma(\lambda)$ is isomorphic to the triangular algebra $T(\lambda^{-2})$ introduced in 4.3. In particular this implies that we must exclude $\lambda = \pm 1$, since then the algebra is not symmetric. We refer to this as a singular algebras, which are excluded in Assumption 3.4(2).

4.5 Three other quivers with three vertices

The following three quivers also have each at least one 3-cycle of f which may or may not belong to \mathcal{T} :

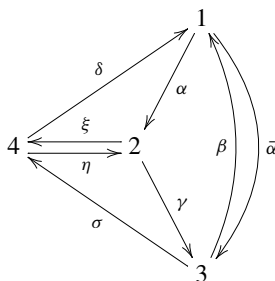


For the first two quivers, there are no virtual arrows of type (b) since there is just one g -orbit of size 6. Consider the third quiver when $f = (\alpha_1 \alpha_2 \alpha_3)(\omega)(\beta \gamma)$. Then $g = (\alpha_1 \beta)(\gamma \alpha_2 \omega \alpha_3)$. We consider the case when $m_\bullet \equiv 1$, then the arrow β is virtual if \mathcal{T} contains $\{\alpha_1, \alpha_2, \alpha_3\}$. However this does not create complications: If β is virtual then relations $\gamma \alpha_2 \alpha_3$ and $\alpha_2 \alpha_3 \gamma$ are excluded in (2), (2') of Definition 3.3. In this case they are, up to non-zero scalars, equal

to $\gamma\beta$ and $\beta\gamma$, which are zero since f has the cycle $(\beta \ \gamma)$. We note that the algebras are not of semidihedral type, as the Cartan matrices are singular.

4.6 An exceptional algebra with four simple modules

Let $H = KQ/I$ where Q is the quiver



with $f = (\bar{\alpha} \ \beta)(\alpha \ \xi \ \delta)(\gamma \ \sigma \ \eta)$, and hence $g = (\alpha \ \gamma \ \beta)(\bar{\alpha} \ \sigma \ \delta)(\xi \ \eta)$. Moreover, we take $m_{\bullet} = 1$ and $c_{\alpha} = c$ and $c_{\beta} = c_{\xi} = 1$. Let $\mathcal{T} = \{\alpha, \xi, \delta, \gamma \ \sigma \ \eta\}$, and let H be the hybrid algebra defined by these data. Then ξ and η are virtual arrows and the Gabriel quiver Q_H is obtained by removing ξ, η .

Lemma 4.3 *The algebra H is special biserial. Let \bar{Q} be the quiver obtained from Q by removing ξ and η , and adding virtual loops ε, ρ of type (a). Then H has a hybrid algebra presentation with this quiver, and with $\bar{\mathcal{T}} = \emptyset$, defined by the data*

$$\bar{f} = (\delta \ \bar{\alpha}' \ \sigma \ \varepsilon)(\alpha \ \rho \ \gamma \ \beta'), \quad \bar{g} = (\delta \ \alpha \ \gamma \ \sigma)(\bar{\alpha}' \ \beta')(\rho)(\varepsilon)$$

with multiplicity $\equiv 1$ and parameter function $\equiv 1$. The loops ε, ρ are virtual of type (a).

Proof Starting with the given presentation, we replace β by $\beta' := \sigma\delta - c\beta$, then $\beta'\alpha = 0$ and $\gamma\beta' = 0$. We also replace $\bar{\alpha}$ by $\bar{\alpha}' := \alpha\gamma - \bar{\alpha}$, and then $\bar{\alpha}'\sigma = 0$ and $\delta\bar{\alpha}' = 0$. We take ε to be the socle monomial $\delta\alpha\gamma\sigma$, and we take ρ to be the socle monomial $\gamma\sigma\delta\alpha$. Then it is straightforward to show that the algebra has the stated presentation. \square

4.7 Singular algebras

In addition to the singular disk, and triangle algebra as we have discussed above, there are two further algebras which were called singular in [8] and [10]. Recall from [8] Example 6.1 the *tetrahedral algebras*. This family contains one algebra, with certain parameters, which is not periodic, and therefore it was called singular in that context. However, it is a hybrid algebra.

Furthermore, in Example 3.6 of [10] we have discussed *spherical algebras*, denoted by $S(\lambda)$ for $\lambda \in K^*$. The quiver has six vertices, and with the smallest multiplicities the algebra has four virtual arrows. When $\lambda = 1$, it is not symmetric and is therefore excluded in Assumption 4.3(4).

5 Hybrid algebras as idempotent algebras of weighted surface algebras

In the first part of this section we will prove that for a weighted surface algebra Λ and an idempotent e of Λ , every block of $e\Lambda e$ is a hybrid algebra. In the second part of this section we will show that every hybrid algebra with $\mathcal{T} \neq Q_1$ occurs in this way. The second part generalizes the main results of [15], which dealt with the hybrid algebras where $\mathcal{T} = \emptyset$, that is, the Brauer graph algebras. Note that we start with a weighted surface algebra, which is not a socle deformation.

Theorem 5.1 *Assume Λ is a weighted surface algebra and let $e \in \Lambda$ be an idempotent. Then each block of the algebra $e\Lambda e$ is a hybrid algebra.*

Proof We fix a weighted surface algebra Λ , and we proceed as in the proof of Proposition 2.7. By general theory, we may assume that $e = \sum_{i \in \Gamma} e_i$ with Γ a subset of the vertices of Q , and we set $R = e\Lambda e$, and we may assume that e is not the identity of Λ . We take the quiver \tilde{Q} with vertices labelled by Γ . For $\alpha \in Q_1$, let $\tilde{\alpha}$ be the shortest path in Q along the g -cycle of α starting with α and ending at some vertex in Γ . We take the set \tilde{Q}_1 of these $\tilde{\alpha}$ as arrows for \tilde{Q} , it is a generating set for R , and we have a surjective algebra map $\psi : K\tilde{Q} \rightarrow R$. As in 2.7, the quiver \tilde{Q} is 2-regular. When $\tilde{\alpha} = \alpha$ then we write for simplicity α . We define the permutation \tilde{f} , and the distinguished set $\tilde{\mathcal{T}}$ of triangles, as in Proposition 2.7. The cycles of the associated permutation \tilde{g} are obtained from the cycles of g by replacing $(\alpha g(\alpha) \dots g^p(\alpha))$ by $\tilde{\alpha}$. We take the multiplicity and parameter functions as for Λ . Then we may write down elements $B_{\tilde{\alpha}}$ of R for each arrow α , and it is clear that these satisfy identity (3) of Definition 3.3. As well we have elements $A_{\tilde{\alpha}}$ such that $A_{\tilde{\alpha}}\tilde{\gamma} = B_{\tilde{\alpha}}$ where $\tilde{\gamma}$ is the last arrow in $B_{\tilde{\alpha}}$. Furthermore, the exceptions in relations (2) and (2') occur precisely when the arrows $\alpha, \tilde{\alpha}$ (or $\alpha, g(\alpha)$) are in $\tilde{\mathcal{T}}$.

We will show that the arrows in \tilde{Q}_1 satisfy the identity (1) of Definition 3.3. For the arrows in $\tilde{\mathcal{T}}$, this follows directly from identity (1) for Λ . Let $\tilde{\alpha}$ be an arrow of \tilde{Q} which is not in $\tilde{\mathcal{T}}$, and let $p := \tilde{\alpha}\tilde{f}(\tilde{\alpha})$. We must show that this is zero in R , (possibly after some adjusting), or possibly that it is a scalar multiple of a socle element, ie we have a socle deformation.

Since $\tilde{\alpha}$ is not in $\tilde{\mathcal{T}}$, we know that p has length $|p| \geq 3$ as a path in Q . If $|p| \geq 5$ then it is zero in Λ , this follows from Lemma 7.5. Suppose now that p is non-zero, then we must have $|p| = 3$ or $|p| = 4$. For the following we exclude the algebras 4.3(2) (this can be done by hand, using Lemma 4.2. Furthermore we exclude 4.2 and 4.3(3), they will be considered below in 5.1.

(a) Assume first that $|p| = 3$, then p is of the form ζ_{α} of ξ_{α} , near a critical or virtual arrow. We start with p near a critical arrow.

(a1) Assume $p = \zeta_{\alpha} = \alpha f(\alpha)g(f(\alpha))$ and $\tilde{\alpha}$ is critical. That is we have $\tilde{\alpha} = \alpha$ and $\tilde{f}(\tilde{\alpha}) = f(\alpha)g(f(\alpha))$. We use diagram 3.1.1, and set $\tau = \tilde{\alpha}$ so that $\alpha : j \rightarrow k$. In this case Γ contains vertices j, k, i but Γ does not contain x . Let $\beta = f(\alpha)$. The cycle of \tilde{f} containing $\tilde{\alpha}$ is

$$(\alpha \tilde{\beta} \tilde{\omega} \gamma \tilde{\alpha})$$

where $\gamma : i \rightarrow j$. Note that $\tilde{\omega} = B_{\omega}$ and $\tilde{\alpha} = B_{\tilde{\alpha}}$ and therefore products along the \tilde{f} cycle with these elements are zero. It remains to adjust the product of $\tilde{\alpha}$ and $\tilde{\beta}$.

By 3.1.1 we have $p = c_{\tilde{\alpha}}c_{\xi}c_{\alpha}A_{\alpha}$, and we see from the diagram that $A_{\alpha} = \alpha C\beta g(\beta)$ where C is a monomial in the arrows of Q of positive length and therefore, as an element of R , it belongs to the radical. We can replace the arrow $\tilde{\alpha}$ by

$$\tilde{\alpha}' := \tilde{\alpha}(1 - c_{\tilde{\alpha}}c_{\xi}c_{\alpha}C)$$

and this has product zero with the arrow $\widetilde{\beta}$.

(a2) Assume $p = \xi_\alpha = \alpha g(\alpha) f(g(\alpha))$ and $f(\alpha)$ is critical. Then we use the diagram of 3.1.1 again, now taking $\alpha : i \rightarrow j$ and we set $\beta = f(g(\alpha))$ so that $p = \alpha \widetilde{\beta}$. In this case, Γ contains i, k, x but not the vertex j . From this we see that the cycle of \widetilde{f} containing $\widetilde{\alpha}$ is

$$(\widetilde{\alpha} \beta \widetilde{f(\beta)} g(\beta) \widetilde{\omega})$$

here $\widetilde{f(\beta)}$ and $\widetilde{\omega}$ are socle elements and products with these along the cycle are zero, also after any adjustment. It remains to deal with the product of $\widetilde{\alpha}$ and β .

We have

$$\xi_\alpha = c_{f(\alpha)} c_\omega c_\alpha A_\alpha$$

In this case, we see from the diagram that $A_\alpha = \widetilde{\alpha} \cdot C \cdot \beta$ where C is a monomial of positive length. We set $\widetilde{\alpha}' = \widetilde{\alpha}(1 - c_{f(\alpha)} c_\omega c_\alpha C)$ and this can be taken as an arrow, and it satisfies $\widetilde{\alpha}' \beta = 0$.

Now consider p near a virtual arrow.

(a3) Assume $p = \zeta_\alpha$ so that $\widetilde{\alpha} = \alpha$, and assume $\bar{\alpha}$ is virtual. We have $\widetilde{\alpha} = \alpha$. Then $s(\alpha)$ and $s(f(\alpha))$ are in Γ but $t(f(\alpha))$ is not in Γ . In this case the virtual arrow $\bar{\alpha}$ cannot be a loop: Otherwise, using part (3) of 3.1.2, we have $\alpha : i \rightarrow j$ and both i, j are in Γ . But then $f(\alpha) : j \rightarrow i$ is an arrow of \widetilde{Q} and $f(\alpha) = \widetilde{f}(\widetilde{\alpha}) \neq f(\alpha)g(f(\alpha))$.

Now we use the diagram (1) of 3.1.2. We can assume that the virtual arrow $\bar{\alpha}$ is equal to ξ , that is we take $\alpha : x \rightarrow k$. The set Γ contains x, k, i but does not contain j . Let $\beta : k \rightarrow i$. We see that \widetilde{f} has the cycle of length four, that is $(\alpha \widetilde{f}(\widetilde{\alpha}) \beta \widetilde{\xi})$. Moreover $\widetilde{\xi} = \xi\omega = B_\xi$ and it belongs to the socle. Therefore $\beta \widetilde{\xi} = 0$ and $\widetilde{\xi}\alpha = 0$. The other two products need to be adjusted. By 3.1.2 we have

$$\alpha \widetilde{f}(\widetilde{\alpha}) = c_\xi c_\alpha A_\alpha \quad \text{and} \quad \widetilde{f}(\widetilde{\alpha})\beta = c_\omega c_{g(\alpha)} A_{g(\alpha)}.$$

Now, we can write $A_\alpha = \alpha \cdot C$ for a monomial C of positive length between vertices in \widetilde{Q} , and luckily, we also have $C \cdot \beta = A_{g(\alpha)}$, moreover $c_\omega = c_\xi$ and $c_\alpha = c_{g(\alpha)}$. Hence we can replace $\widetilde{f}(\widetilde{\alpha})$ by $\widetilde{f}(\widetilde{\alpha})' := \widetilde{f}(\widetilde{\alpha}) - c_\xi c_\alpha C$.

(a4) Assume $p = \xi_\alpha$, so that $\widetilde{\alpha} = \alpha g(\alpha)$, and assume $f(\alpha)$ is virtual. As in (a3), the virtual arrow cannot be a loop. We use the diagram (1) of 3.1.2 and we take α to be the arrow $k \rightarrow j$. Then we have the following arrows of \widetilde{Q}

$$\widetilde{\alpha} : k \rightarrow i, \quad \beta = \widetilde{f}(\widetilde{\alpha}) : i \rightarrow x, \quad \widetilde{\xi} : j \rightarrow j, \quad \gamma : x \rightarrow k$$

and they belong to the cycle of \widetilde{f} of length four

$$(\widetilde{\alpha} \beta \widetilde{\xi} \gamma).$$

Since $\widetilde{\xi} = \xi\omega = B_\xi$ is in the socle, the products with β and γ are zero. We see from 3.1.2 that

$$\widetilde{\alpha}\beta = c_\omega c_{\bar{\alpha}} A_{\bar{\alpha}}, \quad \text{and} \quad \gamma\widetilde{\alpha} = c_\xi c_\gamma A_\gamma$$

Moreover $A_\gamma = \gamma C$ and $C\beta = A_{\bar{\alpha}}$ and as well $c_\gamma = c_{\bar{\alpha}}$ and $c_\xi = c_\omega$. We replace $\widetilde{\alpha}$ by $\widetilde{\alpha}' := \widetilde{\alpha} - c_\omega c_{\bar{\alpha}} C$, then the remaining products along the cycle of \widetilde{f} are zero.

(b) The case when $|p| = 4$ and $p \neq 0$ in Λ : Then by Lemma 7.5 we have $p = \widetilde{\alpha}\widetilde{\beta}$ where $\widetilde{\alpha} = \alpha g(\alpha)$ and $\widetilde{\beta} = \beta g(\beta)$ for $\beta = f(g(\alpha))$. That is we can write $p = \xi_\alpha g(\beta) (= \alpha \zeta_{g(\alpha)})$, and we must have that $\xi_\alpha \neq 0$. This means that the arrow $f(\alpha)$ is critical or virtual.

(b1) Assume $f(\alpha)$ is a virtual loop. In this case we use the diagram (3) of 3.1.2, with $\omega = f(\alpha)$ so that $\alpha : i \rightarrow j$ and $g(\alpha) (= f^2(\alpha)) : j \rightarrow i$. Then $\beta = \alpha$ and therefore $\tilde{\alpha} = \tilde{f}(\tilde{\alpha})$ and it is a loop fixed by \tilde{f} . We compute

$$\tilde{\alpha}^2 = c_\alpha c_\omega B_\alpha$$

which is non-zero in the socle. This means that at $\tilde{\alpha}$ we have a socle deformation.

(b2) Assume $f(\alpha)$ is virtual but not a loop. Then we use the diagram 3.1.2 with $\alpha : k \rightarrow j$, so that $\beta : i \rightarrow x$. Then Γ contains k, i but does not contain j, x . We see that \tilde{f} has a cycle of length two, namely $(\tilde{\alpha} \tilde{\beta})$. Using the formulae in 3.1.2 we compute

$$\tilde{\alpha}\tilde{\beta} = c_{f(\alpha)}c_\alpha B_\alpha = c_{f(\alpha)}c_\alpha \tilde{\alpha}C$$

where C is a monomial of positive length from i to k . Similarly

$$\tilde{\beta}\tilde{\alpha} = c_{f(\beta)}c_\beta B_\beta = c_{f(\beta)}c_\beta \tilde{\beta}B_{\tilde{\beta}}$$

using (3) of Definition 3.3. Now $\tilde{\beta}$ is in the g -orbit of α and we see $c_{\tilde{\beta}} = c_\alpha$ and moreover $B_{\tilde{\beta}} = C\tilde{\beta}$. Furthermore $c_{f(\alpha)} = c_{f(\beta)}$. We replace $\tilde{\beta}$ by $\tilde{\beta}' := \tilde{\beta} - c_\alpha c_{f(\alpha)}C$.

(b3) Assume $f(\alpha)$ is critical. Then we use the diagram 3.1.1 with $\alpha : i \rightarrow j$, and $\beta = f(g(\alpha)) : k \rightarrow x$. Then i, k are in Γ but j, x are not in Γ . The \tilde{f} -cycle of $\tilde{\alpha}$ is

$$(\tilde{\alpha} \tilde{\beta} \tilde{\omega})$$

and $\tilde{\omega} = B_\omega$, hence the product of $\tilde{\omega}$ with any arrow is zero.

Using the calculations in 3.1.1 we have

$$\xi_\alpha g(\beta) = c_{f(\alpha)}c_\omega c_\alpha B_\alpha$$

We factorise $B_\alpha = \tilde{\alpha}C\tilde{\beta}$ and C is a monomial of positive length. We can replace $\tilde{\alpha}$ by $\tilde{\alpha}' := \tilde{\alpha}(1 - c_\alpha c_{f(\alpha)}c_\omega C)$ and the $\tilde{\alpha}'\tilde{\beta} = 0$ (and $\tilde{\omega}\tilde{\alpha}' = 0$).

(c) We determine now when the algebra $R = e\Lambda e$ has only virtual arrows, and then verify that Condition (4) of Definition 3.3 holds.

(i) We show first that in this case, R does not have a virtual arrow of type (b):

Suppose such an arrow $\tilde{\alpha}$ say exists. Then $m_\alpha \tilde{n}_{\tilde{\alpha}} = 2$ and $\tilde{\alpha} = \tilde{\alpha} \in \tilde{T}$. Then $\tilde{\alpha} = \bar{\alpha}$. This must also be virtual and necessarily of type (b). Therefore also $\tilde{\alpha} \in \tilde{T}$ and then $\tilde{\alpha} = \alpha$.

So we have two f -cycles of arrows in Q which all remain arrows of \tilde{Q} . If $\alpha, \bar{\alpha}$ are both loops then Λ must be local and $e = 1$ which is excluded. So say $\alpha : i \rightarrow j$ and $i \neq j$. Then i, j belong to Γ . Since $\tilde{n}_{\tilde{\alpha}} \leq 2$, the g -cycle of α cannot pass through any other vertex of Γ and $g(\alpha)$ is a path from j to i . However $g^{-1}(\alpha) = f^{-1}(\bar{\alpha})$ and it starts at some vertex in Γ . It follows that $g(\alpha) = f^{-1}(\bar{\alpha})$.

Assume (for a contradiction) that $\bar{\alpha}$ is a loop: Then $f(\bar{\alpha}) : i \rightarrow j$ but Q is 2-regular and then $f(\bar{\alpha}) = \alpha$. But then $f(\alpha)$ must be a loop at j and Q has two vertices and moreover $Q = \tilde{Q}$ and $e = 1$ which is excluded.

Then Q has subquiver with three vertices which has arrows $\alpha, \bar{\alpha}, f(\bar{\alpha})$ and $f^2(\bar{\alpha})$. Now we can use the same reasoning for $\bar{\alpha}$ and see that $f^{-1}(\alpha)$ which is $g^{-1}(\bar{\alpha})$ is an arrow $k = t(\bar{\alpha}) \rightarrow i$. Then $f(\alpha) : j \rightarrow k$ and Q is the triangular quiver. The algebra Λ has at least four virtual arrows and this is excluded in 4.3(2).

(ii) We have shown that if R has only virtual arrows then all arrows are virtual of type (a), and hence they are loops, and R is local. Then $\tilde{\alpha} = B_\alpha$ and $\tilde{\alpha} = B_{\bar{\alpha}}$. In particular $\tilde{g}(\tilde{\alpha}) = \tilde{\alpha}$ and therefore $\tilde{f} = (\tilde{\alpha} \tilde{\alpha})$. We see that R is the local algebra as in 4.1(1) with both multiplicities equal to 1. We also see that condition (4) of 3.3 holds. \square

5.1 The proof of 5.1 in the special cases

We consider the algebras which were excluded in the above proof.

5.1.1 Idempotent algebras for a WSA as in 4.2

That is, $\Lambda = KQ/I$ where the quiver Q is of the form

$$\alpha \begin{array}{c} \curvearrowright \\ \end{array} 1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 2 \begin{array}{c} \curvearrowright \\ \end{array} \sigma$$

and $f = (\alpha \beta \gamma)(\sigma)$ so that $g = (\alpha)(\beta \sigma \gamma)$. Let $m_\alpha = t \geq 2$ and $m_\beta = m$, and we can take $c_\alpha = \lambda$ and $c_\beta = 1$. By 4.2 (2a) (and Assumption 3.4(1)), if $t = 2$ then $m \geq 2$. Furthermore, if $(t, m) = (3, 1)$ then $\lambda \neq 1$ (see 4.2(2b)). There are two idempotent algebras $\neq \Lambda$ to be considered, and we describe the result:

(1) Let $R = e_1 \Lambda e_1$. This gives a local algebra as in 4.1(1). In particular for $m = 1$ we have $R \cong K[x]/(x^t)$.

(2) Let $R = e_2 \Lambda e_2$, then $\tilde{T} = \{\sigma\}$ and we get the algebras as in 4.1(2b). When $t > 2$ it is of semidihedral type, and if $t = 2$ it is a socle deformation of an algebra of semidihedral type.

We omit details for (1), but we give details for (2), to show how a socle deformation occurs. Hence let $R := e_2 \Lambda e_2$. This algebra has quiver

$$\tilde{\gamma} \begin{array}{c} \curvearrowright \\ \end{array} 2 \begin{array}{c} \curvearrowright \\ \end{array} \sigma$$

where $\tilde{\gamma} = \gamma\beta$. The permutations are

$$\tilde{f} = (\tilde{\gamma})(\sigma) \text{ and } \tilde{g} = (\tilde{\gamma} \sigma).$$

In this case we have $\tilde{T} = \{\sigma\}$. We write down the type (1) relations of Definition 3.3. The first one is

$$\sigma^2 = A_{\tilde{\gamma}} = (\tilde{\gamma}\sigma)^{m-1}\tilde{\gamma}.$$

Next, $\tilde{\gamma}^2 = \gamma\beta\gamma\beta = 0$ provided $t > 2$, by the zero relations for Λ . Assume now $t = 2$, then using the relations for Λ we see

$$\gamma\beta\gamma\beta = \lambda\gamma\alpha\beta = \lambda\gamma A_\beta = \lambda B_\gamma$$

which is non-zero and spans the socle of R . That is, we get an algebra as in 4.1(2b) when $t > 2$. If $t = 2$ we get a socle deformation of such an algebra.

5.1.2 Idempotent algebras when Λ is a WSA as in 4.3(3)

Then the quiver is triangular, and we have $m_\bullet = (2, 2, 1)$ and $c_\bullet = (\lambda, 1, 1)$. The arrows α_3, β_3 are virtual, and up to labelling we have to consider four idempotent algebra. We describe the result, the details are straightforward and are omitted.

(1) If $e = e_1 + e_2$ and $R = e\Lambda e$ then R is a Brauer graph algebra with one virtual loop.
 (2) If $e = e_1 + e_3$ then again $R = e\Lambda e$ is a Brauer graph algebra. In this case, the virtual arrows of Λ are not virtual as arrows of R ,

(3) The algebra $e_1 \Lambda e_1$ is a local hybrid algebra as in 4.1(1).

(4) The algebra $e_2 \Lambda e_2$ is a 4-dimensional algebra of dihedral type, as in 4.1(1).

Remark 5.2 (a) Suppose $\tilde{\alpha}$ is an arrow of \tilde{Q} starting at i . We must show that $\tilde{m}_{\tilde{\alpha}}\tilde{n}_{\tilde{\alpha}} = 1$ only occurs when the vertex is biserial and $\tilde{\alpha}$ is a loop.

We have $\tilde{n}_{\tilde{\alpha}} = 1$ if and only if $\tilde{\alpha}$ is the product of all arrows in the g -cycle of α , hence is a loop. If in addition $\tilde{m}_{\tilde{\alpha}} = 1$ then $\tilde{\alpha} = B_{\alpha}$ and clearly $\tilde{\alpha}\tilde{\alpha} = 0$ and $\tilde{\alpha}\tilde{\alpha} = 0$. To see that i is biserial we need $\tilde{\alpha}$ is not in \tilde{T} . This is clear if α is a loop since then $\alpha = B_{\alpha}$ and α is virtual of type (a). Suppose α is not a loop and $\tilde{\alpha} = B_{\alpha}$. The last arrow in B_{α} is $f^2(\tilde{\alpha})$ and it does not start at a vertex of Γ and therefore $\tilde{\alpha}$ cannot be in \tilde{T} .

(b) The algebra $e\Lambda e$ is symmetric, therefore the exceptions in Assumption 3.4 cannot occur.

We will now show that every hybrid algebra, such that $T \neq Q_1$, occurs as an idempotent algebra of some weighted surface algebra. This generalizes the main result of [15] where this was done for the case of Brauer graph algebras. As in [15], our tool is the $*$ -construction which we will now introduce.

5.2 The $*$ -construction

Let H be a hybrid algebra such that $T \neq Q_1$, say $H = H_T(Q, f, m_{\bullet}, c_{\bullet})$, and let g be the permutation associated to f . The $*$ -construction gives a triangulation quiver (Q^*, f^*) which contains Q_0 , and furthermore, contains all arrows in T .

The idea is to keep the arrows of T as they are, but split each arrow which is not in T , and add extra arrows in order to create triangles. With this, one has weighted surface algebras with m^*, c^* extending m, c . Explicitly, define

$$Q_0^* := Q_0 \cup \{x_{\alpha}\}_{\alpha \in Q_1 \setminus T}, \quad Q_1^* := T \cup \{\alpha', \alpha'', \varepsilon_{\alpha}\}_{\alpha \in Q_1 \setminus T}$$

For $\beta \in T$ we set $s^*(\beta) = s(\beta)$ and $t^*(\beta) = t(\beta)$. Let α be an arrow which is not in T . Then we set

$$\begin{aligned} s^*(\alpha') &:= s(\alpha), & t^*(\alpha') &:= x_{\alpha}, & s^*(\alpha'') &:= x_{\alpha}, & t^*(\alpha'') &:= t(\alpha) \\ s^*(\varepsilon_{\alpha}) &= x_{f(\alpha)}, & t^*(\varepsilon_{\alpha}) &= x_{\alpha}. \end{aligned}$$

Next we define the permutation f^* on Q^* . If $\beta \in T$ then we take $f^*(\beta) = f(\beta)$, and define

$$f^*(\alpha'') := f(\alpha)', \quad f^*(f(\alpha)') := \varepsilon_{\alpha}, \quad f^*(\varepsilon_{\alpha}) := \alpha'.$$

Then (Q^*, f^*) is a triangulation quiver.

This determines the permutation g^* , explicitly it is as follows. First, if the arrow α of Q is not in T then $g^*(\alpha') = \alpha''$. The arrows starting at $t(\alpha'')$ in Q are $f(\alpha)$ and $g(\alpha)$, and $g^*(\alpha'')$ depends on whether or not $g(\alpha)$ is in T , that is

$$g^*(\alpha'') = \begin{cases} g(\alpha)' & g(\alpha) \notin T, \\ g(\alpha) & \text{else.} \end{cases}$$

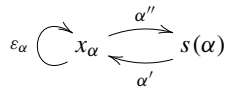
Finally, $g^*(\varepsilon_{\alpha}) = \varepsilon_{f^{-1}(\alpha)}$ for any $\alpha \in Q_1 \setminus T$. The cycles of g^* are obtained from the cycles of g by replacing each α in $Q_1 \setminus T$ by α', α'' , together with cycles only containing arrows of the form ε_{α} . On the cycles without ε -arrows, we take the same multiplicity function and parameter function as for H . On the ε -cycles we may choose multiplicities and parameters arbitrarily. We take them equal to 1 unless when some arrow ε_{γ} is required to be not virtual or critical, then we choose $m_{\varepsilon_{\gamma}} \geq 3$, or when some non-zero scalar factor needs to be specified, we may choose $c_{\varepsilon_{\gamma}}$ differently, depending on the context. This defines then a weighted surface

algebra $\Lambda = \Lambda(Q^*, f^*, m^*, c^*)$. In fact, this is a choice, we could equally well apply the $*$ construction also to triangles in \mathcal{T} .

Note that when $\mathcal{T} = Q_1$, the construction does not do anything, and H is already a weighted surface algebra as in [10]. The case when H is local and $\mathcal{T} = Q_1$ is discussed in 4.1(2a), and this is not a weighted surface algebra by the definition in [10].

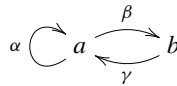
Example 5.3 We illustrate the $*$ -construction.

(1) A loop α in Q fixed by f which does not belong to \mathcal{T} is replaced in Q^* by the subquiver

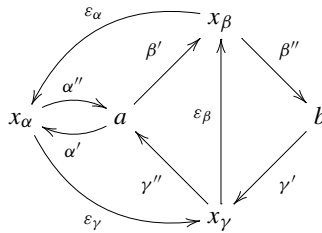


which is an orbit of f^* .

(2) An f -cycle in Q which does not belong to \mathcal{T} of the form

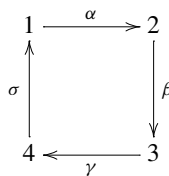


is replaced in Q^* by the quiver

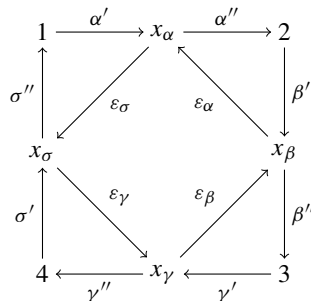


with f^* -orbits $(\alpha'' \beta' \epsilon_\alpha)$, $(\gamma'' \alpha' \epsilon_\gamma)$ and $(\beta'' \gamma' \epsilon_\beta)$.

(3) Suppose f has a 4-cycle



Then the corresponding part of Q^* is of the form



Theorem 5.4 Assume H is a hybrid algebra, such that $\mathcal{T} \neq Q_1$. Then there is a weighted surface algebra Λ and an idempotent e of Λ such that H is isomorphic to a block component of $e\Lambda e$.

Proof Given $H = H_{\mathcal{T}}(Q, f, m_{\bullet}, c_{\bullet})$. We let (Q^*, f^*) and Λ as constructed above. Now let e be the idempotent $e := \sum_{i \in Q_0} e_i$. We want to show that $e\Lambda e$ is isomorphic to H .

We have three algebras, the given algebra is $H = KQ/I$, next we have the weighted surface algebra $\Lambda = KQ^*/I^*$ associated to the triangulation quiver (Q^*, f^*) as introduced above. Furthermore, we have the idempotent algebra $e\Lambda e$. By Theorem 5.1 we know that it has a presentation $K\widetilde{Q}^*/\widetilde{I}^*$ and that it is a hybrid algebra.

Since $e = \sum_{i \in Q_0} e_i$, the quiver \widetilde{Q}^* has vertices $(\widetilde{Q}^*)_0 = Q_0$. The arrows of \widetilde{Q}^* are obtained by contracting paths of Q^* of shortest length between vertices in Q_0 . The arrows of Q^* are

- (1) the arrows of \mathcal{T} ,
- (2) arrows α' , α'' and ε_{α} for each arrow $\alpha \in Q_1 \setminus \mathcal{T}$.

The arrows of Q^* starting at some vertex in Q_0 are therefore the α in \mathcal{T} , and the α' when $\alpha \notin \mathcal{T}$. If $\alpha \in \mathcal{T}$ then $\widetilde{\alpha} = \alpha$, and if $\alpha \notin \mathcal{T}$ then $\alpha' = \alpha'\alpha''$. So \widetilde{Q}^*_1 is the set of $\widetilde{\alpha}$ for $\alpha \in \mathcal{T}$ and α' for $\alpha \in Q_1 \setminus \mathcal{T}$.

The set of triangles $\widetilde{\mathcal{T}}$ of the algebra $K\widetilde{Q}^*/\widetilde{I}^*$ consists therefore of the set $\{\widetilde{\alpha} \mid \alpha \in \mathcal{T}\}$ (see part (c) in the proof of Theorem 5.1). We define a surjective algebra map $\psi : K\widetilde{Q}^* \rightarrow H$ by $\psi(e_i) = e_i$ and if $\widetilde{\gamma}$ is an arrow of \widetilde{Q}^* then

$$\psi(\widetilde{\gamma}) = \begin{cases} \gamma, & \text{if } \widetilde{\gamma} = \gamma \\ \alpha, & \text{if } \widetilde{\gamma} = \alpha', \end{cases}$$

and extending to products and linear combinations.

We show now that $\psi(\widetilde{I}) = 0$ (that is ψ induces an algebra homomorphism from $e\Lambda e$ to H). First we observe that ψ takes any submonomials of $B_{\widetilde{\gamma}}$ starting and ending at vertices in Q_0 to its 'contraction', replacing each subpath of the form $\alpha'\alpha''$ by α , and leaving each $\gamma \in \mathcal{T}$ unchanged.

(a) We consider relation (1) of Definition 3.3. Assume $\widetilde{\gamma} \in \widetilde{\mathcal{T}}$, then we have $\widetilde{\gamma}\widetilde{f}(\widetilde{\gamma}) = \widetilde{\gamma}\widetilde{f}(\alpha) = c_{\widetilde{\gamma}}A_{\widetilde{\gamma}}$. By the above observation we see that ψ preserves this identity. Now consider an arrow of the form $\widetilde{\alpha}'$ for $\alpha \in Q_1$ and not in \mathcal{T} . Then we have

$$\widetilde{\alpha}'\widetilde{f}(\widetilde{\alpha}') = \alpha'\alpha'' \cdot f(\alpha)'f(\alpha)'' \quad (*)$$

By definition, $\psi(\widetilde{\alpha}')\psi(\widetilde{f}(\widetilde{\alpha}')) = \alpha f(\alpha) = 0$. By our convention, we can make sure that $\varepsilon_{f^{-1}(\alpha)} (= f(\alpha'))$ is not virtual or critical. Then the path $\alpha'\alpha'' \cdot f(\alpha)'f(\alpha)''$ is zero, by Lemma 7.1 (see "Appendix").

(b) Next consider a loop of the form $\widetilde{\alpha}'$ for $\alpha \in Q_1$ and $\widetilde{\alpha}'$ not in $\widetilde{\mathcal{T}}$, with $\widetilde{\alpha}' = \widetilde{f}(\widetilde{\alpha}')$. Then we have $f(\alpha) = \alpha$ and $\alpha^2 = 0$. Now

$$\widetilde{\alpha}'^2 = \alpha'\alpha'' \cdot \alpha'\alpha''. \quad (*)$$

By definition $\psi(\widetilde{\alpha}')^2 = \alpha^2 = 0$. The subquiver of Q^* constructed from a loop α fixed by f is shown in Example 5.3(1). We have

$$\alpha'\alpha''\alpha'\alpha'' = c\alpha'A_{\varepsilon_{\alpha}}\alpha'' \quad (\dagger)$$

where $c = c_{\varepsilon_\alpha} \neq 0$. We may choose c and we may also choose m_{ε_α} . We take m_{ε_α} large enough so that ε_α is not virtual or critical, and then (\dagger) is zero.

(c) Now consider the relations (2) and (2') of Definition 3.3 when α and $\tilde{\alpha}$ (respectively $g(\alpha)$) are in \mathcal{T} . Then also $f(\alpha)$ is in \mathcal{T} and this part of the quiver, the map ψ is an identification, so the relations are preserved. Otherwise the elements are mapped to zero by (1) of Definition 3.3. The socle relations (3) follow automatically.

To complete the proof it suffices to establish that $e\Lambda e$ and H have the same dimensions. For any vertex i , the dimension of $e_i H$ is $m_\alpha n_\alpha + m_{\tilde{\alpha}} n_{\tilde{\alpha}}$, and it is the same as that of $e_i (e\Lambda e)$. \square

Example 5.5 Let Λ be the local algebra with arrows α, β and

$$f = (\alpha)(\beta), \quad g = (\alpha \beta).$$

We take $\mathcal{T} = \{\beta\}$ with $m_\bullet = 1$ and $c_\bullet = c$ so that α is virtual. The relations are

$$\beta^2 = cA_\alpha, \quad \alpha^2 = 0$$

and the zero relation $\alpha\beta\alpha = 0$. We apply the $*$ construction to α . This gives the algebra Λ^* with quiver

$$\varepsilon_\alpha \begin{array}{c} \curvearrowright \\ \text{ } \end{array} x_\alpha \begin{array}{c} \xrightarrow{\alpha''} \\ \xleftarrow{\alpha'} \end{array} i \begin{array}{c} \curvearrowleft \\ \text{ } \end{array} \beta$$

Take $m_{\varepsilon_\alpha} = 4$. We may write down the relations defining Λ , for simplicity write $\varepsilon = \varepsilon_\alpha$ and $d = c_\varepsilon$.

$$\alpha''\alpha' = dA_\varepsilon, \quad \alpha'\varepsilon = cA_\beta, \quad \varepsilon\alpha'' = cA_{\alpha''}, \quad \beta^2 = cA_{\alpha'}$$

together with the zero relations, in particular $\alpha'\alpha''\alpha' = 0$.

Now consider the idempotent algebra $e\Lambda e$, we want this to be isomorphic to H . By Theorem 5.1 it has a presentation $K\tilde{Q}/\tilde{I}$ where \tilde{Q} is the quiver with two loops $\tilde{\alpha}'$ and $\tilde{\beta}$, and $\tilde{\alpha}' = \alpha'\alpha''$, and $\tilde{\beta} = \beta$. This has relations

$$\tilde{\beta}^2 = cA_{\tilde{\alpha}'}, \quad (\tilde{\alpha}')^2 = 0$$

Remark 5.6 The algebra Λ in the proof of Theorem 5.4 is a WSA and hence is symmetric, so it is not one of the exceptions in Assumption 3.4.

Lemma 5.7 Assume H is a hybrid algebra. Then H is tame and symmetric.

Proof We have proved that any hybrid algebra is an idempotent algebra of a (general) weighted surface algebra. Weighted surface algebras are tame and symmetric (see [10]), and it is well known that idempotent algebras of tame symmetric algebras are tame and symmetric. \square

6 Stable Auslander–Reiten components

This section is more general, here we assume Λ is a tame symmetric algebra such that its Gabriel quiver is 2-regular. We can take Λ to be basic, with an admissible presentation $\Lambda = KQ/I$ and hence Q is 2-regular.

For background we refer to Chapter 4 in [2].

The Auslander–Reiten (AR) quiver Γ_Λ of an algebra Λ is the graph where the vertices correspond to isomorphism types of indecomposable Λ -modules, and where the arrows are labelled in terms of irreducible maps. For our context it is most relevant that this quiver encodes Auslander–Reiten (AR) sequences, also known as almost split sequences.

A short exact sequence $0 \rightarrow M \rightarrow E \xrightarrow{\sigma} N \rightarrow 0$ is an AR sequence if M and N are indecomposable, the map σ does not split, and moreover given any module N' and a map $\rho : N' \rightarrow N$ which is not a split epimorphism, then $\rho = \psi \circ \sigma$ for some $\psi : N' \rightarrow E$. It was proved by Auslander and Reiten that for any indecomposable non-projective module N , such a sequence exists, and it is unique up to isomorphism of short exact sequences. The module M is denoted by $\tau(N)$ and τ is known as Auslander–Reiten translation. The arrows in Γ_Λ are then as follows: For N indecomposable non-projective, the number of arrows $X \rightarrow N$ is the multiplicity of X as a direct summand of E (which usually is ≤ 1). For M indecomposable and not injective, there is an almost split sequence starting with M . Then the number of arrows from M to X is the multiplicity of X as a direct summand of E .

We assume the algebra is symmetric, so that projectives and injectives are the same. In this case we have $\tau \cong \Omega^2$. The only almost split sequence in which an indecomposable projective P_i corresponding to the simple module S_i can occur, is what we call standard sequence

$$0 \rightarrow \Omega(S_i) \rightarrow P_i \oplus \text{rad}(P_i)/S_i \rightarrow \Omega^{-1}(S_i) \rightarrow 0$$

We assume that Λ is symmetric, then the stable AR-quiver ${}_s\Gamma_\Lambda$ is obtained from Γ_Λ by removing the vertices corresponding to the indecomposable projective modules. The stable AR quiver is a translation quiver, where Ω^2 acts as translation. The graph structure of a component of ${}_s\Gamma_\Lambda$ is described by Riedtmann's structure theorem.

For each component \mathcal{C} of Γ_Λ , its stable part is a component of ${}_s\Gamma_\Lambda$, and for Λ of infinite type, \mathcal{C} is either a stable tube $\mathcal{C} \cong \mathbb{Z}A_\infty/(\tau^r)$ (if it contains a periodic module [17]), or it is an (acyclic) quiver of the form $\mathcal{C} = \mathbb{Z}\Delta$.

The main tool to identify the graph structure of \mathcal{C} are subadditive functions, by applying the classification theorem of [17]. For the case of group algebras of finite groups, this was done by Webb [24], and Okuyama presented a new approach [21]. We use the version from Section 3 of [7] where this is generalized to selfinjective algebras. The identification method is then described as follows.

We say that Λ has enough periodic modules if for each indecomposable non-projective M there is a module W with $W \cong \tau(W)$, such that $\underline{\text{Hom}}_\Lambda(W, M)$ is non-zero. Here $\underline{\text{Hom}}_\Lambda(X, Y) = \text{Hom}_\Lambda(X, Y)/P(X, Y)$ where $P(X, Y)$ is the subspace of maps which factor through some projective module. Note that τ -periodic is the same as Ω -periodic for symmetric algebras.

Proposition 6.1 *Assume Λ has enough periodic modules. Let Θ be the stable component containing some indecomposable non-projective module M , let W be as above. Then $d_W := \dim \underline{\text{Hom}}(W, -)$ defines an additive function on Θ , hence T is either Dynkin or Euclidean or one of the infinite trees $A_\infty, A_\infty^\infty, D_\infty$.*

When Θ contains a periodic module then $T \cong A_\infty$ (for Λ of infinite type), see [17]. If Θ contains no periodic modules then both M and its syzygy $\Omega(M)$ are not summands of W , and then d_W is an additive function, by [7, Lemma 3.2]. The problem is how to find such module W when modules in Θ are not periodic.

6.1 Finding modules W

Assume Λ is tame and symmetric. Furthermore, we assume that the Gabriel quiver of Λ is 2-regular. This means that every component \mathcal{S} of the separated quiver is of the form \tilde{A}_n for some n .

We recall the definition of the separated quiver of an algebra. If Q is the quiver of the algebra and has vertices labelled by $1, 2, \dots, r$ then the separated quiver Q_s has vertices $\{1, 2, \dots, r, 1', 2', \dots, r'\}$. The arrows of Q_s are given by $\alpha : i \rightarrow j'$ whenever $\alpha : i \rightarrow j$ is an arrow in Q . If Q is a 2-regular quiver then there are two arrows starting at each of $1, 2, \dots, r$ of Q_s , and there are two arrows ending at each of $1', 2', \dots, r'$ of Q_s . Hence each component of Q_s is isomorphic to \tilde{A}_n for some n (possibly a Kronecker quiver).

By the well-known classification of indecomposables of such a quiver, there is a 1-parameter family of KS -modules W_λ (for $\lambda \in K^*$) of τ -period 1, all of dimension equal to the number of vertices of \mathcal{S} . Note that they have radical length two.

The modules W_λ can be viewed as Λ -modules (by letting the square of the radical act as zero). By [4] they must be (almost all) periodic as Λ -modules since the algebra is tame, still of τ -period 1, and therefore of Ω -period 2 for Λ . The same holds for an arbitrary component of the separated quiver. There is some $\lambda \in K^*$ such that the W_λ for each component are periodic of period 2 as modules for Λ . Define

$$W_0 := \bigoplus_{\mathcal{S}} W_{\lambda, \mathcal{S}} \quad \text{and} \quad W := W_0 \oplus \Omega_\Lambda(W_0)$$

Then W is a periodic Λ -module with $\Omega(W) \cong W$.

We take this module W , and let d_W as above. By construction, W_0 has radical length = 2 and $\text{soc}(W_0) \cong W_0/\text{rad} W_0 \cong \bigoplus_{i \in Q_0} S_i$. We may take a set of minimal generators $\{v_1, \dots, v_n\}$ of W_0 such that $v_i = v_i e_i$. Then we can take a basis of $\text{soc}(W_0)$, of the form w_1, \dots, w_n such that $w_i = w_i e_i$. Then if for some i the arrows in Q starting at i are $\alpha, \bar{\alpha}$ ending at j, k then $v_i \alpha$ and $v_i \bar{\alpha}$ are non-zero, and are scalar multiples of w_j, w_k respectively (and we may have $j = k$).

Lemma 6.2 *Assume M is indecomposable and not projective, and $\underline{\text{Hom}}(W_0, M) = 0$. Then $\text{Hom}(W_0, M) \cong \text{soc}(M)$.*

Proof (a) We define a homomorphism $\phi : \text{soc}(M) \rightarrow \text{Hom}_\Lambda(W_0, M)$. We fix a K -basis for $\text{soc}(M)$ of the form $\{m_{i, \nu(i)} \mid i \in Q_0, 1 \leq \nu(i) \leq t_i\}$ where $m_{i, \nu(i)} = m_{i, \nu(i)} e_i$. Now define a linear map

$$f_{i\nu(i)} : W_0 \rightarrow M$$

by $f_{i\nu(i)}(v_j) = \delta_{ij} m_{i\nu(i)}$ and $f_{i\nu(i)}(w_x) = 0$. This defines a Λ -module homomorphism. Now define $\phi(m_{i\nu(i)}) = f_{i\nu(i)}$.

(b) We show that ϕ is injective: Suppose $\phi(m) = 0$ where $m = \sum_{i, \nu(i)} c_{i\nu(i)} m_{i\nu(i)}$ with $c_{i\nu(i)} \in K$, so $\phi(m) = \sum_{i, \nu} c_{i\nu(i)} f_{i\nu(i)}$. Applying this to some generator of W_0 gives

$$0 = \phi(m)(v_j) = \sum_{\nu} c_{j\nu(j)} m_{j\nu(j)}$$

and since the $m_{j\nu(j)}$ are linearly independent it follows that all $c_{j\nu(j)}$ are zero. Hence $m = 0$.

(c) We show that ϕ is surjective. Suppose there is some homomorphism $f : W_0 \rightarrow M$. It suffices to show that $f(\text{soc}(W_0)) = 0$: if so then f factors through $W_0/\text{soc}(W_0)$ which is semisimple, and the image is contained in the socle. Then $f(v_j) = \sum_{\nu} c_{j\nu(j)} m_{j\nu(j)}$ with $c_{j\nu(j)} \in K$ for each j and $f = \sum_{i\nu(i)} c_{i\nu(i)} f_{i\nu(i)}$, which is in the image of ϕ .

Assume false, then we may assume $f(w_r)$ is non-zero for some r . We consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_0 & \xrightarrow{\iota} & \Lambda & \longrightarrow & \Omega(W_0) \longrightarrow 0 \\ & & & & & & \downarrow f \\ & & & & & & M \end{array}$$

where ι is the inclusion map. Since $\underline{\text{Hom}}(W_0, M) = 0$, it follows that f must factor through ι , so there is $h : \Lambda \rightarrow M$ such that

$$f = h \circ \iota.$$

Now, $\iota(w_r)$ must span the socle of the copy of $e_r \Lambda$ of Λ and we have $f(w) = h(\iota(w)) \neq 0$. Therefore the restriction of h to $e_r \Lambda$ is non-zero, and then it is a split monomorphism, since $e_r \Lambda$ is also injective. This is not possible since M is indecomposable and not projective. So we have a contradiction. \square

For the next part we will use an explicit injective hull of W_0 . Note that its socle is multiplicity-free, and that every simple module occurs. We know that $W_0 \cong \Omega^2(W_0)$, hence there is an exact sequence

$$0 \rightarrow W_0 \rightarrow \Lambda \rightarrow \Omega(W_0) \rightarrow 0$$

and moreover since W_0 has radical length = 2, it is contained in the second socle of Λ .

Lemma 6.3 *Assume M is indecomposable and not projective, such that $\underline{\text{Hom}}(\Omega(W_0), M) = 0$. Then $\text{Hom}(\Omega(W_0), M) \cong \text{rad}(M)$.*

Proof (a) We show first that every $f : \Omega(W_0) \rightarrow M$ maps into the radical of M . Suppose there is some f and $f(x)$ is not in $\text{rad}(M)$ for some $x \in \Omega(W_0)$, then we may assume $f(x) = f(x)e_i$. Since f is zero in $\underline{\text{Hom}}(\Omega(W_0), M)$, there is $h : \Lambda \rightarrow M$ and $f = h \circ \iota$. In particular there is $z = ze_i \in \Lambda$ and $h(z) = f(x)$. Then z must be a generator of Λ and $z\Lambda \cong e_i \Lambda$. The restriction of h to $z\Lambda$ must split since $e_i \Lambda$ is projective, and M has a projective direct summand, a contradiction.

We identify $\text{Hom}(\Omega(W_0), M)$ with the set of $f : \Lambda \rightarrow \text{rad}(M)$ which take W_0 to zero.

(b) We claim that if f maps into the radical of M then $f(\text{soc}_2(\Lambda)) = 0$, and hence $f(W_0) = 0$. Let $f(e_i) = m = me_i$ in the radical of M . Then we can write $m = z\beta + z^*\beta^*$ where β, β^* are the arrows of Q ending at i , and where z and z^* are elements of M .

Suppose there is some element A in $\text{soc}_2(\Lambda)$ with $mA \neq 0$, say $z\beta A \neq 0$. Then in particular βA is non-zero in the socle of $e_j \Lambda$ (for $j = s(\beta)$). It follows that the submodule $z\Lambda$ of M is isomorphic to $e_j \Lambda$. But $e_j \Lambda$ is injective, and hence is a direct summand of M . This is a contradiction since M is assumed to be indecomposable and not projective (hence injective).

(c) We define a homomorphism $\phi : \text{rad}(M) \rightarrow \text{Hom}_\Lambda(\Omega(W_0), M)$, as in the proof of Lemma 6.2.

Take a basis of $\text{rad}(M)$ of the form $\{m_{iv(i)} \mid i \in Q_0, 1 \leq v(i) \leq s_i\}$ with $m_{iv(i)} \in Me_i$. Then define on the generators of Λ

$$f_{iv(i)}(e_j) = m_{iv(i)}\delta_{ij}$$

By (c), this factors through $\Omega(W_0)$. Now define $\phi(m_{iv(i)}) := f_{iv(i)}$. As in Lemma 6.2 the map ϕ is injective. The map ϕ is surjective: By part (b), the set of all $f_{iv(i)}$ is a basis for $\text{Hom}_\Lambda(\Omega(W_0), M)$. \square

Proposition 6.4 Assume M is indecomposable and not projective. Assume $\underline{\text{Hom}}(W, M) = 0$. Then $\text{top}(M) \cong \text{soc}(M)$.

Proof The modules W_0 and $\Omega(W_0)$ are cyclic since the tops are multiplicity-free. Write $W_0 = \Theta\Lambda$ and $\Omega(W_0) = \Psi\Lambda$, here Θ and Ψ are taken as elements in $\oplus_{i \in Q_0} e_i \Lambda$.

Since $\Omega^2(W_0) \cong W_0$ we have $\Theta\Psi = 0 = \Psi\Theta$, and there are exact sequences

$$0 \rightarrow \Theta\Lambda \rightarrow \Lambda \rightarrow \Psi\Lambda \rightarrow 0, \text{ and } 0 \rightarrow \Psi\Lambda \rightarrow \Lambda \rightarrow \Theta\Lambda \rightarrow 0.$$

We apply the functor $(-, M) := \text{Hom}_\Lambda(-, M)$ to the first exact sequence, it takes it to an exact sequence

$$0 \rightarrow (\Psi\Lambda, M) \rightarrow (\Lambda, M) \rightarrow (\Theta\Lambda, M) \rightarrow 0.$$

We identify the terms, as vector spaces. The middle is M . Furthermore

$$(\Psi\Lambda, M) \cong \{m \in M \mid m\Theta = 0\} \text{ and } (\Theta\Lambda, M) \cong \{m \in M \mid m\Psi = 0\}$$

where we view Θ and Ψ as linear maps $M \rightarrow M$. Hence we have an exact sequence

$$0 \rightarrow \text{Ker}(\Theta) \rightarrow M \rightarrow \text{Ker}(\Psi) \rightarrow 0,$$

which shows that $M/\text{Ker}(\Theta) \cong \text{Ker}(\Psi)$.

By Lemma 6.3, $\text{Ker}(\Theta) \cong \text{rad}M$, and by Lemma 6.2, we have $\text{Ker}(\Psi) \cong \text{soc}(M)$. This shows that $\text{top}(M) = M/\text{rad}(M) \cong \text{soc}(M)$ as vector spaces, as required. \square

Corollary 6.5 If $\underline{\text{Hom}}(W, M) = 0$ then M is Ω -periodic.

Proof We have $\underline{\text{Hom}}(W, M) \cong \underline{\text{Hom}}(W, \Omega^n(M))$ for all $n \in \mathbb{Z}$ since $W \cong \Omega(W)$. Hence by Proposition 6.4, we have $\text{top}(\Omega^n(M)) \cong \text{soc}(\Omega^n(M))$ for all $n \in \mathbb{Z}$. Note that the top of $\Omega^r(M)$ is the socle of $\Omega^{r+1}(M)$. It follows that the dimensions of the tops of the $\Omega^n(M)$ are constant and therefore the dimensions of the $\Omega^n(M)$ are bounded.

Hence there is some integer d such that infinitely many $\Omega^n(M)$ have dimension d . Now we can apply [4] again which shows that some $\Omega^m(M)$ has τ -period 1, that is, Ω -period 2. Therefore M is Ω -periodic. \square

We conclude that on a component of a module M which is not Ω -periodic the additive function d_W above must be non-zero. Hence by Proposition 6.1 we can deduce the graph structure of a component.

6.2 Auslander–Reiten components of simple modules and of some arrow modules

In this part we assume that H is a hybrid algebra (which may have virtual arrows), with distinguished set of triangles \mathcal{T} , and we exclude the local algebra with two virtual loops. We investigate the position of simple modules, and of some modules generated by arrows, in the stable AR quiver of H . We say that a component is of type A if its tree class is one of A_∞^∞ or A_∞ , or A_n , or A_n for some $n \geq 2$, and we say it is of type D if its tree class is one of D_∞ or \tilde{D}_n or D_n . For a vertex i of Q we denote the module $\text{rad}(e_i H)/\text{soc}(e_i H)$ by M_i (the ‘middle’).

6.2.1 Arrow modules for arrows not in \mathcal{T}

Take an arrow $\beta \notin \mathcal{T}$. Then it is easy to see that $\Omega^r(\beta H) \cong f^r(\beta)H$ for $r \geq 1$. Hence βH has Ω -period equal to r_β where r_β is the length of the f -orbit of β . This is also true if some $f^s(\beta)$ is virtual of type (a), in which case the corresponding module is simple. Furthermore all Ω -translates are indecomposable and hence belong to ends of tubes in the stable AR-quiver.

6.2.2 Simple modules at biserial vertices, and at quaternion vertices

(a) Assume i is a biserial vertex. If there is a virtual loop at i then by the previous, the simple module at i is periodic at the end of a tube. Now suppose the arrows starting at i are not virtual. By Lemma 7.8 (see the ‘‘Appendix’’), the ‘middle’ M_i of $e_i H$ is the direct sum of two indecomposable modules. Hence we have an almost split sequence $0 \rightarrow \Omega(S_i) \rightarrow P_i \oplus M_i \rightarrow \Omega^{-1}(S_i) \rightarrow 0$ and $\Omega(S_i)$ has two predecessors in its stable component. This could be in the middle of some component of type A or possible in a component of type D away from the edge. In fact, it might even be in some tube.

(b) If i is a quaternion vertex, with no singular relation close to i (eg excluding the singular tetrahedral, disc, triangle algebra) then S_i is periodic of period four. The proofs in [8], [10] and [3] generalize. This also works for 4.2(2c) and for the algebra in 4.6.

6.2.3 Simple modules at hybrid vertices

Lemma 6.6 *Assume H is a hybrid algebra but is not the algebra 4.2(2c) or the algebra 4.6. Let i be a vertex and $\alpha, \bar{\alpha}$ are arrows starting at i where $\alpha \in \mathcal{T}$ and $\bar{\alpha} \notin \mathcal{T}$. Let $M := M_i = \text{rad}(e_i H) / \text{soc}(e_i H)$.*

(a) The module M_i is indecomposable and it occurs in two different AR-sequences as the non-projective middle term.

(b) If H is not of finite type then the component of S_i is of type D .

Remark 6.7 Consider the algebra 4.2(2c), this has a hybrid vertex. The algebra is special biserial (see Lemma 4.1). Consider the simple module S_2 at the hybrid vertex, by Lemma 4.1 we know that $\text{rad}(e_2 H) / S_2$ is the direct sum of two non-zero modules, and S_2 belongs to a component of tree class A_∞^∞ . Similarly the algebra in 4.6 has hybrid vertices 1, 3 but the modules $\text{rad}(e_i H) / S_i$ for $i = 1, 3$ are decomposable.

Proof (a) Assume $\alpha \in \mathcal{T}$ and $\bar{\alpha} \notin \mathcal{T}$. Note that then $f(\alpha) \neq \bar{\alpha}$. As a preliminary part, we show that always $f(\alpha)f^2(\alpha)\bar{\alpha} = 0$.

If not, then by (2) of Definition 3.3 we have that $f(\alpha), g(\alpha) \in \mathcal{T}$ and $g(\alpha)$ is virtual or critical. Suppose $g(\alpha)$ is virtual, then $n_\alpha = n_{g(\alpha)} \leq 2$. We cannot have $\alpha = g(\alpha)$ since this would imply $f(\alpha) = \bar{\alpha}$. So g must have a 2-cycle $(\alpha, g(\alpha))$, but then $f(g(\alpha)) = \bar{\alpha}$. This gives a contradiction since with $g(\alpha) \in \mathcal{T}$ also $f(g(\alpha)) \in \mathcal{T}$ but $\bar{\alpha} \notin \mathcal{T}$. This shows that $g(\alpha)$ is not virtual.

Suppose $g(\alpha)$ is critical, consider first the case when the g -cycle of $g(\alpha)$ does not have a loop, then we use the diagram 3.1.1 with $\tau = g(\alpha)$. Then ξ must be virtual and therefore the arrow $y \rightarrow x$ must be in \mathcal{T} , and then also $\bar{\alpha}$ is in \mathcal{T} , a contradiction. Similarly one gets a contradiction in the other case, ie where H is the algebra 4.2(2c). Hence $g(\alpha)$ is not critical.

The module M is indecomposable by Lemma 7.8. Therefore it is the indecomposable non-projective middle term of the AR-sequence starting with $\Omega(S_i)$. Moreover we have a

non-split short exact sequence

$$0 \rightarrow V \rightarrow M \rightarrow U \rightarrow 0 \quad (*)$$

where $V = \bar{\alpha}H/\langle B_{\bar{\alpha}} \rangle$ and $U = \alpha H/\langle A_{\bar{\alpha}} \rangle$ (we take $M = (\alpha H + \bar{\alpha}H)/\langle B_{\alpha} \rangle$, and then $V \rightarrow M$ is the inclusion and $M \rightarrow U$ is the canonical surjection). Note that this is true also when $\bar{\alpha}$ is virtual. We show first $V \cong \Omega^2(U)$, and next that $\text{Ext}^1(U, V) \cong K$. With these, it will follow that $(*)$ is an AR-sequence. Let $j = t(\alpha)$ and $y = t(f(\alpha))$.

(i) We claim that U is isomorphic to $e_j H/f(\alpha)H$: Consider the projective cover $\pi : e_j H \rightarrow U$ given by $\pi(x) = \alpha x + \langle A_{\bar{\alpha}} \rangle$. Then $\pi(f(\alpha)) = 0$ and hence $f(\alpha)H \subseteq \text{Ker}(\pi)$. We can compare dimensions, applying Lemma 7.8. The dimension of U is $m_{\alpha}n_{\alpha} - 1$ and we have $\dim e_j H = m_{\alpha}n_{\alpha} + m_{f(\alpha)}n_{f(\alpha)}$. Hence the kernel of π has dimension $n_{f(\alpha)}m_{f(\alpha)} + 1 = \dim f(\alpha)H$, and we have equality. This implies that $\Omega(U) \cong f(\alpha)H$.

(ii) We claim that $\Omega(f(\alpha)H) \cong f^2(\alpha)\bar{\alpha}H$, and that it is isomorphic to V : Let $\pi : e_y H \rightarrow f(\alpha)H$ be the projective cover, given by $e_y x \mapsto f(\alpha)x$. As we have shown in the preliminary step, we always have $f(\alpha)f^2(\alpha)\bar{\alpha} = 0$, so $f^2(\alpha)\bar{\alpha}H$ is contained in the kernel of π . By comparing dimensions we see that it is equal. To show that this is isomorphic to V , consider left multiplication with $f^2(\alpha)$ from $\bar{\alpha}H$ to $f^2(\alpha)\bar{\alpha}H$. This is a surjective H -module homomorphism. By Lemma 7.8, $f^2(\alpha)\bar{\alpha}H$ has dimension $m_{f^2(\alpha)}n_{f^2(\alpha)} - 1$ and $\dim \bar{\alpha}H = m_{\bar{\alpha}}n_{\bar{\alpha}} = m_{f^2(\alpha)}n_{f^2(\alpha)}$ noting $\bar{\alpha} = g(f^2(\alpha))$. So the kernel is equal to $\langle B_{\bar{\alpha}} \rangle$.

(iii) It remains to show that $\text{Ext}^1(U, V)$ is at most 1-dimensional (we know already that it is non-zero). We have an exact sequence

$$V e_j \cong \text{Hom}(e_j H, V) \xrightarrow{\iota^*} \text{Hom}(f(\alpha)H, V) \rightarrow \text{Ext}^1(U, V) \rightarrow 0$$

where $0 \rightarrow f(\alpha)H \xrightarrow{\iota} e_j H$ is the inclusion map.

Assume first that $\bar{\alpha}$ is virtual. Then V is 1-dimensional and spanned by the coset of $\alpha f(\alpha)$, so it is isomorphic to the simple module S_y . In particular $V e_y = V$ is 1-dimensional, and hence the quotient $\text{Ext}^1(U, V)$ is at most 1-dimensional.

Now assume $\bar{\alpha}$ is not virtual. We have $\text{Hom}(f(\alpha)H, V) \cong \{v \in V e_y \mid v f^2(\alpha)\bar{\alpha} = 0\}$. The space $V e_y$ is spanned by the (cosets of) initial submonomials of $A_{\bar{\alpha}}$ which end at vertex y , that is which end in either $f(\alpha)$ or in $\beta := g^{-1}(f^2(\alpha))$.

Suppose p is an initial submonomial of $A_{\bar{\alpha}}$ ending in $f(\alpha)$. By the preliminary fact, we know that $p f^2(\alpha)\bar{\alpha} = 0$, and we deduce that there is a homomorphism $\theta_p : f(\alpha)H \rightarrow V$ taking $f(\alpha)$ to p . We claim that this is in the image of ι^* : Such a monomial p has a factorisation $p = \tilde{p} \cdot f(\alpha)$ with \tilde{p} a monomial of positive length. There is a homomorphism $\tilde{\theta} : e_j H \rightarrow V$ taking e_j to \tilde{p} and hence $\theta = \tilde{\theta} \circ \iota$.

Now consider an initial submonomial p of $A_{\bar{\alpha}}$ ending in β . If $p \neq A_{\bar{\alpha}}$ then $p f^2(\alpha)\bar{\alpha}$ is again an initial submonomial of $A_{\bar{\alpha}}$ and is non-zero in the algebra. This means that we do not have a homomorphism taking $f(\alpha)$ to p . This leaves only the case $p = A_{\alpha}$ so that the ext space is at most 1-dimensional. (In fact, this last case gives rise to the non-split short exact sequence).

(b) By assumption, $\bar{\alpha}H$ is Ω -periodic. Let W be the direct sum of the distinct Ω -translates of $\bar{\alpha}H$. Then $W \cong \Omega(W)$ and $d_W(-)$ is an additive function on any non-periodic component on which it does not vanish. By assumption, H is of infinite type and then the summands of W belong to tubes. On the other hand, since H has infinite type, by part (a) the component of M_i cannot be a tube. The inclusion $\bar{\alpha}H \rightarrow \Omega(S_i)$ is nonzero in the stable category. Therefore d_W is non-zero on this component. We have $d_W(M_i) = 2d_W(\Omega(S_i)) \neq 0$ by exactness. Comparing with a general additive function on components as described in [17], it follows that the component is of type D . \square

Remark 6.8 We see from the proof that U or V can be simple, or even both. Consider the algebra H with triangular quiver. We use the notation as in 4.3, and take $\mathcal{T} = \{\alpha_i\}$ and we take $m_\bullet = (2, 1, 1)$. Then β_2 and β_3 are virtual, and we have $\Omega(S_1) \cong \Omega^{-1}(S_2)$. In this case, all three simple modules are of type D , in fact they are all in the same component which has tree class \tilde{D}_5 . Consider $M_3 = \text{rad}(e_3 H)/S_3$, in this case both U and V are simple.

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Data availability Not applicable.

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7 Appendix: Consistency, bases and dimensions

This extends to the general case what was done for regular hybrid algebras in Sect. 2.

7.1 Consistency

In this Section we assume throughout that H is a hybrid algebra, which is not local, and is not an algebra considered in detail in Sect. 4. With this assumption, we can use the diagrams in 3.1, see also Corollary 3.9.

Lemma 7.1 *Assume $\tilde{\alpha}$ is a virtual arrow, and $\alpha, \tilde{\alpha} \in \mathcal{T}$. If $\tilde{\alpha}$ is not a loop then there are six relations of type ζ or ξ in which $\tilde{\alpha}$ occurs. If $\tilde{\alpha}$ is a loop then there are four relations of type ζ or ξ in which $\tilde{\alpha}$ occurs. In both cases, each of these is zero in H .*

The proof is the same as that of Lemma 3.3 in [14], using the diagrams displayed in 3.1. See also Corollary 3.9.

Lemma 7.2 *Assume $|A_\alpha| \geq 2$ but α is not critical. Let $\zeta = \zeta_\alpha := \alpha f(\alpha)g(f(\alpha))$.*

(a) *If $\alpha, \tilde{\alpha} \in \mathcal{T}$ and $\tilde{\alpha}$ is virtual or critical, then $\zeta \equiv A_\alpha$. Moreover*

$$\zeta f^2(\tilde{\alpha}) \equiv B_\alpha, \quad \zeta g(f(\tilde{\alpha})) = 0, \quad g^{-1}(\alpha)\zeta \equiv B_{g^{-1}(\alpha)}, \quad f^{-1}(\alpha)\zeta = 0.$$

Furthermore $B_\alpha J = 0 = JB_\alpha$ and $B_{g^{-1}(\alpha)} J = 0 = JB_{g^{-1}(\alpha)}$.

(b) *Otherwise $\zeta = 0$.*

Proof Part (b) is a direct consequence of part (2) in Definition 3.3.

(a) By the assumptions, α is not virtual or critical. We know from 3.1.1 and 3.1.2 that $\zeta \equiv A_\alpha$. It is clear that $\zeta f^2(\tilde{\alpha}) \equiv B_\alpha$ and $g^{-1}(\alpha)\zeta \equiv B_{g^{-1}(\alpha)}$. Furthermore, since $\zeta \equiv A_\alpha \equiv \tilde{\alpha} f(\tilde{\alpha})$, any monomial of length three having this as a factor, and which has 'type ζ or type ξ ' must be zero in H , by Lemma 7.1. We will use this throughout the proof (without further comments).

(i) $\zeta g(f(\tilde{\alpha})) = 0$ and $f^{-1}(\alpha)\zeta = 0$: By the preamble,

$$f^{-1}(\alpha)\zeta = g^{-1}(\tilde{\alpha})\zeta \equiv g^{-1}(\tilde{\alpha})\tilde{\alpha} f(\tilde{\alpha}) = 0 \quad \text{and} \quad \zeta g(f(\tilde{\alpha})) \equiv \tilde{\alpha} f(\tilde{\alpha})g(f(\tilde{\alpha})) = 0.$$

Note that these imply $B_{g^{-1}\alpha} \cdot g(f\bar{\alpha}) = 0$ and $f^{-1}(\alpha)B_{\alpha} = 0$.

(ii) $B_{\alpha}J = 0 = JB_{\alpha}$: first we have

$$B_{\alpha}\bar{\alpha} \equiv A_{\alpha}g^{-1}(\alpha)\bar{\alpha} = A'\beta g^{-1}(\alpha)\bar{\alpha} = A'\xi_{\beta} = 0$$

where $\beta = g^{-2}(\alpha)$ that is $A'\beta = A_{\alpha}$ (which has length ≥ 2 by assumption). Next, $B_{\alpha}\alpha \equiv B_{\bar{\alpha}}\alpha$. If $\bar{\alpha}$ is virtual we can write this as

$$B_{\bar{\alpha}}\alpha = \bar{\alpha}g(\bar{\alpha})\alpha = \bar{\alpha}f^2(\alpha)\alpha = \xi_{\bar{\alpha}} = 0.$$

Suppose $\bar{\alpha}$ is critical, then we have, since $\xi_{\bar{\alpha}} = 0$,

$$B_{\bar{\alpha}}\alpha = \bar{\alpha}g(\bar{\alpha})g^2(\bar{\alpha})\alpha = \bar{\alpha}g(\bar{\alpha})f^2(\alpha)\alpha \equiv \bar{\alpha}g(\bar{\alpha})A_{f(g\bar{\alpha})} = 0.$$

It remains to show $g^{-1}(\alpha)B_{\alpha} = 0$ which is $\equiv g^{-1}(\alpha)B_{\bar{\alpha}}$. If $\bar{\alpha}$ is virtual we have

$$g^{-1}(\alpha)B_{\bar{\alpha}} = g^{-1}(\alpha)\bar{\alpha}g(\bar{\alpha}) = \zeta_{g^{-1}(\alpha)} = 0$$

If $\bar{\alpha}$ is critical

$$g^{-1}(\alpha)B_{\bar{\alpha}} = g^{-1}(\alpha)\bar{\alpha}g(\bar{\alpha})g^2(\bar{\alpha}) = \zeta_{g^{-1}(\alpha)}g^2(\bar{\alpha}) = 0$$

(iii) $B_{g^{-1}(\alpha)}J = 0 = JB_{g^{-1}(\alpha)}$: This is similar to (ii). We omit details. \square

Lemma 7.3 Assume α is an arrow with $|A_{\alpha}| \geq 2$ but α not critical. Let $\xi = \xi_{\alpha} := \alpha g(\alpha) f(g(\alpha))$.

(a) Suppose $\alpha, \bar{\alpha} \in \mathcal{T}$ and $f(\alpha)$ is virtual. Then $\xi \equiv A_{\bar{\alpha}}$. Moreover

$$g^{-1}(\alpha)\xi = 0, \quad f^2(\alpha)\xi = B_{f^2(\alpha)}, \quad \xi f^2(\alpha) = B_{\bar{\alpha}}, \quad \xi f^2(g(\alpha)) = 0.$$

We have $B_{\bar{\alpha}}J = 0 = JB_{\bar{\alpha}}$ and $B_{f^2(\alpha)}J = 0 = JB_{f^2(\alpha)}$.

(b) Suppose $\alpha, \bar{\alpha} \in \mathcal{T}$ and $f(\alpha)$ is critical. Then $\xi \equiv A_{\alpha}$. Moreover

$$\xi g^{-1}(\alpha) = B_{\alpha}, \quad \xi g^{-1}(f(\alpha)) = 0, \quad g^{-1}(\alpha)\xi = B_{g^{-1}(\alpha)}, \quad f^2(\alpha)\xi = 0.$$

We have $B_{\alpha}J = 0 = JB_{\alpha}$ and $B_{g^{-1}(\alpha)}J = 0 = JB_{g^{-1}(\alpha)}$. (c) Otherwise $\xi = 0$.

Proof This is similar to the proof of Lemma 7.2. We omit the details. \square

The following deals with another special case.

Lemma 7.4 Assume that either α is virtual and $\alpha \in \mathcal{T}$, or α is critical and $\alpha, g(\alpha) \in \mathcal{T}$. Then $A_{\alpha}J = \langle B_{\alpha}, A_{\bar{\alpha}} \rangle$ and $A_{\alpha}J^2 = \langle B_{\alpha} \rangle$ and $B_{\alpha}J = 0$.

Proof Assume first that α is virtual, that is $\alpha = A_{\alpha}$ and $\bar{\alpha} \in \mathcal{T}$. Then $A_{\alpha}J = \langle \alpha g(\alpha), \alpha f(\alpha) \rangle = \langle B_{\alpha}, A_{\bar{\alpha}} \rangle$. By considering the diagrams in 3.1.2 we see that $\bar{\alpha}$ is not virtual or critical. We apply Lemma 7.2 with $\alpha, \bar{\alpha}$ interchanged and get

$$A_{\bar{\alpha}} \equiv \zeta_{\bar{\alpha}}, \quad A_{\bar{\alpha}}J = B_{\bar{\alpha}}, \quad B_{\bar{\alpha}}J = 0.$$

Therefore $A_{\alpha}J^2 = \langle A_{\bar{\alpha}}J \rangle = \langle B_{\bar{\alpha}} \rangle = \langle B_{\alpha} \rangle$.

Now assume α is critical with $g(\alpha) \in \mathcal{T}$. We have $A_{\alpha} = \alpha g(\alpha)$ and

$$A_{\alpha}J = \langle \alpha g(\alpha)g^2(\alpha), \alpha g(\alpha)f(g(\alpha)) \rangle = \langle B_{\alpha}, \xi_{\alpha} \rangle,$$

and we have $\xi_{\alpha} \equiv A_{\bar{\alpha}}$ (see 3.1.2(1)(b)). By Lemma 7.3 we have that $A_{\bar{\alpha}}J = B_{\bar{\alpha}}$, and $B_{\bar{\alpha}}J = 0$ which implies the statement. \square

Lemma 7.5 Assume α is any arrow, then (i) $B_\alpha J = 0$ and $J B_\alpha = 0$. (ii) B_α is non-zero.

Proof (i) It suffices to show that for an arbitrary arrow α we have $\alpha B_{g(\alpha)}$, that is $\alpha B_{f(\alpha)} = 0$. Then part (i) follows using identity (3) of the definition 3.3, and an identity such as $\alpha B_{g(\alpha)} = B_\alpha \alpha$.

If $\alpha \notin \mathcal{T}$ then $\alpha B_{f(\alpha)} = 0$ by identity (1) of Definition 3.3, so we assume now that $\alpha \in \mathcal{T}$. Then $f(\alpha)$ cannot be virtual of type (a) and therefore $|B_{f(\alpha)}| \geq 2$.

(1) Assume $|B_{f(\alpha)}| = 2$. Then $\alpha B_{f(\alpha)} = \zeta_\alpha$. If $f(\alpha)$ is virtual then $\zeta_\alpha = 0$ by Lemma 7.1. Assume now that $f(\alpha)$ is not virtual, it also is not critical (since $|B_{f(\alpha)}| \neq 3$). Therefore $\zeta_\alpha = 0$ by identity (2) of Definition 3.3.

(2) Assume $|B_{f(\alpha)}| = 3$, then $\alpha B_{f(\alpha)} = \zeta_\alpha g^{-1}(f(\alpha))$. This is zero unless $\bar{\alpha} \in \mathcal{T}$ and $\bar{\alpha}$ is critical or virtual. Suppose $\bar{\alpha}$ is critical or virtual. Note first that we see from 3.1.1, 3.1.2 that α is not a loop. Therefore $\alpha B_{f(\alpha)}$ is not a cyclic path. We also see from 3.1.1 and 3.1.2 that α cannot be virtual or critical. That is, the assumption of Lemma 7.2 holds. It follows that $\zeta_\alpha g^{-1}(f(\alpha))$ is zero by 7.2 (it is not cyclic and cannot be $\equiv B_\alpha$).

(3) Now assume $|B_{f(\alpha)}| \geq 4$. Then $\alpha B_{f(\alpha)} = \zeta_\alpha C$ where C is a monomial of length ≥ 2 . Suppose $\bar{\alpha}$ is virtual or critical, then α is not virtual or critical (see 3.1.1 or 3.1.2). By Lemma 7.2 we know $\zeta_\alpha J = \langle B_\alpha \rangle$ and $B_\alpha J = 0$ and hence $\alpha B_{f(\alpha)} = 0$.

(ii) When the vertex $i = s(\alpha)$ is quaternion, the statement is proved in 4.5 of [10]. Suppose i is biserial. From the relations, the only submonomials of B_α which occur in a minimal relation are B_α itself and A_α and $A_{g(\alpha)}$. In general, A_α occurs in a relation $\tilde{\alpha} f(\tilde{\alpha}) - c_\alpha A_\alpha$ but this is not the case when i is biserial. Similarly $A_{g(\alpha)}$ could occur in a relation $f(\alpha) f^2(\alpha) - c_{g(\alpha)} A_{g(\alpha)}$ but not if i is biserial since in that case $f(\alpha) f^2(\alpha)$ is zero (or a scalar multiple of $B_{f(\alpha)}$). Hence B_α is non-zero in H .

Now assume that i is hybrid, say $\alpha \in \mathcal{T}$ and $\bar{\alpha} \notin \mathcal{T}$. Then A_α does not occur in a defining relation. We have the relation $f(\alpha) f^2(\alpha) = c_\alpha A_{g(\alpha)}$ but this does not give a relation which forces B_α to be zero in H . \square

Lemma 7.6 Consider a path of length four of the form $p := \alpha g(\alpha) \beta g(\beta)$ where $\beta = f(g(\alpha))$. (a). If $f(\alpha)$ is virtual or critical then p is a non-zero scalar multiple of B_α . (b) Otherwise it is zero.

Proof We can write $p = \alpha \zeta_{g(\alpha)}$ and also $p = \xi_\alpha g(\beta)$. By Lemma 7.2 we know $\alpha \zeta_{g(\alpha)} \neq 0$ if and only if $f(\alpha)$ is virtual or critical and if so it is $\equiv B_\alpha$ (which is $\equiv B_{\bar{\alpha}}$).

By Lemma 7.3 we have $\xi_\alpha g(\beta) \neq 0$ if and only if $f(\alpha)$ is virtual or critical, and if so then p is a cyclic path of length four $\equiv B_\alpha$. \square

7.2 Bases and dimension

In the following write $|A_\alpha| = \ell$ and $|A_{\bar{\alpha}}| = \bar{\ell}$. We also write $[A_\alpha]_j$ for the initial submonomial of A_α of length j .

Lemma 7.7 Assume α is an arrow of Q . Then the set $\{[A_\alpha]_j \mid 1 \leq j \leq \ell, B_\alpha\}$ is linearly independent.

This is similar to proofs in Sect. 2, we omit details.

Lemma 7.8 Assume i is a vertex which is either biserial or hybrid. Then

(a) $e_i H$ has basis consisting of all proper initial submonomials of B_α , $B_{\bar{\alpha}}$ together with e_i and B_α . (b) $\dim e_i H = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$.

(c) If $\alpha \in \mathcal{T}$ then $\dim \alpha H = m_\alpha n_\alpha + 1$, otherwise αH has dimension $m_\alpha n_\alpha$. The module $\alpha g(\alpha)H$ has dimension $m_\alpha n_\alpha - 1$.

(d) Let $M_i = \text{rad}(e_i H)/\text{soc}(e_i H)$. If i is biserial and $\alpha, \bar{\alpha}$ are not virtual of type (a) then M_i is the direct sum of two uniserial modules. If one of $\alpha, \bar{\alpha}$ is virtual of type (a) then M_i is uniserial. If i is hybrid, and H is not the algebra in 4.2(2c) or 4.6 then M_i is indecomposable.

Proof We prove part (a), then parts (b) and (c) follow directly. We may assume $\bar{\alpha} \notin \mathcal{T}$. The given set spans $e_i H$ by Lemmas 7.2 and 7.3. We show linear independence. Take a linear combination

$$\sum_{j=1}^{\ell} a_j [A_\alpha]_j + \sum_{t=1}^{\bar{\ell}} d_t [A_{\bar{\alpha}}]_t + s B_\alpha = 0. \quad (*)$$

Let $\beta = f^{-1}(\bar{\alpha}) = g^{-1}(\alpha)$, then $\beta \bar{\alpha} = 0$, unless possibly $\beta = f(\bar{\alpha})$, a loop, and $\beta \bar{\alpha} = b_\beta B_\beta$. But then noting that $\alpha \in \mathcal{T}$ it follows that $|Q_0| \leq 2$, which we have excluded. Therefore we have $\beta \bar{\alpha} = 0$. We premultiply (*) with β and obtain $\sum_{j=1}^{\ell} a_j [A_\beta]_{j+1} = 0$, and by Lemma 7.7 it follows that $a_j = 0$ for $1 \leq j \leq \ell$. Now applying Lemma 7.7 again implies $d_t = 0$ for all t , and $s = 0$.

(d) When i is biserial, the claim also follows from part (a). Now suppose i is a hybrid vertex, so $\alpha \notin \mathcal{T}$. If $f(\alpha)$ is not virtual then M_i can be viewed as a string module (see [5, II.3]), hence it is indecomposable. If $f(\alpha)$ is virtual then by the assumption that H is not the algebra in 4.2(2c) or 4.6 one checks that at least one of A_α and $A_{\bar{\alpha}}$ has length > 2 , and then one verifies directly that M_i is indecomposable. \square

Lemma 7.9 Assume i is a periodic vertex so that $\alpha, \bar{\alpha}$ are both in \mathcal{T} . Then the set $\{[A_\alpha]_j, (j \leq \ell), [A_{\bar{\alpha}}]_t, (t \leq \bar{\ell}), B_\alpha\}$ is linearly independent, except when H is the singular spherical algebra, or H is the singular triangle algebra.

This is proved in [10] (see Proposition 4.9).

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