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Multiple normalized solutions for the planar Schrödinger–Poisson system with critical exponential growth

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Abstract

The paper deals with the existence of normalized solutions for the following Schrödinger–Poisson system with L^2-constraint:

$$\begin{cases} -\Delta u + \lambda u + \mu \left(\log|\cdot| * u^2\right) u = \left(e^{u^2} - 1 - u^2\right) u, \ x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases}$$

where $\mu > 0$, $\lambda \in \mathbb{R}$ will arise as a Lagrange multiplier and the nonlinearity enjoys critical exponential growth of Trudinger-Moser type. By specifying explicit conditions on the energy level c, we detect a geometry of local minimum and a minimax structure for the corresponding energy functional, and prove the existence of two solutions, one being a local minimizer and one of mountain-pass type. In particular, to catch a second solution of mountain-pass type, some sharp estimates of energy levels are proposed, suggesting a new threshold of compactness in the L^2-constraint. Our study extends and complements the results of Cingolani–Jeanjean (SIAM J Math Anal 51(4): 3533-3568, 2019) dealing with the power nonlinearity $a|u|^{p-2}u$ in the case of $a > 0$ and $p > 4$, which seems to be the first contribution in the context of normalized solutions. Our model presents some new difficulties due to the intricate interplay between a logarithmic convolution potential and a nonlinear

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term of critical exponential type and requires a novel analysis and the implementation of new ideas, especially in the compactness argument. We believe that our approach will open the door to the study of other L^2-constrained problems with critical exponential growth, and the new underlying ideas are of future development and applicability.

Keywords Planar Schrödinger–Poisson system · Logarithmic convolution potential · Normalized solution · Critical exponential growth · Trudinger–Moser inequality

Mathematics Subject Classification 35J20 · 35J62 · 35Q55

1 Introduction

In this paper, we study the following planar Schrödinger–Poisson equation with L^2-constraint

$$\begin{cases} -\Delta u + \lambda u + \mu \left(\log|\cdot| * u^2\right) u = \left(e^{u^2} - 1 - u^2\right) u, \ x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases} \tag{1.1}$$

where $\mu > 0$, $c > 0$ is a given constant, $\lambda \in \mathbb{R}$ appears as a Lagrange parameter and is part of the unknowns. Particularly, the nonlinearity has critical exponential growth in the sense of Trudinger–Moser, which is a novelty for L^2-constrained problems. Here, we recall that the nonlinear term f is said to have *critical exponential growth* if f satisfies

(f1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0, \end{cases}$$

which is the maximal growth allowing to treat the problem variationally in $H^1(\mathbb{R}^2)$, see Adimurthi and Yadava [2] and also de Figueiredo, Miyagaki and Ruf [22].

Solutions having *a priori* prescribed L^2-norm are referred to as normalized solutions in the literatures. Physicists are often interested in normalized solutions because the L^2-norm of such solutions is a preserved quantity of the evolution and their variational characterization can help to analyze the orbital stability or instability, see, for example, [5, 39, 40]. Besides that its solutions have *a priori* prescribed mass, another interesting feature of (1.1) is that a logarithmic convolution potential appears, which is unbounded and changes sign. As one will see, for prescribed $c > 0$, a solution of problem (1.1) can be obtained as a critical point of the functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{\mu}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y \\ - \frac{1}{2}\int_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 - \frac{u^4}{2}\right) \mathrm{d}x \tag{1.2}$$

on the constraint

$$\mathcal{S}_c = \left\{u \in X : \|u\|_2^2 = c\right\}, \tag{1.3}$$

where

$$X := \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|) u^2 \mathrm{d}x < \infty\right\}. \tag{1.4}$$

This problem arises when one looks for solutions of the Schrödinger-Poisson system of the type

$$\begin{cases} -\Delta u + \lambda u + \mu\phi u = f(u), & x \in \mathbb{R}^N, \\ \Delta\phi = u^2, & x \in \mathbb{R}^N \end{cases} \tag{1.5}$$

with $N \geq 2$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, which has a strong physical meaning because it originates in quantum mechanics models (see e.g. [8, 13, 33]) and in semiconductor theory [7, 34, 35]. The second equation (1.5) determines only up to harmonic functions, and it is natural to choose ϕ as the negative Newton potential of u^2, i.e., the convolution of u^2 with the fundamental solution Γ_N of the Laplacian, which is given by

$$\Gamma_N(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & N = 2, \\ \frac{1}{N(2-N)\omega_N}|x|^{2-N}, & N \geq 3, \end{cases}$$

and ω_N is the volume of the unit N-ball. With this formal inversion, system (1.5) is converted into an equivalent nonlocal equation

$$-\Delta u + \lambda u + \mu(\Gamma_N * u^2)u = f(u), \quad x \in \mathbb{R}^N. \tag{1.6}$$

In the last decades, this equation has been extensively investigated by using variational methods. The majority of the literature focuses on the study of (1.6) with $N = 3$, it seems that it is impossible to summarize it for the case that $\lambda > 0$ is a fixed and assigned a parameter since the related literature is too large, we just refer to [5, 6, 28, 38] for the case that λ appears as a Lagrange parameter.

In contrast with the higher-dimensional case $N = 3$, much less is known for (1.6) with $N = 2$. In this case, the applicability of variational methods is not straightforward because the corresponding energy functional is not well-defined on $H^1(\mathbb{R}^2)$ under the effect of the logarithmic convolution potential. This direction of research was likely brought to the attention of the community of nonlinear PDEs by the paper [21] published in 2016. In that paper, Cingolani and Weth, inspired by Stubbe [41], developed a variational framework to deal with (1.6) with $N = 2$, within the smaller Hilbert space X defined by (1.4), and proved the existence of ground state solutions when $f(u) = |u|^{q-2}u$ for $q > 4$. The key tool to prove the compactness is a new smart strong compactness condition (modulo translation) for Cerami sequences in the periodic setting. This tool was subsequently used by Du-Weth [23] for the case that $f(u) = |u|^{q-2}u$ for $2 < q \leq 4$, and by Chen-Shi-Tang [14] for the more general case that $f(u) \sim |u|^{q-2}u$ for $q > 2$. In recent papers [17] and [18], we introduced another axially symmetric variational framework within a natural constraint

$$E_{as} := X \cap \left\{u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|),\ \forall\, x \in \mathbb{R}^2\right\}, \tag{1.7}$$

and proved respectively the existence of axially symmetric solutions for (1.6) with $N = 2$ when $f(u) \sim |u|^{q-2}u$ for $q > 2$ and when f has critical exponential growth satisfying (f1).

Compared with the above case where $\lambda > 0$ is fixed, the search of normalized solutions for (1.6) with $N = 2$ is more challenging due to the extra need to respect the L^2-constraint, which is our focus of the present paper. It seems that the first contribution to this topic was made recently by Cingolani-Jeanjean [20], in which the existence of normalized solutions for the following equation with the power nonlinearity

$$\begin{cases} -\Delta u + \lambda u + \mu\left(\log|\cdot| * u^2\right)u = a|u|^{p-2}u, \ x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases} \tag{1.8}$$

was established, and a complete analysis of the various cases on parameters $\mu, a \in \mathbb{R}$ and $p > 2$ that may happen for (1.8) was provided. In the study of (1.8), an important role is played by the so-called L^2-critical exponent 4. If $p > 4$ or $2 < p < 4$, one speaks of an L^2-supercritical case or an L^2-subcritical case. Precisely, it was proved that (1.8) has a ground state provided that $\mu > 0$ and one of the following three conditions: i) $a \leq 0$ and $p > 2$; ii) $a > 0$ and $p < 4$; iii) $a > 0$, $p = 4$ and $c < 2/(a\mathcal{C}_4)$, under which the associated energy functional is bounded from below on the constraint $\mathcal{S}_c$ for any $c > 0$ and a global minimum on $\mathcal{S}_c$ can be achieved, where the constant $\mathcal{C}_4 > 0$ comes from the Gagliardo-Nirenberg inequality (see (2.9) later). In all the other cases, although it is not possible to find a global minimizer, the interplay between a logarithmic convolution potential and a power function adds some richness to the geometric picture of the associated energy functional. In particular, when $\mu, a > 0$ and $p > 4$, it was proved that there exists an explicit value $c_0 = c_0(\mu, a, p)$ such that for $c \in (0, c_0)$, (1.8) has two normalized solutions, one being a local minimizer and one of mountain-pass type. This is reminiscent of the recent work by Soave [39], where a similar structure has been observed for the following Schrödinger equation with combined nonlinearities of power type:

$$\begin{cases} -\Delta u + \lambda u = \gamma |u|^{q-2}u + |u|^{p-2}u, \ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \mathrm{d}x = c, \end{cases} \tag{1.9}$$

with $N \geq 1, \gamma > 0$ and $2 < q < 2+4/N < p < 2^* := \begin{cases} 2N/(N-2), & N \geq 3, \\ +\infty, & N = 1, 2, \end{cases}$ see also subsequent papers [26, 27, 30, 42] for extensions from $p < 2^*$ to Sobolev critical exponent $p = 2^*$. However, the appearance of the nonlocal convolution term $\left(\log |\cdot| * u^2\right) u$ in (1.8) exhibits some serious mathematical differences to a local nonlinear term of the form $|u|^{q-2}u$. To address this trouble, Cingolani and Jeanjean used the combination of the fibration method of Pohozaev (relying on the decomposition of L^2-Pohozaev manifold used in [39]) and the strong compactness condition developed by Cingolani–Weth [21], where some new estimates of energy on the dilated function $su(s\cdot)$ for $u \in L^2$ and $s > 0$ belonging to $\mathcal{S}_c$ were given. It is worth mentioning that the argument strongly depends on the order p of power function and is not adequate for the following problem

$$\begin{cases} -\Delta u + \lambda u + \mu \left(\log |\cdot| * u^2\right) u = f(u), \ x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases} \tag{1.10}$$

with the more general nonlinear term f, even the sum of power functions with super-cubic growth. In the recent preprint paper, Alves–Böer–Miyagaki [3] considered (1.10) with critical exponential growth satisfying (f1) with $\alpha_0 = 4\pi$. In particular, if f also satisfies

(f2) $f(0) = 0$ and there exists $\tau > 3$ such that $\lim_{t \to 0} \frac{|f(t)|}{|t|^\tau} = 0$;
(f3) there exists $\theta > 6$ such that $f(t)t \geq \theta F(t) > 0, \ \forall\, t \neq 0$, where $F(t) := \int_0^t f(s)\mathrm{d}s$;
(f4) there exist $q > 4$ and $\nu > \nu_0$ such that $F(t) \geq \nu |t|^q, \ \forall\, t \in \mathbb{R}$,

it was proved that for any $c \in (0, 1)$, there are implicit parameters $\mu_0, \nu_0 > 0$ such that problem (1.10) has a solution for $\mu \in (0, \mu_0)$ and $\nu > \nu_0$. Note that this statement is of perturbative nature in two respects: i) μ_0 is sufficiently small such that (1.10) can be viewed as a perturbative form of the planar Schrödinger equation; ii) ν_0 is sufficiently large such that the obtained mountain-pass level is small enough from which the compactness can be obtained in the same way as that of (1.8).

Clearly, the perturbative argument excludes many concrete models, and circumvents the added difficulties arising from the logarithmic nature of convolution kernel and the critical exponential growth of nonlinearity compared to (1.8) and (1.10) with $\mu = 0$. To our knowledge, **it still remains open exactly how the interplay between the logarithmic convolution term** $\left(\log|\cdot| * u^2\right) u$ **and the nonlinear term** f **satisfying (f1) effects the geometry structure of the corresponding functional**, which is unbounded from below on $\mathcal{S}_c$ for all $c > 0$ since $\lim_{|u|\to\infty} \frac{|f(u)|}{|u|^3} = +\infty$ if (f1) holds.

Motivated by the study of (1.8) in the L^2-critical case that $a|u|^{p-2}u$ with $a > 0$ and $p > 4$, considered in [20], a natural question arises:

(Q) *Is it possible to obtain an analogous structure of local minima for planar Schrödinger–Poisson problems with critical exponential growth?*

In the present paper, we will give an affirmative answer to above question. More precisely, after the search for a structure of local minima, differently from the perturbative argument of [3], by specifying explicit conditions on c, we study the existence of multiple normalized solutions for (1.10) with critical exponential growth, and **achieve a significant extension of nonlinearity from the power type to the critical exponential type**. To better illustrate our approach, we provide a concrete nonlinear model $f(u) = \left(e^{u^2} - 1 - u^2\right) u$, which clearly satisfies condition (f1). This model is somehow inspired by Cassani–Tavares–Zhang [12] for the study of positive solutions to the Bose–Einstein type systems in $\mathbb{R}^2$.

Compared to (1.8) with $a > 0$ and $p > 4$ considered in [20], additional difficulties arise in the study of (1.1) since the combination of the logarithmic nature of convolution kernel and the critical exponential growth of nonlinearity mixes things up.

Indeed, first, a nonlinear term of exponential type behaves like infinite series of powers nonlinear interactions, **the interplay between it and the nonlocal term** $\left(\log|\cdot| * u^2\right) u$ **is more intricate**, which strongly effects the geometry structure of Φ on $\mathcal{S}_c$. Even if such a geometry may somehow be expected for sufficiently small values of $c > 0$ along the research lines of [20] considering (1.8) with $a|u|^{p-2}u$ ($a > 0$ and $p > 4$), **the arguments of** [20] **are insufficient to find an explicit existence range of** c for (1.1). It requires us to **develop more robust arguments in the search for a geometry of local minima** for Φ on $\mathcal{S}_c$. Note that such a structure suggests the possibility to search for another solution lying at a mountain pass level, as well as a solution characterizing as a local minima. If such a structure exists, then the next most complicated part lies in **the compactness analysis** for minimizing sequences and (PS) sequences, since it is not clear whether

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^s \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x = \int_{\mathbb{R}^2} |\bar{u}|^s \left(e^{\alpha \bar{u}^2} - 1\right) \mathrm{d}x \tag{1.11}$$

for $s \geq 2$ if $u_n \rightharpoonup u$ in X, despite the compactness of embedding $X \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \geq 2$. **This fact prevents us from using the compactness argument of** [20]. These difficulties enforce the implementation of new ideas since the approach due to Cingolani–Jeanjean [20], treating the power case $f(u) = a|u|^{p-2}u$, is not available for (1.1).

In particular, instead of working directly in space X, we shall take advantage of the axially symmetric variational framework within E_{as} defined by (1.7), endowed with the norm given by

$$\|u\|_{E_{as}} := (\|\nabla u\|_2^2 + \|u\|_*^2)^{1/2}, \text{ where } \|u\|_*^2 = \int_{\mathbb{R}^2} \log(2 + |x|) u^2(x) \mathrm{d}x, \tag{1.12}$$

and work with the constraint

$$\hat{\mathcal{S}}_c := E_{as} \cap \mathcal{S}_c = \left\{u \in E_{as} : \|u\|_2^2 = c\right\}. \tag{1.13}$$

As in our paper [18], if u is a critical point of Φ restricted to $\hat{\mathcal{S}}_c$, then u is a critical point of Φ on $\mathcal{S}_c$. As one will observe, besides helping to overcome the lack of compactness caused by the critical exponential growth, this type of axially symmetric setting is of extremely benefit to the proof of the L^2-convergence of minimizing sequences and (PS) sequences, which is a well-identified obstacle dealing with the L^2-constrained problems due to the lack of compactness for the embedding $H^1_{\text{rad}}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$.

Our main results read as follows.

Theorem 1.1 *For any $\mu > 0$, there exists $c_1 = c_1(\mu) > 0$ such that, for any $c \in (0, c_1)$, (1.1) has a couple solution $(u_c, \lambda_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$u_c \in \hat{\mathcal{S}}_c, \quad u_c \geq 0, \quad \Phi(u_c) = m(c) := \inf \left\{ \Phi(u) : u \in \hat{\mathcal{S}}_c, \ \|\nabla u\|_2^2 < \pi/3 \right\}. \tag{1.14}$$

Theorem 1.2 *For any $\mu > 0$, there exists $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, (1.1) has a second couple solution $(\hat{u}_c, \hat{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$0 < \Phi(\hat{u}_c) < m(c) + 2\pi. \tag{1.15}$$

Remark 1.3 The condition $c \in (0, c_1)$ in Theorem 1.1 enters in the study of a geometry of local minima of Φ, while the condition $c \in (0, c_0)$ in Theorem 1.2, which appears to be more delicate, is used in order to further ensure that a minimax structure of the mountain-pass type exists and the obtained energy level is less than $m(c) + 2\pi$ that is a threshold of compactness, which is an essential and striking ingredient in our compactness argument.

Define the L^2-Pohozaev functional $\mathcal{P} : X \to \mathbb{R}$ by

$$\mathcal{P}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \frac{\mu c^2}{4} - \int_{\mathbb{R}^2} \left[\left(u^2 - 1\right) e^{u^2} + 1 - \frac{u^4}{2} \right] \mathrm{d}x. \tag{1.16}$$

As one will see in Lemma 3.4, any solution to (1.1) satisfies the L^2-Pohozaev identity $\mathcal{P}(u) = 0$.

Let us now sketch our research strategies and point out key elements for the proofs of Theorems 1.1 and 1.2.

First, we search for a geometry of local minima for Φ on $\hat{\mathcal{S}}_c = \mathcal{S}_c \cap E_{as}$ under explicit conditions on c. For this, we introduce a crucial set $A_{\pi/3} = \{u \in E_{as} : \|\nabla u\|_2^2 < \pi/3\}$ such that for any $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$, $\mathcal{P}(u) > 0$ and there exists $t_u \in (0, 1)$ such that $\mathcal{P}(t_u u_{t_u}) = 0$, with this important property and subtle estimates of energy, for any $\mu > 0$, we succeed in finding an explicit value $c_1 = c_1(\mu) > 0$ such that for any $c \in (0, c_1)$, Φ has a geometry of local minima

$$m(c) := \inf_{\hat{\mathcal{S}}_c \cap A_{\pi/3}} \Phi < \inf_{\hat{\mathcal{S}}_c \cap \partial A_{\pi/3}} \Phi. \tag{1.17}$$

In our argument, the upper bound $\pi/3$ is not essential but brings a convenience in obtaining the explicit existence range $c \in (0, c_1)$ and proving the compactness in the following discussion. In this regard, our strategy is totally different from that of [20], since the boundary of the corresponding auxiliary set used in [20] depends on the mass $\|u\|_2^2 = c$ as well as the order p of power function in (1.8), instead of being given in advance like us.

Second, we prove that the local minima $m(c)$ defined by (1.17) is attained, that is, letting $\{u_n\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ be such that $\Phi(u_n) \to m(c)$, we verify that $u_n \to u$ in E_{as}, proving Theorem 1.1. The crucial ingredient of the proof is to obtain the boundedness of $\{\|u_n\|_X\}$, or more precisely, prove that $\|u_n\|_*^2 = \int_{\mathbb{R}^2} \log(2 + |x|) u_n^2(x) \mathrm{d}x \leq C$ for some $C > 0$ due to the fact that $\{u_n\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ and the definition of $\| \cdot \|_X$ given by (1.12). Note that at this stage, the sign of $m(c)$ can not be judged under the unpleasant effect of a nonlinear term of exponential type. This fact results in the failure of the method used in [20] relying on the strong compactness condition. Indeed, following the lines of [20], it is essential to verify that $u_n \to u \in L^2(\mathbb{R}^2)\backslash\{0\}$ pointwise a.e. on $\mathbb{R}^2$ such that the strong compactness condition works which leads to the boundedness of $\{\|u_n\|_*\}$ up to translations. But it seems impossible to make it in our case since the vanishing of $\{u_n\}$ can not be ruled out when the situation of $m(c) = 0$ may occur. Somewhat surprisingly, our axially symmetric variational framework allows us to avoid the obstacle since the boundedness of $\{\|u_n\|_*^2\}$ follows directly from the specific inequality related with the coupling term of equation restricted on E_{as}

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(2+|x-y|)\, u^2(x)v^2(y)\mathrm{d}x\mathrm{d}y \geq \frac{1}{4}\int_{\mathbb{R}^2} u^2(x)\mathrm{d}x \int_{\mathbb{R}^2} \log(2+|x|)v^2(x)\mathrm{d}x. \tag{1.18}$$

This also explains why we work with $E_{as} \cap \mathcal{S}_c$ at the beginning. The remaining proof of convergence is standard, since (1.11) follows directly from Trudinger-Moser inequality (see Lemma 2.3) due to the fact that $\|\nabla u_n\|_2^2 \leq \pi/3 < 2\pi$ for all $n \in \mathbb{N}$.

Last but not least, we further specify an explicit range on c to guarantee the existence of another solution of mountain-pass type, proving Theorem 1.2, which is the heart of the paper. Several crucial steps are summarized as follows.

Step 1. Construct a (PS) sequence $\{u_n\} \subset \hat{\mathcal{S}}_c$ of $\Phi\big|_{\hat{\mathcal{S}}_c}$ possessing additional property $\mathcal{P}(u_n) \to 0$ at a mountain pass level $M(c)$. The condition that $\mathcal{P}(u_n) \to 0$ helps to deduce the boundedness of $\{\|\nabla u_n\|_2\}$.

This step is reminiscent of the one developed by Jeanjean [25] but here the fact that Φ has a structure of local minimum instead of a direct mountain-pass geometry and the appearance of a logarithmic convolution potential make the proof more delicate.

To detect a minimax structure of $\Phi\big|_{\hat{\mathcal{S}}_c}$, we use several critical point theorems on a manifold, developed recently by us in [16] considering problem (1.9). Our approach is applicable to more general nonlinearities and totally different from that of [20] dealing with $f(u) = a|u|^{p-2}u$ in the case of $a > 0$ and $p > 4$. Noting that the situation $m(c) = 0$ can not be ruled out in advance, it is from the specific inequality (1.18) that we deduce the boundedness of $\{\|u_n\|_*^2\}$, and thus $\{\|u_n\|_X\}$ is bounded and there exists $\bar{u} \in \hat{\mathcal{S}}_c$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in E_{as} and $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$. However, it is insufficient to show that $\bar{u}$ is a solution to (1.1) since it is unclear whether

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_n^2(x)[u_n(y) - \bar{u}(y)]v(y)\mathrm{d}x\mathrm{d}y = 0, \quad \forall\, v \in \mathcal{C}_0^\infty(\mathbb{R}^2). \tag{1.19}$$

This requires to further prove the strong convergence. Inspired by the Brezis-Nirenberg problem, the crucial point in proving the compactness is to obtain a good energy estimate of the obtained (PS) sequence, which is what to do next.

Step 2. Establish a precise upper estimate of the energy level $M(c)$, given by

$$M(c) < m(c) + 2\pi, \tag{1.20}$$

such that the compactness of the obtained (PS) sequences still holds.

In the unconstrained case (1.5), this kind of sharp upper estimate is known, see our papers [15, 17], and the usual way to derive such strict inequality is through the use of testing functions, that is a sequence of Moser-type functions introduced by de Figueiredo, Miyagaki and Ruf [22], related with the Trudinger–Moser inequality. But, there seems no an analogue in our case due to the need to respect L^2-constraint and the logarithmic nature of convolution kernel. **This step gives firstly a counterpart in that direction, whose proof is rather complicated, and requires a lot of subtle energy estimates as well as a better understanding of structure for Φ on $\hat{\mathcal{S}}_c$**, see Remark 1.4 for further description.

Step 3. Prove the limit (1.11) and then $u_n \to \bar{u}$ in E_{as}, up to a subsequence.

To ensure that the Trudinger–Moser inequality ii) of Lemma 2.3 works in the proof of (1.11), one needs to control appropriately the value of $\|\nabla u_n\|_2^2$ from above, which is why one requires a sharp upper estimate of $M(c)$ before. Unfortunately, it seems impossible to obtain $\|\nabla u_n\|_2^2 < 4\pi$ for large n. Instead, we prove $\|\nabla(u_n - \bar{u})\|_2^2 < 4\pi$ for large n in a tactfully round-about way, of these arguments, two main difficulties are to prove $\mathcal{P}(\bar{u}) \geq 0$ and $\Phi(\bar{u}) \geq m(c)$, see the proof of (4.98), and then show indirectly (1.11) with the Young's inequality.

Remark 1.4

i) To obtain a constrained (PS) sequence with additional property, the approach in [20], treating (1.8) with $f(u) = a|u|^{p-2}u$ in the case of $a > 0$ and $p > 4$, not only relied on the decomposition of the L^2-Pohozaev manifold into three disjoint subsets, but used the Ghoussoub minimax principle [24], where the former just works for an **easy calculating** form of nonlinearity, and the later requires technical topological, very complicated, arguments based on σ-homotopy stable family of compact subsets. This approach was also applied to problem (1.9) with mixed nonlinearities, see [27, 29, 30, 39, 40, 42], nevertheless, it is not available in our case due to the **complex behaviors** of the terms $\left(\log|\cdot| * u^2\right)u$ and $\left(e^{u^2} - 1 - u^2\right)u$. In contrast, our method does not require the decomposition of the L^2-Pohozaev manifold, and our tool to detect the minimax structures just depends on the general deformation lemma on a manifold, and is **technically simpler** than topological arguments involved in the Ghoussoub minimax principle [24].

ii) Note that (1.20) gives a new threshold of compactness for planar Schrödinger–Poisson systems with critical exponential growth in the L^2-constraint. To obtain the strict inequality (1.20), roughly speaking, we use a nice superposition of a minima obtained in Theorem 1.1 and a modified sequence of Moser-type functions with finer supports where the supports would be disjoint, see Lemma 4.4 for more details. The idea behind the proof is that the interaction decreases the involved energy value. Even if this idea is somehow inspired by [42] concerning the Sobolev critical situation in the higher dimensions, the mathematical strategies and proof techniques are different, for example, the variational characterizations of a minima are various in the use of testing functions; our tool of energy estimate is the neatly combination of the Gagliardo-Nirenberg inequality and the Trudinger-Moser inequality instead of the Sobolev inequality; extra efforts are always required to overcome the unpleasant effect due to the logarithmic nature of convolution kernel.

iii) We believe that our approach may be adapted to attack more L^2-constrained problems with critical exponential growth, and the new underlying ideas and the strategy of energy estimates are of future development and applicability.

The paper is organized as follows. Section 2 is devoted to some preliminaries. In particular, we present several critical point theorems on a manifold, we have developed recently in [16], which play a crucial role in the proofs of theorems. In Sect. 3, we consider the existence of a local minima for Φ on $E_{as} \cap \mathcal{S}_c$, and give the proof of Theorem 1.1. In Sect. 4, we study the existence of a critical point of mountain-pass type for Φ on $E_{as} \cap \mathcal{S}_c$, and finish the proof of Theorem 1.2.

Throughout the paper, we make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\, u, v \in H^1(\mathbb{R}^2);$$

- $H^1_{\mathrm{rad}}(\mathbb{R}^2)$ denotes the space of spherically symmetric functions belonging to $H^1(\mathbb{R}^2)$:

$$H^1_{\mathrm{rad}}(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^2\};$$

- $L^s(\mathbb{R}^2)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^2)\backslash\{0\}$, $u_t(x) := u(tx)$ for $t > 0$;
- For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$ and $B_r = B_r(0)$;
- $C_1, C_2, \cdots$ denote positive constants possibly different in different places, which are dependent on $c > 0$.

2 Preliminary results

As in [17], we define the following symmetric bilinear forms

$$(u, v) \mapsto A_1(u, v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(2 + |x - y|)\, u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

$$(u, v) \mapsto A_2(u, v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{2}{|x - y|}\right) u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

$$(u, v) \mapsto A_0(u, v) := A_1(u, v) - A_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x - y| u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.3}$$

where the definition is restricted, in each case, to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Noting that $0 \le \log(1 + r) \le r$ for $r \ge 0$, it follows from the Hardy–Littlewood–Sobolev inequality(see [31] or [32, p. 98]) that

$$|A_2(u, v)| \le 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x - y|}|u(x)v(y)|\mathrm{d}x\mathrm{d}y \le \mathcal{C}_0\|u\|_{4/3}\|v\|_{4/3} \tag{2.4}$$

with a constant $\mathcal{C}_0 > 0$. Using (2.1), (2.2) and (2.3), we define the following energy functionals:

$$\begin{aligned}
&I_1 : H^1(\mathbb{R}^2) \to [0, \infty],\\
&\quad I_1(u) := A_1(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(2 + |x - y|)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\\
&I_2 : L^{8/3}(\mathbb{R}^2) \to [0, \infty),\\
&\quad I_2(u) := A_2(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{2}{|x - y|}\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\\
&I_0 : H^1(\mathbb{R}^2) \to \mathbb{R} \cup \{\infty\},\\
&\quad I_0(u) := A_0(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x - y|\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Here I_2 only takes finite values on $L^{8/3}(\mathbb{R}^2)$. Indeed, (2.4) implies

$$|I_2(u)| \le \mathcal{C}_0\|u\|^4_{8/3}, \quad \forall\, u \in L^{8/3}(\mathbb{R}^2). \tag{2.5}$$

Let $\|\cdot\|_*$ be defined by (1.12). Then $\|u\|_X := (\|\nabla u\|_2^2 + \|u\|_*^2)^{1/2}$ is a norm on X, where X is defined in (1.4). Moreover, it follows from Rellich's Theorem (see [37, Theorem XIII.65]) that the embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $s \in [2, \infty)$, and so the embedding $E_{as} \hookrightarrow L^s(\mathbb{R}^2)$ is also compact for $s \in [2, \infty)$. Since

$$\log(2 + |x - y|) \leq \log(2 + |x| + |y|) \leq \log(2 + |x|) + \log(2 + |y|), \quad \forall\, x, y \in \mathbb{R}^2, \tag{2.6}$$

we have

$$\begin{aligned} |A_1(uv, wz)| &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big[\log(2 + |x|) + \log(2 + |y|)\big]\, |u(x)v(x)||w(y)z(y)| \mathrm{d}x\mathrm{d}y \\ &\leq \|u\|_*\|v\|_*\|w\|_2\|z\|_2 + \|u\|_2\|v\|_2\|w\|_*\|z\|_*, \quad \forall\, u, v, w, z \in X. \end{aligned} \tag{2.7}$$

According to [21, Lemma 2.2], we have I_0, I_1 and I_2 are of class $\mathcal{C}^1$ on X, $I_0 = I_1 - I_2$ and

$$\langle I_i'(u), v\rangle = 4A_i(u^2, uv), \quad \forall\, u, v \in X, \; i = 0, 1, 2. \tag{2.8}$$

Lemma 2.1 [4, Gagliardo–Nirenberg inequality] *There holds*

$$\|u\|_s^s \leq \mathcal{C}_s^s \|u\|_2^2 \|\nabla u\|_2^{s-2} \;\; \textit{for} \;\; s > 2, \tag{2.9}$$

where $\mathcal{C}_s > 0$ is a constant determined by s.

Lemma 2.2 [43] *For any $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ and $r_0 > 0$,*

$$|u(x)| \leq \frac{1}{\sqrt{\pi |x|}} \|u\|_2^{1/2} \|\nabla u\|_2^{1/2}, \quad \forall\, |x| \geq r_0. \tag{2.10}$$

Lemma 2.3 [1, 10, 11]

i) *If $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left(e^{u^2} - 1\right) \mathrm{d}x < \infty;$$

ii) *if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq \alpha < 4\pi$ and $\|u\|_2 \leq \beta < \infty$, then there exists a constant $C(\alpha, \beta)$, which depends only on α and β, such that*

$$\int_{\mathbb{R}^2} \left(e^{u^2} - 1\right) \mathrm{d}x \leq C(\alpha, \beta). \tag{2.11}$$

Lemma 2.4 [17, Lemma 2.2] *There holds*

$$A_1(u^2, v^2) \geq \frac{1}{4} \|u\|_2^2 \|v\|_*^2, \quad \forall\, u, v \in E_{as}. \tag{2.12}$$

Corollary 2.5 [17, Corollary 2.3] *There holds*

$$I_1(u) \geq \frac{1}{4} \|u\|_2^2 \|u\|_*^2, \quad \forall\, u \in E_{as}. \tag{2.13}$$

Lemma 2.6 [17, Lemma 2.4] *If u is a critical point of Φ restricted to E_{as}, then u is a critical point of Φ on X.*

Lemma 2.7 [21, Lemma 2.6] *Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in X such that $u_n \rightharpoonup \bar{u} \in X$. Then for every $\phi \in X$, we have*

$$A_1(v_n w_n, \phi(u_n - \bar{u})) \to 0.$$

Lemma 2.8 *If $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} |u|^{2k} \mathrm{d}x \leq \frac{2 + 2^{2k-1}(k-2)}{(k-2)\pi^{k-1}} \|u\|_2^k \|\nabla u\|_2^k + \frac{k!}{2\pi^{k-1}} \|\nabla u\|_2^{2k}, \quad \forall\, k \in \mathbb{N}, \tag{2.14}$$

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 - \frac{u^4}{2} \right) \mathrm{d}x \\ &\leq \frac{\|\nabla u\|_2^6}{2\pi(\pi - \|\nabla u\|_2^2)} + \frac{2c^{3/2}\|\nabla u\|_2^3}{\pi^2} \sum_{k=0}^{\infty} \frac{4^{k+2}(k+1)+1}{(k+1)(k+3)!} \left(\frac{\|u\|_2 \|\nabla u\|_2}{\pi} \right)^k, \\ &\forall\, u \in \mathcal{S}_c, \quad \|\nabla u\|_2^2 < \pi \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^2} \left[\left(u^2 - 1\right) e^{u^2} + 1 - \frac{u^4}{2} \right] \mathrm{d}x \\ &\leq \frac{\left(2\pi - \|\nabla u\|_2^2\right) \|\nabla u\|_2^6}{2\pi \left(\pi - \|\nabla u\|_2^2\right)^2} + \frac{2c^{3/2}\|\nabla u\|_2^3}{\pi^2} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\frac{\|u\|_2 \|\nabla u\|_2}{\pi} \right)^k, \\ &\forall\, u \in \mathcal{S}_c, \quad \|\nabla u\|_2^2 < \pi. \end{aligned} \tag{2.16}$$

Proof : For the proof of Lemma 2.8, we refer to [19, Lemma 2.4]. □

Let H be a real Hilbert space whose norm and scalar product will be denoted respectively by $\|\cdot\|_H$ and $(\cdot,\cdot)_H$. Let E be a real Banach space with norm $\|\cdot\|_E$. We assume throughout this section that

$$E \hookrightarrow H \hookrightarrow E^* \tag{2.17}$$

with continuous injections, where E^* is the dual space of E. Thus H is identified with its dual space. We will always assume in the sequel that E and H are infinite dimensional spaces. We consider the manifold

$$M := \{u \in E : \|u\|_H = 1\}. \tag{2.18}$$

M is the trace of the unit sphere of H in E and is, in general, unbounded. Throughout the paper, M will be endowed with the topology inherited from E. Moreover M is a submanifold of E of codimension 1 and its tangent space at a given point $u \in M$ can be considered as a closed subspace of E of codimension 1, namely

$$T_u M := \{v \in E : (u, v)_H = 0\}. \tag{2.19}$$

We consider a functional $\varphi : E \to \mathbb{R}$ which is of class $\mathcal{C}^1$ on E. We denote by $\varphi|_M$ the trace of φ on M. Then $\varphi|_M$ is a $\mathcal{C}^1$ functional on M, and for any $u \in M$,

$$\langle \varphi|_M'(u), v\rangle = \langle \varphi'(u), v\rangle, \quad \forall\, v \in T_u M. \tag{2.20}$$

In the sequel, for any $u \in M$, we define the norm $\left\|\varphi|_M'(u)\right\|$ by

$$\left\|\varphi|_M'(u)\right\| = \sup_{v \in T_u M, \|v\|_E = 1} |\langle \varphi'(u), v\rangle|. \tag{2.21}$$

Let $E \times \mathbb{R}$ be equipped with the scalar product

$$((u,\tau),(v,\sigma))_{E\times\mathbb{R}} := (u,v)_H + \tau\sigma, \quad \forall\, (u,\tau),(v,\sigma) \in E \times \mathbb{R},$$

and corresponding norm

$$\|(u,\tau)\|_{E\times\mathbb{R}} := \sqrt{\|u\|_H^2 + \tau^2}, \quad \forall\, (u,\tau) \in E \times \mathbb{R}.$$

Next, we consider a functional $\tilde{\varphi} : E \times \mathbb{R} \to \mathbb{R}$ which is of class $\mathcal{C}^1$ on $E \times \mathbb{R}$. We denote by $\tilde{\varphi}|_{M\times\mathbb{R}}$ the trace of $\tilde{\varphi}$ on $M \times \mathbb{R}$. Then $\tilde{\varphi}|_{M\times\mathbb{R}}$ is a $\mathcal{C}^1$ functional on $M \times \mathbb{R}$, and for any $(u, \tau) \in M \times \mathbb{R}$,

$$\langle \tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau), (v,\sigma)\rangle := \langle \tilde{\varphi}'(u,\tau), (v,\sigma)\rangle, \ \ \forall\, (v,\sigma) \in \tilde{T}_{(u,\tau)}(M\times\mathbb{R}), \tag{2.22}$$

where

$$\tilde{T}_{(u,\tau)}(M\times\mathbb{R}) := \{(v,\sigma) \in E\times\mathbb{R} : (u,v)_H = 0\}. \tag{2.23}$$

In the sequel, for any $(u,\tau) \in M\times\mathbb{R}$, we define the norm $\left\|\tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau)\right\|$ by

$$\left\|\tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau)\right\| = \sup_{(v,\sigma)\in\tilde{T}_{(u,\tau)}(M\times\mathbb{R}), \|(v,\sigma)\|_{E\times\mathbb{R}}=1} |\langle \tilde{\varphi}'(u,\tau), (v,\sigma)\rangle|. \tag{2.24}$$

Lemma 2.9 [9] *Let $\{u_n\} \subset M$ be a bounded sequence in E. Then the following are equivalent:*

(i) $\|\varphi|'_M(u_n)\| \to 0$ *as* $n \to \infty$*;*
(ii) $\varphi'(u_n) - \langle \varphi'(u_n), u_n\rangle u_n \to 0$ *in* E' *as* $n \to \infty$.

Lemma 2.10 [16] *Let $\varphi \in \mathcal{C}^1(E,\mathbb{R})$ and $K \subset E$. If there exists $\rho > 0$ such that*

$$a := \inf_{v\in M\cap K} \varphi(v) < b := \inf_{v\in M\cap(K_\rho\setminus K)} \varphi(v), \tag{2.25}$$

where $K_\rho := \{v \in E : \|v-u\|_E < \rho, \ \forall\, u \in K\}$, then, for every $\delta \in (0,\rho/2)$ *and* $w \in M\cap K$ *such that*

$$\varphi(w) \le a + \varepsilon, \tag{2.26}$$

there exists $u \in M$ such that

(i) $a - 2\varepsilon \le \varphi(u) \le a + 2\varepsilon$*;*
(ii) $\|u - w\|_E \le 2\delta$*;*
(iii) $\left\|\varphi|'_M(u)\right\| \le 8\varepsilon/\delta$.

Corollary 2.11 [16] *Let $\varphi \in \mathcal{C}^1(E,\mathbb{R})$ and $K \subset E$. If there exist $\rho > 0$ and $\bar{u} \in M\cap K$ such that*

$$\varphi(\bar{u}) = \inf_{v\in M\cap K} \varphi(v) < \inf_{v\in M\cap(K_\rho\setminus K)} \varphi(v), \tag{2.27}$$

then $\varphi|'_M(\bar{u}) = 0$.

Lemma 2.12 [16] *Assume that $\tilde{\theta} \in \mathbb{R}$, $\tilde{\varphi} \in \mathcal{C}^1(E\times\mathbb{R},\mathbb{R})$ and $\tilde{\Upsilon} \subset M\times\mathbb{R}$ is a closed set. Let*

$$\tilde{\Gamma} := \left\{\tilde{\gamma} \in \mathcal{C}([0,1], M\times\mathbb{R}) : \tilde{\gamma}(0) \in \tilde{\Upsilon}, \ \tilde{\varphi}(\tilde{\gamma}(1)) < \tilde{\theta}\right\}. \tag{2.28}$$

If $\tilde{\varphi}$ satisfies

$$\tilde{a} := \inf_{\tilde{\gamma}\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{\varphi}(\tilde{\gamma}(t)) > \tilde{b} := \sup_{\tilde{\gamma}\in\tilde{\Gamma}} \max\{\tilde{\varphi}(\tilde{\gamma}(0)), \tilde{\varphi}(\tilde{\gamma}(1))\}, \tag{2.29}$$

then, for every $\varepsilon \in (0, (\tilde{a}-\tilde{b})/2)$, $\delta > 0$ and $\tilde{\gamma}_* \in \tilde{\Gamma}$ such that

$$\sup_{t\in[0,1]} \tilde{\varphi}(\tilde{\gamma}_*(t)) \le \tilde{a} + \varepsilon, \tag{2.30}$$

there exists $(v,\tau) \in M\times\mathbb{R}$ such that

(i) $\tilde{a} - 2\varepsilon \le \tilde{\varphi}(v,\tau) \le \tilde{a} + 2\varepsilon$*;*
(ii) $\min_{t\in[0,1]} \|(v,\tau) - \tilde{\gamma}_*(t)\|_{E\times\mathbb{R}} \le 2\delta$*;*
(iii) $\left\|\tilde{\varphi}|'_{M\times\mathbb{R}}(v,\tau)\right\| \le \frac{8\varepsilon}{\delta}$.

3 Proof of Theorem 1.1

In this section, we consider the existence of a local minima for Φ on $E_{as} \cap \mathcal{S}_c$, and give the proof of Theorem 1.1.

Let $c_1 = c_1(\mu) > 0$ be the unique root of the following equation with respect to τ:

$$\frac{19\pi}{72} = \frac{\mu\tau^2}{4} + \frac{2\tau^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{\tau}{3\pi}}\right)^k. \tag{3.1}$$

Let $c_2 = c_2(\mu) > 0$ be the unique root of the following equation:

$$c = \sqrt{\frac{2}{\mu}}\, \eta(c), \tag{3.2}$$

where $\eta(c) > 0$ is the unique root of the following equation with respect to τ:

$$\frac{2c}{\pi} \sum_{k=3}^{\infty} \frac{(k-1)^2\left[4^{k-1}(k-2)+1\right]}{(k-2)k!} \left(\frac{\tau^2\sqrt{c}}{\pi}\right)^{k-2} + \frac{\tau^2\left(4\pi^2 - 3\pi\tau^2 + \tau^4\right)}{2\pi\left(\pi - \tau^2\right)^3} = 1. \tag{3.3}$$

Lemma 3.1 *Let $\mu > 0$. If $\|u\|_2^2 = c$, $\|\nabla u\|_2^2 = \frac{\pi}{3}$ and $\mathcal{P}(u) \le 0$, then $c \ge c_1$.*

Proof Since $\|u\|_2^2 = c$, $\|\nabla u\|_2^2 = \frac{\pi}{3}$ and $\mathcal{P}(u) \le 0$, then it follows from (1.16) and (2.16) that

$$\begin{aligned}
\frac{\pi}{3} = \|\nabla u\|_2^2 &\le \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left[\left(u^2 - 1\right) e^{u^2} + 1 - \frac{u^4}{2}\right] \mathrm{d}x \\
&\le \frac{\mu c^2}{4} + \frac{\left(2\pi - \|\nabla u\|_2^2\right) \|\nabla u\|_2^6}{2\pi\left(\pi - \|\nabla u\|_2^2\right)^2} \\
&\quad + \frac{2c^{3/2}\|\nabla u\|_2^3}{\pi^2} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\frac{\sqrt{c}\|\nabla u\|_2}{\pi}\right)^k \\
&= \frac{\mu c^2}{4} + \frac{5\pi}{72} + \frac{2c^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{c}{3\pi}}\right)^k,
\end{aligned} \tag{3.4}$$

which implies

$$\frac{19\pi}{72} \le \frac{\mu c^2}{4} + \frac{2c^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{c}{3\pi}}\right)^k, \tag{3.5}$$

which, together with (3.1), implies that $c \ge c_1$. □

For any $\rho > 0$, we define

$$A_\rho := \{u \in E_{as} : \|\nabla u\|_2^2 < \rho\} \tag{3.6}$$

and

$$m(c) := \inf_{\hat{\mathcal{S}}_c \cap A_{\pi/3}} \Phi. \tag{3.7}$$

By Lemma 3.1, we have the following corollary immediately.

Corollary 3.2 *Let $\mu > 0$ and $c \in (0, c_1)$. If $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$, then $\mathcal{P}(u) > 0$.*

For any $u \in \hat{\mathcal{S}}_c$, we denote $g_u : (0, +\infty) \to \mathbb{R}$ the function defined by

$$g_u(t) := \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{\mu}{4} I_0(u) - \frac{\mu c^2}{4}\log t - \frac{1}{2t^2}\int_{\mathbb{R}^2}\left(e^{t^2u^2} - 1 - t^2u^2 - \frac{t^4u^4}{2}\right)\mathrm{d}x. \quad (3.8)$$

Clearly g_u is $\mathcal{C}^2$ on $(0, +\infty)$, and we obviously have

$$\begin{aligned} g_u'(t) &= \frac{1}{t}\left\{t^2\|\nabla u\|_2^2 - \frac{\mu c^2}{4} - \frac{1}{t^2}\int_{\mathbb{R}^2}\left[\left(t^2u^2 - 1\right)e^{t^2u^2} + 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x\right\} \\ &= \frac{1}{t}\mathcal{P}(tu_t), \quad \forall\, t > 0. \end{aligned} \quad (3.9)$$

Lemma 3.3 *Let $\mu > 0$ and $c \in (0, c_1)$. If $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$, then there exists $t_u \in (0, 1)$ such that $\mathcal{P}(t_u u_{t_u}) = 0$.*

Proof Since $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$, then Corollary 3.2 shows that $g_u'(1) > 0$. On the other hand, by (3.9) t, $g_u'(t) \to -\frac{\mu c^2}{4} < 0$ as $t \to 0^+$. Hence, there exists $t_u \in (0, 1)$ such that $g_u'(t_u) = 0$, that is $\mathcal{P}(t_u u_{t_u}) = 0$. □

Lemma 3.4 *If there exist $u \in E_{as}$ and $\lambda \in \mathbb{R}$ satisfy (1.1), then $\mathcal{P}(u) = 0$, where $\mathcal{P}$ is defined by (1.16).*

Proof By (1.1) and the Pohozaev identity in [15, Theorem 1.4], we have

$$\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x + \lambda\|u\|_2^2 + \mu\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2}\left(e^{u^2} - 1 - u^2\right)u^2\mathrm{d}x = 0 \quad (3.10)$$

and

$$\mu\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y + \frac{\mu}{4}\|u\|_2^4 + \lambda\|u\|_2^2 - \int_{\mathbb{R}^2}\left(e^{u^2} - 1 - u^2 - \frac{u^4}{2}\right)\mathrm{d}x = 0. \quad (3.11)$$

Combining (3.10) with (3.11), we obtain

$$\mathcal{P}(u) = \int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x - \frac{\mu c^2}{4} - \int_{\mathbb{R}^2}\left[\left(u^2 - 1\right)e^{u^2} + 1 - \frac{u^4}{2}\right]\mathrm{d}x = 0. \quad (3.12)$$

The proof is now complete. □

Proof of Theorem 1.1 Let $\{u_n\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ be a minimizing sequence of Φ for $m(c)$. Clearly, $\{|u_n|\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ is also a minimizing sequence for $m(c)$. Without loss of generality, we can assume that $u_n \geq 0$. Then we have

$$\|u_n\|_2^2 = c, \quad \|\nabla u_n\|_2^2 \leq \frac{\pi}{3}, \quad \Phi(u_n) = m(c) + o(1). \quad (3.13)$$

Next, we split the proof into several steps.

Step 1. By (1.2), (2.5), (3.13) and Lemma 2.8, it is easy to verify that $I_1(u_n)$ is bounded. Hence, we then deduce by Corollary 2.5 that $\{\|u_n\|_*\}$ is bounded, so $\{u_n\}$ is bounded in E_{as}. We may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E_{as}, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$, $s \in [2, \infty)$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. Furthermore, we have

$$\|\bar{u}\|_2^2 = \|u_n\|_2^2 = c, \quad \|\nabla\bar{u}\|_2^2 \leq \frac{\pi}{3}. \quad (3.14)$$

Step 2. Set $v_n := u_n - \bar{u}$. Then by Step 1, $\|v_n\|_s \to 0$ for $s \in [2, +\infty)$ and $v_n \rightharpoonup 0$ in E_{as}. It follows from (2.1)-(2.4), (2.7), (3.13), (3.14) and Lemma 2.3 ii) that

$$\|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2 - \|\nabla \bar{u}\|_2^2 + o(1), \tag{3.15}$$
$$I_0(u_n) = I_0(\bar{u}) + I_0(v_n) + 2A_0(\bar{u}^2, v_n^2) + o(1) \tag{3.16}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|\left(e^{u_n^2} - 1\right) - \left(e^{v_n^2} - 1\right) - \left(e^{\bar{u}^2} - 1\right)\right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^2} (|u_n| + |\bar{u}|)|v_n| e^{u_n^2 + \bar{u}^2} \mathrm{d}x + \int_{\mathbb{R}^2} e^{v_n^2} |v_n|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} (|u_n| + |\bar{u}|)|v_n| \left(e^{u_n^2 + \bar{u}^2} - 1\right) \mathrm{d}x + \int_{\mathbb{R}^2} (|u_n| + |\bar{u}|)|v_n| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1\right) |v_n|^2 \mathrm{d}x + \|v_n\|_2^2 \\
&\leq \left[\int_{\mathbb{R}^2} \left(e^{2u_n^2 + 2\bar{u}^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{2}} (\|u_n\|_4 + \|\bar{u}\|_4) \|v_n\|_4 \\
&\quad + \left[\int_{\mathbb{R}^2} \left(e^{2v_n^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{2}} \|v_n\|_4^2 + o(1) \\
&= o(1).
\end{aligned} \tag{3.17}$$

Hence, by (1.2), (3.15), (3.16), (3.17) and the Brezis–Lieb lemma, we have

$$\Phi(u_n) = \Phi(\bar{u}) + \Phi(v_n) + \frac{\mu}{2} A_0(\bar{u}^2, v_n^2) + o(1). \tag{3.18}$$

Step 3. By (2.4), (3.13), (3.14) and (3.18), we deduce

$$\begin{aligned}
m(c) + o(1) = \Phi(u_n) &= \Phi(\bar{u}) + \Phi(v_n) + \frac{\mu}{2} A_0(\bar{u}^2, v_n^2) + o(1) \\
&\geq m(c) + \Phi(v_n) + \frac{\mu}{2} A_0(\bar{u}^2, v_n^2) + o(1) \\
&\geq m(c) + \Phi(v_n) + \frac{\mu}{2} A_1(\bar{u}^2, v_n^2) + o(1).
\end{aligned} \tag{3.19}$$

(3.19) shows that $\Phi(v_n) \leq o(1)$. Hence it follows from (1.2), (2.5), (2.9) and Lemma 2.3 that

$$\begin{aligned}
\|\nabla v_n\|_2^2 + \frac{\mu}{2} I_1(v_n) &\leq o(1) + \frac{\mu}{2} I_2(v_n) + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1 - v_n^2 - \frac{v_n^4}{2}\right) \mathrm{d}x \\
&\leq o(1) + \frac{\mu}{2} \mathcal{C}_0 \|v_n\|_{8/3}^4 + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1 - v_n^2 - \frac{v_n^4}{2}\right) \mathrm{d}x \\
&\leq \frac{\mu}{2} \mathcal{C}_0 \mathcal{C}_{8/3}^4 \|\nabla v_n\|_2 \|v_n\|_2^3 + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1\right) v_n^2 \mathrm{d}x + o(1) \\
&\leq \frac{\mu}{2} \mathcal{C}_0 \mathcal{C}_{8/3}^4 \|\nabla v_n\|_2 \|v_n\|_2^3 \\
&\quad + \mathcal{C}_4^2 \left[\int_{\mathbb{R}^2} \left(e^{2v_n^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{2}} \|\nabla v_n\|_2 \|v_n\|_2 + o(1) \\
&= o(1).
\end{aligned} \tag{3.20}$$

It follows from (3.20) that $\|\nabla v_n\|_2^2 = o(1)$ and $I_1(v_n) = o(1)$, and so $\Phi(v_n) = o(1)$. It follows from (3.19) that $A_1(\bar{u}^2, v_n^2) = o(1)$, and so by Lemma 2.4, $v_n \to 0$ in E_{as}, i.e. $u_n \to \bar{u}$ in E_{as}. Since $u_n \geq 0$, it follows that $\bar{u} \geq 0$.

Step 4. Obviously $u \in \bar{A}_{\pi/3}$ and $\Phi(\bar{u}) = m(c)$. Next, we show that $\|\nabla\bar{u}\|_2^2 < \frac{\pi}{3}$. Let us assume by contradiction that $\|\nabla\bar{u}\|_2^2 = \frac{\pi}{3}$. Then we see directly from Corollary 3.2 that necessarily $\mathcal{P}(\bar{u}) > 0$. But then we consider t_0 with $t_0 < 1$ close to 1. Recording (3.9), it follows that $t_0\bar{u}_{t_0} \in A_{\pi/3}$ and $\Phi(t_0\bar{u}_{t_0}) < \Phi(\bar{u}) = m(c)$, providing a contradiction. Hence, Corollary 2.11 implies that $\Phi|'_{\hat{\mathcal{S}}_c}(\bar{u}) = 0$, and so there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $\langle \Phi'(\bar{u}) + \lambda_c\bar{u}, \phi\rangle = 0$ for any $\phi \in E_{as}$. By Lemma 2.6, we have $\langle \Phi'(\bar{u}) + \lambda_c\bar{u}, \phi\rangle = 0$ for any $\phi \in X$, that is

$$-\Delta\bar{u} + \mu\left(\log|x| * \bar{u}^2\right)\bar{u} - \left(e^{\bar{u}^2} - 1 - \bar{u}^2\right)\bar{u} = -\lambda_c\bar{u}, \quad x \in \mathbb{R}^2. \tag{3.21}$$

This completes the proof. □

4 Proof of Theorem 1.2

In this section, we consider the existence of a critical point of mountain-pass type for Φ on $\hat{\mathcal{S}}_c = E_{as} \cap \mathcal{S}_c$, and give the proof of Theorem 1.2.

Lemma 4.1 *Let $\mu > 0$ and $c \in (0, c_2)$. For any $u \in \hat{\mathcal{S}}_c$, the following exist:*

(i) *A unique $s_u^+ > 0$ such that s_u^+ is a strict local minimum point for g_u.*
(ii) *A unique $s_u^- > 0$ such that s_u^- is a strict local maximum point for g_u.*

Proof For any $u \in \hat{\mathcal{S}}_c$, let $\tau := 1/\|\nabla u\|_2$ and $\hat{u} := \tau u_\tau$. Then $\|\nabla\hat{u}\|_2^2 = 1$ and $t\hat{u}_t = (t\tau)u_{t\tau}$ for $t > 0$. Therefore, we only prove this lemma for $u \in \hat{\mathcal{S}}_c$ with $\|\nabla u\|_2^2 = 1$.

Fix $u \in \hat{\mathcal{S}}_c$ with $\|\nabla u\|_2^2 = 1$, we have

$$g_u'(t) = \frac{1}{t}\mathcal{P}(tu_t), \quad \forall\, t > 0. \tag{4.1}$$

Let $t_* > 0$ such that

$$t_*^{-4}\int_{\mathbb{R}^2}\left[\left(1 - t_*^2u^2 + t_*^4u^4\right)e^{t_*^2u^2} - 1 - \frac{t_*^4u^4}{2}\right]\mathrm{d}x = 1. \tag{4.2}$$

It follows that

$$t^2 > t^{-2}\int_{\mathbb{R}^2}\left[\left(1 - t^2u^2 + t^4u^4\right)e^{t^2u^2} - 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x, \quad 0 < t < t_* \tag{4.3}$$

and

$$t^2 < t^{-2}\int_{\mathbb{R}^2}\left[\left(1 - t^2u^2 + t^4u^4\right)e^{t^2u^2} - 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x, \quad t_* < t < +\infty. \tag{4.4}$$

By (3.9) and (4.3), one has

$$\begin{aligned} g_u'(t) &= \frac{1}{t}\left\{t^2 - \frac{\mu c^2}{4} - t^{-2}\int_{\mathbb{R}^2}\left[\left(t^2u^2 - 1\right)e^{t^2u^2} + 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x\right\} \\ &\geq \frac{1}{t}\left\{t^2 - \frac{\mu c^2}{4} - \frac{t^{-2}}{2}\int_{\mathbb{R}^2}\left[\left(1 - t^2u^2 + t^4u^4\right)e^{t^2u^2} - 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x\right\} \end{aligned}$$

$$> \frac{1}{2t}\left(t^2 - \frac{\mu c^2}{2}\right), \quad 0 < t < t_*. \tag{4.5}$$

If $t_* < \pi$, then from (2.14) and (4.2), we have

$$\begin{aligned}
1 &= t_*^{-4} \int_{\mathbb{R}^2} \left[\left(1 - t_*^2 u^2 + t_*^4 u^4\right) e^{t_*^2 u^2} - 1 - \frac{t_*^4 u^4}{2}\right] \mathrm{d}x \\
&= \sum_{k=3}^{\infty} \frac{(k-1)^2}{k!} \|u\|_{2k}^{2k} t_*^{2(k-2)} \\
&\le \frac{2c}{\pi} \sum_{k=3}^{\infty} \frac{(k-1)^2 \left[4^{k-1}(k-2)+1\right]}{(k-2)k!} \left(\frac{t_*^2 \sqrt{c}}{\pi}\right)^{k-2} + \frac{1}{2\pi} \sum_{k=3}^{\infty} (k-1)^2 \left(\frac{t_*^2}{\pi}\right)^{k-2} \\
&= \frac{2c}{\pi} \sum_{k=3}^{\infty} \frac{(k-1)^2 \left[4^{k-1}(k-2)+1\right]}{(k-2)k!} \left(\frac{t_*^2 \sqrt{c}}{\pi}\right)^{k-2} + \frac{t_*^2 \left(4\pi^2 - 3\pi t_*^2 + t_*^4\right)}{2\pi \left(\pi - t_*^2\right)^3}.
\end{aligned} \tag{4.6}$$

Combining (3.3) with (4.6), we deduce $t_* \ge \eta(c)$. It follows from (3.3) that $\eta(c)$ is decreasing on $c > 0$. Hence, by (3.2), we have

$$\frac{\mu c^2}{2} < \frac{\mu c_2^2}{2} = \eta^2(c_2) < \eta^2(c) \le t_*^2, \quad \forall\, c \in (0, c_2). \tag{4.7}$$

Hence, (4.7) shows that there exists $\delta > 0$ such that $t^2 - \frac{\mu c^2}{2} > 0$ for any $t \in (t_* - \delta, t_*)$. Hence, by (4.5), we infer that $g_u'(t) > 0$ for any $t \in (t_* - \delta, t_*)$, and thus $g_u(t)$ is increasing in $(t_* - \delta, t_*)$.

Taking into account that the function $g_u(t) \to +\infty$ as $t \to 0+$ and $g_u(t) \to -\infty$ as $t \to +\infty$, we conclude that there exists at least a critical point $s_u^+ < t_*$ which is a local minimum point of g_u and a critical point $s_u^- > t_*$ which is a local maximum point of g_u. Since $s_u^- > t_*$, from (4.4) we derive that

$$\begin{aligned}
(s_u^-)^2 &< (s_u^-)^{-2} \int_{\mathbb{R}^2} \left[\left(1 - (s_u^-)^2 u^2 + (s_u^-)^4 u^4\right) e^{(s_u^-)^2 u^2} - 1 - \frac{(s_u^-)^4 u^4}{2}\right] \mathrm{d}x \\
&= \sum_{k=3}^{\infty} \frac{(k-1)^2}{k!} \|u\|_{2k}^{2k} (s_u^-)^{2(k-1)}.
\end{aligned} \tag{4.8}$$

Moreover, from (3.9), (4.8) and the fact that $g_u'(s_u^-) = 0$, we derive that

$$\begin{aligned}
g_u''(s_u^-) &= \frac{1}{(s_u^-)^2} \left[(s_u^-)^2 - \sum_{k=3}^{\infty} \frac{(k-1)(2k-3)}{k!} \|u\|_{2k}^{2k} (s_u^-)^{2(k-1)} + \frac{\mu c^2}{4}\right] \\
&= \frac{2}{(s_u^-)^2} \left[(s_u^-)^2 - \sum_{k=3}^{\infty} \frac{(k-1)^2}{k!} \|u\|_{2k}^{2k} (s_u^-)^{2(k-1)}\right] < 0.
\end{aligned} \tag{4.9}$$

Therefore s_u^- is a strict maximum point for g_u.

We have to show that s_u^- is unique. By contradiction we assume that there exists $\hat{s}_u^- > 0$, another critical point of g_u which is a local maximum point.

First, we observe that if $0 < \hat{s}_u^- < t_*$, then from $g_u'(\hat{s}_u^-) = 0$ and (4.3) we obtain

$$g_u''(\hat{s}_u^-) = \frac{1}{(\hat{s}_u^-)^2} \left[(\hat{s}_u^-)^2 - \sum_{k=3}^{\infty} \frac{(k-1)(2k-3)}{k!} \|u\|_{2k}^{2k} (\hat{s}_u^-)^{2(k-1)} + \frac{\mu c^2}{4}\right]$$

$$= \frac{2}{(\hat{s}_u^-)^2}\left[(\hat{s}_u^-)^2 - \sum_{k=3}^{\infty}\frac{(k-1)^2}{k!}\|u\|_{2k}^{2k}(\hat{s}_u^-)^{2(k-1)}\right] > 0, \tag{4.10}$$

which is a contradiction. This implies that $\hat{s}_u^- > t_*$, and thus arguing as before we have $g_u''(\hat{s}_u^-) < 0$. We derive the existence of another critical point: $\theta_u \in (\hat{s}_u^-, s_u^-)$ or $\theta_u \in (s_u^-, \hat{s}_u^-)$, which is a local minimum for g_u. Taking into account (4.4), we again deduce $g_u''(\theta_u) < 0$, which is a contradiction. Therefore the point s_u^- is unique.

Now a direct adaptation of the argument used for s_u^- leads us to conclude that s_u^+ is the unique local minimum point for g_u. □

Lemma 4.2 *Let $\mu > 0$. For any $c \in (0, c_1)$, there exists $\kappa_c > 0$ such that*

$$M(c) := \inf_{\gamma\in\Gamma_c}\max_{t\in[0,1]}\Phi(\gamma(t)) \geq \kappa_c > \sup_{\gamma\in\Gamma_c}\max\{\Phi(\gamma(0)), \Phi(\gamma(1))\}, \tag{4.11}$$

where

$$\Gamma_c = \left\{\gamma \in \mathcal{C}([0,1], \hat{\mathcal{S}}_c) : \gamma(0) = u_c, \Phi(\gamma(1)) < m(c) - 1\right\}, \tag{4.12}$$

and u_c is determined by Theorem 1.1.

Proof Set $\kappa_c := \inf_{u\in\partial(\hat{\mathcal{S}}_c\cap A_{\pi/3})}\Phi(u)$. By Theorem 1.1 and Corollary 3.2, $\kappa_c > m(c) = \Phi(u_c)$. Let $\gamma \in \Gamma_c$ be arbitrary. Since $\gamma(0) = u_c$, and $\Phi(\gamma(1)) < m(c) - 1$, necessarily in view of Theorem 1.1, $\gamma(1) \notin \hat{\mathcal{S}}_c \cap A_{\pi/3}$. By continuity of $\gamma(t)$ on $[0, 1]$, there exists a $t_0 \in (0, 1)$ such that $\gamma(t_0) \in \partial(\hat{\mathcal{S}}_c \cap A_{\pi/3})$, and so $\max_{t\in[0,1]}\Phi(\gamma(t)) \geq \kappa_c$. Thus, (4.11) holds. □

To apply Lemma 2.12, we let $E = E_{as}$ and $H = L^2(\mathbb{R}^2)$. Define the norms of E and H by

$$\|u\|_E := \left(\|\nabla u\|^2 + \|u\|_*^2\right)^{1/2}, \quad \|u\|_H^2 := \frac{1}{\sqrt{c}}\left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x\right)^{1/2}, \quad \forall\, u \in E. \tag{4.13}$$

By Lemma 2.6, after identifying H with its dual, we have $E \hookrightarrow H \hookrightarrow E^*$ with continuous injections. Set

$$M := \left\{u \in E : \|u\|_2^2 = \int_{\mathbb{R}^2} u^2 \mathrm{d}x = c\right\}. \tag{4.14}$$

Obviously, Lemma 2.3 shows that $\Phi \in \mathcal{C}^1(E, \mathbb{R})$, and

$$\langle \Phi'(u), u\rangle = \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \mu I_0(u) - \int_{\mathbb{R}^2}\left(e^{u^2} - 1 - u^2\right)u^2 \mathrm{d}x. \tag{4.15}$$

Set $F(u) := \frac{1}{2}\left(e^{u^2} - 1 - u^2 - \frac{u^4}{2}\right)$ and $f(u) := \left(e^{u^2} - 1 - u^2\right)u$. Inspired by [25], let us define a continuous map $\beta : E_{as} \times \mathbb{R} \to E_{as}$ by

$$\beta(v, t)(x) := e^t v(e^t x) \text{ for } v \in E_{as},\ t \in \mathbb{R},\ x \in \mathbb{R}^2, \tag{4.16}$$

and consider the following auxiliary functional:

$$\tilde{\Phi}(v, t) := \Phi(\beta(v, t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 + \frac{\mu}{4}I_0(v) - \frac{\mu c^2 t}{4} - \frac{1}{e^{2t}}\int_{\mathbb{R}^2} F(e^t v)\mathrm{d}x. \tag{4.17}$$

We see that $\tilde{\Phi}'$ is of class $\mathcal{C}^1$, and for any $(w, s) \in E_{as} \times \mathbb{R}$,

$$\left\langle\tilde{\Phi}'(v, t), (w, s)\right\rangle = \left\langle\tilde{\Phi}'(v, t), (w, 0)\right\rangle + \left\langle\tilde{\Phi}'(v, t), (0, s)\right\rangle$$

$$
\begin{aligned}
&= e^{2t}\int_{\mathbb{R}^2}\nabla v\cdot\nabla w\mathrm{d}x+\mu\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|v^2(x)v(y)w(y)\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{1}{e^{2t}}\int_{\mathbb{R}^2}f(e^tv)e^tw\mathrm{d}x+e^{2t}s\|\nabla v\|_2^2-\frac{\mu c^2 s}{4}\\
&\quad+\frac{s}{e^{2t}}\int_{\mathbb{R}^2}\left[2F(e^tv)-f(e^tv)e^tv\right]\mathrm{d}x\\
&=\left\langle\Phi'(\beta(v,t)),\beta(w,t)\right\rangle+s\mathcal{P}(\beta(v,t)). \qquad (4.18)
\end{aligned}
$$

Let

$$
u(x):=\beta(v,t)(x)=e^tv(e^tx),\quad \phi(x):=\beta(w,t)(x)=e^tw(e^tx). \qquad (4.19)
$$

Then

$$
(u,\phi)_H=\frac{1}{c}\int_{\mathbb{R}^2}u(x)\phi(x)\mathrm{d}x=\frac{1}{c}\int_{\mathbb{R}^2}v(x)w(x)\mathrm{d}x=(v,w)_H. \qquad (4.20)
$$

This shows that

$$
\phi\in T_u(\hat{\mathcal{S}}_c)\ \Leftrightarrow\ (w,s)\in\tilde{T}_{(v,t)}(\hat{\mathcal{S}}_c\times\mathbb{R}),\quad\forall\, t,s\in\mathbb{R}. \qquad (4.21)
$$

It is easy to verify that

$$
\log\left(2+e^{-|t|}r\right)\geq e^{-|t|}\log(2+r),\quad\forall\, r>0,\ t\in\mathbb{R}. \qquad (4.22)
$$

It follows from (1.12), (4.18), (4.19), (4.21) and (4.22) that

$$
|\mathcal{P}(u)|=\left|\left\langle\tilde{\Phi}'(v,t),(0,1)\right\rangle\right|\leq\left\|\tilde{\Phi}|'_{\hat{\mathcal{S}}_c\times\mathbb{R}}(v,t)\right\| \qquad (4.23)
$$

and

$$
\begin{aligned}
\left\|\Phi|'_{\hat{\mathcal{S}}_c}(u)\right\| &= \sup_{\phi\in T_u(\hat{\mathcal{S}}_c)}\frac{1}{\|\phi\|_E}\left|\left\langle\Phi'(u),\phi\right\rangle\right|\\
&= \sup_{\phi\in T_u(\hat{\mathcal{S}}_c)}\frac{1}{\sqrt{\|\nabla\phi\|_2^2+\|\phi\|_*^2}}\left|\left\langle\Phi'(\beta(v,t)),\beta(w,t)\right\rangle\right|\\
&= \sup_{\phi\in T_u(\hat{\mathcal{S}}_c)}\frac{1}{\sqrt{\|\nabla\phi\|_2^2+\|\phi\|_*^2}}\left|\left\langle\tilde{\Phi}'(v,t),(w,0)\right\rangle\right|\\
&\leq \sup_{(w,0)\in\tilde{T}_{(v,t)}(\hat{\mathcal{S}}_c\times\mathbb{R})}\frac{e^{|t|}}{\|(w,0)\|_{E\times\mathbb{R}}}\left|\left\langle\tilde{\Phi}'(v,t),(w,0)\right\rangle\right|\\
&\leq e^{|t|}\left\|\tilde{\Phi}|'_{\hat{\mathcal{S}}_c\times\mathbb{R}}(v,t)\right\|. \qquad (4.24)
\end{aligned}
$$

Lemma 4.3 *Let $\mu>0$. Then for any $c\in(0,c_1)$, there exists a sequence $\{u_n\}\subset\hat{\mathcal{S}}_c$ such that*

$$
\Phi(u_n)\to M(c)>m(c),\quad \Phi|'_{\hat{\mathcal{S}}_c}(u_n)\to 0\ \text{ and }\ \mathcal{P}(u_n)\to 0. \qquad (4.25)
$$

Proof Set

$$
\tilde{\Gamma}_c:=\left\{\tilde{\gamma}\in\mathcal{C}([0,1],\hat{\mathcal{S}}_c\times\mathbb{R}):\tilde{\gamma}(0)=(u_c,0),\,\tilde{\Phi}(\tilde{\gamma}(1))<m(c)-1\right\} \qquad (4.26)
$$

and

$$
\tilde{M}(c):=\inf_{\tilde{\gamma}\in\tilde{\Gamma}_c}\max_{t\in[0,1]}\tilde{\Phi}(\tilde{\gamma}(t)). \qquad (4.27)
$$

For any $\tilde{\gamma} \in \tilde{\Gamma}_c$, it is easy to see that $\gamma = \beta \circ \tilde{\gamma} \in \Gamma_c$ defined by (4.12). Let $\kappa_c' := \sup_{\gamma \in \Gamma_c} \max \{\Phi(\gamma(0)), \Phi(\gamma(1))\}$. Then it follows from (4.11) that

$$\begin{aligned} \max_{t\in[0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) = \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \kappa_c > \kappa_c' \geq \max \{\Phi(\gamma(0)), \Phi(\gamma(1))\} \\ = \max \left\{\tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1))\right\}. \end{aligned}$$

It follows that $\tilde{M}(c) \geq M(c)$, and

$$\tilde{M}(c) = \inf_{\tilde{\gamma}\in\tilde{\Gamma}_c} \max_{t\in[0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) \geq \kappa_c > \kappa_c' \geq \sup_{\tilde{\gamma}\in\tilde{\Gamma}_c} \max \left\{\tilde{\Phi}(\tilde{\gamma}(0)), \Phi(\tilde{\gamma}(1))\right\}. \tag{4.28}$$

This shows that (2.29) holds.

On the other hand, for any $\gamma \in \Gamma_c$, let $\tilde{\gamma}(t) := (\gamma(t), 0)$. It is easy to verify that $\tilde{\gamma} \in \tilde{\Gamma}_c$ and $\Phi(\gamma(t)) = \tilde{\Phi}(\tilde{\gamma}(t))$, and so, we trivially have $\tilde{M}(c) \leq M(c)$. Thus $\tilde{M}(c) = M(c)$.

For any $n \in \mathbb{N}$, (4.11) implies that there exists $\gamma_n \in \Gamma_c$ such that

$$\max_{t\in[0,1]} \Phi(\gamma_n(t)) \leq M(c) + \frac{1}{n}. \tag{4.29}$$

Set $\tilde{\gamma}_n(t) := (\gamma_n(t), 0)$. Then apply Lemma 2.12 to $\tilde{\Phi}$, there exists a sequence $\{(v_n, t_n)\} \subset \hat{\mathcal{S}}_c \times \mathbb{R}$ satisfying

(i) $M(c) - \frac{2}{n} \leq \tilde{\Phi}(v_n, t_n) \leq M(c) + \frac{2}{n}$;
(ii) $\min_{t\in[0,1]} \|(v_n, t_n) - (\gamma_n(t), 0)\|_{E\times\mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
(iii) $\left\| \tilde{\Phi}|'_{\hat{\mathcal{S}}_c\times\mathbb{R}}(v_n, t_n) \right\| \leq \frac{8}{\sqrt{n}}$.

Let $u_n = \beta(v_n, t_n)$. It follows from (4.23), (4.24) and (i)-(iii) that (4.25) holds. □

Now we define the following Moser type functions $w_n(x)$ supported in $B_1(0)$

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq 1/n; \\ \frac{\log(1/|x|)}{\sqrt{\log n}}, & 1/n \leq |x| \leq 1; \\ 0, & |x| \geq 1. \end{cases} \tag{4.30}$$

Computing directly, we get that

$$\|\nabla w_n\|_2^2 = \int_{\mathbb{R}^2} |\nabla w_n|^2 \mathrm{d}x = 1, \tag{4.31}$$

$$\begin{aligned} \|w_n\|_2^2 &= \int_{\mathbb{R}^2} |w_n|^2 \mathrm{d}x = \log n \int_0^{1/n} r\mathrm{d}r + \int_{1/n}^1 \frac{\log^2(1/r)}{\log n} r\mathrm{d}r \\ &= \frac{1}{4\log n} - \frac{1}{4n^2 \log n} - \frac{1}{2n^2}, \end{aligned} \tag{4.32}$$

$$\|w_n\|_{8/3}^{8/3} = \int_{\mathbb{R}^2} |w_n|^{8/3} \mathrm{d}x = O\left(\frac{1}{\log^{4/3} n}\right), \quad n \to \infty, \tag{4.33}$$

$$\|w_n\|_*^2 = \int_{\mathbb{R}^2} \log(2 + |x|)|w_n|^2 \mathrm{d}x = O\left(\frac{1}{\log n}\right), \quad n \to \infty \tag{4.34}$$

and

$$|I_0(w_n)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x - y| w_n^2(x) w_n^2(y) \mathrm{d}x\mathrm{d}y \right| \leq O\left(\frac{1}{\log^2 n}\right), \quad n \to \infty. \tag{4.35}$$

Lemma 4.4 *Let $\mu > 0$. Then for any $c \in (0, c_1)$, there holds*

$$M(c) < m(c) + 2\pi. \tag{4.36}$$

Proof Let u_c be determined by Theorem 1.1. By Theorem 1.1 and Lemma 3.4, we have

$$\|u_c\|_2^2 = c, \quad \Phi(u_c) = m(c), \quad u_c(x) \geq 0, \quad \forall\, x \in \mathbb{R}^2 \tag{4.37}$$

and

$$-\lambda_c c = \frac{\mu c^2}{4} + \mu I_0(u_c) - \int_{\mathbb{R}^2} \left(e^{u_c^2} - 1 - u_c^2 - \frac{u_c^4}{2}\right) \mathrm{d}x. \tag{4.38}$$

Since $u_c \in E_{as}$, it follows from (2.4), (2.7), (4.30), (4.32), (4.33) and (4.34) that

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c(x) w_n(x) u_c(y) w_n(y) \mathrm{d}x\mathrm{d}y\right| = O\left(\frac{1}{\log n}\right), \quad n \to \infty, \tag{4.39}$$

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c^2(x) w_n^2(y) \mathrm{d}x\mathrm{d}y\right| = O\left(\frac{1}{\log n}\right), \quad n \to \infty, \tag{4.40}$$

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c(x) w_n(x) w_n^2(y) \mathrm{d}x\mathrm{d}y\right| = O\left(\frac{1}{\log^{3/2} n}\right), \quad n \to \infty, \tag{4.41}$$

$$\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x = O\left(\frac{1}{\sqrt{\log n}}\right), \quad n \to \infty \tag{4.42}$$

and

$$\int_{\mathbb{R}^2} \left(e^{u_c^2} - 1 - u_c^2\right) u_c w_n \mathrm{d}x = O\left(\frac{1}{\sqrt{\log n}}\right), \quad n \to \infty. \tag{4.43}$$

By (1.1), (4.32) and (4.37), one has

$$\int_{\mathbb{R}^2} \nabla u_c \cdot \nabla w_n \mathrm{d}x = \int_{\mathbb{R}^2} \left[-\mu \int_{\mathbb{R}^2} \log|x-y| u_c^2(y)\mathrm{d}y + \left(e^{u_c^2} - 1 - u_c^2\right) - \lambda_c\right] u_c w_n \mathrm{d}x \tag{4.44}$$

and

$$\begin{aligned}\|u_c + t w_n\|_2^2 &= c + t^2\|w_n\|_2^2 + 2t\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x \\ &= c + 2t\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right], \quad n \to \infty.\end{aligned} \tag{4.45}$$

Let $\tau := \|u_c + t w_n\|_2/\sqrt{c}$. Then

$$\tau^2 = 1 + \frac{2t}{c}\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right], \quad n \to \infty \tag{4.46}$$

and for any $p \geq 1$,

$$\tau^{-2p} = 1 - \frac{2pt}{c}\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right], \quad n \to \infty. \tag{4.47}$$

Now, we define

$$W_{n,t}(x) := u_c(\tau x) + t w_n(\tau x). \tag{4.48}$$

Then one has

$$\|\nabla W_{n,t}\|_2^2 = \|\nabla(u_c + t w_n)\|_2^2, \quad \|W_{n,t}\|_2^2 = \tau^{-2}\|u_c + t w_n\|_2^2 = c, \tag{4.49}$$

$$
\begin{aligned}
I_0(W_{n,t}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|[u_c(\tau x)+tw_n(\tau x)]^2[u_c(\tau y)+tw_n(\tau y)]^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\tau^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|[u_c(x)+tw_n(x)]^2[u_c(y)+tw_n(y)]^2\mathrm{d}x\mathrm{d}y - c^2\log\tau
\end{aligned}
\tag{4.50}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(e^{W_{n,t}^2}-1-W_{n,t}^2-\frac{W_{n,t}^4}{2}\right)\mathrm{d}x \\
&\quad = \frac{1}{\tau^2}\int_{\mathbb{R}^2}\left[e^{(u_c+tw_n)^2}-1-(u_c+tw_n)^2-\frac{(u_c+tw_n)^4}{2}\right]\mathrm{d}x.
\end{aligned}
\tag{4.51}
$$

From (4.42) and (4.46), one has

$$
\tau^2 = 1+\frac{2t}{c}\int_{\mathbb{R}^2} u_c w_n \mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right] \le 1+t+t^2, \quad \text{for large } n\in\mathbb{N}. \tag{4.52}
$$

Now, we define $\Psi_n(t)$ by

$$
\Psi_n(t) = \frac{t^2}{2} - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2w_n^2}-1-t^2w_n^2-\frac{1}{2}t^4w_n^4\right)\mathrm{d}x, \quad \forall\, t>0. \tag{4.53}
$$

We claim that

$$
\begin{aligned}
&\sup_{t>0}\left[\Psi_n(t)+t^2\left(O\left(\frac{1}{\log n}\right)\right)+t^4\left(O\left(\frac{1}{\log^2 n}\right)\right)\right] \\
&\quad \le 2\pi - \frac{\pi}{2\log n}\log\frac{\log n}{32\pi}, \quad \text{for large } n\in\mathbb{N}.
\end{aligned}
\tag{4.54}
$$

There are three cases to distinguish. In the sequel, we agree that all inequalities hold for large $n\in\mathbb{N}$ without mentioning.

Case i) $t\in\left[0,\sqrt{2\pi}\right]$. Then by (4.35) and (4.53), we have

$$
\Psi_n(t) = \frac{t^2}{2} - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2w_n^2}-1-t^2w_n^2-\frac{1}{2}t^4w_n^4\right)\mathrm{d}x \le \frac{t^2}{2} \le \frac{3\pi}{2}. \tag{4.55}
$$

It follows that

$$
\begin{aligned}
&\sup_{0<t\le\sqrt{2\pi}}\left[\Psi_n(t)+t^2\left(O\left(\frac{1}{\log n}\right)\right)+t^4\left(O\left(\frac{1}{\log^2 n}\right)\right)\right] \\
&\quad \le 2\pi - \frac{\pi}{2\log n}\log\frac{\log n}{32\pi}, \quad \text{for large } n\in\mathbb{N}.
\end{aligned}
\tag{4.56}
$$

Case ii) $t\in\left[\sqrt{2\pi},\sqrt{6\pi}\right)$. Then it follows from (4.30), (4.35) and (4.52) that

$$
\begin{aligned}
&\frac{1}{\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2w_n^2}-1-t^2w_n^2-\frac{1}{2}t^4w_n^4\right)\mathrm{d}x \\
&\quad \ge \frac{1}{2\tau^2}\int_{B_{1/n}} e^{t^2w_n^2}\mathrm{d}x \ge \frac{1}{16n^2}e^{(2\pi)^{-1}t^2\log n}.
\end{aligned}
\tag{4.57}
$$

Using (4.53) and (4.57), we are led to

$$\begin{aligned}\Psi_n(t) &= \frac{t^2}{2} - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2 w_n^2} - 1 - t^2 w_n^2 - \frac{1}{2}t^4 w_n^4\right)\mathrm{d}x \\ &\le \frac{t^2}{2} - \frac{1}{32n^2}e^{(2\pi)^{-1}t^2\log n} := \varphi_n(t). \end{aligned} \tag{4.58}$$

Choosing $t_n > 0$ be such that $\varphi_n'(t_n) = 0$, then we have

$$1 = \frac{\log n}{32\pi n^2}e^{(2\pi)^{-1}t_n^2\log n}. \tag{4.59}$$

It follows that

$$t_n^2 = 4\pi\left[1 + \frac{\log(32\pi) - \log(\log n)}{2\log n}\right] \tag{4.60}$$

and

$$\varphi_n(t) \le \varphi_n(t_n) = \frac{t_n^2}{2} - \frac{\pi}{\log n}, \quad \forall\, t \ge 0. \tag{4.61}$$

Using (4.60) and (4.61), we are led to

$$\varphi_n(t) \le \frac{t_n^2}{2} - \frac{\pi}{\log n} = 2\pi - \frac{\pi}{\log n}\log\frac{e\log n}{32\pi},$$

which, together with (4.58), yields

$$\Psi_n(t) \le 2\pi - \frac{\pi}{\log n}\log\frac{e\log n}{32\pi}.$$

It follows that

$$\begin{aligned}&\sup_{\sqrt{2\pi}<t\le\sqrt{6\pi}}\left[\Psi_n(t) + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right)\right] \\ &\le 2\pi - \frac{\pi}{2\log n}\log\frac{\log n}{32\pi}, \quad \text{for large } n \in \mathbb{N}. \end{aligned} \tag{4.62}$$

Case iii) $t \in \left(\sqrt{6\pi}, +\infty\right)$. Then it follows from (4.30),(4.35) and (4.52) that

$$\begin{aligned}&\Psi_n(t) + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right) \\ &\le \frac{t^2}{2} - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2\omega_n^2} - 1 - t^2\omega_n^2 - \frac{1}{2}t^4\omega_n^4\right)\mathrm{d}x \\ &\quad + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right) \\ &\le \frac{t^2}{2} - \frac{\pi}{4n^2\tau^2}e^{(2\pi)^{-1}t^2\log n} + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right) \\ &\le \frac{t^2}{2} - \frac{\pi}{4n^2(1+t+t^2)}e^{(2\pi)^{-1}t^2\log n} + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right) \\ &:= \frac{t^2}{2} - \frac{\pi}{4n^2(1+t+t^2)}e^{(2\pi)^{-1}t^2\log n} + a_n t^2 + b_n t^4 \end{aligned} \tag{4.63}$$

$$\leq 3\pi - \frac{\pi}{2n^2(1+\sqrt{6\pi}+6\pi)}e^{3\log n} + 6\pi a_n + 36\pi^2 b_n \leq \frac{3}{2}\pi, \tag{4.64}$$

where we have used the fact that the function

$$\phi_n(t) := \frac{t^2}{2} - \frac{\pi}{4n^2(1+t+t^2)}e^{(2\pi)^{-1}t^2\log n} + a_n t^2 + b_n t^4$$

is decreasing on $t \in \left(\sqrt{6\pi}, +\infty\right)$ for large n. In fact,

$$\phi_n'(t) = (1+2a_n)t + 4b_n t^3 - \frac{\left(1+t+t^2\right)t\log n - (1+2t)\pi}{4n^2\left(1+t+t^2\right)^2}e^{(2\pi)^{-1}t^2\log n}.$$

Assume that $s_n > 0$ such that $\phi_n'(s_n) = 0$ for large n. Then

$$4\left[(1+2a_n)s_n+4b_n s_n^3\right]\left(1+s_n+s_n^2\right)^2 = \frac{\left(1+s_n+s_n^2\right)s_n\log n - (1+2s_n)\pi}{n^2}e^{(2\pi)^{-1}s_n^2\log n},$$

which yields

$$s_n^2 = 4\pi\left\{1 + \frac{\log\left[4\left((1+2a_n)s_n+4b_n s_n^3\right)\left(1+s_n+s_n^2\right)^2\right]}{2\log n} - \frac{\log\left[\left(1+s_n+s_n^2\right)s_n\log n - (1+2s_n)\pi\right]}{2\log n}\right\}. \tag{4.65}$$

This implies that $\lim_{n\to\infty} s_n^2 = 4\pi$. So $\phi_n(t)$ is decreasing on $t \in \left(\sqrt{6\pi}, +\infty\right)$ for large n. From (4.64), one has

$$\sup_{\sqrt{6\pi}\leq t<+\infty}\left[\Psi_n(t) + t^2\left(O\left(\frac{1}{\log n}\right)\right) + t^4\left(O\left(\frac{1}{\log^2 n}\right)\right)\right] \leq 2\pi - \frac{\pi}{2\log n}\log\frac{\log n}{32\pi}, \quad \text{for large } n \in \mathbb{N}. \tag{4.66}$$

Cases i)–iii) show that (4.54) holds. It is easy to verify the following inequality:

$$(1+t)^q \geq 1 + qt^{q-1} + t^q, \quad \forall\, t \geq 0,\ q \geq 2. \tag{4.67}$$

By (4.35), (4.39)-(4.41), we have

$$\begin{aligned}
I_0(u_c + tw_n) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|[u_c(x)+tw_n(x)]^2[u_c(y)+tw_n(y)]^2\mathrm{d}x\mathrm{d}y \\
&= I_0(u_c) + t^4 I_0(w_n) + 4t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y \\
&\quad + 4t^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c(x)w_n(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y \\
&\quad + 2t^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)w_n^2(y)\mathrm{d}x\mathrm{d}y \\
&\quad + 4t^3\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c(x)w_n(x)w_n^2(y)\mathrm{d}x\mathrm{d}y \\
&= I_0(u_c) + 4t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$+t^2\left[O\left(\frac{1}{\log n}\right)\right]+t^3\left[O\left(\frac{1}{\log^{3/2} n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right]. \tag{4.68}$$

From (1.2), (4.31), (4.37)–(4.44), (4.46)–(4.53), (4.54),(4.67) and (4.68), we have

$$\begin{aligned}
&\Phi(W_{n,t})\\
&=\frac{1}{2}\|\nabla W_{n,t}\|_2^2+\frac{\mu}{4}I_0(W_{n,t})-\frac{1}{2}\int_{\mathbb{R}^2}\left(e^{W_{n,t}^2}-1-W_{n,t}^2-\frac{W_{n,t}^4}{2}\right)\mathrm{d}x\\
&=\frac{1}{2}\|\nabla(u_c+tw_n)\|_2^2+\frac{\mu}{4\tau^4}I_0(u_c+tw_n)-\frac{\mu c^2}{4}\log\tau\\
&\quad-\frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left[e^{(u_c+tw_n)^2}-1-(u_c+tw_n)^2-\frac{(u_c+tw_n)^4}{2}\right]\mathrm{d}x\\
&\le\frac{1}{2}\|\nabla u_c\|_2^2+\frac{\mu\tau^{-4}}{4}I_0(u_c)-\frac{\mu c^2}{4}\log\tau-\frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2-\frac{u_c^4}{2}\right)\mathrm{d}x\\
&\quad+\frac{t^2}{2}\|\nabla w_n\|_2^2-\frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2w_n^2}-1-t^2w_n^2-\frac{t^4w_n^4}{2}\right)\mathrm{d}x+t\int_{\mathbb{R}^2}\nabla u_c\cdot\nabla w_n\mathrm{d}x\\
&\quad+\mu\tau^{-4}t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y-\tau^{-2}t\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_cw_n\mathrm{d}x\\
&\quad+t^2\left[O\left(\frac{1}{\log n}\right)\right]+t^3\left[O\left(\frac{1}{\log^{3/2} n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right]\\
&=\Phi(u_c)+\Psi_n(t)-\frac{\mu\left(1-\tau^{-4}\right)}{4}I_0(u_c)-\frac{\mu c^2}{4}\log\tau\\
&\quad+\frac{1-\tau^{-2}}{2}\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2-\frac{u_c^4}{2}\right)\mathrm{d}x\\
&\quad-\mu\left(1-\tau^{-4}\right)t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y\\
&\quad+\left(1-\tau^{-2}\right)t\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_cw_n\mathrm{d}x-\lambda_ct\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x\\
&\quad+t^2\left[O\left(\frac{1}{\log n}\right)\right]+t^3\left[O\left(\frac{1}{\log^{3/2} n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right]\\
&\le m(c)+\Psi_n(t)-\lambda_ct\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x-\frac{\mu c^2}{4}\left[\frac{t}{c}\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\\
&\quad-\mu I_0(u_c)\left[\frac{t}{c}\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\\
&\quad+\left[\frac{t}{c}\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2-\frac{u_c^4}{2}\right)\mathrm{d}x\\
&\quad-\mu\left[\frac{4t}{c}\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y\\
&\qquad+\left[\frac{2t}{c}\int_{\mathbb{R}^2}u_cw_n\mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]t\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_cw_n\mathrm{d}x\\
&\qquad+t^2\left[O\left(\frac{1}{\log n}\right)\right]+t^3\left[O\left(\frac{1}{\log^{3/2} n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\le m(c)+\Psi_n(t)+t^2\left(O\left(\frac{1}{\log n}\right)\right)+t^3\left[O\left(\frac{1}{\log^{3/2} n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right]\\
&\quad -\frac{4\mu t^2}{c}\left(\int_{\mathbb{R}^2}u_c w_n \mathrm{d}x\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)w_n(y)\mathrm{d}x\mathrm{d}y\\
&\quad +\frac{2t^2}{c}\left[\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_c w_n \mathrm{d}x\right]\left(\int_{\mathbb{R}^2}u_c w_n \mathrm{d}x\right)\\
&\le m(c)+\Psi_n(t)+t^2\left[O\left(\frac{1}{\log n}\right)\right]+t^4\left[O\left(\frac{1}{\log^2 n}\right)\right],\quad \forall\, t>0,
\end{aligned}
\tag{4.69}
$$

which, together with (4.54), implies that there exists $\bar{n}\in\mathbb{N}$ such that

$$
\sup_{t>0}\Phi(W_{\bar{n},t})<m(c)+2\pi. \tag{4.70}
$$

It follows from (4.46), (4.48), (4.49), (4.53) and (4.69) that $W_{\bar{n},t}\in\hat{\mathcal{S}}_c$ for all $t>0$, $W_{\bar{n},0}=u_c$ and $\Phi(W_{\bar{n},t})<m(c)-1$ for large $t>0$. Thus, there exists $\bar{t}>0$ such that

$$
\Phi(W_{\bar{n},\bar{t}})<m(c)-1. \tag{4.71}
$$

Let $\gamma_{\bar{n}}(t):=W_{\bar{n},t\bar{t}}$. Then $\gamma_{\bar{n}}\in\Gamma_c$ defined by (4.12). Hence, it follows from (4.11) and (4.70) that (4.36) holds. □

Lemma 4.5 *Let $u_n\rightharpoonup\bar{u}$ in $H^1(\mathbb{R}^2)$ and*

$$
\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n^2\mathrm{d}x\le K_0 \tag{4.72}
$$

for some constant $K_0>0$. Then there hold.

(i) *For any $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$,*

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n\phi\mathrm{d}x=\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}\phi\mathrm{d}x. \tag{4.73}
$$

(ii) *Suppose $u_n\to\bar{u}$ in $L^q(\mathbb{R}^2)$ for some $q\ge 2$. Then*

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\mathrm{d}x=\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x. \tag{4.74}
$$

Proof (i) is a direct consequence of [22, Lemma 2.1]. (ii) can be proved by a similar fashion as [17, Assertion 2]. □

Proof of Theorem 1.2 In view of Lemmas 4.3 and 4.4, we can deduce that for any $c\in(0,c_0)$, there exists a sequence $\{u_n\}\subset\hat{\mathcal{S}}_c$ such that

$$
\Phi(u_n)\to M(c)\in(0,2\pi+m(c)),\quad \Phi|'_{\hat{\mathcal{S}}_c}(u_n)\to 0\ \text{ and }\ \mathcal{P}(u_n)\to 0. \tag{4.75}
$$

From (1.2), (1.16) and (4.75), one has

$$
\frac{1}{2}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_0(u_n)-\frac{1}{2}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\mathrm{d}x=M(c)+o(1) \tag{4.76}
$$

and

$$
\|\nabla u_n\|_2^2-\frac{\mu c^2}{4}-\int_{\mathbb{R}^2}\left[\left(u_n^2-1\right)e^{u_n^2}+1-\frac{u_n^4}{2}\right]\mathrm{d}x=o(1). \tag{4.77}
$$

By (2.5), (2.9), (4.76) and (4.77), one deduces

$$\begin{aligned} M(c)+o(1) &= \frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_0(u_n)+\frac{\mu c^2}{16}+\frac{1}{4}\int_{\mathbb{R}^2}\left[\left(u_n^2-3\right)e^{u_n^2}+3+2u_n^2+\frac{u_n^4}{2}\right]\mathrm{d}x \\ &\geq \frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-\frac{\mu}{4}I_2(u_n)+\frac{1}{4}\sum_{k=4}^{\infty}\frac{k-3}{k!}\int_{\mathbb{R}^2}u_n^{2k}\mathrm{d}x \\ &\geq \frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-C_1c^{3/2}\sqrt{\|\nabla u_n\|_2}, \end{aligned} \tag{4.78}$$

which implies $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$ are bounded. Hence, we then deduce by Corollary 2.5 that $\{\|u_n\|_*\}$ is bounded, so the sequence $\{u_n\}$ is bounded in E_{as}. We may thus assume, passing to a subsequence again if necessary, that $u_n \rightharpoonup \bar{u}$ in E_{as}, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \in [2,\infty)$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. Furthermore, we have

$$u_n, u \in \hat{\mathcal{S}}_c,\quad u_n \rightharpoonup \bar{u}\ \text{ in } E_{as},\quad u_n \to \bar{u}\ \text{ in } L^s(\mathbb{R}^2) \text{ for } s\geq 2,\quad u_n \to \bar{u} \text{ a.e. in } \mathbb{R}^2. \tag{4.79}$$

From (2.5), (4.76), (4.77) and the boundedness of $\{\|u_n\|_{E_{as}}\}$, one deduces

$$\begin{aligned} \int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n^2\mathrm{d}x &\leq 3\int_{\mathbb{R}^2}\left[\left(e^{u_n^2}-1-u_n^2\right)u_n^2-2\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\right]\mathrm{d}x \\ &= 6M(c)-\frac{3\mu}{2}I_0(u_n)-\frac{3\mu c^2}{4}+o(1)\leq C_2 \end{aligned} \tag{4.80}$$

and

$$I_1(u_n)\leq C_3,\quad \int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\mathrm{d}x\leq C_4, \tag{4.81}$$

By Lemma 2.9, one has

$$\Phi'(u_n)+\lambda_n u_n \to 0, \tag{4.82}$$

where

$$-\lambda_n=\frac{1}{\|u_n\|_2^2}\langle\Phi'(u_n),u_n\rangle=\frac{1}{c}\left[\|\nabla u_n\|_2^2+\mu I_0(u_n)-\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n^2\mathrm{d}x\right]. \tag{4.83}$$

Since $\{\|u_n\|_{E_{as}}\}$ is bounded, it follows from (4.80), (4.81) and (4.83) that $\{|\lambda_n|\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that $\lambda_n \to \lambda_c$. Taking into account that $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$, it follows from (ii) of Lemma 4.5 that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\mathrm{d}x=\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x. \tag{4.84}$$

To prove that $\bar{u}$ is a solution to (1.1), it suffices to show that $\Phi'(\bar{u})+\lambda_c\bar{u}=0$. For this, we prove below five assertions in turn.

Assertion 1.

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n u\mathrm{d}x=\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}^2\mathrm{d}x. \tag{4.85}$$

The strategy of the proof is come from [17, Assertion 3]. Noting that $\bar{u}\in E_{as}$, for any given $\varepsilon>0$, we can choose $\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^2)\subset E_{as}$ such that $\|\phi_\varepsilon-\bar{u}\|_{E_{as}}<\varepsilon$. Hence, from (2.7) and the fact that $\{\|u_n\|_{E_{as}}^2\}=\{\|\nabla u_n\|_2^2+\|u_n\|_*^2\}$ is bounded, we have

$$\left|A_1(u_n^2, u_n(\phi_\varepsilon-\bar{u}))\right|$$

$$\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [log(2+|x|)+log(2+|y|)]\, u_n^2(x)|u_n(y)||\phi_\varepsilon(y)-\bar{u}(y)|\mathrm{d}x\mathrm{d}y$$
$$\leq \|u_n\|_*^2\|u_n\|_2\|\phi_\varepsilon-\bar{u}\|_2+\|u_n\|_2^2\|u_n\|_*\|\phi_\varepsilon-\bar{u}\|_* < C_5\varepsilon. \tag{4.86}$$

By (2.4), we also have

$$\left|A_2(u_n^2, u_n(\phi_\varepsilon-\bar{u}))\right| < C_6\varepsilon. \tag{4.87}$$

Combining (1.2) with (4.82), we obtain

$$\begin{aligned} o(1) &= \langle \Phi'(u_n)+\lambda_n u_n, \phi_\varepsilon-\bar{u}\rangle \\ &= \int_{\mathbb{R}^2} [\nabla u_n\cdot\nabla(\phi_\varepsilon-\bar{u})+\lambda_n u_n(\phi_\varepsilon-\bar{u})]\,\mathrm{d}x \\ &\quad +\mu A_1(u_n^2, u_n(\phi_\varepsilon-\bar{u}))-\mu A_2(u_n^2, u_n(\phi_\varepsilon-\bar{u})) \\ &\quad -\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x. \end{aligned} \tag{4.88}$$

From (4.86), (4.87) and (4.88), one has

$$\begin{aligned} \left|\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x\right| &\leq \left|\int_{\mathbb{R}^2}[\nabla u_n\cdot\nabla(\phi_\varepsilon-\bar{u})+\lambda_n u_n(\phi_\varepsilon-\bar{u})]\,\mathrm{d}x\right| \\ &\quad +\mu\left|A_1(u_n^2, u_n(\phi_\varepsilon-\bar{u}))\right| \\ &\quad +\mu\left|A_2(u_n^2, u_n(\phi_\varepsilon-\bar{u}))\right|+o(1) \\ &\leq \|u_n\|_{E_{as}}\|\phi_\varepsilon-\bar{u}\|_{E_{as}}+C_7\varepsilon+o(1) \\ &\leq C_8\varepsilon+o(1). \end{aligned} \tag{4.89}$$

On the other hand, by Lemma 2.3 i), we have

$$\left|\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}(\phi_\varepsilon-\bar{u})\mathrm{d}x\right| \leq \left[\int_{\mathbb{R}^2}\left(e^{2\bar{u}^2}-1\right)\mathrm{d}x\right]^{\frac{1}{2}}\|\bar{u}\|_4\|\phi_\varepsilon-\bar{u}\|_4 \leq C_9\varepsilon. \tag{4.90}$$

Since $\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, then by Lemma 4.5 i), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n\phi_\varepsilon\mathrm{d}x = \int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}\phi_\varepsilon\mathrm{d}x. \tag{4.91}$$

From (4.89), (4.90) and (4.91), one has

$$\begin{aligned} &\left|\int_{\mathbb{R}^2}\left[\left(e^{u_n^2}-1-u_n^2\right)u_n-\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}\right]\bar{u}\mathrm{d}x\right| \\ &\leq \left|\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x\right|+\left|\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}(\phi_\varepsilon-\bar{u})\mathrm{d}x\right| \\ &\quad +\left|\int_{\mathbb{R}^2}\left[\left(e^{u_n^2}-1-u_n^2\right)u_n-\left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}\right]\phi_\varepsilon\mathrm{d}x\right| \\ &\leq (C_8+C_9)\varepsilon+o(1). \end{aligned} \tag{4.92}$$

Due to the arbitrariness of $\varepsilon > 0$, we deduce Assertion 1 from (4.92).

Assertion 2. $\mathcal{P}(\bar{u}) \geq 0$.

By (1.2), (2.5), (4.82), (4.85) and Lemma 2.7, we have

$$0 = \lim_{n\to\infty}\langle \Phi'(u_n)+\lambda_n u_n, \bar{u}\rangle$$

$$= \|\nabla \bar{u}\|_2^2 + \lambda_c \|\bar{u}\|_2^2 + \mu \lim_{n\to\infty} A_0(u_n^2, u_n\bar{u}) - \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2\right) \bar{u}^2 \mathrm{d}x$$
$$= \|\nabla \bar{u}\|_2^2 + \lambda_c \|\bar{u}\|_2^2 + \mu \lim_{n\to\infty} A_0(u_n^2, \bar{u}^2) - \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2\right) \bar{u}^2 \mathrm{d}x. \tag{4.93}$$

By (2.1) and Lemma 2.7, one can deduce

$$\lim_{n\to\infty} A_1(u_n^2, u_n^2) = \lim_{n\to\infty} \left[A_1(u_n^2, (u_n - \bar{u})^2) + 2A_1(u_n^2, u_n\bar{u}) - A_1(u_n^2, \bar{u}^2)\right]$$
$$\geq \lim_{n\to\infty} A_1(u_n^2, \bar{u}^2). \tag{4.94}$$

It follows from (2.5), (4.77), (4.83), (4.84), (4.93) and (4.94) that

$$0 = \lim_{n\to\infty} \left\{\|\nabla u_n\|_2^2 - \frac{\mu c^2}{4} - \int_{\mathbb{R}^2} \left[\left(u_n^2 - 1\right) e^{u_n^2} + 1 - \frac{u_n^4}{2}\right] \mathrm{d}x\right\}$$
$$= \lim_{n\to\infty} \left[\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^2} \left(e^{u_n^2} - 1 - u_n^2\right) u_n^2 \mathrm{d}x\right] - \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right) \mathrm{d}x$$
$$= \lim_{n\to\infty} \left[-\lambda_n \|u_n\|_2^2 - \mu I_0(u_n)\right] - \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right) \mathrm{d}x$$
$$= -\lambda_c \|\bar{u}\|_2^2 - \mu \lim_{n\to\infty} I_0(u_n) - \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right) \mathrm{d}x$$
$$\leq -\lambda_c \|\bar{u}\|_2^2 - \mu \lim_{n\to\infty} A_0(u_n^2, \bar{u}^2) - \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right) \mathrm{d}x$$
$$= \|\nabla \bar{u}\|_2^2 - \frac{\mu c^2}{4} - \int_{\mathbb{R}^2} \left[\left(\bar{u}^2 - 1\right) e^{\bar{u}^2} + 1 - \frac{\bar{u}^4}{2}\right] \mathrm{d}x$$
$$= \mathcal{P}(\bar{u}). \tag{4.95}$$

Assertion 3. $\Phi(\bar{u}) \geq m(c)$.

Since $\mathcal{P}(\bar{u}) \geq 0$, then $g'_{\bar{u}}(1) \geq 0$, where $g_{\bar{u}}(t) = \Phi(t\bar{u}_t)$. By Lemma 4.1, there exists unique $0 < s_{\bar{u}}^+ \leq 1$ and $1 \leq s_{\bar{u}}^- < +\infty$ such that

$$g_{\bar{u}}(s_{\bar{u}}^+) < g_{\bar{u}}(t) < g_{\bar{u}}(s_{\bar{u}}^-) < +\infty, \quad \forall\, t \in (s_{\bar{u}}^+, s_{\bar{u}}^-).$$

That is

$$\Phi\left(s_{\bar{u}}^+ u_{s_{\bar{u}}^+}\right) < \Phi(t\bar{u}_t) < \Phi\left(s_{\bar{u}}^- \bar{u}_{s_{\bar{u}}^-}\right) < +\infty, \quad \forall\, t \in (s_{\bar{u}}^+, s_{\bar{u}}^-). \tag{4.96}$$

Set $\bar{v} := \tau \bar{u}_\tau$ with $\tau = \frac{\sqrt{\pi}}{\sqrt{3}\|\nabla \bar{u}\|_2}$. Then $\bar{v} \in \hat{\mathcal{S}}_c \cap A_{\pi/3}$. By Lemma 3.3, there exists $t_{\bar{v}} \in (0, 1)$ such that $\mathcal{P}(t_{\bar{v}} \bar{v}_{t_{\bar{v}}}) = 0$. Noting that $t_{\bar{v}} \bar{v}_{t_{\bar{v}}} = (t_{\bar{v}}\tau)\bar{u}_{t_{\bar{v}}\tau}$. Therefore, $s_{\bar{u}}^+ = t_{\bar{v}}\tau$, which, together with (4.96), implies that

$$\Phi(\bar{u}) \geq \Phi\left(s_{\bar{u}}^+ u_{s_{\bar{u}}^+}\right) = \Phi\left(t_{\bar{v}} \bar{v}_{t_{\bar{v}}}\right) \geq m(c).$$

Assertion 4. $\int_{\mathbb{R}^2} \left(e^{u_n^2} - 1 - u_n^2\right) u_n(u_n - \bar{u}) \mathrm{d}x = o(1)$.

Set $v_n := u_n - \bar{u}$. By (2.4), (2.5), (3.15), (3.16), (4.76) and (4.84), we have

$$M(c) + o(1) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_0(u_n) - \frac{1}{2}\int_{\mathbb{R}^2} \left(e^{u_n^2} - 1 - u_n^2 - \frac{u_n^4}{2}\right) \mathrm{d}x$$
$$= \frac{1}{2}\left(\|\nabla \bar{u}\|_2^2 + \|\nabla v_n\|_2^2\right) + \frac{\mu}{4}\left[I_0(\bar{u}) + I_0(v_n) + 2A_0(\bar{u}^2, v_n^2)\right]$$

$$
\begin{aligned}
& -\frac{1}{2}\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x+o(1)\\
&=\Phi(\bar{u})+\frac{1}{2}\|\nabla v_n\|_2^2+\frac{\mu}{4}I_0(v_n)+\frac{\mu}{2}A_0(\bar{u}^2,v_n^2)+o(1)\\
&\geq \frac{1}{2}\|\nabla(u_n-\bar{u})\|_2^2+m(c)+\frac{\mu}{2}A_1(\bar{u}^2,v_n^2)+o(1).
\end{aligned}
\tag{4.97}
$$

Since $0<M(c)<m(c)+2\pi$ for any $c\in(0,c_0)$, it follows from (4.97) that there exists $\bar{\varepsilon}>0$ such that

$$
\|\nabla(u_n-\bar{u})\|_2^2\leq 4\pi(1-3\bar{\varepsilon}),\quad \text{for large } n\in\mathbb{N}. \tag{4.98}
$$

Choose $q\in(1,2)$ such that $q^2(1-3\bar{\varepsilon})\leq(1-\bar{\varepsilon})$. Then by (4.98), the Young's inequality and Lemma 2.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)^q\mathrm{d}x\leq\int_{\mathbb{R}^2}\left(e^{qu_n^2}-1\right)\mathrm{d}x\\
&\leq\int_{\mathbb{R}^2}\left[e^{(1+\bar{\varepsilon}^{-1})q\bar{u}^2}e^{(1+\bar{\varepsilon})q(u_n-\bar{u})^2}-1\right]\mathrm{d}x\\
&\leq\frac{(q-1)}{q}\int_{\mathbb{R}^2}\left[e^{(1+\bar{\varepsilon}^{-1})q^2(q-1)^{-1}\bar{u}^2}-1\right]\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^2}\left[e^{(1+\bar{\varepsilon})q^2(u_n-\bar{u})^2}-1\right]\mathrm{d}x\\
&\leq C_{10}.
\end{aligned}
\tag{4.99}
$$

Noting that $q/(q-1)>1$, by (4.79), (4.99) and the Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(u_n-\bar{u})\mathrm{d}x\\
&\leq\left[\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)^q\mathrm{d}x\right]^{1/q}\|u_n\|_{2q/(q-1)}\|u_n-\bar{u}\|_{2q/(q-1)}=o(1).
\end{aligned}
\tag{4.100}
$$

Hence, Assertion 4 follows directly from (4.100).

Assertion 5. $u_n\to\bar{u}$ in E_{as}.

By (1.2), (2.1)–(2.3), (2.5), (4.79), (4.82), Lemma 2.7 and Assertion 4, we have

$$
\begin{aligned}
o(1)&=\langle\Phi'(u_n)+\lambda_n u_n,u_n-\bar{u}\rangle\\
&=\|\nabla u_n\|_2^2-\|\nabla\bar{u}\|_2^2+\mu A_1\left(u_n^2,(u_n-\bar{u})^2\right)+\mu A_1\left(u_n^2,(u_n-u)\bar{u}\right)\\
&\quad-\mu A_2\left(u_n^2,u_n(u_n-\bar{u})\right)-\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(u_n-\bar{u})\mathrm{d}x+o(1)\\
&=\|\nabla(u_n-\bar{u})\|_2^2+A_1\left(u_n^2,(u_n-\bar{u})^2\right)+o(1),
\end{aligned}
\tag{4.101}
$$

which, together with Lemma 2.4, implies that $u_n\to\bar{u}$ in E_{as}. □

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