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"The Sierpinski gasket minus its bottom line" as a tree of Sierpinski gaskets

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Abstract
The Sierpinski gasket K has three line segments constituting a regular triangle as its border. This paper studies what will happen if one of them, which is called the bottom line and is denoted by I, is removed from K. At a glance, "the Sierpinski gasket minus the bottom line" $K \backslash I$ has a structure of a tree of Sierpinski gaskets. This observation leads us to the results showing that the boundary of $K \backslash I$ is not the line segment I but a Cantor set from viewpoints of geometry and analysis. As a by-product, we have an explicit expression of the jump kernel of the trace of the Brownian motion of K on the bottom line I.

Keywords Sierpinski gasket · Shortest path metric · Trace · Jump kernel

Mathematics Subject Classification Primary 31E05 · 31C25; Secondary 28A80 · 60J45

1 Introduction

This paper concerns a tree structure of "the Sierpinski gasket minus its bottom line" shown in the left-hand side of Fig. 2 and its consequences from the viewpoints of both geometry and analysis.

The Sierpinski gasket is defined as the unique non-empty compact set satisfying

$$K = F_0(K) \cup F_1(K) \cup F_2(K),$$

where $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i$$

for $x \in \mathbb{R}^2$ with $p_0 = (0, 0)$, $p_1 = (1, 0)$ and $p_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The bottom line I of the Sierpinski gasket K is $I = \overline{p_0 p_1} = [0, 1] \times \{0\}$, which is naturally identified with the unit interval $[0, 1]$. See Fig. 1.

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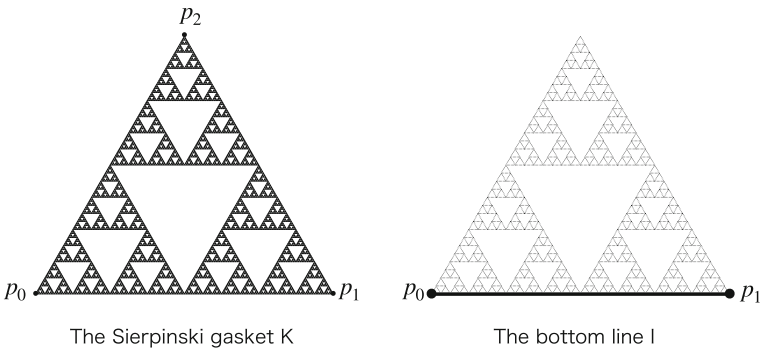

Fig. 1 The Sierpinski gasket and its bottom line

Once the bottom line I is removed from the Sierpinski gasket K, the resulting set has infinitely many cuts at every dyadic rational point in I and one can observe a tree structure illustrated in the left-hand side of Fig. 2, where the horizontal scale is modified to visualize the cuts. In other words, $K\backslash I$ has infinitely many "loose ends" towards I. More precisely, for example, let p_{ij} be the midpoint of p_i and p_j. Originally, the line segments $\overline{p_{20}p_{01}}$ and $\overline{p_{21}p_{01}}$ have the same end p_{01} but they will not meet without $p_{01} \in I$. See the right-hand side of Fig. 2. The same phenomena happen at every dyadic rationals.

Geometrically, the tree structure of $K\backslash I$ becomes clearer by introducing the shortest path metric $\widetilde{D}$ on $K\backslash I$ defined as follows: for $x, y \in K\backslash I$,

$$\widetilde{D}(x, y) = \inf\{L(\gamma)|\gamma \text{ is a rectifiable curve between } x \text{ and } y \text{ in } K\backslash I\},$$

where $L(\gamma)$ is the length of a rectifiable curve γ with respect to the Euclidean metric d_*. Let $\gamma_i : [0, \frac{1}{2}] \to \overline{p_{2i}p_{01}}$ be a curve starting from p_{2i} and converging to p_{01} as $t \to \frac{1}{2}$ for $i = 0, 1$. See Fig. 2 for a graphic representation of γ_1 and γ_2. Then $\lim_{t\to\frac{1}{2}} \widetilde{D}(\gamma_0(t), \gamma_1(t)) = \frac{3}{2}$, while $\lim_{t\to\frac{1}{2}} d_*(\gamma_0(t), \gamma_1(t)) = 0$. This shows that the geometry of $K\backslash I$ under $\widetilde{D}$ and that under d_* are essentially different because the shortest path metric $\widetilde{D}$ captures the tree structure of $K\backslash I$ but the Euclidean metric does not. Indeed, Theorem 2.15 shows that the "boundary" of $K\backslash I$ under $\widetilde{D}$ is not I but a Cantor set $\Sigma_T = \{0, 1\}^{\mathbb{N}}$, where T is an infinite binary tree illustrated in Fig. 5. This corresponds to the well-known fact that the hyperbolic boundary of T is the Cantor set Σ_T.

Analytically, the tree structure is reflected in the resistance metric $\widetilde{R}$ associated with a resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $K\backslash I$ defined in Sect. 4. In fact, it will be shown in Sect. 5 that the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a suitable extension of the standard resistance form $(\mathcal{E}, \mathcal{F})$ on K, which corresponds to the Brownian motion on K.

Analysis on the Sierpinski gasket was initiated by Goldstein [3], Kusuoka [13], and Barlow-Perkins [1]. They have constructed and studied the Brownian motion of the Sierpinski gasket. Later the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$, which is now called the standard resistance form, was constructed in [8].

After the removal of I, the paths of the Brownian motion exhibit a similar nature as the paths γ_0 and γ_1 above. Namely, consider two paths approaching to $p_{01} \in I$, one from inside $F_0(K)$ and the other from inside $F_1(K)$. They will not meet after the removal of I. The extended resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $K\backslash I$ reflects such phenomena of the limits of paths. In fact, Theorem 4.5 shows that the resistance metric $\widetilde{R}$ is biLipschitz equivalent to a power of

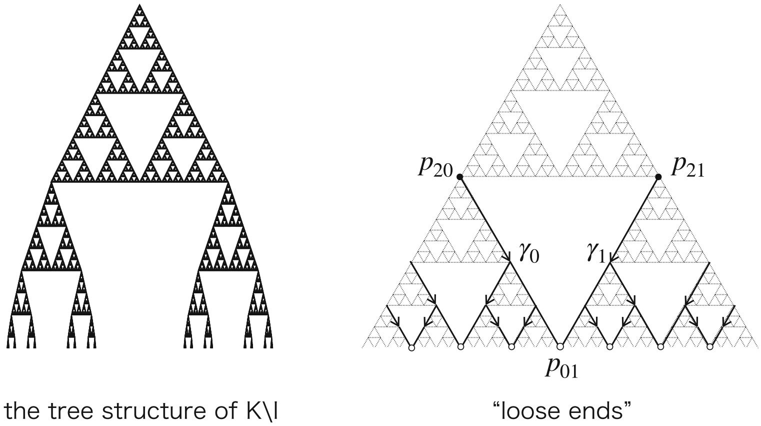

Fig. 2 Tree structure of $K\backslash I$ and loose ends

the shortest path metric $\widetilde{D}$, i.e. there exist $c_1, c_2 > 0$ such that

$$c_1\widetilde{D}(x, y)^\alpha \le \widetilde{R}(x, y) \le c_2\widetilde{D}(x, y)^\alpha \tag{1.1}$$

for any $x, y \in K\backslash I$, where $\alpha = \frac{\log 5 - \log 3}{\log 2}$. Consequently, the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is naturally regarded as a resistance form on $(K\backslash I) \cup \Sigma_T$.

As a by-product of the above results, we will show an exact expression of the jump kernel J_* of the trace $(\mathcal{E}|_I, \mathcal{F}|_I)$ of $(\mathcal{E}, \mathcal{F})$ on the bottom line I, which is defined as

$$\mathcal{F}_I = \{f|_I | f \in \mathcal{F}\} \quad \text{and} \quad \mathcal{E}|_I(\psi, \psi) = \mathcal{E}(h(\psi), h(\psi))$$

for $\psi \in \mathcal{F}_I$, where $h(\psi) \in \mathcal{F}$ is the harmonic function on K with the boundary value ψ on I. See Appendix 1 for the exact definitions. The map $h : \mathcal{F}|_I \to \mathcal{F}$ gives the solution of the Dirichlet problem of the Poisson equation on K with the boundary I, which is

$$\Delta f = 0 \quad \text{on } K\backslash I,$$
$$f|_I = \psi.$$

From the probabilistic point of view, the trace $(\mathcal{E}|_I, \mathcal{F}|_I)$ corresponds to the jump process on I that only sees the hits of the Brownian motion on I, i.e. let $\{X_t\}_{t\ge 0}$ be the Brownian motion. Define $\{t_i\}_{i\ge 0}$ inductively as $t_0 = 0$ and $t_{n+1} = \inf\{t | t > t_n, X_t \in I\}$. Roughly speaking, the trace on I is the process given by $\{Y_t\}_{t\ge 0}$ defined as $Y_t = X_{t_n}$ for any $t \in [t_n, t_{n+1})$.

The first study on the trace $(\mathcal{E}|_I, \mathcal{F}|_I)$ was due to A. Jonsson who identified $\mathcal{F}|_I$ with a Besov space $B^{2,2}_\beta(I)$ where $\beta = \frac{1}{2}(\alpha + 1)$ in [6]. Also, R. Stricharz obtained an exact expression of the harmonic map h in [16]. Moreover, one can find detailed study of boundary values problems of harmonic functions on certain domains of the Sierpinski gasket in [15] and [5].

In this paper, we obtain an expression of the jump kernel $J_*(x, y)$ of $(\mathcal{E}_I, \mathcal{F}_I)$ as follows:

Theorem 1.1 [Corollary 6.2] *For $x, y \in I$, if the binary expressions of x and y are $0.i_1 i_2 \ldots$ and $0.j_1 j_2 \ldots$ respectively, where $i_1 i_2 \ldots$ and $j_1 j_2 \ldots$ are infinite sequences of 0 and 1, define*

$$n_*(x, y) = \min\{n | n \ge 1, i_n \ne j_n\} - 1$$

and

$$J_*(x,y) = \frac{35}{16}\left(\frac{14}{17}\left(\frac{20}{3}\right)^{n_*(x,y)} + \frac{3}{17}\right).$$

Then

$$\mathcal{F}|_I = \left\{ f \,\middle|\, f \in C(I, d_*), \int_{I\times I} J_*(x,y)(f(x)-f(y))^2 dxdy < \infty \right\},$$

where d_* is the Euclidean metric on I and $C(I, d_*)$ is the collection of real-valued continuous functions on I, and

$$\mathcal{E}|_I(f,f) = \int_{I\times I} J_*(x,y)(f(x)-f(y))^2 dxdy$$

for any $f \in \mathcal{F}|_I$.

Moreover, using this exact expression, we will obtain an upper and a near-diagonal lower estimates of the transition density of the associated jump process in Corollary 6.2.

The exact expression of J_* above is made possible by three ingredients. First, we will show that the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $(K\backslash I) \cup \Sigma_T$ can be reduced to that on $T \cup \Sigma_T$ associated with a random walk on T. Second, applying the results in [11], we will obtain an exact expression of the jump kernel of the trace of the random walk on its "boundary" Σ_T. Third, using the fact that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is an extension of $(\mathcal{E}, \mathcal{F})$, we identify the jump kernel J_* with what is obtained in the second step.

The organization of this paper is as follows. In Sect. 2, we give the exact definition and the fundamental properties of the Sierpinski gasket. Also, later in Sect. 2, we identify the "boundary" of $K\backslash I$ with respect to the shortest path metric $\widetilde{D}$ with the Cantor set Σ_T. In Sect. 3, we introduce the definition and the basic properties of the standard resistance form $(\mathcal{E}, \mathcal{F})$ on the Sierpinski gasket K. In Sect. 4, we introduce the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $K\backslash I$ and show (1.1). In Sect. 5, we characterize the standard resistance form $(\mathcal{E}, \mathcal{F})$ on the Sierpinski gasket by means of the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $K\backslash I$. As a consequence, $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is shown to be an extension of $(\mathcal{E}, \mathcal{F})$. Finally in Sect. 6, we show an explicit expression of the jump kernel of the trace $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T})$. Then through the results in Sect. 5, we show results on the trace $(\mathcal{E}|_I, \mathcal{F}|_I)$ including Theorem 1.1. Finally, in Appendix A, we review the definitions and the fundamental facts about resistance forms, their traces, and weighted graphs.

Remark In this paper, we often define a quadratic form $\mathcal{Q}$, which would be a resistance form or a Dirichlet form, on a vector space V only on the diagonal values, i.e. $\mathcal{Q}(f, f)$ for $f \in V$. As a quadratic form, $\mathcal{Q}(f, g)$ is always given by the following polarizing identity

$$\mathcal{Q}(f,g) = \frac{1}{4}(\mathcal{Q}(f+g, f+g) - \mathcal{Q}(f-g, f-g)).$$

2 Geometry of the Sierpinski gasket

In this section, we study the geometries of the Sierpinski gasket and "the Sierpinski gasket minus the bottom line". As mentioned in the introduction, they are the same under the Euclidean metric but become quite different under the shortest path metrics.

First, we give an explicit definition of the Sierpinski gasket. The points p_0, p_1, and p_2 and the maps F_0, F_1, and F_2 are those given in the introduction. By [9, Theorem 1.1.4], we have the following theorem.

Theorem 2.1 *There exists a unique non-empty compact set satisfying*

$$K = F_0(K) \cup F_1(K) \cup F_2(K). \tag{2.1}$$

The non-empty compact set K is called the Sierpinski gasket. Let d_* be the restriction of the Euclidean metric on K. Then the Hausdorff dimension of (K, d_*) is $\frac{\log 3}{\log 2}$.

Other than the (restriction of) the Euclidean metric d_*, we often use the shortest path metric D on the Sierpinski gasket.

Definition 2.2 Define the shortest path metric $D(\cdot, \cdot)$ on K as

$$D(x, y) = \inf\{L(\gamma) | \gamma \text{ is a rectifiable curve in } K \text{ between } x \text{ and } y\}$$

for $x, y \in K$, where $L(\gamma)$ is the length of a rectifiable curve. A rectifiable curve between x and y attaining the above infimum is called a shortest path between x and y.

It is easy to see that the Euclidean metric and the shortest path metric are biLipschitz equivalent.

Proposition 2.3 *There exists a constant $c > 0$ such that*

$$d_*(x, y) \leq D(x, y) \leq c d_*(x, y)$$

for any $x, y \in K$. Moreover, a shortest path between x and y exists for any $x, y \in K$.

The followings are standard notations regarding word and shift spaces.

Definition 2.4 (1) Let $S = \{0, 1, 2\}$. For any $m \geq 0$, define

$$W_m = S^m = \{w_1 \dots w_m | w_1, \dots, w_m \in S\},$$

where $W_0 = \{\phi\}$, and

$$W_* = \bigcup_{m \geq 0} W_m$$

For $i \in S$ and $n \geq 1$, set $(i)_n = \underbrace{i \dots i}_{n\text{-times}} \in W_n$.

(2) Define

$$\Sigma(S) = S^{\mathbb{N}} = \{\omega_1 \omega_2 \dots | \omega_i \in S \text{ for any } i \in \mathbb{N}\}.$$

For simplicity, we use Σ in place of $\Sigma(S)$. For $i \in S$, set $(i)_\infty = ii \dots \in \Sigma$, which is also denoted by $\bar{i}$ in Fig. 3. For $\omega = \omega_1 \omega_2 \dots \in \Sigma$ and $i \in S$, define $\sigma(\omega)$ and $\sigma_i(\omega)$ by

$$\sigma(\omega) = \omega_2 \omega_3 \dots \quad \text{and} \quad \sigma_i(\omega) = i\omega.$$

The map σ is called the shift map.

(3) For $w \in W_*$ and $v \in W_* \cup \Sigma(S)$, the concatenation of w and v is denoted by wv. For $w \in W_* \cup \Sigma$,

$$|w| = \begin{cases} \text{the unique } m \text{ satisfying } w \in W_m & \text{if } w \in W_*, \\ \infty & \text{if } w \in \Sigma(S). \end{cases}$$

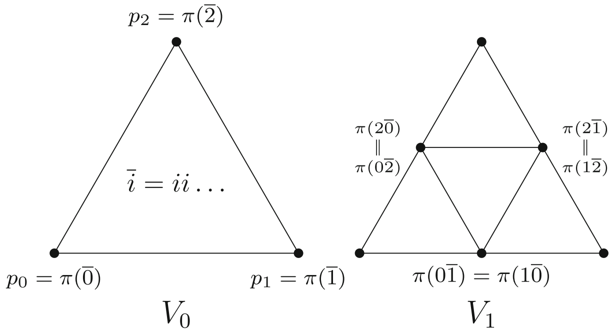

Fig. 3 Graph approximation of the Sierpinski gasket

For $w \in W_* \cup \Sigma$ and $n \leq |w|$, we define $[w]_n$ as the unique $u \in W_n$ satisfying $w = uv$ for some $v \in W_* \cup \Sigma$. For $w, v \in W_* \cup \Sigma$ with $w \neq v$, define

$$n(w, v) = \min\{i | 1 \leq i \leq \min\{|w|, |v|\}, [w]_i \neq [v]_i\} - 1$$

and

$$w \wedge v = [w]_{n(w,v)},$$

which is called the confluence of w and v. If $w = v$, we define $w \wedge v = w$.

Note that $\{\sigma_i\}_{i \in S}$ is the collection of branches of the inverse of σ.

Using $n(\omega, \tau)$, we define a family of metrics on Σ.

Proposition 2.5 [9, Theorem 1.2.2] *For $\omega, \tau \in \Sigma$ and $r \in (0, 1)$, define*

$$\delta_r(\omega, \tau) = \begin{cases} r^{n(\omega,\tau)} & \text{if } \omega \neq \tau, \\ 0 & \text{if } \omega = \tau. \end{cases}$$

Then δ_r is a metric on Σ and the metric space (Σ, δ_r) is a Cantor set, i.e. it is compact, totally disconnected and perfect. Moreover σ and σ_i are continuous maps. In particular,

$$\delta_r(\sigma_i(\omega), \sigma_i(\tau)) \leq r\delta_r(\omega, \tau)$$

for any $\omega, \tau \in \Sigma$.

Definition 2.6 For $w = w_1 \dots w_m \in W_*$, define

$$F_w = F_{w_1} \circ \dots \circ F_{w_m} \quad \text{and} \quad K_w = F_w(K).$$

Furthermore, define $V_0 = \{p_0, p_1, p_2\}$,

$$V_m = \bigcup_{w \in W_m} F_w(V_0) \quad \text{and} \quad V_* = \bigcup_{m \geq 0} V_m$$

The followings are the basic properties of the Sierpinski gasket. See [9, Chapter 1] ([9, Examples 1.2.8 and 1.3.15] in particular) for details.

Proposition 2.7 (1) *For any $m \geq 0$, $V_m \subseteq V_{m+1}$. Moreover, V_* is a dense subset of K.*
(2) *Let μ_* be the normalized $\frac{\log 3}{\log 2}$-dimensional Hausdorff measure on (K, d_*). Then*

$$\mu_*(K_w) = \left(\frac{1}{3}\right)^{|w|}$$

for any $w \in W_*$. In particular, μ_* is the self-similar measure with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
(3) *For any $\omega \in \Sigma$ and $m \geq 0$, $K_{[\omega]_m} \supseteq K_{[\omega]_{m+1}}$ and*

$$\bigcap_{m \geq 0} K_{[\omega]_m}$$

is a single point. Let $\pi(\omega)$ be the single point. Then $\pi : \Sigma \to K$ is a continuous surjection satisfying

$$\pi(\sigma_i(\omega)) = F_i(\pi(\omega))$$

for any $\omega \in \Sigma$ and $i \in S$. In particular, $\pi((i)_\infty) = p_i$ for any $i \in S$ and

$$\pi(i(j)_\infty) = F_j(p_i) = F_i(p_j) = \pi(j(i)_\infty) \tag{2.2}$$

if $i, j \in S$ and $i \neq j$. Moreover, $\pi^{-1}(x)$ is not a single point if and only if $x = F_{wj}(p_i)$ for some $w \in W_*$ and $i \neq j \in S$ when $\pi^{-1}(x) = \{wi(j)_\infty, wj(i)_\infty\}$.

See Fig. 3 for an illustration of (2.2), where $(i)_\infty$ is denoted by $\bar{i}$.

Hereafter in this section, we consider the geometry of $K \backslash I$ where I is the line segment $\overline{p_0 p_1}$. One of the notable properties of $K \backslash I$ is that it has the structure of a binary tree. To give further explanations, we need to introduce several notions.

Definition 2.8 (1) For $n \geq 0$, define

$$T_n = \bigcup_{m=1}^{n} \{0, 1\}^{m-1},$$

where $T_1 = \{\phi\}$. Furthermore, define

$$T = \bigcup_{n \geq 1} T_n \quad \text{and} \quad \Sigma_T = \{0, 1\}^{\mathbb{N}} = \{i_1 i_2 \ldots | i_j \in \{0, 1\} \text{ for any } j \in \mathbb{N}\}$$

(2) Define $I = [0, 1] \times \{0\}$.

See Fig. 5 for an illustration of T_3.

We naturally identify I with the unit interval $[0, 1]$. Under this identification,

$$\pi(i_1 i_2 \ldots) = \sum_{n \geq 1} \frac{i_n}{2^n}.$$

for any $i_1 i_2 \ldots \in \Sigma_T$. This is exactly the binary expansion of $x = \pi(i_1 i_2 \ldots) \in [0, 1]$. In particular, $\pi(\Sigma_T) = I$.

The next proposition states that $K \backslash I$ can be regarded as a tree of Sierpinski gaskets.

Proposition 2.9

$$K \backslash I = \bigcup_{w \in T} K_{w2}.$$

Moreover, for $w, v \in T$, define $E = \{(w, v) | w, v \in T, w \neq v, K_{w2} \cap K_{v2} \neq \emptyset\}$. Then (T, E) is a binary tree with the root ϕ.

Note that Σ_T equipped with the metric $\delta_r|_{\Sigma_T \times \Sigma_T}$ is a Cantor set, which is the "boundary" of the binary tree (T, E).

Proof Note that $(w, v) \in E$ if and only if there exists $u \in T$ such that $(w, v) \in \{(u, ui), (ui, u)\}$ for some $i \in \{0, 1\}$. So, every $w \in T$ has two children $w0$ and $w1$. So (T, E) is exactly the infinite binary tree as defined. □

Geometrically, the shortest path metric on $K\backslash I$ introduced below reflects this structure of the binary tree (T, E). As we will see in Theorem 2.15, the Cantor set Σ_T appears as the boundary of $K\backslash I$ under the shortest path metric.

Definition 2.10 (1) Define $\widetilde{D}$ as the shortest path metric on $K\backslash I$, i.e. for $x, y \in K\backslash I$,

$$\widetilde{D}(x, y) = \inf\{L(\gamma)|\gamma \text{ is a rectifiable curve between } x \text{ and } y \text{ in } K\backslash I\},$$

where the rectifiability and the length of a curve are with respect to the Euclidean metric.
(2) For $w \in T$ and $\omega \in \Sigma_T$, define $p(w) = F_w(p_2)$ and $p_m(\omega) = p([\omega]_m)$.

Topologically, there is no difference between D and $\widetilde{D}$.

Proposition 2.11 *The identity map $\iota : K\backslash I \to K\backslash I$ is a homeomorphism between $(K\backslash I, D)$ and $(K\backslash I, \widetilde{D})$.*

To show this proposition, we need several lemmas.

Lemma 2.12 $\mathrm{diam}(K, D) = 1$.

Proof **Claim**: $D(p_i, x) \leq 1$ for any $x \in K$ and $i \in S$.
Proof of Claim: Without loss of generality, we may assume that $i = 0$. Choose $\omega \in \Sigma$ such that $x = \pi(\omega)$. Define $\{q_m\}_{m\geq 0}$ inductively as follows: let $q_0 = p_0$ and let q_{m+1} be the unique element in $F_{[\omega]_{m+1}}(V_0)$ which attains $\min\{d_*(q_m, q)|q \in F_{[\omega]_{m+1}}(V_0)\}$. Then $\bigcup_{m\geq 0} \overline{q_m q_{m+1}} \cup \{x\}$ is a rectifiable curve between p_0 and x and its length is no greater than

$$\sum_{m\geq 0} L(\overline{p_m p_{m+1}}) \leq \sum_{m\geq 0} 2^{-(m+1)} \leq 1.$$

Thus we have obtained the claim.

Let $x, y \in K$ with $x \neq y$. Then there exist $n \geq 1$ and $w, v \in W_n$ such that $w \neq v$, $K_w \cap K_v \neq \emptyset$, $x \in K_w$ and $y \in K_v$. Let $\{q\} = K_w \cap K_v$. Then by the above claim, $D(x, p) \leq 2^{-n}$ and $D(y, p) \leq 2^{-n}$. Hence $D(x, y) \leq D(x, p) + D(y, p) \leq 1$. □

Lemma 2.13 *For any $w \in W_*$ and $x, y \in K_w$, there exists a rectifiable curve γ_{xy} between x and y included in K_w such that*

$$L(\gamma_{xy}) = D(x, y).$$

Proof Let γ_{xy} be a shortest path between x and y. Then γ_{xy} does not have any loop. So, once it get out from K_w at some point in $F_w(V_0)$, it returns to K_w at a different point in $F_w(V_0)$. So, if γ_{xy} is not included in K_w, then it must pass two distinct points of $F_w(V_0)$ and $L(\gamma_{xy}) > 2^{-|w|}$. On the other hand, Lemma 2.12 yields that $\mathrm{diam}(K_w, D) = 2^{-|w|}$, so that $L(\gamma_{xy}) \leq 2^{-|w|}$. Therefore, γ_{xy} is included in K_w. □

Lemma 2.14 *For any $w \in T$ and $x, y \in K_{w2}$,*

$$\widetilde{D}(x, y) = D(x, y).$$

Proof By Lemma 2.13, there exists a rectifiable curve γ_{xy} between x and y included in K_{w2} such that $L(\gamma_{xy}) = D(x, y)$. Hence $\widetilde{D}(x, y) = D(x, y) = L(\gamma_{xy})$. □

Proof of Proposition 2.11 Since $D(x, y) \leq \widetilde{D}(x, y)$ for any $x, y \in K\backslash I$, the identity map from $(K\backslash I, \widetilde{D})$ to $(K\backslash I, D)$ is continuous. Conversely, suppose that $\{x_n\}_{n\geq 1} \subseteq K\backslash I$, $x \in K\backslash I$ and $D(x_n, x) \to 0$ as $n \to \infty$. If $x \in K_{w2}\backslash F_{w2}(V_0)$ for some $w \in T$, then $x_n \in K_{w2}$ for sufficiently large n. Hence by Lemma 2.14, we see that $\widetilde{D}(x_n, x) = D(x_n, x) \to 0$ as $n \to \infty$. Otherwise, $x = K_{w2} \cap K_{wi2}$ for some $w \in T$ and $i \in \{0, 1\}$. Then it follows that $\{x_n, x\} \subseteq K_{w2}$ or $\{x_n, x\} \subseteq K_{wi2}$. In either case, Lemma 2.14 shows $\widetilde{D}(x_n, x) = D(x_n, x)$ and hence $D(x_n, x) \to 0$ as $n \to \infty$. Thus we have shown that the identity map from $(K\backslash I, D)$ to $(K\backslash I, \widetilde{D})$ is continuous as well. □

Although $\widetilde{D}$ gives the same topology as the restriction of D to $K\backslash I$, they are not biLipshitz equivalent. For example, $D(p(0(1)_n), p(1(0)_n)) = 2^{-n}$ while $\widetilde{D}(p(0(1)_n), p(1(0)_n)) = 3/2 - 2^{-n}$. This discrepancy is due to the lack of the point $(\frac{1}{2}, 0) \in I$. The same phenomena happens at a point $(\frac{i}{2^m}, 0) \in I$ for any $m \geq 1$ and $i \in \{1, \ldots, 2^n - 1\}$, so that we have the following fact.

Theorem 2.15 *The completion of* $(K\backslash I, \widetilde{D})$ *is (homeomorphic to)* $(K\backslash I) \cup \Sigma_T$. *In particular,* $\widetilde{D}|_{\Sigma_T \times \Sigma_T} = \frac{3}{2}\delta_{\frac{1}{2}}|_{\Sigma_T \times \Sigma_T}$.

Proof The shortest path between $p_m(\omega)$ and $p_{m+n}(\omega)$ is the union of line segments $\cup_{i=0}^{n-1}\overline{p_{m+i}(\omega)p_{m+i+1}(\omega)}$ and so $\widetilde{D}(p_m(\omega), p_{m+n}(\omega)) = 2^{-m}(1 - 2^{-n})$. This shows that $\{p_m(\omega)\}_{m\geq 1}$ is a Cauchy sequence with respect to $\widetilde{D}$. Through the correspondence between the equivalence class of $\{p_m(\omega)\}_{m\geq 0}$ and $\omega \in \Sigma_T$, we identify Σ_T as a subset of the completion. At the same time, the shortest path between $p_m(\omega)$ and $\omega \in \Sigma_T$ is the combination of infinite line segments $\{\overline{p_i(\omega)p_{i+1}(\omega)}\}_{i\geq m}$, which is denoted by $\mathbf{p}_m(\omega)$, and $\widetilde{D}(p_m(\omega), \omega) = 2^{-m}$. Moreover, let $\omega, \tau \in \Sigma_T$. Then the shortest path between ω and τ consists of $\mathbf{p}_{k+1}(\omega)$, the line segment $\overline{p_{k+1}(\omega)p_{k+1}(\tau)}$, and $\mathbf{p}_{k+1}(\tau)$, where $k = n(\omega, \tau)$. Hence we see that $\widetilde{D}(\omega, \tau) = 3 \cdot 2^{-n(\omega,\tau)-1} = \frac{3}{2}\delta_{\frac{1}{2}}(\omega, \tau)$. The rest of arguments are entirely the same as in Sect. 4, where we will show that the completion of $K\backslash I$ with respect to the resistance metric $\widetilde{R}$ equals $(K\backslash I) \cup \Sigma_T$, if we replace the exponent $\left(\frac{3}{5}\right)^{|w|}$ by $\left(\frac{1}{2}\right)^{|w|}$. In fact, some of the arguments become even simpler because it is easier to handle the shortest path metric rather than the resistance metric. □

3 Standard resistance form on the Sierpinski gasket

From this section on, we study the difference between K and $K\backslash I$ from the viewpoint of analysis, in particular, resistance forms, whose very basics are given in Appendix A. First of all, in this section, we introduce the standard resistance form $(\mathcal{E}, \mathcal{F})$ on the Sierpinski gasket, which is the local regular Dirichlet form on $L^2(K, \mu_*)$ associated with the Brownian motion on the Sierpinski gasket.

The standard resistance form $(\mathcal{E}, \mathcal{F})$ is defined as the limit of a compatible sequence of weighted graphs, $\{(V_m, C_m)\}_{m\geq 1}$, defined below. See Appendix A for the definitions and the basic facts on weighted graphs.

Definition 3.1 For $m \geq 0$, define $C_m : V_m \times V_m \to [0, \infty)$ by

$$C_m(x, y) = \begin{cases} \left(\frac{5}{3}\right)^m & \text{if } (x, y) \in \{(F_w(p_i), F_w(p_j)) | w \in W_m, i, j \in S, i \neq j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notation For a set A, we define

$$\ell(A) = \{f | f : A \to \mathbb{R}\}.$$

The pair (V_m, C_m) is a connected weighted graph defined in Definition A.5. For simplicity, we denote the energy $\mathcal{E}_{C_m}$ associated with (V_m, C_m) by $\mathcal{E}_m$. Then we have

$$\mathcal{E}_{m+1}(f, f) = \frac{5}{3} \sum_{i \in S} \mathcal{E}_m(f \circ F_i, f \circ F_i)$$

for any $f \in \ell(V_{m+1})$. A straightforward calculation shows

$$\mathcal{E}_m(f, f) = \min\{\mathcal{E}_{m+1}(g, g) | g \in \ell(V_{m+1}), g|_{V_m} = f\}$$

for any $m \geq 0$ and $f \in \ell(V_m)$. See [9, Example 3.1.5] for details. This shows that $\{(V_m, C_m)\}_{m \geq 0}$ is a compatible sequence, so that Theorems A.9 and A.2 yield the following theorem.

Notation Let (X, d) be a metric space. Define $C(X, d)$. as the collection of real-valued continuous functions on (X, d). Moreover, define $B_d(x, r) = \{y | y \in X, d(x, y) < r\}$ for $x \in X$ and $r > 0$.

Theorem 3.2 *Define*

$$\mathcal{F} = \left\{ f \;\middle|\; f \in \ell(V_*), \lim_{m \to \infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m}) < \infty \right\}$$

and

$$\mathcal{E}(f, f) = \lim_{m \to \infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m})$$

for $f \in \mathcal{F}$.
(1) *$\mathcal{F}$ is naturally identified as a subset of $C(K, d_*)$ and $(\mathcal{E}, \mathcal{F})$ is a resistance form on K. Let R be the resistance metric on K associated with $(\mathcal{E}, \mathcal{F})$. Set $\alpha = \frac{\log 5 - \log 3}{\log 2}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 d_*(x, y)^\alpha \leq R(x, y) \leq c_2 d_*(x, y)^\alpha \tag{3.1}$$

for any $x, y \in K$.
(2) *For any $i \in S$ and $f \in \mathcal{F}$, $f \circ F_i \in \mathcal{F}$ and*

$$\mathcal{E}(f, f) = \frac{5}{3} \sum_{i \in S} \mathcal{E}(f \circ F_i, f \circ F_i).$$

(3) *For any Radon measure on K satisfying $\nu(O) > 0$ for any non-empty open subset of K, $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \nu)$.*

$(\mathcal{E}, \mathcal{F})$ is called the standard resistance form on the Sierpinski gasket. The diffusion process associated with the local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu_*)$ is called the Brownian motion on the Sierpinski gasket, which was originally introduced and studied by [1, 3, 13].

4 Resistance form on "the Sierpinski gasket minus the bottom line"

As is observed in the latter half of Sect. 2, once we remove the bottom line I from the Sierpinski gasket K, then the limits of the paths towards the bottom line I form the Cantor set Σ_T instead of the line segment I. In association with this phenomenon, we can extend the standard resistance form $(\mathcal{E}, \mathcal{F})$ on K to a resistance form on $K\backslash I$ whose associated resistance metric reflects the geometry of $(K\backslash I, \widetilde{D})$. To construct such an extension, we replace the original compatible sequence $\{(V_m, C_m)\}_{m\geq 1}$ by a new one $\{(\widetilde{V}_m, \widetilde{C}_m)\}_{m\geq 1}$, which is illustrated in Fig. 4, as follows.

Definition 4.1 Define $\widetilde{V}_m = V_m\backslash(V_m \cap I)$ and $\widetilde{V}_* = \cup_{m\geq 0}\widetilde{V}_m$. Define $\widetilde{C}_m = C_m|_{\widetilde{V}_m\times\widetilde{V}_m}$ and let $\widetilde{R}_m$ be the resistance metric on $\widetilde{V}_m$ associated with $(\widetilde{V}_m, \widetilde{C}_m)$. Moreover, define $\rho_m : \widetilde{V}_m \to V_m$ as the natural inclusion map.

The following lemma is straightforward.

Lemma 4.2 $\{(\widetilde{V}_m, \widetilde{C}_m)\}_{m\geq 0}$ *is a compatible sequence.*

By the above lemma and Theorem A.9, we have the following counterpart of Theorem 3.2.

Theorem 4.3 *Define*

$$\widetilde{\mathcal{F}} = \left\{ f \,\middle|\, f \in \ell(\widetilde{V}_*), \lim_{m\to\infty} \widetilde{\mathcal{E}}_m(f|_{\widetilde{V}_m}, f|_{\widetilde{V}_m}) < \infty \right\}$$

and

$$\widetilde{\mathcal{E}}(f, f) = \lim_{m\to\infty} \widetilde{\mathcal{E}}_m(f|_{\widetilde{V}_m}, f|_{\widetilde{V}_m})$$

for $f \in \widetilde{\mathcal{F}}$. Then $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a resistance form on $\widetilde{V}_*$. Furthermore, let $\widetilde{R}$ be the associated resistance metric on $\widetilde{V}_*$ and let $(\widetilde{K}, \widetilde{R})$ be the completion of $(\widetilde{V}_*, \widetilde{R})$. Then $f \in \widetilde{\mathcal{F}}$ is naturally extended to a continuous function on $\widetilde{K}$, and $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is regarded as a resistance form on $\widetilde{K}$ whose associated resistance metric is $\widetilde{R}$.

The resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ will be shown to be an extension of $(\mathcal{E}, \mathcal{F})$ in the next section. More precisely, the inequality (4.2) in the next lemma is upgraded to an equality in Theorem 5.4.

Lemma 4.4 *There exists a continuous map $\rho : \widetilde{K} \to K$ such that $\rho|_{\widetilde{V}_m} = \rho_m$ for any $m \geq 0$,*

$$R(\rho(x), \rho(y)) \leq \widetilde{R}(x, y) \tag{4.1}$$

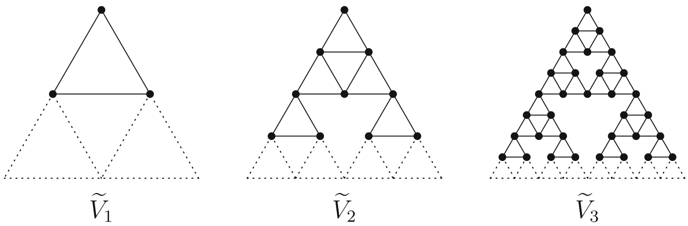

Fig. 4 Graph approximation of $K\backslash I$

for any $x, y \in \widetilde{K}$ *and, for any* $f \in \mathcal{F}$, $f \circ \rho \in \widetilde{\mathcal{F}}$ *and*

$$\widetilde{\mathcal{E}}(f \circ \rho, f \circ \rho) \leq \mathcal{E}(u, u). \tag{4.2}$$

Proof By the definition of $\widetilde{C}_m$,

$$\widetilde{\mathcal{E}}_m(f \circ \rho_m, f \circ \rho_m) \leq \mathcal{E}_m(u, u) \tag{4.3}$$

for any $m \geq 0$ and $f \in \ell(V_m)$. Define $\rho : \widetilde{V}_* \to V_*$ as $\rho|_{V_m} = \rho_m$. For any $x, y \in \widetilde{V}_*$, choose m such that $x, y \in \widetilde{V}_m$. Then by (4.1), for any $f \in \mathcal{F}$,

$$\frac{|f(\rho_m(x)) - f(\rho_m(y))|^2}{\mathcal{E}(f, f)} \leq \frac{|f(\rho_m(x)) - f(\rho_m(y))|^2}{\widetilde{\mathcal{E}}(f \circ \rho_m, f \circ \rho_m)}$$

This shows that

$$R_m(\rho_m(x), \rho_m(y)) \leq \widetilde{R}(x, y).$$

for any $x, y \in \widetilde{V}_m$. Hence we see that (4.1) is satisfied for any $x, y \in \widetilde{V}_*$. This shows that ρ can be naturally extended to a map from $\widetilde{K}$ to K and it satisfies (4.1) for any $x, y \in \widetilde{K}$.

Finally, by (4.3), if $f \in \mathcal{F}$, then

$$\lim_{m\to\infty} \widetilde{\mathcal{E}}_m(f \circ \rho_m, f \circ \rho_m) \leq \lim_{m\to\infty} \mathcal{E}_m(f, f).$$

Hence $f \circ \rho \in \widetilde{\mathcal{F}}$ and (4.2) holds. □

The next theorem is one of the main results of this paper. It concerns the geometry of $K \backslash I$ under the resistance metric $\widetilde{R}$.

Theorem 4.5 (1) $\widetilde{K}$ *is homeomorphic to* $(K \backslash I) \cup \Sigma_T$. *Furthermore, there exists* $c_1, c_2 > 0$ *such that*

$$c_1 \widetilde{D}(x, y)^\alpha \leq \widetilde{R}(x, y) \leq c_2 \widetilde{D}(x, y)^\alpha \tag{4.4}$$

for any $x, y \in \widetilde{K}$, *where* α *is the exponent appearing in Theorem* 3.2.
(2) *The map* $\rho : \widetilde{K} \to K$ *is surjective and*

$$\rho(x) = \begin{cases} x & \text{if } x \in K \backslash I, \\ \pi(x) & \text{if } x \in \Sigma_T. \end{cases}$$

The rest of this section is filled with a proof of Theorem 4.5. The arguments seem lengthy but are indispensable as we have to deal with the equivalence classes of the collection of Cauchy sequences. Nevertheless, the essence is the tree structure of $K \backslash I$ illustrated in Fig. 5.

Definition 4.6 Define

$$G = \bigcup_{w \in T} F_{w2}(V_0)$$

and

$$G_n = \bigcup_{m=1}^{n} \bigcup_{w \in \{0,1\}^{m-1}} F_{w2}(V_0)$$

for $n \geq 1$.

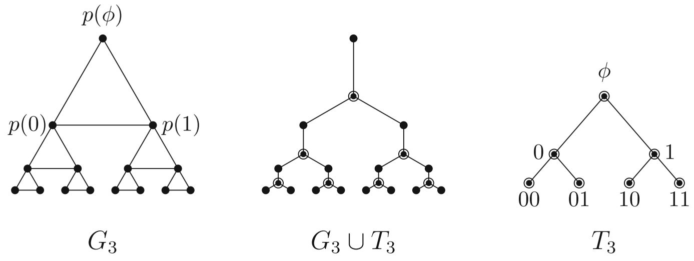

Fig. 5 Tree structures behind $K \backslash I$

Remark By (2.2) and Definition 2.10

$$\begin{aligned} F_{w2}(V_0) &= \{F_{w2}(p_2), F_{w2}(p_0), F_{w2}(p_1)\} = \{F_w(p_2), F_{w0}(p_2), F_{w1}(p_2)\} \\ &= \{p(w), p(w0), p(w1)\}. \end{aligned}$$

Hence

$$G = \{p(w)|w \in T\} \quad \text{and} \quad G_n = \{p(w)|w \in \cup_{k=0}^n \{0, 1\}^n\}.$$

The next lemma is a collection of immediate observations concerning G_n and G.

Lemma 4.7 (1) *For any $m \geq 1$, $G_m \subseteq \widetilde{V}_m$.*
(2) *For any $x \in \widetilde{V}_* \backslash G$, there exists a unique $w \in T$ such that $x \in K_{w2}$.*

First, we are going to show that R and $\widetilde{R}$ are uniformly biLipschitz equivalent on K_{w2} for any $w \in T$.

Lemma 4.8 *There exists $c_* > 0$ such that, for any $w \in T$ and $x, y \in K_{w2} \cap V_*$,*

$$R(x, y) \leq \widetilde{R}(x, y) \leq c_* R(x, y)$$

Remark For any $w \in T$ and $m \geq 0$, $K_{w2} \cap V_m = K_{w2} \cap \widetilde{V}_m$ and hence $K_{w2} \cap V_* = K_{w2} \cap \widetilde{V}_*$.

Proof Let $x, y \in K_{w2} \cap V_m$. Then

$$\left(\frac{5}{3}\right)^{|w|+1} \mathcal{E}_{m-|w|-1}(f \circ F_{w2}, f \circ F_{w2}) \leq \widetilde{\mathcal{E}}_m(f, f)$$

for any $f \in \ell(\widetilde{V}_m)$. Hence

$$\frac{|f(x) - f(y)|^2}{\widetilde{\mathcal{E}}_m(f, f)} \leq \left(\frac{5}{3}\right)^{|w|+1} \frac{|f \circ F_{w2}\big((F_{w2})^{-1}(x)\big) - f \circ F_{w2}\big((F_{w2})^{-1}(y)\big)|^2}{\mathcal{E}_{m-|w|-1}(f \circ F_{w2}, f \circ F_{w2})}.$$

This implies

$$\widetilde{R}_m(x, y) \leq \left(\frac{5}{3}\right)^{|w|+1} R_{m-|w|-1}((F_{w2})^{-1}(x), (F_{w2})^{-1}(y)).$$

Letting $m \to \infty$, we obtain

$$\widetilde{R}(x, y) \leq \left(\frac{5}{3}\right)^{|w|+1} R((F_{w2})^{-1}(x), (F_{w2})^{-1}(y)).$$

Finally, we have the desired inequality by [10, Theorem A.1]. □

Lemma 4.9 *Let $w \in T$. Set $\widetilde{K}_{w2}$ be the closure of $K_{w2} \cap V_*$ with respect to the metric $\widetilde{R}$. Then $\rho(\widetilde{K}_{w2}) = K_{w2}$ and*

$$R(\rho(x), \rho(y)) \le \widetilde{R}(x, y) \le c_* R(\rho(x), \rho(y)) \tag{4.5}$$

for any $x, y \in \widetilde{K}_{w2}$, where c_* is the same constant as in Lemma 4.8. In particular, $\rho|_{\widetilde{K}_{w2}} : \widetilde{K}_{w2} \to K_{w2}$ is a biLipschitz homeomorphism.

This lemma is a counterpart of Lemma 2.14 where we study shortest path metrics D and $\widetilde{D}$ in place of R and $\widetilde{R}$.

Proof Let $x \in K_{w2}$. Then there exists $\{x_n\}_{n\ge1} \subseteq K_{w2} \cap V_* = K_{w2} \cap \widetilde{V}_*$ such that $R(x, x_n) \to 0$ as $n \to \infty$. By Lemma 4.8,

$$\widetilde{R}(x_n, x_m) \le c_* R(x_n, x_m)$$

for any $n, m \ge 1$. This shows that $\{x_n\}_{n\ge1}$ is a Cauchy sequence in $(K_{w2} \cap V_*, \widetilde{R})$. Therefore there exists $y \in \widetilde{K}_{w2}$ such that $\rho(y) = x$. Thus we have shown $\rho(\widetilde{K}_{w2}) \supseteq K_{w2}$.

Next let $y \in \widetilde{K}_w$. Then there exists $\{y_n\}_{n\ge1} \subseteq K_{w2} \cap \widetilde{V}_*$ such that $\widetilde{R}(y_n, y) \to 0$ as $n \to \infty$. This implies that $R(y_n, \rho(y)) \to 0$ as $n \to \infty$ and hence $\rho(y) \in K_w$. Thus we see that $\rho(\widetilde{K}_w) = K_w$.

Now by Lemma 4.8, we have (4.5) for any $x, y \in \widetilde{K}_{w2}$. The rest of the statements are straightforward from (4.5). □

Next, we are going to estimate $\widetilde{R}(x, y)$ when x and y belongs distinct K_{w2}'s. In the following lemmas, if we replace $\widetilde{R}$ and the exponent $\frac{3}{5}$ by $\widetilde{D}$ and the exponent $\frac{1}{2}$ respectively, the statements and the proofs still hold with minor modifications of constants. Consequently, they constitute parts of the proof of Theorem 2.15 as mentioned in its proof.

Lemma 4.10 *Let $w, v \in T$. Assume that $K_w \cap K_v = \emptyset$. Then for any $x \in \widetilde{K}_{w2}$ and $y \in \widetilde{K}_{v2}$,*

$$\frac{2}{5}\left(\frac{3}{5}\right)^{|w\wedge v|+1} \le \widetilde{R}(x, y) \tag{4.6}$$

Proof Suppose that $|w| \le |v|$. First, assume that $x \in K_{w2} \cap \widetilde{V}_*$ and $y \in K_{v2} \cap \widetilde{V}_*$. Then there exists $m \ge 0$ such that $x \in K_{w2} \cap \widetilde{V}_m$ and $y \in K_{v2} \cap \widetilde{V}_m$.

Case 1; $|w \wedge v| = |w|$.

In this case, $w \wedge v = w$ and $v = wi_1 \ldots i_k$ for some $k \ge 2$ and $i_1, \ldots, i_k \in \{0, 1\}$. Without loss of generality, we may assume that $i_1 = i_2 = 0$. Now if we remove $(K_{w02} \cap \widetilde{V}_m) \backslash F_{w02}(V_0)$ from $\widetilde{V}_m$, then the connected graph $(\widetilde{V}_m, E_m)$ is divided into three connected components. U_0, U_1 and U_2, where $F_{w02}(p_i) \in U_i$ for each $i = 0, 1, 2$. Define

$$\begin{aligned}\mathcal{U} = \{f | f \in \ell(\widetilde{V}_m), &\text{ there exist } a_1, a_2, a_3 \in \mathbb{R} \\ &\text{such that } f|_{U_i} \equiv a_i \text{ for each } i = 1, 2, 3.\}\end{aligned}$$

Then

$$\begin{aligned}\sup_{f \in \ell(K_{w02} \cap \widetilde{V}_m)} \frac{|f(a_2) - f(a_0)|^2}{\mathcal{E}_{m, K_{w02} \cap \widetilde{V}_m}(f, f)} &= \sup_{f \in \mathcal{U}} \frac{|u(x) - u(y)|^2}{\widetilde{\mathcal{E}}_m(u, u)} \\ &\le \sup_{f \in \ell(\widetilde{V}_m), f(x) \ne f(y)} \frac{|f(x) - f(y)|^2}{\widetilde{\mathcal{E}}_m(f, f)} = \widetilde{R}(x, y)\end{aligned}$$

On the other hand, for $f \in \ell(K_{w02} \cap \widetilde{V}_m)$,

$$\mathcal{E}_{m,K_{w02}\cap\widetilde{V}_m}(f, f) = \left(\frac{5}{3}\right)^{|w|+2} \mathcal{E}_{m-|w|-2}(f \circ F_{w02}, f \circ F_{w02})$$

Hence

$$\begin{aligned}\sup_{f\in\ell(K_{w02}\cap\widetilde{V}_m)} \frac{|f(a_2) - f(a_0)|^2}{\mathcal{E}_{m,K_{w02}\cap\widetilde{V}_m}(f, f)} &= \left(\frac{3}{5}\right)^{|w|+2} \sup_{f\in\ell(V_{m-|w|-2}), f(p_0)\neq f(p_2)} \frac{|f(p_0) - f(p_2)|^2}{\mathcal{E}_{m-|w|-2}(f, f)} \\ &= \left(\frac{3}{5}\right)^{|w|+2} R(p_0, p_2) = \frac{2}{5}\left(\frac{3}{5}\right)^{|w|+1}.\end{aligned}$$

Thus we have obtained the desired inequality in this case.
Case 2; $|w \wedge v| < |w|$
Let $u = w \wedge v$. In this case, without loss of generality, we may assume that $w = u0i_1 \ldots i_k$ and $v = u1j_1 \ldots j_l$. Then, exchanging $w02$ and a_2 for $u2$ and a_1 respectively in the arguments of Case 1, we obtain

$$\frac{2}{5}\left(\frac{3}{5}\right)^{|u|} \leq \widetilde{R}(x, y),$$

so that (4.6) has been shown in this case as well.

Finally, taking the completion, we have (4.6) for any $x \in K_w$ and $y \in K_v$. □

Lemma 4.11 *There exists $c_0 > 0$ such that*

$$\sup_{x\in\widetilde{K}_{w2}, y\in\widetilde{K}_{v2}} \widetilde{R}(x, y) \leq c_0\left(\frac{3}{5}\right)^{|w\wedge v|}$$

for any $w, v \in T$.

Proof Without loss of generality, we may assume that $|w| \leq |v|$. By (4.5),

$$\mathrm{diam}(\widetilde{K}_{u2}, \widetilde{R}) \leq c_*\mathrm{diam}(K_{u2}, R) \leq c_*\left(\frac{3}{5}\right)^{|u|}\mathrm{diam}(K, R)$$

for any $u \in T$.
Case 1: $|w \wedge v| = |w|$.
In this case, $w \wedge v = w$ and $v = wi_1 \ldots i_k$. Thus

$$\sup_{x\in\widetilde{K}_w, y\in\widetilde{K}_v} \widetilde{R}(x, y) \leq \sum_{j=0}^{k} \mathrm{diam}(\widetilde{K}_{wi_1\ldots i_j}, \widetilde{R}) \leq c_*\frac{5}{2}\left(\frac{3}{5}\right)^{|w|}\mathrm{diam}(K, R).$$

Case 2: $|w \wedge v| < |w|$.
In this case, let $u = w \wedge v$. Then $w = ui_1 \ldots i_k$ and $v = uj_1 \ldots j_l$. This shows

$$\begin{aligned}&\sup_{x\in\widetilde{K}_w, y\in\widetilde{K}_v} \widetilde{R}(x, y) \\ &\leq \sum_{m=1}^{k} \mathrm{diam}(\widetilde{K}_{wi_1\ldots i_m}, \widetilde{R}) + \mathrm{diam}(\widetilde{K}_{u2}, \widetilde{R}) + \sum_{m=1}^{l} \mathrm{diam}(\widetilde{K}_{wj_1\ldots j_m}, \widetilde{R}) \\ &\leq 4c_*\left(\frac{3}{5}\right)^{|u|}\mathrm{diam}(K, R).\end{aligned}$$

So, combining the above two cases, we obtain the desired inequality with $c_0 = 4c_*\mathrm{diam}(K, R)$. □

Finally, we start to deal with Cauchy sequences converging to the "boundary" of $K\backslash I$.

Definition 4.12 (1) For $x \in \widetilde{V}_*$, define $\xi(x)$ as the unique $w \in T$ satisfying $x \in (K_{w2} \cap V_*)\backslash\{F_{w2}(p_0), F_{w2}(p_1)\}$.
(2) Let $\mathcal{C}$ be the totality of Cauchy sequences of $(\widetilde{V}_*, \widetilde{R})$. Define an equivalence relation $\sim$ on $\mathcal{C}$ in the following way: $\{x_n\}_{n\geq 1} \sim \{y_n\}_{n\geq 1}$ if $\lim_{n\to\infty} \widetilde{R}(x_n, y_n) = 0$. For $\{x_n\}_{n\geq 1} \in \mathcal{C}$, we denote the equivalence class of $\{x_n\}_{n\geq 1}$ with respect to $\sim$ by $[\{x_n\}_{n\geq 1}]$. Set

$$\mathcal{C}_1 = \{\{x_n\}_{n\geq 1} | \{x_n\}_{n\geq 1} \in \mathcal{C}, \\ \text{there exists } w \in T \text{ such that } \{i | i \geq 1, w = \xi(x_i)\} \text{ is an infinite set.}\}$$

and $\mathcal{C}_2 = \mathcal{C}\backslash\mathcal{C}_1$.

Note that the completion of $\widetilde{K} = \mathcal{C}/\sim$.

In the followings, we are going to show that $\mathcal{C}_1$ is the collection of Cauchy sequences converging to a point in $K\backslash I$ and that $\mathcal{C}_2$ is the collection of those converging to Σ_T.

Lemma 4.13 *Let $\{x_n\}_{n\geq 1} \in \mathcal{C}$. Set $x = [\{x_n\}_{n\geq 1}]$. If $\rho(x) \in K\backslash I$, then $\{x_n\}_{n\geq 1} \in \mathcal{C}_1$. In particular, $\rho^{-1}(K\backslash I) \subseteq \mathcal{C}_1/\sim$.*

Proof There exist $w, v \in T$ such that $K_{w2} \cap K_{v2} \neq \emptyset$ and $K_{w2} \cup K_{v2}$ is a neighborhood of $\rho(x)$. Since $R(x_n, \rho(x)) \leq \widetilde{R}(x_n, x) \to 0$ as $n \to \infty$, if follows that $\xi(x_n) \in \{w, v\}$ for sufficiently large n. Thus $\{x_n\}_{n\geq 1} \in \mathcal{C}_1$. □

Lemma 4.14

$$\mathcal{C}_1/\sim = \bigcup_{w\in T} \widetilde{K}_{w2}$$

Moreover, let $\widetilde{K}_1 = \bigcup_{w\in T} \widetilde{K}_{w2}$. Then $\rho(\widetilde{K}_1) = K\backslash I$ and $\rho|_{\widetilde{K}_1} : \widetilde{K}_1 \to K\backslash I$ is a homeomorphism.

Proof Let $\{x_n\}_{n\geq 1} \in \mathcal{C}_1$ and let $x = [\{x_n\}_{n\geq 1}]$. Then there exists $w \in T$ such that $\{i | \xi(x_i) = w\}$ is an infinite set. So, there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ such that $\xi(x_{n_j}) = w$ for any $j \geq 1$. Since $\lim_{j\to\infty} x_{n_j} = x$, we see that $x \in \widetilde{K}_{w2}$ and hence $\mathcal{C}_1/\sim \subseteq \widetilde{K}_1$.

If $z \in \widetilde{K}_{w2}$ for some $w \in T$, then there exists $\{z_n\}_{n\geq 1} \in \mathcal{C}$ such that $\{z_n\}_{n\geq 1} \subseteq K_{w2} \cap \widetilde{V}_*$ and $z = [\{z_n\}_{n\geq 1}]$. Obviously, $\{z_n\}_{n\geq 1} \in \mathcal{C}_1$ and hence $\mathcal{C}_1/\sim = \widetilde{K}_1$. Now Lemma 4.9 suffices to show $\rho(\widetilde{K}_1) = K\backslash I$.

Suppose that $\rho(x) = \rho(y)$ for $x, y \in \widetilde{K}_1$. Then there exist $w, v \in T$ and $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \in \mathcal{C}$ such that $\{x_n\}_{n\geq 1} \subseteq K_{w2} \cap \widetilde{V}_*$, $\{y_n\}_{n\geq 1} \subseteq K_{v2} \cap \widetilde{K}_{v2}$, $x = [\{x_n\}_{n\geq 1}]$ and $y = [\{y_n\}_{n\geq 1}]$. If $w = v$, then Lemma 4.9 shows that $x = y$. Assume that $w \neq v$. Let $z = \rho(x) = \rho(y)$. Then $\lim_{n\to\infty} R(x_n, z) = \lim_{n\to\infty} R(y_n, z) = 0$. Hence $z \in K_{w2} \cap K_{v2} = F_{w2}(V_0) \cap F_{v2}(V_0)$. By (4.5), we see that $\lim_{n\to\infty} \widetilde{R}(x_n, z) = \lim_{n\to\infty} \widetilde{R}(y_n, z) = 0$ and hence $x = y = z$. Thus $\rho|_{\widetilde{K}_1}$ is injective.

Suppose that $\{z_n\}_{n\geq 1} \subseteq K\backslash I$ and $\lim_{n\to\infty} R(z, z_n) = 0$ for some $z \in K\backslash I$. Then there exist $w, v \in T$ such that $z_n \in K_w \cup K_v$ for sufficiently large n, and hence $z \in K_w \cap K_v$. Applying (4.5) for both w and v, we see that $\widetilde{R}(\rho^{-1}(z_n), \rho^{-1}(z)) \leq CR(z_n, z) \to 0$ as $n \to \infty$. Thus it follows that $(\rho|_{\widetilde{K}_1})^{-1} : K\backslash I \to \widetilde{K}_1$ is continuous. □

Lemma 4.15 *Let $\{x_n\}_{n\geq 1} \in \mathcal{C}_2$ and set $w_n = \xi(x_n)$. Then there exists a unique $\omega = \omega_1\omega_2 \ldots \in \Sigma_T$ such that $|\omega \wedge w_n| \to \infty$ as $n \to \infty$. Moreover, if we define a map $\widetilde{\varphi} : \mathcal{C}_2 \to \Sigma_T$ by this correspondence, then $\widetilde{\varphi}(\{x_n\}_{n\geq 1}) = \widetilde{\varphi}(\{y_n\}_{n\geq 1})$ if and only if $\{x_n\}_{n\geq 1} \sim \{y_n\}_{n\geq 1}$.*

Proof Since $\{x_n\}_{n\geq 1} \in \mathcal{C}_2$, it follows that $|w_n| \to \infty$ as $n \to \infty$. Therefore, for any $N \geq 1$, there exists $M_N \geq 1$ such that $|w_n| \geq N$ and $\widetilde{R}(x_n, x_m) < \frac{2}{5}\left(\frac{3}{5}\right)^{N+1}$ whenever $n, m \geq M_N$. Suppose $n \geq M_N$. If $K_{w_{M_N}2} \cap K_{w_n2} = \emptyset$, then (4.6) implies that $|w_{M_N} \wedge w_n| \geq N$. If $K_{w_{M_N}2} \cap K_{w_n2} \neq \emptyset$, the fact that $|w_n| \geq N$ and $|w_{M_N}| \geq N$ shows that $|w_{M_N} \wedge w_n| \geq N$. So, we see that $|w_{M_N} \wedge w_n| \geq N$ for any $n \geq M_N$. Set $w^{(N)} = [w_{M_N}]_N$. Then $[w_n]_N = w^{(N)}$ for any $n \geq M_N$. It follows that if $N_1 \geq N_2$, then $[w^{(N_1)}]_{N_2} = w^{(N_2)}$. Thus there exists $\omega \in \Sigma_T$ such that $[\omega]_N = w^{(N)}$. Since $[w_n]_N = [\omega]_N$ for any $n \geq M_N$, we see that $|\omega \wedge w_n| \to \infty$ as $n \to \infty$. The uniqueness of such an ω is obvious.

Let $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \in \mathcal{C}_2$. Set $\omega = \widetilde{\varphi}(\{x_n\}_{n\geq 1})$ and $\tau = \widetilde{\varphi}(\{y_n\}_{n\geq 1})$. Assume that $\omega \neq \tau$. Then for sufficiently large n, $\xi(x_n) \wedge \xi(y_n) = \omega \wedge \tau$, $|\xi(y_n)| > |\omega \wedge \tau|$ and $|\xi(x_n)| > |\omega \wedge \tau|$. By (4.6), we see that $\frac{2}{5}\left(\frac{3}{5}\right)^{|\omega\wedge\tau|+1} \leq \widetilde{R}(x_n, y_n)$ for sufficiently large n. This implies $[\{x_n\}_{n\geq 1}] \neq [\{y_n\}_{n\geq 1}]$. Thus if $\{x_n\}_{n\geq 1} \sim \{y_n\}_{n\geq 1}$, then $\omega = \tau$. Conversely, assume that $\omega = \tau$. Since $\lim_{n\to\infty} |\omega \wedge \xi(x_n)| = \lim_{n\to\infty} |\tau \wedge \xi(y_n)| = \infty$, it follows that $\lim_{n\to\infty} |\xi(x_n) \wedge \xi(y_n)| = \infty$. By Lemma 4.11, we see that $\widetilde{R}(x_n, y_n) \to 0$ as $n \to \infty$. $\square$

By Lemma 4.15, the map $\widetilde{\varphi}$ induces a natural bijection $\varphi : \mathcal{C}_2/\sim \to \Sigma_T$.

Lemma 4.16 *The map $\varphi : (\mathcal{C}_2/\sim, \widetilde{R}) \to (\Sigma_T, \delta_{\frac{3}{5}})$ is a biLipschitz homeomorphism.*

Proof Let $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \in \mathcal{C}_2$. Set $x = [\{x_n\}_{n\geq 1}]$, $y = [\{y_n\}_{n\geq 1}]$, $\omega = \widetilde{\varphi}(\{x_n\}_{n\geq 1})$ and $\tau = \widetilde{\varphi}(\{y_n\}_{n\geq 1})$. Then for sufficiently large n, we see that $\xi(x_n) \wedge \xi(y_n) = \omega \wedge \tau$. Thus Lemma 4.11 yields

$$\widetilde{R}(x_n, y_n) \leq c_0\left(\frac{3}{5}\right)^{|\omega\wedge\tau|} = c_0\delta_{\frac{3}{5}}(\omega, \tau).$$

Taking $n \to \infty$, we see that

$$\widetilde{R}(x, y) \leq c_0\delta_{\frac{3}{5}}(\varphi(x), \varphi(y))$$

Assume that $\omega \neq \tau$. Then $\widetilde{K}_{\xi(x_n)2} \cap \widetilde{K}_{\xi(y_n)2} = \emptyset$ for sufficiently large n. Hence by (4.6)

$$\frac{6}{25}\delta_{\frac{3}{5}}(\omega, \tau) = \frac{2}{5}\left(\frac{3}{5}\right)^{|\omega\wedge\tau|+1} \leq \widetilde{R}(x_n, y_n).$$

Thus we have

$$\frac{6}{25}\delta_{\frac{3}{5}}(\varphi(x), \varphi(y)) \leq \widetilde{R}(x, y).$$

$\square$

Through φ, we identify $\mathcal{C}_2/\sim$ with Σ_T.

Lemma 4.17 *$\rho|_{\Sigma_T} = \pi|_{\Sigma_T}$. In particular, $\rho(\Sigma_T) = I$.*

Proof Let $\omega \in \Sigma_T$. For each $n \geq 1$, choose $x_n \in \widetilde{K}_{[\omega]_n2}$. Then $\{x_n\}_{n\geq 1} \in \mathcal{C}_2$ and $[\{x_n\}_{n\geq 1}] = \omega$. Now $x_n \in K_{[\omega]_n}$, it follows that

$$R(x_n, \pi(\omega)) \leq \mathrm{diam}(K_{[\omega]_n}, R) \to 0$$

as $n \to \infty$. Hence $\rho(\omega) = \pi(\omega)$. $\square$

Lemma 4.18 *$(\widetilde{K}, \widetilde{R})$ is compact.*

Proof Since $(\widetilde{K}, \widetilde{R})$ is complete, it is enough to show that $(\widetilde{K}, \widetilde{R})$ is totally bounded. Let $v \in W_*$. Define $\widetilde{K}_v$ as the closure of $K_v \cap \widetilde{V}_*$ with respect to $\widetilde{R}$. Note that

$$\widetilde{K} = \bigcup_{v \in W_n} \widetilde{K}_v$$

for any $n \geq 1$. If $v \notin T$, then there exists $w \in T$ such that $K_v \subseteq K_{w2}$ and hence $\widetilde{K}_v \subseteq \widetilde{K}_w$. By (3.1) and (4.5), there exists a constant c' which is independent of v such that

$$\mathrm{diam}(\widetilde{K}_v, \widetilde{R}) \leq c_* \mathrm{diam}(K_v, R) \leq c' \Big(\frac{3}{5}\Big)^{|v|} \mathrm{diam}(K, R).$$

Next assume that $v \in T$. For any $x, y \in K_v \cap \widetilde{V}_*$, it follows that $\xi(x) \wedge \xi(y) = vi_1 \ldots i_k$ for some $i_1, \ldots, i_k \in \{0, 1\}$. Thus by Lemma 4.11,

$$\widetilde{R}(x, y) \leq c_0 \Big(\frac{3}{5}\Big)^{|v|},$$

so that $\mathrm{diam}(\widetilde{K}_v, \widetilde{R}) \leq c_0 \big(\frac{3}{5}\big)^{|v|}$. Consequently, for any $\epsilon > 0$, $\{\widetilde{K}_v\}_{v \in W_n}$ is an ϵ-covering of $\widetilde{K}$ for sufficiently large n. This shows that $(\widetilde{K}, \widetilde{R})$ is totally bounded. □

Now we are ready to give a proof of Theorem 4.5.

Proof of Theorem 4.5 (1) By Lemmas 4.14 and 4.16, it follows that $\widetilde{K} = \mathcal{C}/\!\sim$ is homeomorphic to $(K \backslash I) \cup \Sigma_T$. To show (4.4), we consider the following three cases:
Case A: $x, y \in \Sigma_T$.
In this case,Theorem 2.15 and Lemma 4.16 suffice.
Case B: $x, y \in K \backslash I$ and $K_{\xi(x)2} \cap K_{\xi(y)2} = \emptyset$.
Lemmas 4.10 and 4.11 show that

$$\frac{2}{5}\Big(\frac{3}{5}\Big)^{|\xi(x) \wedge \xi(y)|+1} \leq \widetilde{R}(x, y) \leq c_0 \Big(\frac{3}{5}\Big)^{|\xi(x) \wedge \xi(y)|}.$$

On the other hand, modifying the proofs of Lemmas 4.10 and 4.11, we see that there exist $c_1, c_2 > 0$, which are independent of x and y, such that

$$c_1 \Big(\frac{1}{2}\Big)^{|\xi(x) \wedge \xi(y)|} \leq \widetilde{D}(x, y) \leq c_2 \Big(\frac{1}{2}\Big)^{|\xi(x) \wedge \xi(y)|}.$$

Thus we have (4.4).
Case C: There exists $w \in T$ such that $x, y \in K_{w2}$.
In this case, using Lemma 2.14, we have

$$D(x, y) = \widetilde{D}(x, y).$$

This equality with Proposition 2.3, (3.1) and (4.5) shows (4.4) in this case.
Case D: $x, y \in K \backslash I$ and $K_{\xi(x)2} \cap K_{\xi(y)2} \neq \emptyset$.
Without loss of generality, we may assume that $\xi(x) = w$ and $\xi(y) = w0$ for some $w \in T$. Then $K_{w2} \cap K_{w02} = \{p(w0)\}$. If we remove $p(w0)$, then $K \backslash I$ breaks up into two connected components. Therefore,

$$\widetilde{D}(x, y) = \widetilde{D}(x, p(w0)) + \widetilde{D}(p(w0), y) \text{ and } \widetilde{R}(x, y) = \widetilde{R}(x, p(w0)) + \widetilde{R}(p(w0), y).$$

Now by Case C, we obtain (4.4) in this case.
Case E: $x \in \Sigma_T$ and $y \in K \backslash I$.
Choose $\omega \in \Sigma_T$ such that $x = \pi(\omega)$. Applying Case B for $p([\omega]_m)$ and y and taking $m \to \infty$, we have (4.4) in this case.

Thus we have (4.4) for any $x, y \in (K\backslash I) \cup \Sigma_T$.
(2) Since $\widetilde{K}$ and K are compact and $\rho(\widetilde{V}_*)$ is dense in K, we see that $\rho(\widetilde{K}) = K$. The rest follows from Lemmas 4.14 and 4.17. □

5 Relation between two resistance forms

In this section, we give an alternative expression of the domain $\widetilde{\mathcal{F}}$ of the resistance form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $(K\backslash I) \cup \Sigma_T$. Through the expression, we obtain a characterization of the domain $\mathcal{F}$ of the resistance form $(\mathcal{E}, \mathcal{F})$ on $(K\backslash I) \cup I = K$ in terms of $\widetilde{\mathcal{F}}$.

To start with, the following lemma shows a relation between continuous functions on K and $\widetilde{K}$.

Lemma 5.1

$$C(K, R) = \{f | f : K \to \mathbb{R}, f \circ \rho \in C(\widetilde{K}, \widetilde{R})\}$$

Proof Obviously $C(K, R) \subseteq \{f | f : K \to \mathbb{R}, f \circ \rho \in C(\widetilde{K}, \widetilde{R})\}$. Conversely, let $f : K \to \mathbb{R}$ satisfying $f \circ \rho \in C(\widetilde{K}, \widetilde{R})$. Assume that $\{x_n\}_{n \geq 1} \subseteq K$ and $R(x_n, x) \to 0$ as $n \to \infty$ for some $x \in K$. Since $\rho^{-1}(x)$ consists of two points at most, let $\rho^{-1}(x) = \{z_1, z_2\}$. Choose $y_n \in \rho^{-1}(x_n)$ for each $n \geq 1$. Suppose

$$\limsup_{n \to \infty} \min\{\widetilde{R}(y_n, z_1), \widetilde{R}(y_n, z_2)\} > 0.$$

Then there exists a subsequence $\{y_{n_i}\}$ and $z \notin \rho^{-1}(x)$ such that $\widetilde{R}(y_{n_i}, z) \to 0$ as $i \to \infty$. This contradicts the fact that $\rho(y_{n_i}) = x_{n_i} \to x \neq \rho(z)$ as $i \to \infty$. Hence

$$\lim_{n \to \infty} \min\{\widetilde{R}(y_n, z_1), \widetilde{R}(y_n, z_2)\} = 0$$

Since $\widetilde{K}$ is compact and $f \circ \rho$ is uniformly continuous, this implies

$$\lim_{n \to \infty} \min\{|f \circ \rho(y_n) - f \circ \rho(z_1)|, |f \circ \rho(y_n) - f \circ \rho(z_2)|\} = 0.$$

This immediately yields that $\lim_{n \to \infty} |f(x_n) - f(x)| = 0$. Hence we have shown $f \in C(K, R)$. □

The following notions are used in an alternative expression of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$.

Definition 5.2 Let

$$\mathcal{A} = \{f | f : K\backslash I \to \mathbb{R}, f \circ F_{w2} \in \mathcal{F} \text{ for any } w \in T\}.$$

For $f \in \mathcal{A}$, define

$$\widehat{\mathcal{E}}^{(n)}(f, f) = \sum_{m=1}^{n} \left(\frac{5}{3}\right)^m \sum_{w \in \{0,1\}^{m-1}} \mathcal{E}(f \circ F_{w2}, f \circ F_{w2}).$$

Theorem 5.3

$$\widetilde{\mathcal{F}} = \left\{ f \,\middle|\, f \in \mathcal{A}, \lim_{n \to \infty} \widehat{\mathcal{E}}^{(n)}(f, f) < \infty \right\}$$

and

$$\widetilde{\mathcal{E}}(f, f) = \lim_{n \to \infty} \widehat{\mathcal{E}}^{(n)}(f, f) \quad (5.1)$$

for any $f \in \widetilde{\mathcal{F}}$. *Moreover, for any* $f \in \widetilde{\mathcal{F}}$ *and* $w \in T$, $f \circ F_w \in \widetilde{\mathcal{F}}$ *and*

$$\widetilde{\mathcal{E}}(f, f) = \widehat{\mathcal{E}}^{(n)}(f, f) + \left(\frac{5}{3}\right)^n \sum_{w \in \{0,1\}^n} \widetilde{\mathcal{E}}(f \circ F_w, f \circ F_w) \tag{5.2}$$

Remark The above theorem shows that if $f \in \mathcal{A}$ and $\lim_{n\to\infty} \widehat{\mathcal{E}}^{(n)}(f, f) < \infty$, then f can be extended to a continuous function on $\widetilde{K}$.

Proof For $f \in \ell(\widetilde{V}_m)$, it follows that

$$\widetilde{\mathcal{E}}_m(f, f) = \sum_{k=1}^{m} \left(\frac{5}{3}\right)^k \sum_{w \in \{0,1\}^{k-1}} \mathcal{E}_{m-k}(f \circ F_{w2}, f \circ F_{w2}). \tag{5.3}$$

Since $\mathcal{E}_m(g, g) \leq \mathcal{E}(g, g)$ for any $g \in \mathcal{F}$, the above equality implies

$$\widetilde{\mathcal{E}}_m(f, f) \leq \widehat{\mathcal{E}}^{(m)}(f, f) \tag{5.4}$$

for any $f \in \mathcal{A}$ and $m \geq 1$. Assume that $f \in \mathcal{A}$ and $\lim_{n\to\infty} \widehat{\mathcal{E}}^{(n)}(f, f) < \infty$. Taking $m \to \infty$ in (5.4), we see that

$$\widetilde{\mathcal{E}}(f, f) = \lim_{m\to\infty} \widehat{\mathcal{E}}^{(m)}(f, f) < \infty.$$

Thus it follows that $f \in \widetilde{\mathcal{F}}$. Again by (5.3), for a fixed n,

$$\sum_{k=1}^{n} \left(\frac{5}{3}\right)^k \sum_{w \in \{0,1\}^{k-1}} \mathcal{E}_{m-k}(f \circ F_{w2}, f \circ F_{w2}) \leq \widetilde{\mathcal{E}}_m(f, f).$$

This implies

$$\sum_{k=1}^{n} \left(\frac{5}{3}\right)^k \sum_{w \in \{0,1\}^{k-1}} \mathcal{E}(f \circ F_{w2}, f \circ F_{w2}) \leq \widetilde{\mathcal{E}}(f, f).$$

Therefore, we have (5.1).

Conversely, assume that $f \in \widetilde{\mathcal{F}}$. Then by (5.3),

$$\limsup_{n\to\infty} \mathcal{E}_n(f \circ F_{w2}, f \circ F_{w2}) \leq \widetilde{\mathcal{E}}(f, f)$$

for any $w \in T$. Hence $f \in \mathcal{A}$. The deduction of (5.1) is entirely the same as above.

Note that if $w \in \{0, 1\}^{m-1}$ and $m \geq n + 1$, then $w = uv$ for some $u \in \{0, 1\}^n$ and $v \in \{0, 1\}^{k-1}$ with $n + k = m$. Hence

$$\begin{aligned}
\widetilde{\mathcal{E}}(f, f) &= \widehat{\mathcal{E}}^{(n)}(f, f) + \sum_{m \geq n+1} \left(\frac{5}{3}\right)^m \sum_{w \in \{0,1\}^{m-1}} \mathcal{E}(f \circ F_{w2}, f \circ F_{w2}) \\
&= \widehat{\mathcal{E}}^{(n)}(f, f) + \sum_{u \in \{0,1\}^n} \sum_{k \geq 1} \left(\frac{5}{3}\right)^{n+k} \sum_{v \in \{0,1\}^{k-1}} \mathcal{E}(f \circ F_{uv2}, f \circ F_{uv2}) \\
&= \widehat{\mathcal{E}}^{(n)}(f, f) + \left(\frac{5}{3}\right)^n \sum_{u \in \{0,1\}^n} \widetilde{\mathcal{E}}(f \circ F_u, f \circ F_u).
\end{aligned}$$

Hence we have (5.2). □

The next characterization of $(\mathcal{E}, \mathcal{F})$ in terms of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is one of the main results of this paper.

Theorem 5.4

$$\mathcal{F} = \{f | f : K \to \mathbb{R}, f \circ \rho \in \widetilde{\mathcal{F}}\}$$

and

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \widehat{\mathcal{E}}^{(n)}(f, f) = \widetilde{\mathcal{E}}(f\circ\rho, f\circ\rho)$$

for any $f \in \mathcal{F}$.

Since $\rho(x) = x$ on $K\backslash I$, which is dense in (K, D) and in $(\widetilde{K}, \widetilde{D})$, the above theorem says that $(\mathcal{E}, \mathcal{F})$ is an extension of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$, i.e. $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$ and $\mathcal{E} = \widetilde{\mathcal{E}}|_{\mathcal{F}\times\mathcal{F}}$.

To give a proof of the above theorem, we need the notion of energy measures associated with a local regular Dirichlet form. For a moment, let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form on $L^2(X, \mu)$. For simplicity, we assume that a metric space (X, d) is compact and $\mu(X) < \infty$. Then for any $f \in \mathcal{F}$, it is known that there exists a Radon measure ν_f on X such that

$$\int_X g d\nu_f = 2\mathcal{E}(fg, f) - \mathcal{E}(f^2, g)$$

for any $g \in \mathcal{F}$. The measure ν_f is called the energy measure of f. See [2, Sect. 3.2] for details and the general theory. In our case, the energy measures associated with the standard resistance form $(\mathcal{E}, \mathcal{F})$ were thoroughly studied initially by Kusuoka in [14]. It is known that there exists a Radon measure ν_*, which is now called the Kusuoka measure, on K such that ν_f is absolutely continuous with respect to ν_* for any $f \in \mathcal{F}$.

Lemma 5.5 *Define* ν_f *as the energy measure of* $f \in \mathcal{F}$. *Then* $\nu_f(I) = 0$ *for any* $f \in \mathcal{F}$.

Proof As is mentioned above, for any $f \in \mathcal{F}$, the energy measure ν_f is absolutely continuous with respect to ν_*. So, it is enough to show that $\nu_*(I) = 0$. Furthermore, $\nu_* = \nu_{h_1} + \nu_{h_2}$ for some harmonic functions h_1 and h_2. See [7, (3.2) and Proposition 5.4] for details. Consequently, if $\nu_h(I) = 0$ for any harmonic function h, then the lemma is shown. Note that the space of harmonic functions $\mathcal{H}$ is three-dimensional. Let ψ_i be the harmonic function of K with $\psi_i(p_i) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then

$$\mathcal{H} = \{a_1\psi_1 + a_2\psi_2 + a_3\psi_3 | a_1, a_2, a_3 \in \mathbb{R}\}.$$

Now for any $h \in \mathcal{H}$ with $\mathcal{E}(h, h) \neq 0$,

$$\begin{aligned}\mathcal{E}(h, h) &= \frac{5}{3}\big(\mathcal{E}(h\circ F_0, h\circ F_0) + \mathcal{E}(h\circ F_1, h\circ F_1) + \mathcal{E}(h\circ F_2, h\circ F_2)\big)\\ &> \frac{5}{3}\big(\mathcal{E}(h\circ F_0, h\circ F_0) + \mathcal{E}(h\circ F_1, h\circ F_1)\big)\end{aligned}$$

Let $\mathcal{H}_1 = \{h | h \in \mathcal{H}, \mathcal{E}(h, h) = 1, h(p_1) = 0\}$. Then $\mathcal{H}_1$ is compact. Replacing h by $(h - h(p_1))/\sqrt{\mathcal{E}(h, h)}$ in the above inequality and taking the supremum of the right hand side over $h \in \mathcal{H}_1$, we see that there exists $c \in (0, 1)$ such that

$$\frac{5}{3}\big(\mathcal{E}(h\circ F_0, h\circ F_0) + \mathcal{E}(h\circ F_1, h\circ F_1)\big) \leq c\mathcal{E}(h, h).$$

Iterating this, we obtain

$$\left(\frac{5}{3}\right)^m \sum_{w\in\{0,1\}^m} \mathcal{E}(f\circ F_w.f\circ F_w) \leq c^m\mathcal{E}(h, h)$$

Set $I_m = \cup_{w\in\{0,1\}^m} K_w$. Then the left-hand side of the above inequality coincides with $\nu_h(I_m)$. Thus

$$\nu_h(I_m) \le c^m \nu_h(K).$$

Letting $m \to \infty$, we see that $\nu_h(I) = 0$. □

One of the ingredients of our proof of Theorem 5.4 is the use of the energies associated with G and $G \cup T$, which are illustrated in Fig. 5. In particular, $G \cup T$ is a tree and hence the calculation of effective resistances between points is straightforward.

Definition 5.6 (1) Define

$$\mathcal{E}_G^{(n)}(f, f) = \sum_{k=1}^{n} \left(\frac{5}{3}\right)^k \sum_{w\in\{0,1\}^{k-1}} \mathcal{E}_0(f \circ F_{w2}, f \circ F_{w2})$$

for $f \in \ell(G_n)$.
(2) Define

$$\mathcal{E}_{T\cup G}^{(n)}(f, f) = \sum_{w\in T_n} \left(\frac{5}{3}\right)^{|w|} 5 \sum_{i=0,1,2} (f(F_{w2}(p_i)) - f(w))^2$$

for $f \in \ell(G_n \cup T_n)$.

Lemma 5.7 (1) *If $m \ge n$, then $\widetilde{\mathcal{E}}_m|_{G_n} = \mathcal{E}_G^{(n)}$.*
(2) *For any $n \ge 1$, $\mathcal{E}_{T\cup G}^{(n)}|_{G_n} = \mathcal{E}_G^{(n)}$.*
(3) *Set $q_{n,i} = F_{(i)_{n-1}2}(p_i)$. Let $G_n^b = \{p_2, q_{n,0}, q_{n,1}\}$. Define*

$$r_n = \frac{4}{5} - \left(\frac{3}{5}\right)^n,\ R_n = r_n + \frac{2}{5},\ \text{and}\ S_n = 5r_nR_n.$$

Then

$$\mathcal{E}_G^{(n)}|_{G_n^b}(f, f) = \frac{1}{R_n}\big((f(p_2) - f(q_{n,0}))^2 + (f(p_2) - f(q_{n,1}))^2\big) + \frac{1}{S_n}(f(q_{n,0}) - f(q_{n,1}))^2 \tag{5.5}$$

Proof (1) Note that $\mathcal{E}_m|_{V_0} = \mathcal{E}_0$ for any $m \ge 1$. Therefore, (5.3) suffices.
(2) Applying Δ-Y transform ([9, Lemma 2.1.15]), we obtain the desired result.
(3) By (2), it follows that $\mathcal{E}_G^{(n)}|_{G_n^b} = \mathcal{E}_{T\cup G}^{(n)}|_{G_n^b}$. Note that the weighted graph associated with $\mathcal{E}_{T\cup G}^{(n)}$ is a tree. Let $\widetilde{G}_n^b = G_n^b \cup \{\phi\}$. Then

$$\mathcal{E}_{T\cup G}^{(n)}|_{\widetilde{G}_n^b}(f, f) = 5(f(p_2) - f(\phi))^2 + \frac{1}{\alpha_n}\big((f(\phi) - f(q_{n,0}))^2 + (f(\phi) - f(q_{n,1})^2\big),$$

where $\alpha_n = \frac{1}{5} + \frac{2}{5}\left(\frac{3}{5} + \cdots + \left(\frac{3}{5}\right)^{n-1}\right) = r_n$. Applying the Δ-Y transform, we verify (5.5). □

Lemma 5.8 *For $f : \widetilde{K} \to \mathbb{R}$ and $w \in T$, define*

$$Q_w(f, f) = \frac{5}{6}((f(F_w(p_2)) - f(w\bar{0}))^2 + (f(F_w(p_2)) - f(w\bar{1}))^2) + \frac{5}{24}(f(w\bar{0}) - f(w\bar{1}))^2.$$

Then for any $f \in \widetilde{\mathcal{F}}$,

$$\widehat{\mathcal{E}}^{(n)}(f,f) + \left(\frac{5}{3}\right)^n \sum_{w\in\{0,1\}^n} Q_w(f,f) \le \widetilde{\mathcal{E}}(f,f).$$

In particular, for any $f \in \widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}_n(f,f) + \left(\frac{5}{3}\right)^n \sum_{w\in\{0,1\}^n} Q_w(f,f) \le \widetilde{\mathcal{E}}(f,f). \tag{5.6}$$

Proof By Lemma 5.7, $\widetilde{\mathcal{E}}_n|_{G_n^b} = \mathcal{E}_G^{(n)}|_{G_n^b}$. Hence by (5.5),

$$\begin{aligned}&\widetilde{\mathcal{E}}_n(f,f)\\ &\ge \frac{1}{R_n}\big((f(p_2)-f(q_{n,0}))^2 + (f(p_2)-f(q_{n,1}))^2\big) + \frac{1}{S_n}(f(q_{n,0})-f(q_{n,1}))^2.\end{aligned}$$

Letting $n \to \infty$, we see that $\widetilde{\mathcal{E}}(f,f) \ge Q_\phi(f,f)$ for any $f \in \widetilde{\mathcal{F}}$. Combining this with (5.2), we have the desired inequality. □

Proof of Theorem 5.4 If $f \in \mathcal{F}$, then by (5.3),

$$\widetilde{\mathcal{E}}_m(f\circ\rho, f\circ\rho) = \sum_{k=1}^m \left(\frac{5}{3}\right)^k \sum_{w\in\{0,1\}^{k-1}} \mathcal{E}_{m-k}(f\circ F_{w2}, f\circ F_{w2}) \le \mathcal{E}_m(f,f).$$

Taking $m \to \infty$, we see that

$$\widetilde{\mathcal{E}}(f\circ\rho, f\circ\rho) = \lim_{n\to\infty} \mathcal{E}^{(n)}(f,f) \le \mathcal{E}(f,f)$$

and hence $f\circ\rho \in \widetilde{\mathcal{F}}$. Define

$$K^{(n)} = \bigcup_{m\ge 1}^{n} \bigcup_{v\in\{0,1\}^{m-1}} K_{v2}.$$

Then

$$\widehat{\mathcal{E}}^{(n)}(f,f) = \int_{K^{(n)}} \nu_f(dx).$$

Lemma 5.5 implies

$$\mathcal{E}(f,f) = \nu(K) = \nu\left(\bigcup_{n\ge 0} K^{(n)}\right) = \lim_{n\to\infty} \widehat{\mathcal{E}}^{(n)}(f,f).$$

Next assume $f : K \to \mathbb{R}$ and $f\circ\rho \in \widetilde{\mathcal{F}}$. Then Lemma 5.1 implies that $f \in C(K,R)$. Define

$$Q_0(f,f) = \frac{5}{6}((f(p_2)-fp_0)^2 + (f(p_2)-f(p_1))^2) + \frac{5}{24}(f(p_0)-f(p_1))^2,$$

Then for any $w \in T$,

$$Q_w(f\circ\rho, f\circ\rho) = Q_0(f\circ F_w, f\circ F_w).$$

By (5.6)

$$\frac{5}{24}\mathcal{E}_m(f,f) \le \widetilde{\mathcal{E}}_m(f,f) + \left(\frac{5}{3}\right)^m \sum_{w\in\{0,1\}^m} Q_0(f\circ F_w, f\circ F_w) \le \widetilde{\mathcal{E}}(f\circ\rho, f\circ\rho).$$

Thus $\lim_{m\to\infty}\mathcal{E}_m(f,f) < \infty$, so that $f \in \mathcal{F}$. □

6 Traces of two resistance forms on the boundaries

The main purpose of this section is to determine the jump kernel $J_*(x,y)$ of the trace of $(\mathcal{E},\mathcal{F})$ on I. Due to Theorem A.4, $(\mathcal{E}|_I, \mathcal{F}|_I)$ and $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T})$ are resistance forms on I and Σ_T respectively. So by Theorem A.2, both resistance forms induce Hunt processes, which are jump processes in fact, on I and Σ_T respectively. In light of Theorem 5.4, we see that

$$\mathcal{F}|_I = \{f | f : I \to \mathbb{R}, f\circ\rho \in \widetilde{\mathcal{F}}|_{\Sigma_T}\}$$

and

$$\mathcal{E}|_I(f,f) = \widetilde{\mathcal{E}}|_{\Sigma_T}(f\circ\rho, f\circ\rho).$$

Hence to know $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T})$ is to know $(\mathcal{E}|_I, \mathcal{F}|_I)$, and we do know $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}_{\Sigma_T})$ rather well as follows.

Theorem 6.1 *For $\omega, \tau \in \Sigma_T$ with $\omega \ne \tau$, define*

$$J(\omega,\tau) = \frac{35}{16}\left(\frac{14}{17}\left(\frac{20}{3}\right)^{n(\omega,\tau)} + \frac{3}{17}\right).$$

and let ν be the self-similar measure on Σ_T with weight $\left(\frac{1}{2},\frac{1}{2}\right)$. Then

$$\widetilde{\mathcal{F}}|_{\Sigma_T} = \left\{ f \,\middle|\, f \in L^2(\Sigma_T,\mu), \int_{\Sigma_T\times\Sigma_T} J(\omega,\tau)(f(\omega)-f(\tau))^2\nu(d\omega)\nu(d\tau) < \infty \right\}$$

and

$$\widetilde{\mathcal{E}}|_{\Sigma_T}(f,f) = \int_{\Sigma_T\times\Sigma_T} J(\omega,\tau)(f(\omega)-f(\tau))^2\nu(d\omega)\nu(d\tau).$$

Moreover, let $p_{\Sigma_T} : (0,\infty)\times\Sigma_T\times\Sigma_T \to [0,\infty)$ be the jointly continuous transition density associated with the Dirichlet form $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T})$ on $L^2(\Sigma_T,\nu)$. Then there exist $c_1, c_2 > 0$ such that

$$c_1 \min\left\{\frac{t}{\delta_{\frac{1}{2}}(\omega,\tau)^{\alpha+2}}, t^{-\frac{1}{\alpha+1}}\right\} \le p_{\Sigma_T}(t,\omega,\tau) \le c_2 \min\left\{\frac{t}{\delta_{\frac{1}{2}}(\omega,\tau)^{\alpha+2}}, t^{-\frac{1}{\alpha+1}}\right\}$$

for any $(t,\omega,\tau) \in (0,\infty)\times\Sigma_T\times\Sigma_T$, where $\alpha = \frac{\log 5 - \log 3}{\log 2}$ is the exponent appearing in (3.1).

The existence of the transition density p_{Σ_T} is included in Theorem A.2.

Remark The same expression of $J(\omega,\tau)$ was obtained in [17].

Note that there exist $c_3, c_4 > 0$ such that

$$c_3 \frac{1}{\delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+2}} \leq J(\omega, \tau) \leq c_4 \frac{1}{\delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+2}}$$

for any $\omega, \tau \in \Sigma_T$. Moreover,

$$\min\left\{\frac{t}{\delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+2}}, t^{-\frac{1}{\alpha+1}}\right\} = \begin{cases} \dfrac{t}{\delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+2}} & \text{if } t \leq \delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+1}, \\ t^{-\frac{1}{\alpha+1}} & \text{if } t \geq \delta_{\frac{1}{2}}(\omega, \tau)^{\alpha+1}. \end{cases}$$

A proof of the above theorem will be given later in this section. Meanwhile, we present an expression of $(\mathcal{E}|_I, \mathcal{F}|_I)$ as an immediate corollary of Theorems 5.4 and 6.1.

Corollary 6.2 *Define* $J_* : I \times I \to [0, \infty)$ *by*

$$J_*(x, y) = \max_{\omega \in \pi^{-1}(x), \tau \in \pi^{-1}(y)} J(\omega, \tau)$$

for $x, y \in I$. *Then*

$$\mathcal{F}|_I = \left\{ f \,\middle|\, f \in L^2(I, \nu_*), \int_{I\times I} J_*(x, y)(f(x) - f(y))^2 \nu_*(dx)\nu_*(dy) < \infty \right\}$$

and

$$\mathcal{E}|_I(f, f) = \int_{I\times I} J_*(x, y)(f(x) - f(y))^2 \nu_*(dx)\nu_*(dy).$$

for any $f \in \mathcal{F}|_I$. *Moreover, let* $p_I : (0, \infty) \times I \times I \to [0, \infty)$ *be the jointly continuous transition density associated with the Dirichlet form* $(\mathcal{E}|_I, \mathcal{F}|_I)$ *on* $L^2(I, \nu_*)$. *Then there exist* $c_5, c_6 > 0$ *such that*

$$p_I(t, x, y) \leq c_5 \min\left\{\frac{t}{|x - y|^{\alpha+2}}, t^{-\frac{1}{\alpha+1}}\right\} \tag{6.1}$$

for any $(t, x, y) \in (0, \infty) \times I \times I$ *and*

$$c_6 t^{-\frac{1}{\alpha+1}} \leq p_I(t, x, y) \tag{6.2}$$

if $t \geq |x - y|^{\alpha+1}$.

Set $B = \{\frac{i}{2^n} | n \geq 0, 0 \leq i \leq 2^n\}$. If both x and y do not belong to B, then $\pi^{-1}(x)$ and $\pi^{-1}(y)$ consist of a single point and we do not need to take the minimus in the above definition of J_*. Note that $\nu_*(B) = 0$. So, even if we define

$$J_*(x, y) = J(\pi^{-1}(x), \pi^{-1}(y)),$$

J_* makes sense as an element of $L^2(I \times I, \nu_* \times \nu_*)$.

Proof of Corollary 6.2 The expressions of $\mathcal{F}|_I$ and $\mathcal{E}|_I$ are immediate from Theorems 5.3 and 6.1. The existence and the continuity of $p_I(t, x, y)$ is due to Theorem A.2. Since there exists $c > 0$ such that

$$J_*(x, y) \leq \frac{c}{|x - y|^{\alpha+2}}$$

for any $x, y \in I$, we obtain (6.1) by using [4, Theorem 6.13]. The lower estimate (6.2) follows from [12, Theorems 15.6 and 15.13]. □

The rest of this section is devoted to a proof of Theorem 6.1. The main idea is to identify the trace of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on Σ_T as that of a weighted tree and to use the results of [11].

Definition 6.3 Define

$$\mathcal{F}_G = \left\{ f \,\middle|\, f : G \to \mathbb{R}, \lim_{n\to\infty} \mathcal{E}_G^{(n)}(f, f) < \infty \right\}$$

and

$$\mathcal{E}_G(f, f) = \lim_{n\to\infty} \mathcal{E}_G^{(n)}(f, f)$$

for $f \in \mathcal{F}_G$.

Note that

$$\mathcal{E}_G(f, f) = \sum_{k=1}^{\infty} \left(\frac{5}{3}\right)^k \sum_{w\in\{0,1\}^{k-1}} \mathcal{E}_0(f \circ F_{w2}, f \circ F_{w2}).$$

Lemma 6.4 *The closure of $(G, \widetilde{R})$ is $G \cup \Sigma_T$. In particular, if $f|_G = g|_G$ for $f, g \in \widetilde{\mathcal{F}}$, then $f|_{\Sigma_T} = g|_{\Sigma_T}$.*

Proof For any $\omega \in \Sigma_T$, let $x_n = F_{[\omega]_n 2}(p_2)$. Then $\widetilde{R}(x_n, \omega) \to 0$ as $n \to \infty$. Hence $\overline{G} \supseteq G \cup \Sigma_T$. Assume that there exists $x \in \overline{G} \backslash G \cup \Sigma_T$. Since $x \in (K\backslash I)\backslash G$, x belongs to $K_{w2}\backslash G$ for some $w \in T$. So, $K_{w2}\backslash G$ is an open neighborhood of x and this contradicts the fact that x is an accumulating point of G. Hence we have $\overline{G} = G \cup \Sigma_T$. □

Lemma 6.5 *For any $f : G \to \mathbb{R}$, define $h_G(f) : K\backslash I \to \mathbb{R}$ as*

$$h_G(f) \circ F_{w2} = \sum_{i\in\{0,1,2\}} f(F_{w2}(p_i))\psi_i$$

for each $w \in T$. Then the following conditions (1), (2)*, and* (3) *are equivalent to each other:*
(1) $f \in \mathcal{F}_G$*,*
(2) $h_G(f) \in \widetilde{\mathcal{F}}$*,*
(3) *There exists $g \in \widetilde{\mathcal{F}}$ such that $g|_G = f$.*
Furthermore, if $f = g|_G$ for some $g \in \widetilde{\mathcal{F}}$, then

$$\mathcal{E}_G(f, f) = \widetilde{\mathcal{E}}(h_G(f), h_G(f)) = \min\{\widetilde{\mathcal{E}}(g, g)|g \in \widetilde{\mathcal{F}}, g|_G = f\}. \tag{6.3}$$

Proof (1) ⇔ (2); By the definitions of $\mathcal{E}_G^{(n)}$ and $\widehat{\mathcal{E}}^{(n)}$, we have

$$\mathcal{E}_G^{(n)}(f, f) = \widehat{\mathcal{E}}^{(n)}(h_G(f), h_G(f)). \tag{6.4}$$

This immediately shows the equivalence of (1) and (2).
(2) ⇒ (3): Since $h_G(f)|_G = f$, this is obvious.
(3) ⇒ (2): Since $\mathcal{E}_0(g \circ F_{w2}, g \circ F_{w2}) \le \mathcal{E}(g \circ F_{w2}, g \circ F_{w2})$ for any $w \in T$, we see that

$$\mathcal{E}_G^{(n)}(f, f) \le \widehat{\mathcal{E}}^{(n)}(g, g)$$

for any $n \ge 1$. Letting $n \to \infty$, we obtain

$$\mathcal{E}_G(f, f) \le \widetilde{\mathcal{E}}(g, g) < \infty. \tag{6.5}$$

Hence $f \in \mathcal{F}_G$.

Finally, (6.4) and (6.5) suffice (6.3). □

By Lemmas 6.5, $f = h_G(f)|_G$ for any $f \in \mathcal{F}_G$. Since $h_G(f) \in \widetilde{\mathcal{F}}$, $h_G(f) \in C(\widetilde{K}, \widetilde{R})$. Hence f can be naturally regarded as an element of $C(G \cup \Sigma_T, \widetilde{R})$. In this manner, we think $\mathcal{F}_G$ as a subspace of $C(G \cup \Sigma_T, \widetilde{R})$ hereafter.

Lemma 6.6 $(\widetilde{\mathcal{E}}|_{G\cup\Sigma_T}, \widetilde{\mathcal{F}}|_{G\cup\Sigma_T}) = (\mathcal{E}_G, \mathcal{F}_G)$.

Definition 6.7 Define

$$\mathcal{F}_{G\cup T} = \left\{ f \,\middle|\, f : G \cup T \to \mathbb{R}, \lim_{n\to\infty} \mathcal{E}^{(n)}_{G\cup T}(f, f) < \infty \right\}$$

and

$$\mathcal{E}_{G\cup T} = \lim_{n\to\infty} \mathcal{E}^{(n)}_{G\cup T}(f, f).$$

The structure of the graph $G \cup T$ is illustrated in Fig. 5.

Lemma 6.8 *$(\mathcal{E}_{G\cup T}, \mathcal{F}_{G\cup T})$ is a resistance form on $G \cup T \cup \Sigma_T$ and*

$$(\mathcal{E}_{G\cup T}|_{G\cup\Sigma_T}, \mathcal{F}_{G\cup T}|_{G\cup\Sigma_T}) = (\mathcal{E}_G, \mathcal{F}_G).$$

Proof It is straightforward to see that $(\mathcal{E}_{G\cup T}, \mathcal{F}_{G\cup T})$ is a resistance form on $G\cup T$. Moreover, by the Δ-Y transform, it follows that $(\mathcal{E}_{G\cup T}|_G, \mathcal{F}_{G\cup T}|_G) = (\mathcal{E}_G, \mathcal{F}_G)$. Hence let R_0 be the resistance metric on $G \cup T$ associated with $(\mathcal{E}_{G\cup T}, \mathcal{F}_{G\cup T})$. Then $R_0|_{G\times G} = \widetilde{R}|_{G\times G}$. Since $\overline{G} = G \cup \Sigma_T$. we see that $\overline{G \cup T} \subseteq G \cup T \cup \Sigma_T$. Assume that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence of $(G \cup T, R_0)$ and that $\lim_{n\to\infty} x_n \notin G \cup T \cup \Sigma_T$. Then there exists a subsequence $\{x_{n_i}\}_{i\geq 1}$ such that $x_{n_i} \in T$ for any $i \geq 1$ and $|x_{n_i}| \to \infty$ as $i \to \infty$. Set $y_i = F_{x_{n_i}2}(p_2)$. Then $R_0(x_{n_i}, y_i) \leq \frac{1}{5}\left(\frac{3}{5}\right)^{|x_{n_i}|+1}$. Therefore $\lim_{i\to\infty} x_{n_i} = \lim_{n\to\infty} y_i$ and the limit belongs to $G \cup \Sigma_T$ because $\{y_i\}_{i\geq 1} \subseteq G$ and $\overline{G} = G \cup \Sigma_T$. This contradiction shows that $\overline{G \cup T} = G \cup T \cup \Sigma_T$. Therefore, $(\mathcal{E}_{G\cup T}, \mathcal{F}_{G\cup T})$ is a resistance form on $G \cup T \cup \Sigma_T$. □

Definition 6.9 For $f : T \to \mathbb{R}$, define

$$\mathcal{E}_T(f, f) = \sum_{w\in T} \frac{25}{8}\left(\frac{5}{3}\right)^{|w|}\big((f(w) - f(w0))^2 + (f(w) - f(w1))^2\big),$$

whose value can be ∞. Moreover, define

$$\mathcal{F}_T = \{f | f : T \to \mathbb{R}, \mathcal{E}_T(f, f) < \infty\}.$$

The structure of the tree T is illustrated in Fig. 5.

Lemma 6.10 $(\mathcal{E}_{G\cup T}|_T, \mathcal{F}_{G\cup T}|_T) = (\mathcal{E}_T, \mathcal{F}_T)$. *Moreover, $(\mathcal{E}_T, \mathcal{F}_T)$ is a resistance form on $T \cup \Sigma_T$ and $(\mathcal{E}_{G\cup T}|_{T\cup\Sigma_T}, \mathcal{F}_{G\cup T}|_{T\cup\Sigma_T}) = (\mathcal{E}_T, \mathcal{F}_T)$.*

Proof Note that $F_{w2}(p_i) = F_{wi2}(p_2)$ for any $w \in T$ and $i \in \{0, 1\}$. So, we have

$$\begin{aligned}\mathcal{E}_{G\cup T}(f, f) &= 5(f(p_2) - f(\phi))^2 \\ &+ 5\sum_{w\in T}\sum_{i\in\{0,1\}}\left(\frac{5}{3}\right)^{|w|}\left((f(w) - f(F_{w2}(p_i)))^2 + \frac{5}{3}(f(F_{w2}(p_i)) - f(wi))^2\right) \\ &\geq 5\sum_{w\in T}\sum_{i\in\{0,1\}}\left(\frac{5}{3}\right)^{|w|}\frac{5}{8}(f(w) - f(wi))^2 = \mathcal{E}_T(f, f),\end{aligned}$$

where the equality holds when $f(p_2) = f(\phi)$ and $f(F_{w2}(p_i)) = \frac{3}{8}f(w) + \frac{5}{8}f(wi)$ for any $w \in T$ and $i \in \{0, 1\}$. This yields that $(\mathcal{E}_{G\cup T}|_T, \mathcal{F}_{G\cup T}|_T) = (\mathcal{E}_T, \mathcal{F}_T)$. Since the closure of $G \cup T$ with respect to R_0 is $G \cup T \cup \Sigma_T$, it follows that $\overline{T} \subseteq G \cup T \cup \Sigma_T$, where $\overline{T}$ is the closure of T with respect to the metric R_0. Let $\{w(n)\}_{n\geq 1}$ be a Cauchy sequence in (T, R_0). If $\liminf_{n\to\infty} |w(n)| < \infty$, then there exists $w_* \in T$ such that $w(n) = w_*$ for infinitely many n. Therefore $\lim_{n\to\infty} w(n) = w_* \in T$. In case $|w(n)| \to \infty$ as $n \to \infty$, the limit does not belong to $G \cup T$ and hence it must belong to Σ_T. Thus we have shown that $\overline{T} = T \cup \Sigma_T$. This implies that $(\mathcal{E}_T, \mathcal{F}_T)$ is a resistance form on $T \cup \Sigma_T$. □

Proof of Theorem 6.1 By Lemmas 6.6, 6.8 and 6.10, we see that $(\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T}) = (\mathcal{E}_T, \mathcal{F}_T)$. Note that $(\mathcal{E}_T, \mathcal{F}_T)$ is a resistance form associated with a weighted tree (T, C), where $C : T \times T \to [0, \infty)$ given by

$$C(x, y) = \begin{cases} \frac{25}{8}\left(\frac{5}{3}\right)^{|w|} & \text{if } (x, y) \in \{(w, wi), (wi, w) | w \in T \text{ and } i \in \{0, 1\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

This weighted tree coincides with a constant multiple of a self-similar binary tree C_S with $(r_1, r_2) = \left(\frac{3}{5}, \frac{3}{5}\right)$ studied in [11, Sect. 9]. More precisely, $C = \frac{15}{8}C_S$. Set $r_w = \left(\frac{3}{5}\right)^{|w|}$ for $w \in \mathcal{T}$. Then $\sum_{n\geq 1} r_{[\omega]_n} < \infty$ for any $\omega \in \Sigma_T$. Therefore, by [11, Corollary 8.2], we see that $\mathrm{Cap}(\{\omega\}) > 0$ for any $\omega \in \Sigma_T$. Hence by [11, Theorem 8.3], it follows that Σ_T is identified as the Martin boundary of the random walk on T associated with the weighted tree (T, C). Moreover, let $(\mathcal{E}_{\Sigma_T}, \mathcal{F}_{\Sigma_T})$the natural quadartic form on the Martin boundary associated with the random walk. Then $(\mathcal{E}_{\Sigma_T}, \mathcal{F}_{\Sigma_T})$ is a resistance form on Σ_T and $(\mathcal{E}_{\Sigma_T}, \mathcal{F}_{\Sigma_T}) = (\mathcal{E}_T|_{\Sigma_T}, \mathcal{F}_T|_{\Sigma_T})$. Consequently we have $(\mathcal{E}_{\Sigma_T}, \mathcal{F}_{\Sigma_T}) = (\widetilde{\mathcal{E}}|_{\Sigma_T}, \widetilde{\mathcal{F}}|_{\Sigma_T})$. Using [11, Theorem 5.6], we may obtain an explicit expression of the jump kernel $J(\omega, \tau)$ associated with $(\mathcal{E}_{\Sigma_T}, \mathcal{F}_{\Sigma_T})$ as follows:

$$J(\omega, \tau) = \frac{1}{2}\left(\lambda_\phi + \sum_{m=0}^{n(\omega,\tau)-1} \frac{\lambda_{[\omega\wedge\tau]_{m+1}} - \lambda_{[\omega\wedge\tau]_m}}{\nu(\Sigma_{[\omega\wedge\tau]_m})}\right),$$

where ingredients λ_w and $\nu(\Sigma_w)$ can be obtained from the results in [11, Sect. 9]. In fact, we have

$$R_w = \frac{8}{35}\left(\frac{3}{5}\right)^{|w|}, \nu(\Sigma_w) = \left(\frac{1}{2}\right)^{|w|},$$
$$D_w = \nu(\Sigma_w)R_w = \frac{8}{35}\left(\frac{3}{10}\right)^{|w|}, \text{ and } \lambda_w = 1/D_w = \frac{35}{8}\left(\frac{10}{3}\right)^{|w|}.$$

As a result, we obtain the expression of $J(\omega, \tau)$. The results on $p_{\Sigma_T}(t, \omega, \tau)$ are due to [11, Theorems 7.3, 7.6 and 9.5]. □

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Appendix: Resistance forms and weighted graphs

This section presents some of the basics of resistance forms and weighted graphs used in this paper. First, we give the definition of resistance forms.

Definition A.1 Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if the following conditions (RF1) through (RF5) are satisfied:
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(f, f) = 0$ if and only if f is a constant function on X.
(RF2) Let $\sim$ be an equivalence relation on $\mathcal{F}$ defined by $f \sim g$ if and only if $f - g$ is a constant function on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a real Hilbert space.
(RF3) If $x \neq y \in X$, then there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.
(RF4) For any $x, y \in X$,

$$\sup_{f \in \mathcal{F}, \mathcal{E}(f,f) \neq 0} \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} < \infty.$$

(RF5) For any $f \in \mathcal{F}$, $\overline{f} \in \mathcal{F}$ and $\mathcal{E}(\overline{f}, \overline{f}) \leq \mathcal{E}(f, f)$, where $\overline{f}$ is given by

$$\overline{f}(x) = \begin{cases} 1 & \text{if } f(x) \geq 1, \\ 0 & \text{if } f(x) \leq 0, \\ f(x) & \text{if } 0 < f(x) < 1. \end{cases}$$

For a resistance form $(\mathcal{E}, \mathcal{F})$ on X, we denote the supremum in (RF4) by $R(x, y)$ and call it the resistance metric associated with $(\mathcal{E}, \mathcal{F})$.

The resistance metric R associated with a resistance form $(\mathcal{E}, \mathcal{F})$ on X is indeed a metric on X. See [9, Chapter 2] for details. The following theorem is a collection of important facts on resistance forms and associated Dirichlet forms given in [12, Part 1].

Theorem A.2 *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X, and let R be the associated resistance metric. Assume that (X, R) is compact. Let μ be a Borel regular measure on (X, R) satisfying $\mu(B_R(x, r)) > 0$ for any $x \in X$ and $r > 0$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(X, \mu)$. Moreover, let $(\{X_t\}_{t>0}, \{P_x\}_{x \in X})$ be the Hunt process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Then there exists a jointly continuous transition density $p(t, x, y)$, i.e.*

$$E_x(f(X_t)) = \int_X p(t, x, y) f(y) \mu(dy)$$

for any bounded measurable function f on X, $x \in X$, and $t > 0$. Furthermore, if $(\mathcal{E}, \mathcal{F})$ has the local property, i.e. $\mathcal{E}(f, g) = 0$ whenever $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$ for $f, g \in \mathcal{F}$, then $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(X, \mu)$.

One of the notions playing an important role in this paper is a trace of a resistance form.

Lemma A.3 [12, Lemma 8.2] *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For a non-empty subset $Y \subseteq X$, define*

$$\mathcal{F}|_Y = \{f|_Y | f \in \mathcal{F}\}.$$

Then for any $f \in \mathcal{F}|_Y$, there exists a unique $f_* \in \mathcal{F}$ such that $f_*|_Y = f$ and

$$\mathcal{E}(f_*, f_*) = \min\{\mathcal{E}(g, g) | g \in \mathcal{F}, g|_Y = f\}.$$

Moreover, if we define $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ be $h_Y(f) = f_*$, then h_Y is linear.

Theorem A.4 [12, Theorem 8.4] *Let $(\mathcal{E}, \mathcal{F})$ be resistance form on X and let R be the associated resistance metric. Define*

$$\mathcal{E}|_Y(f, g) = \mathcal{E}(h_Y(f), h_Y(g))$$

Then, $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a resistance form on Y and the associated resistance metric is the restriction of R to $Y \times Y$. $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is called the trace of the resistance form $(\mathcal{E}, \mathcal{F})$ on Y.

The rest of this appendix is devoted to a brief review of weighted graphs, which correspond to resistance forms on finite sets.

Definition A.5 Let V be a countable set and let $C : V \times V \to [0, \infty)$. (V, C) is said to be a weighted graph if $C(x, y) = C(y, x)$ for any $x, y \in V$ and $C(x, x) = 0$ for any $x \in V$. For a weighted graph (V, C), define

$$\mathcal{F}_C = \{f | f \in \ell(V), \sum_{x,y \in V} C(x, y)(f(x) - f(y))^2 < \infty\}.$$

and

$$\mathcal{E}_C(f, f) = \frac{1}{2} \sum_{x,y \in V} C(x, y)(f(x) - f(y))^2$$

for $f \in \mathcal{F}_C$. $(\mathcal{E}_C, \mathcal{F}_C)$ is called the Dirichlet form associated with (V, C). A weighted graph (V, C) is called connected if for any $x, y \in V$, there exist $x_1, \ldots, x_n \in V$ such that $x_1 = x$, $x_n = y$ and $C(x_i, x_{i+1}) > 0$ for any $i = 1, \ldots, n - 1$. Moreover, (V, C) is locally finite if $\{y | y \in V, C(x, y) > 0\}$ is a finite set for any $x \in V$.

Weighted graphs appearing in this paper are all connected and locally finite.

Verifying the conditions (RF1) through (RF5) in Definition A.1, we immediately obtain the following proposition.

Proposition A.6 *Let (V, C) be a connected and locally finite weighted graph. Then*

$$\sup_{f \in \mathcal{F}_C, \mathcal{E}_C(f,f) \neq 0} \frac{|f(x) - f(y)|^2}{\mathcal{E}_C(f, f)} < \infty$$

for any $x, y \in V$. If $R_C(x, y)$ is the above supremum, then R_C is a metric on V. In particular, $(\mathcal{E}_C, \mathcal{F}_C)$ is a resistance form on K and R_C is the associated resistance metric.

For the time being, we only deal with the case where V is a finite set. In such a case, $\mathcal{F}_C = \ell(V)$ and (V, C) is always locally finite.

Remark Assume that V is a finite set. For a weighted graph (V, C), define $H_C = (H_C(x, y))_{x,y\in V}$ as the non-negative matrix satisfying

$$\mathcal{E}_C(f, f) = (f, H_C f)_V$$

for any $f \in \ell(V)$, where $(\cdot, \cdot)_V$ is the standard inner-product defined by $(f, g) = \sum_{x\in V} f(x)g(x)$. In [9], H_C is called a Laplacian on V and the pair (V, H_C) is called a resistance network on V. See [9, Definition 2.1.2]. In fact, the correspondence $C \leftrightarrow \mathcal{E}_C \leftrightarrow H_C$ gives a natural one-to-one correspondence between weighted graphs, Dirichlet forms, and Laplacians on a finite set. See [9, Sect. 2.1] for precise statements on the correspondence between Laplacians and Dirichlet forms.

Definition A.7 (1) Let (U_1, C_1) and (U_2, C_2) be connected weighted graphs on finite sets. We write $(U_1, C_1) \leq (U_2, C_2)$ if $U_1 \subseteq U_2$ and

$$\mathcal{E}_{C_1}(f, f) = \min\{\mathcal{E}_{C_2}(g, g)|g \in \ell(U_2), g|_{U_1} = f\}$$

for any $f \in \ell(U_1)$.

(2) Let $\{(U_m, C_m)\}_{m\geq 0}$ be a sequence of connected weighted graphs on finite sets. The sequence $\{(U_m, C_m)\}_{m\geq 0}$ is said to be compatible if $(U_m, C_m) \leq (U_{m+1}, C_{m+1})$ for any $m \geq 0$.

Proposition A.8 *Let (U_1, C_1) and (U_2, C_2) be connected weighted graphs on finite sets. If $(U_1, C_1) \leq (U_2, C_2)$, then*

$$R_{C_1}(x, y) = R_{C_2}(x, y)$$

for any $x, y \in U_1$.

Combining [9, Theorem 2.2.6] and [9, Theorem 2.3.10], we have the following theorem, which is a principal tool to construct a resistance form out of a sequence of weighted graphs.

Theorem A.9 *Let $\{(U_m, C_m)\}_{m\geq 0}$ be a compatible sequence of connected weighted graphs on finite sets. Set $U_* = \cup_{m\geq 0} U_m$. Define*

$$\mathcal{F} = \left\{ f \,\middle|\, f \in \ell(U_*), \lim_{m\to\infty} \mathcal{E}_{C_m}(f|_{U_m}, f|_{U_m}) < \infty \right\}$$

and

$$\mathcal{E}(f, f) = \lim_{m\to\infty} \mathcal{E}_{C_m}(f, f)$$

for $f \in \mathcal{F}$. Furthermore, for $x, y \in U_*$, define $R(x, y) = R_m(x, y)$, where we choose m such that $x, y \in U_m$. Then R is a metric on U_* and if (X, R) be the completion of (U_*, R), then any $f \in \mathcal{F}$ is extended to a continuous function on (X, R), $(\mathcal{E}, \mathcal{F})$ is a resistance form on X, and the resistance metric associated with $(\mathcal{E}, \mathcal{F})$ is R.

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