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Bergman projection and BMO in hyperbolic metric: improvement of classical result

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Abstract

The Bergman projection P_α, induced by a standard radial weight, is bounded and onto from L^∞ to the Bloch space $\mathcal{B}$. However, $P_\alpha : L^\infty \to \mathcal{B}$ is not a projection. This fact can be emended via the boundedness of the operator $P_\alpha : \mathrm{BMO}_2(\Delta) \to \mathcal{B}$, where $\mathrm{BMO}_2(\Delta)$ is the space of functions of bounded mean oscillation in the Bergman metric. We consider the Bergman projection P_ω and the space $\mathrm{BMO}_{\omega,p}(\Delta)$ of functions of bounded mean oscillation induced by $1 < p < \infty$ and a radial weight $\omega \in \mathcal{M}$. Here $\mathcal{M}$ is a wide class of radial weights defined by means of moments of the weight, and it contains the standard and the exponential-type weights. We describe the weights such that $P_\omega : \mathrm{BMO}_{\omega,p}(\Delta) \to \mathcal{B}$ is bounded. They coincide with the weights for which $P_\omega : L^\infty \to \mathcal{B}$ is bounded and onto. This result seems to be new even for the standard radial weights when $p \neq 2$.

Keywords Bergman projection · Bergman metric · Bergman space · Bloch space · Doubling weight · Hankel operator · Mean oscillation

Mathematics Subject Classification Primary 30H20 · 30H35 · 47B34

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1 Introduction and main results

It is well-known that the Bergman projection P_α, induced by the standard weight $(\alpha+1)(1-|z|^2)^\alpha$, is bounded and onto from L^∞ to the Bloch space $\mathcal{B}$ [6, Section 5.1]. This is a very useful result with a large variety of applications in the operator theory on spaces of analytic functions on $\mathbb{D}$. However, the operator $P_\alpha : L^\infty \to \mathcal{B}$ is in fact not a projection because of the strict inclusion $H^\infty \subsetneq \mathcal{B}$. This downside can be emended by replacing L^∞ by the space $\mathrm{BMO}_2(\Delta)$ of functions of bounded mean oscillation in the Bergman metric [6, Section 8.1]. It is known that the analytic functions in $\mathrm{BMO}_2(\Delta)$ constitute the Bloch space $\mathcal{B}$ [6, Theorem 8.7], and it is a folklore result that $P_\alpha : \mathrm{BMO}_2(\Delta) \to \mathcal{B}$ is bounded. Professor Kehe Zhu kindly offered us the following proof:

If $f \in \mathrm{BMO}_2(\Delta)$, then the big Hankel operators $H_f^\alpha(g) = (I-P_\alpha)(fg)$ and $H_{\overline{f}}^\alpha(g) = (I-P_\alpha)(\overline{f}g)$ are both bounded on the Bergman space A^2_α by [6, Section 8.1], and therefore so are the little Hankels $h_f^\alpha(g) = \overline{P_\alpha(fg)}$ and $h_{\overline{f}}^\alpha(g) = \overline{P_\alpha(\overline{f}g)}$. Now that $h_{\overline{f}}^\alpha = h^\alpha_{\overline{P_\alpha(f)}}$, and the little Hankel operator $h_{\overline{\varphi}}^\alpha$, induced by an analytic symbol φ, is bounded on A^2_α if and only if $\varphi \in \mathcal{B}$ by [6, Section 8.7], it follows that $P_\alpha(f) \in \mathcal{B}$, whenever $f \in \mathrm{BMO}_2(\Delta)$. Since this argument preserves the information on the norms, it follows that $P_\alpha : \mathrm{BMO}_2(\Delta) \to \mathcal{B}$ is bounded.

In this paper we are interested in understanding the nature of a space X of complex-valued functions such that $X \cap \mathcal{H}(\mathbb{D}) = \mathcal{B}$, and radial weights ω for which the Bergman projection $P_\omega : X \to \mathcal{B}$ is bounded. Here, as usual, $\mathcal{H}(\mathbb{D})$ stands for the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We proceed towards the statements via necessary notation.

For a non-negative function $\omega \in L^1([0,1))$, its extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and such an ω, the Lebesgue space L^p_ω consists of complex-valued measurable functions f on $\mathbb{D}$ such that

$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$. For a radial weight ω, the orthogonal Bergman projection P_ω from L^2_ω to A^2_ω is

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\,\omega(\zeta) dA(\zeta),$$

where B_z^ω are the reproducing kernels of the Hilbert space A^2_ω. It has been recently shown in [5, Theorems 1-2-3] that the Bergman projection P_ω, induced by a radial weight ω, is bounded from L^∞ to the Bloch space $\mathcal{B}$ if and only if $\omega \in \widehat{\mathcal{D}}$, while the Bloch space is continuously embedded into $P_\omega(L^\infty)$ if and only if $\omega \in \mathcal{M}$. Therefore, $P_\omega : L^\infty \to \mathcal{B}$ is bounded and onto if and only $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \mathcal{M}$. Recall that a radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \le C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \le r < 1$, while $\omega \in \mathcal{M}$ if $\omega_x \ge C\omega_{Kx}$, for all $x \ge 1$, for some $C = C(\omega) > 1$ and $K = K(\omega) > 1$. Here and from now on $\omega_x = \int_0^1 r^x \omega(r)\, dr$, for all $x \ge 0$.

Let $\beta(z,\zeta)$ denote the hyperbolic distance between the points z and ζ in $\mathbb{D}$, and let $\Delta(z,r)$ stand for the hyperbolic disc of center $z \in \mathbb{D}$ and radius $0 < r < \infty$. Further, let ω be a radial weight and $0 < r < \infty$ such that $\omega(\Delta(z,r)) > 0$ for all $z \in \mathbb{D}$. Then, for $f \in L^p_\omega$

with $1 \le p < \infty$, write

$$\mathrm{MO}_{\omega,p,r}(f)(z) = \left(\frac{1}{\omega(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \widehat{f}_{r,\omega}(z)|^p \omega(\zeta)\, dA(\zeta) \right)^{\frac{1}{p}},$$

where

$$\widehat{f}_{r,\omega}(z) = \frac{\int_{\Delta(z,r)} f(\zeta)\omega(\zeta)\, dA(\zeta)}{\omega(\Delta(z,r))}, \quad z \in \mathbb{D}.$$

The space $\mathrm{BMO}(\Delta)_{\omega,p,r}$ consists of $f \in L^p_\omega$ such that

$$\|f\|_{\mathrm{BMO}(\Delta)_{\omega,p,r}} = \sup_{z\in\mathbb{D}} \mathrm{MO}_{\omega,p,r}(f)(z) < \infty.$$

It is known by [4, Theorem 11] that for each $\omega \in \mathcal{D}$ there exists $r_0 = r_0(\omega) > 0$ such that

$$\mathrm{BMO}(\Delta)_{\omega,p,r} = \mathrm{BMO}(\Delta)_{\omega,p,r_0}, \quad r \ge r_0. \tag{1.1}$$

We call this space $\mathrm{BMO}(\Delta)_{\omega,p}$ whenever (1.1) holds, and assume that the norm is always calculated with respect to a fixed $r \ge r_0$. However, in contrast to the class $\mathcal{D}$, for each prefixed $r > 0$, the quantity $\omega(\Delta(z,r))$ may equal to zero for some z arbitrarily close to the boundary if $\omega \in \widehat{\mathcal{D}}$, by Proposition 3 below. Therefore the space $\mathrm{BMO}(\Delta)_{\omega,p,r}$ is not necessarily well-defined if $\omega \in \widehat{\mathcal{D}}$, and consequently, we consider the class $\mathcal{D}$ in the main results of this paper.

It is clear that the space $\mathrm{BMO}(\Delta)_{\omega,p}$ depends on $\omega \in \mathcal{D}$, but for $\omega \in \mathcal{I}nv$, straightforward calculations show that for each $r_1, r_2 \in (0,\infty)$, we have $\mathrm{BMO}(\Delta)_{\omega,p,r_1} = \mathrm{BMO}(\Delta)_{\nu,p,r_2}$ where $\nu(z) \equiv 1$. Therefore we call this space $\mathrm{BMO}(\Delta)_p$. Recall that ω is invariant, denoted by $\omega \in \mathcal{I}nv$, if for some (equivalently for all) $r \in (0,\infty)$ there exists a constant $C = C(r) \ge 1$ such that such that $C^{-1}\omega(\zeta) \le \omega(z) \le C\omega(\zeta)$ for all $\zeta \in \Delta(z,r)$. That is, an invariant weight is essentially constant in each hyperbolically bounded region. The class $\mathcal{R}$ of regular weights, which is a large subclass of smooth weights in $\mathcal{D}$, satisfies $\mathcal{R} \subset \mathcal{I}nv \cap \mathcal{D}$ by [1, Section 1.3]. The space $\mathrm{BMO}(\Delta)_{\omega,p}$ certainly depends on p as is seen by considering the function $f(z) = |z|^{-\frac{2}{p}}$ which satisfies $f \in \mathrm{BMO}(\Delta)_q \setminus \mathrm{BMO}(\Delta)_p$ for $q < p$.

We recall one last thing before stating the main result of this paper. Namely, an analytic function f belongs to $\mathcal{B}$ if and only if it is Lipschitz continuous in the hyperpolic metric [6, Theorem 5.5]. Therefore $\mathcal{B} \subset \mathrm{BMO}(\Delta)_{\omega,p,r}$ for each $1 \le p < \infty$, $0 < r < \infty$ and a radial weight ω such that $\omega(\Delta(z,r)) > 0$ for all $z \in \mathbb{D}$.

Theorem 1 *Let $1 < p < \infty$ and $\omega \in \mathcal{M}$. Then the following statements are equivalent:*

(i) *There exists $r_0 = r_0(\omega) \in (0,\infty)$ such that $\mathrm{BMO}(\Delta)_{\omega,p,r}$ does not depend on r, provided $r \ge r_0$. Moreover, $P_\omega : \mathrm{BMO}(\Delta)_{\omega,p} \to \mathcal{B}$ is bounded;*
(ii) *$P_\omega : L^\infty \to \mathcal{B}$ is bounded;*
(iii) *$\omega \in \widehat{\mathcal{D}}$.*

As far as we know, the statement in Theorem 1 is new even for the standard weights when $p \ne 2$. The class $\mathcal{M}$ is a wide class of radial weights containing the standard radial weights as well as exponential-type weights [1, Chapter 1]. It is also worth observing that weights in $\mathcal{M}$ may admit a substantial oscillating behavior. In fact, a careful inspection of the proof of [5, Proposition 14] reveals the existence of a weight $\omega \in \mathcal{M}$ such that $\mathrm{BMO}(\Delta)_{\omega,p,r}$ is not well-defined for any $r > 0$ and $1 < p < \infty$, and therefore we cannot get rid of the first statement in the case (i) in Theorem 1. However, each weight ω in the class $\check{\mathcal{D}}$ has the property that $\omega(\Delta(z,r)) > 0$ for all $z \in \mathbb{D}$ and for all r sufficiently large depending on ω. The class $\check{\mathcal{D}}$ consists of radial weights ω for which there exist constants $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \ge C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $0 \le r < 1$. Recall that $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$ but $\check{\mathcal{D}} \subsetneq \mathcal{M}$ by [5, Proof of Theorem 3 and Proposition 14].

As for the proof of Theorem 1, the equivalence between (ii) and (iii) is already known by [5, Theorem 3], so our contribution here consists of showing that (iii) implies (i). Our approach to this implication does not involve the Hankel operators, is direct and based on the decomposition $\mathrm{BMO}(\Delta)_{\omega,p} = \mathrm{BA}(\Delta)_{\omega,p} + \mathrm{BO}(\Delta)$, provided in [4, Theorem 11(ii)]. For continuous $f : \mathbb{D} \to \mathbb{C}$ and $0 < r < \infty$, we define

$$\Omega_r f(z) = \sup\{|f(z) - f(\zeta)| : \beta(z,\zeta) < r\}, \quad z \in \mathbb{D},$$

and let $\mathrm{BO}(\Delta)$ denote the space of those f such that

$$\|f\|_{\mathrm{BO}(\Delta)} = \sup_{z\in\mathbb{D}} \Omega_r f(z) < \infty.$$

It is known that the definition of $\mathrm{BO}(\Delta)$ is independent of the choice of r by [6, Lemma 8.1]. Further, if ω is a radial weight such that $\omega(\Delta(z,r)) > 0$ for all $z \in \mathbb{D}$, then, for $0 < p < \infty$, the space $\mathrm{BA}(\Delta)_{\omega,p,r}$ consists of $f \in L^p_\omega$ such that

$$\|f\|_{\mathrm{BA}(\Delta)_{\omega,p,r}} = \sup_{z\in\mathbb{D}} \left(\frac{1}{\omega(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta)|^p \omega(\zeta)\, dA(\zeta) \right)^{\frac{1}{p}} < \infty.$$

If $\omega \in \mathcal{D}$, then the space $\mathrm{BA}(\Delta)_{\omega,p,r}$ depends on p and ω but, by [4, Lemma 10], there exists an $r_0 = r_0(\omega) \in (0,\infty)$ such that it is independent of r as long as $r \ge r_0$, so we write $\mathrm{BA}(\Delta)_{\omega,p}$ for short. With these definitions and observations the decomposition $\mathrm{BMO}(\Delta)_{\omega,p} = \mathrm{BA}(\Delta)_{\omega,p} + \mathrm{BO}(\Delta)$ gets explained.

The rest of the paper consists of the proof of Theorem 1. But before getting to that, we finish the section with couple of words about the notation used. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable.

2 Preliminary results on radial weights

We begin with a known characterization of weights in $\widehat{\mathcal{D}}$, proved in [2, Lemma 2.1].

Lemma A *Let ω be a radial weight. Then, $\omega \in \widehat{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \le C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \le r \le t < 1.$$

The following simple lemma is useful for our purposes. It reveals that $\mathcal{D}$ is closed under multiplication by a suitably small negative power of the hat of another weight in $\widehat{\mathcal{D}}$.

Lemma 2 *Let $\omega \in \mathcal{D}$ and $\nu \in \widehat{\mathcal{D}}$. Then there exists $\gamma_0 = \gamma_0(\omega,\nu) > 0$ such that, for each $\gamma \in (0,\gamma_0]$, we have $(\widehat{\nu})^{-\gamma}\omega \in \mathcal{D}$, and*

$$\int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)^\gamma}\, ds \asymp \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^\gamma}, \quad 0 \le r < 1. \tag{2.1}$$

Proof By [5, (2.27)], $\omega \in \check{\mathcal{D}}$ if and only if there exist constants $C = C(\omega) > 0$ and $\alpha_0 = \alpha_0(\omega) > 0$ such that

$$\widehat{\omega}(t) \leq C\left(\frac{1-t}{1-r}\right)^{\alpha} \widehat{\omega}(r), \quad 0 \leq r \leq t < 1, \tag{2.2}$$

for all $\alpha \in (0, \alpha_0]$. Let $\gamma = \gamma(\omega, \nu) \in (0, \alpha_0/\beta)$, where $\beta = \beta(\nu) > 0$ is that of Lemma A. Then

$$\lim_{s\to 1^-} \frac{\widehat{\omega}(s)}{\widehat{\nu}(s)^{\gamma}} \lesssim \lim_{s\to 1^-} \frac{(1-s)^{\alpha_0}}{\widehat{\nu}(s)^{\gamma}} = 0.$$

Two integrations by parts together with (2.2) and Lemma A yield

$$\begin{aligned}
\frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}} &\leq \int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)^{\gamma}}\,ds = \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}} + \gamma \int_r^1 \frac{\widehat{\omega}(s)}{\widehat{\nu}(s)^{\gamma+1}} \nu(s)\,ds \\
&\lesssim \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}} + \frac{\widehat{\omega}(r)}{(1-r)^{\alpha_0}} \gamma \int_r^1 (1-s)^{\alpha_0} \frac{\nu(s)}{\widehat{\nu}(s)^{\gamma+1}}\,ds \\
&\lesssim \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}} + \frac{\widehat{\omega}(r)}{(1-r)^{\alpha_0}} \int_r^1 \left(\frac{(1-s)^{\beta}}{\widehat{\nu}(s)}\right)^{\gamma} (1-s)^{\alpha_0-1-\gamma\beta}\,ds \\
&\lesssim \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}} + \frac{\widehat{\omega}(r)}{(1-r)^{\alpha_0}} \left(\frac{(1-r)^{\beta}}{\widehat{\nu}(r)}\right)^{\gamma} \int_r^1 (1-s)^{\alpha_0-1-\gamma\beta}\,ds \lesssim \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)^{\gamma}}, \quad 0 \leq r < 1.
\end{aligned}$$

Therefore (2.1) is satisfied, and standard arguments show that $(\widehat{\nu})^{-\gamma}\omega \in \mathcal{D}$.

We finish the section by showing that the space $\mathrm{BMO}(\Delta)_{\omega,p,r}$ is not necessarily well-defined for all $r \in (0, 1)$ if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$. This serves us as a justification for the initial hypotheses on ω in our study.

Proposition 3 *Let $\psi : [0, 1) \to [(\log 3)/4, \infty)$ be arbitrary. Then there exist an $\omega = \omega_\psi \in \widehat{\mathcal{D}} \setminus \mathcal{D}$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of points on $(0, 1)$ depending on ψ only such that $\lim_{n\to\infty} r_n = 1$ and $\omega_\psi(\Delta(r_n, \psi(r_{n-1})) = 0$ for all $n \in \mathbb{N}$.*

Proof Let us consider the increasing sequence $\{t_n\}_{n=1}^{\infty} \in [0, 1)$ defined inductively by the identities $t_1 = 0$ and $\beta(t_n, t_{n+1}) = 2\psi(t_n)$ for all $n \in \mathbb{N}$. Since the range of ψ is $[(\log 3)/4, \infty)$, we have

$$\frac{e^{4\psi(r)} - 1}{e^{4\psi(r)} + 1} \geq \frac{1}{2} \geq \frac{1}{2+r}, \quad r \in [0, 1).$$

Therefore $t_{n+1} \geq \frac{1+t_n}{2}$, and consequently, $\lim_{n\to\infty} t_n = 1$. Then, for $n \geq 2$, the annulus $\{z \in \mathbb{D} : t_n \leq |z| \leq t_{n+1}\}$ contains $\Delta(s_n, \psi(s_{n-1}))$, where s_n is the hyperbolic midpoint of (t_n, t_{n+1}). Define $\omega = \sum_{n=1}^{\infty} a_n \chi_{\{z: t_{2n} \leq |z| \leq t_{2n+1}\}}$, where $\{a_n\}_{n=1}^{\infty}$ are chosen such that $a_n(t_{2n+1} - t_{2n}) = 2^{-n}$ for all $n \in \mathbb{N}$. Then $\widehat{\omega}(t_{2n}) = \sum_{j=n}^{\infty} 2^{-j} = 2^{1-n}$ for all $n \in \mathbb{N}$, and it follows that $\omega \in \widehat{\mathcal{D}}$ because $\beta\left(r, \frac{1+r}{2}\right) \asymp 1$ for all $0 \leq r < 1$, and $\beta(t_{2n}, t_{2(n+1)}) = 2(\psi(t_{2n}) + \psi(t_{2n+1})) \to \infty$, as $n \to \infty$. Further, by setting $r_n = s_{2n+1}$, we have $\omega(\Delta(r_n, \psi(r_{n-1}))) = 0$ for all $n \in \mathbb{N}$. This also implies that $\omega \notin \mathcal{D}$.

3 Proof of Theorem 1

The statements (ii) and (iii) are equivalent by [5, Theorem 1], and the fact that (i) implies (ii) is an immediate consequence of the continuous embedding $L^\infty \subset \mathrm{BMO}(\Delta)_{\omega,p}$. Assume now (iii), that is, $\omega \in \mathcal{D}$. In the proof we will use the fact that $f \in \mathrm{BMO}_{\omega,p}(\Delta)$ if and only if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in \mathrm{BA}(\Delta)_{\omega,p}$ and $f_2 \in \mathrm{BO}(\Delta)$ such that $\|f_1\|_{\mathrm{BA}(\Delta)_{\omega,p}} + \|f_2\|_{\mathrm{BO}(\Delta)} \lesssim \|f\|_{\mathrm{BMO}(\Delta)_{\omega,p}}$. This statement follows from [4, Theorem 11(ii)] and its proof. Consequently, it is enough to prove that $P_\omega : \mathrm{BA}(\Delta)_{\omega,p} \to \mathcal{B}$ and $P_\omega : \mathrm{BO}(\Delta) \to \mathcal{B}$ are bounded operators.

We first show that $P_\omega : \mathrm{BA}(\Delta)_{\omega,p} \to \mathcal{B}$ is bounded. To do this, choose $0 < r_0 < \infty$ such that $\mathrm{BA}(\Delta)_{\omega,p,r} = \mathrm{BA}(\Delta)_{\omega,p}$ is independent of r as long as $r \geq r_0$. Further, let $f_1 \in \mathrm{BA}(\Delta)_{\omega,p}$ and $r \geq r_0$, and let $\{a_k\}_{k=1}^\infty$ be an r-lattice. Then Hölder's inequality and the definition of $\mathrm{BA}(\Delta)_{\omega,p}$ yield

$$
\begin{aligned}
|(P_\omega(f_1))'(z)| &\leq \int_{\mathbb{D}} |f_1(\zeta)||(B_\zeta^\omega)'(z)|\,\omega(\zeta)\,dA(\zeta)\\
&\leq \sum_{k=1}^\infty \int_{\Delta(a_k,r)} |f_1(\zeta)||(B_\zeta^\omega)'(z)|\,\omega(\zeta)\,dA(\zeta)\\
&\leq \sum_{k=1}^\infty \left(\int_{\Delta(a_k,r)} |f_1(\zeta)|^p\,\omega(\zeta)\,dA(\zeta)\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\Delta(a_k,r)} |(B_\zeta^\omega)'(z)|^{p'}\,\omega(\zeta)\,dA(\zeta)\right)^{\frac{1}{p'}}\\
&\leq \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \sum_{k=1}^\infty (\omega(\Delta(a_k,r)))^{\frac{1}{p}} \left(\int_{\Delta(a_k,r)} |(B_\zeta^\omega)'(z)|^{p'}\,\omega(\zeta)\,dA(\zeta)\right)^{\frac{1}{p'}}\\
&\leq \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \sum_{k=1}^\infty \omega(\Delta(a_k,r)) \sup_{\zeta\in\Delta(a_k,r)} |(B_\zeta^\omega)'(z)|, \quad z \in \mathbb{D},
\end{aligned}
$$

from which the subharmonicity and standard estimates give

$$
\begin{aligned}
|(P_\omega(f_1))'(z)| &\lesssim \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \sum_{k=1}^\infty \omega(\Delta(a_k,r)) \frac{\int_{\Delta(a_k,2r)} |(B_\zeta^\omega)'(z)|\,dA(\zeta)}{(1-|a_k|)^2}\\
&\lesssim \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \sum_{k=1}^\infty \int_{\Delta(a_k,2r)} |(B_\zeta^\omega)'(z)| \frac{\omega(\Delta(\zeta,3r))}{(1-|\zeta|)^2}\,dA(\zeta)\\
&\lesssim \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \int_{\mathbb{D}} |(B_\zeta^\omega)'(z)| \frac{\omega(\Delta(\zeta,3r))}{(1-|\zeta|)^2}\,dA(\zeta), \quad z \in \mathbb{D}.
\end{aligned}
\tag{3.1}
$$

Next, for each $a \in \mathbb{D} \setminus \{0\}$, consider the interval $I_a = \left\{e^{i\theta} : |\arg(ae^{-i\theta})| \leq \frac{(1-|a|)}{2}\right\}$, and let $S(a) = \{z \in \mathbb{D} : |z| \geq |a|,\ e^{it} \in I_a\}$ denote the Carleson square induced by a. Then Fubini's theorem yields

$$
\begin{aligned}
&\int_{S(a)} \frac{\omega(\Delta(\zeta,3r))}{(1-|\zeta|)^2}\,dA(\zeta)\\
&= \int_{\{z\in\mathbb{D}:S(a)\cap\Delta(z,3r)\neq\emptyset\}} \left(\int_{S(a)\cap\Delta(z,3r)} \frac{dA(\zeta)}{(1-|\zeta|)^2}\right)\omega(z)\,dA(z)\\
&\leq \int_{S(b)} \left(\int_{\Delta(z,3r)} \frac{dA(\zeta)}{(1-|\zeta|)^2}\right)\omega(z)\,dA(z) \asymp \omega(S(b)), \quad |a| > R',
\end{aligned}
$$

where $R' = R'(r) \in (0, \infty)$ and $b = b(a, r) \in \mathbb{D}$ satisfies $\arg b = \arg a$ and $1-|b| \asymp 1-|a|$ for all $a \in \mathbb{D} \setminus \overline{D(0, R')}$. Since $\omega \in \widehat{\mathcal{D}}$ by the hypothesis, we have $\omega(S(b)) \lesssim \omega(S(a))$ by Lemma A. Therefore (3.1) and [2] [Theorem 3.3] imply

$$|\,(P_\omega(f_1))'\,(z)| \lesssim \|f\|_{\mathrm{BA}(\Delta)_{\omega,p}} \int_{\mathbb{D}} |(B_\zeta^\omega)'(z)|\omega(\zeta)\,dA(\zeta), \quad z \in \mathbb{D}.$$

Since [3, Theorem 1] yields

$$\int_{\mathbb{D}} |(B_\zeta^\omega)'(z)|\omega(\zeta)\,dA(\zeta) \asymp 1 + \int_0^{|z|} \frac{dt}{(1-t)^2} \asymp \frac{1}{1-|z|}, \quad z \in \mathbb{D},$$

we deduce that $P_\omega : \mathrm{BA}(\Delta)_{\omega,p} \to \mathcal{B}$ is bounded.

It remains to show that $P_\omega : \mathrm{BO}(\Delta) \to \mathcal{B}$ is bounded. Let $f_2 \in \mathrm{BO}(\Delta)$. First, observe that an application of Lemma 2 yields

$$\begin{aligned}
|(P_\omega(f_2)(z)| &\le |f_2(0)|\omega(\mathbb{D}) + \int_{\mathbb{D}} |f_2(\zeta) - f_2(0)|B_z^\omega(\zeta)|\omega(\zeta)\,dA(\zeta) \\
&\le |f_2(0)|\omega(\mathbb{D}) + \|f_2\|_{\mathrm{BO}(\Delta)} \int_{\mathbb{D}} \log\frac{1}{1-|\zeta|}|B_z^\omega(\zeta)|\omega(\zeta)\,dA(\zeta) \\
&\le |f_2(0)|\omega(\mathbb{D}) + C_z\|f_2\|_{\mathrm{BO}(\Delta)} \int_{\mathbb{D}} \log\frac{1}{1-|\zeta|}\omega(\zeta)\,dA(\zeta) < \infty, \quad z \in \mathbb{D}.
\end{aligned}$$

Further, since $1 = \langle 1, B_z^\omega\rangle_{A^2_\omega}$ and $0 = \langle 1, (B_z^\omega)'\rangle_{A^2_\omega}$, we have

$$\begin{aligned}
(P_\omega(f_2))'\,(z) &= \langle f_2, (B_z^\omega)'\rangle_{A^2_\omega} = \langle f_2, (B_z^\omega)'\rangle_{A^2_\omega} - f_2(z)\langle 1, (B_z^\omega)'\rangle_{A^2_\omega} \\
&= \int_{\mathbb{D}} \frac{z}{\overline{\zeta}}\,(f_2(\zeta) - f_2(z))\,(B_\zeta^\omega)'(z)\omega(\zeta)\,dA(\zeta), \quad z \in \mathbb{D}.
\end{aligned}$$

By Lemma 2 there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, we have $\omega_{[-\varepsilon]} = (1-|z|)^{-\varepsilon}\omega(z) \in \mathcal{D}$ and $\widehat{\omega_{[-\varepsilon]}} \asymp \widehat{\omega}_{[-\varepsilon]}$ on $\mathbb{D}$. Take $0 < \varepsilon < \min\left\{\frac{1}{2+\beta}, \frac{\varepsilon_0}{1+\varepsilon_0}\right\}$, where β is that from Lemma A. Since $f_2 \in \mathrm{BO}(\Delta)$, we have

$$|f_2(z) - f_2(\zeta)| \lesssim (1 + \beta(z,\zeta))\|f_2\|_{\mathrm{BO}(\Delta)} \lesssim \frac{|1-\overline{\zeta}z|^{2\varepsilon}}{(1-|z|)^\varepsilon(1-|\zeta|)^\varepsilon}\|f\|_{\mathrm{BO}(\Delta)}, \quad z, \zeta \in \mathbb{D}.$$

Therefore Hölder's inequality yields

$$\begin{aligned}
\left|(P_\omega(f_2))'\,(z)\right| &\lesssim (1-|z|)^{-\varepsilon} \int_{\mathbb{D}} |1-\overline{\zeta}z|^{2\varepsilon}\left|(B_\zeta^\omega)'(z)\right|\omega_{[-\varepsilon]}(\zeta)\,dA(\zeta) \\
&\le (1-|z|)^{-\varepsilon} I_1(z)^\varepsilon I_2(z)^{1-\varepsilon}, \quad z \in \mathbb{D},
\end{aligned} \tag{3.2}$$

where

$$I_1(z) = \int_{\mathbb{D}} \left|(1-\overline{\zeta}z)(B_\zeta^\omega)'(z)\right|^2 \omega(\zeta)\,dA(\zeta), \quad z \in \mathbb{D},$$

and

$$I_2(z) = \int_{\mathbb{D}} \left|(B_\zeta^\omega)'(z)\right|^{\frac{1-2\varepsilon}{1-\varepsilon}} \omega_{[-\frac{\varepsilon}{1-\varepsilon}]}(\zeta)\,dA(\zeta), \quad z \in \mathbb{D}.$$

By Lemma 2, [3, Theorem 1], Lemma A and our choice of ε, we have

$$\begin{aligned} I_2(z) &\lesssim 1+\int_0^{|z|}\frac{\widehat{\omega_{[-\frac{\varepsilon}{1-\varepsilon}]}}(t)}{\left(\widehat{\omega(t)}(1-t)^2\right)^{\frac{1-2\varepsilon}{1-\varepsilon}}}\,dt \asymp 1+\int_0^{|z|}\frac{\widehat{\omega}(t)^{\frac{\varepsilon}{1-\varepsilon}}}{(1-t)^{\frac{2-3\varepsilon}{1-\varepsilon}}}\,dt \\ &\lesssim 1+\left(\frac{\widehat{\omega}(z)}{(1-|z|)^{\beta}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\int_0^{|z|}\frac{dt}{(1-t)^{\frac{2-(3+\beta)\varepsilon}{1-\varepsilon}}} \qquad (3.3)\\ &\asymp 1+\left(\frac{\widehat{\omega}(z)^{\varepsilon}}{(1-|z|)^{1-2\varepsilon}}\right)^{\frac{1}{1-\varepsilon}} \asymp \left(\frac{\widehat{\omega}(z)^{\varepsilon}}{(1-|z|)^{1-2\varepsilon}}\right)^{\frac{1}{1-\varepsilon}}, \quad z\in\mathbb{D}. \end{aligned}$$

Let us now bound $I_1(z)$. To do this we first observe that

$$\begin{aligned} 2(1-\overline{\zeta}z)(B_\zeta^\omega)'(z) &= \overline{\zeta}\left(\sum_{n=1}^{\infty}\frac{n(\overline{\zeta}z)^{n-1}}{\omega_{2n+1}}-\sum_{n=1}^{\infty}\frac{n(\overline{\zeta}z)^{n}}{\omega_{2n+1}}\right)\\ &= \overline{\zeta}\left(\frac{1}{\omega_3}+\sum_{n=1}^{\infty}\frac{(n+1)(\overline{\zeta}z)^{n}}{\omega_{2n+3}}-\sum_{n=1}^{\infty}\frac{n(\overline{\zeta}z)^{n}}{\omega_{2n+1}}\right)\\ &= \overline{\zeta}\left(\frac{1}{\omega_3}+\sum_{n=1}^{\infty}\frac{(\overline{\zeta}z)^{n}}{\omega_{2n+3}}+\sum_{n=1}^{\infty}\frac{n(\omega_{2n+1}-\omega_{2n+3})}{\omega_{2n+1}\omega_{2n+3}}(\overline{\zeta}z)^{n}\right)\\ &= \overline{\zeta}\left(J_1+J_2(z,\overline{\zeta})+J_3(z,\overline{\zeta})\right), \quad z,\zeta\in\mathbb{D}. \end{aligned}$$

By [3, Theorem 1] we have

$$\begin{aligned} \int_{\mathbb{D}}|J_2(z,\overline{\zeta})|^2\omega(\zeta)\,dA(\zeta) &= \sum_{n=1}^{\infty}\frac{\omega_{2n+1}}{\omega_{2n+3}^2}|z|^{2n} \asymp \sum_{n=1}^{\infty}\frac{1}{\omega_{2n+1}}|z|^{2n}\\ &\lesssim \|B_z^\omega\|_{A^2_\omega}^2 \asymp \frac{1}{(1-|z|)\widehat{\omega}(z)}, \quad z\in\mathbb{D}. \end{aligned}$$

Further, we have $n(\omega_{2n+1}-\omega_{2n+3}) = n\int_0^1 s^{2n+1}(1-s^2)\omega(s)\,ds \lesssim \omega_{2n+1}$ for all $n\in\mathbb{N}$ by [5, (1.3)]. Therefore another application of [3, Theorem 1] gives

$$\int_{\mathbb{D}}|J_3(z,\overline{\zeta})|^2\omega(\zeta)\,dA(\zeta) \lesssim \sum_{n=1}^{\infty}\frac{1}{\omega_{2n+1}}|z|^{2n} \lesssim \|B_z^\omega\|_{A^2_\omega}^2 \asymp \frac{1}{(1-|z|)\widehat{\omega}(z)}, \quad z\in\mathbb{D},$$

and it follows that

$$I_1(z) \lesssim \frac{1}{(1-|z|)\widehat{\omega}(z)}, \quad z\in\mathbb{D}.$$

This estimate, (3.2) and (3.3) yield

$$\left|(P_\omega(f_2))'(z)\right| \lesssim (1-|z|)^{-\varepsilon}I_1(z)^{\varepsilon}I_2(z)^{1-\varepsilon} \lesssim \frac{1}{1-|z|}, \quad z\in\mathbb{D}.$$

Consequently, $P_\omega:\mathrm{BO}(\Delta)\to\mathcal{B}$ is bounded. This finishes the proof of the theorem.

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