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Exact Morse index of radial solutions for semilinear elliptic equations with critical exponent on annuli

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Abstract

Let $N \geq 3$, $R > \rho > 0$ and $A_\rho := \{x \in \mathbb{R}^N;\ \rho < |x| < R\}$. Let $U^{\pm}_{n,\rho}$, $n \geq 1$, be a radial solution with n nodal domains of

$$\begin{cases} \Delta U + |x|^\alpha |U|^{p-1}U = 0 & \text{in } A_\rho, \\ U = 0 & \text{on } \partial A_\rho. \end{cases}$$

We show that if $p = \frac{N+2+2\alpha}{N-2}$, $\alpha > -2$ and $N \geq 3$, then $U^{\pm}_{n,\rho}$ is nondegenerate for small $\rho > 0$ and the Morse index $\mathsf{m}(U^{\pm}_{n,\rho})$ satisfies

$$\mathsf{m}(U^{\pm}_{n,\rho}) = n\frac{(N+2\ell-1)(N+\ell-1)!}{(N-1)!\ell!} \quad \text{for small } \rho > 0,$$

where $\ell = [\frac{\alpha}{2}] + 1$. Using Jacobi elliptic functions, we show that if $(p, \alpha) = (3, N-4)$ and $N \geq 3$, then the Morse index of a positive and negative solutions $\mathsf{m}(U^{\pm}_{1,\rho})$ is completely determined by the ratio $\rho/R \in (0, 1)$. Upper and lower bounds for $\mathsf{m}(U^{\pm}_{n,\rho})$, $n \geq 1$, are also obtained when $(p, \alpha) = (3, N-4)$ and $N \geq 3$.

Keywords Hénon equation · Emden–Fowler equation · Critical Sobolev exponent · Jacobi elliptic functions

Mathematics Subject Classification Primary 35B05 · 34L15; Secondary 35J9 · 35J20

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1 Introduction and main results

Let $N \geq 3$, $R > \rho > 0$ and $A_\rho := \{x \in \mathbb{R}^N;\ \rho < |x| < R\}$. We are concerned with radial solutions of the Dirichlet problem

$$\begin{cases} \Delta U + |x|^\alpha |U|^{p-1} U = 0 & \text{in } A_\rho, \\ U = 0 & \text{on } \partial A_\rho. \end{cases} \tag{1.1}$$

The equation in (1.1) is called the Hénon equation arising in astrophysics. Hénon [16] studied radial solutions of (1.1) on a ball. We are particularly interested in the case

$$p = p_c := \frac{N + 2 + 2\alpha}{N - 2} \quad \text{and} \quad \alpha > -2.$$

The exponent p_c becomes the critical Sobolev exponent provided that $\alpha = 0$.

Let B denote a unit ball in $\mathbb{R}^N$. First we consider the problem

$$\begin{cases} \Delta U + |x|^\alpha |U|^{p-1} U = 0 & \text{in } B, \\ U = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

When $p \geq p_c$, (1.2) has no nontrivial radial solution due to Pohozaev's identity. On the other hand, Ni [21] proved that (1.2) has a positive radial solution provided that $1 < p < p_c$. This result was generalized by Bartsch–Willem [7]. They proved that, for each $n \geq 2$, (1.2) has radial solutions $U_n^\pm(r)$ that have n nodal domains, i.e., the set $\{x \in B;\ U_n^\pm(x) \neq 0\}$ has exactly n connected components. The uniqueness of $U_n^+(r)$ follows from Ni–Nussbaum [22], and $U_n^-(r) = -U_n^+(r)$. Therefore, p_c is a threshold between existence and nonexistence of nontrivial radial solutions of (1.2).

The situation, however, is different in annular domains. It is known that for each $n \geq 1$, (1.1) has exactly two classical radial solutions $U_{n,\rho}^\pm$ with n nodal domains, which satisfies $U_{n,\rho}^-(r) = -U_{n,\rho}^+(r)$. We let $(U_{n,\rho}^+)'(R) < 0$. In particular, $U_{1,\rho}^+$ and $U_{1,\rho}^-$ are a positive and negative solution, respectively. Existence and uniqueness results hold for arbitrary $p > 1$ and $\alpha \in \mathbb{R}$. See [22, Theorem 3.8] for existence of radial solutions $U_{n,\rho}^\pm$ and see [22, Theorem 3.1] for the uniqueness of $U_{n,\rho}^+$. Thus, it is interesting to study what happens in the case $p = p_c$ as $\rho \to 0$. Bandle-Peletier [6] obtained, among other things, an asymptotic expansion of $\left\| U_{1,\rho}^\pm \right\|_{L^\infty(A_\rho)}$ as $\rho \to 0$ in the case $\alpha = 0$. Specifically, as $\rho \to 0$,

$$\left\| U_{1,\rho}^\pm \right\|_{L^\infty(A_\rho)} = \left\{ \frac{N(N-2)}{\rho R} \right\}^{\frac{N-2}{4}} (1 + o(1)).$$

In this paper we study the Morse index of $U_{n,\rho}^\pm$ when $\rho > 0$ is small. We call the Morse index $\mathsf{m}(U_{n,\rho}^\pm)$ of a solution $U_{n,\rho}^\pm$ of (1.1) the maximal dimension of a subspace $X \subset H_0^1(A_\rho)$ where the quadratic form associated to the linearization operator at $U_{n,\rho}^\pm$ is negative definite. In our problem (1.1) $\mathsf{m}(U_{n,\rho}^\pm)$ is equal to the number of the negative eigenvalues of the problem

$$\begin{cases} \Delta \Phi + p|x|^\alpha |U_{n,\rho}^\pm|^{p-1} \Phi = -\lambda \Phi & \text{in } A_\rho, \\ \Phi = 0 & \text{on } \partial A_\rho. \end{cases} \tag{1.3}$$

counted with their multiplicity, i.e.,

$$\mathsf{m}(U_{n,\rho}^\pm) = \sharp\{\text{negative eigenvalues of (1.3) counted with their multiplicity}\}. \tag{1.4}$$

We call that $U^{\pm}_{n,\rho}$ is nondegenerate if (1.3) does not have a zero eigenvalue.

We prepare some notation to state our main results. Let $[\xi]$, $\xi \in \mathbb{R}$, denote the largest integer that does not exceed ξ. Let $\{\nu_j\}_{j=0}^{\infty}$ denote the set of the eigenvalues of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on the sphere $\mathbb{S}^{N-1}$, i.e.,

$$\nu_j := j(N+j-2), \quad j = 0, 1, 2, \dots. \tag{1.5}$$

The multiplicity of the eigenvalue ν_j is given by

$$M_j(N) := \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!} \quad \text{for } N \geq 3 \text{ and } j = 0, 1, 2, \dots.$$

The first main result is about the exact Morse index for small $\rho > 0$ and the divergence of the Morse index as $\rho \to R$.

Theorem 1.1 *Let $N \geq 3$, $\alpha > -2$ and $p = p_c$. Let $U^{\pm}_{n,\rho}$, $n \geq 1$, denote radial solutions with n nodal domains of (1.1). Then the following hold:*

(i) *Let $\ell := \left[\frac{\alpha}{2}\right] + 1$. Then,*

$$\mathsf{m}(U^{\pm}_{n,\rho}) \begin{cases} = n \sum_{j=0}^{\ell} M_j(N) = n \dfrac{(N+2\ell-1)(N+\ell-2)!}{(N-1)!\ell!} & \text{for small } \rho > 0, \\ \geq n \sum_{j=0}^{\ell} M_j(N) = n \dfrac{(N+2\ell-1)(N+\ell-2)!}{(N-1)!\ell!} & \text{for } \rho < R, \end{cases}$$

and $U^{\pm}_{n,\rho}$ is nondegenerate for small $\rho > 0$. In particular, if $\alpha = 0$, then

$$\mathsf{m}(U^{\pm}_{n,\rho}) = n(N+1) \quad \text{for small} \quad \rho > 0.$$

(ii) *As $\rho \to R$,*

$$\mathsf{m}(U^{\pm}_{n,\rho}) \to +\infty.$$

(iii) *Let $R > \rho > 0$ be fixed. As $\alpha \to \infty$ (and hence $p = p_c \to \infty$),*

$$\mathsf{m}(U^{\pm}_{n,\rho}) \to +\infty.$$

The Morse index formula given in Theorem 1.1 (i) has the following simpler form

$$\mathsf{m}(U^{\pm}_{n,\rho}) = nM_\ell(N+1) \text{ for small } \rho > 0.$$

Explicit formulas for $N = 3, 4, 5$ are given in the following:

Example 1.2 Let $\alpha > -2$, $p = p_c$ and $\ell = \left[\frac{\alpha}{2}\right] + 1$. Then the following hold:

(i) If $N = 3$, then $\mathsf{m}(U^{\pm}_{n,\rho}) = \frac{n}{2!}(2\ell+2)(\ell+1)$ for small $\rho > 0$.
(ii) If $N = 4$, then $\mathsf{m}(U^{\pm}_{n,\rho}) = \frac{n}{3!}(2\ell+3)(\ell+1)(\ell+2)$ for small $\rho > 0$.
(iii) If $N = 5$, then $\mathsf{m}(U^{\pm}_{n,\rho}) = \frac{n}{4!}(2\ell+4)(\ell+1)(\ell+2)(\ell+3)$ for small $\rho > 0$.

Remark 1.3 Let $\rho = a$, $R = a+1$, $\lambda \in (-\infty, \lambda_1)$ and $p > 1$. Here, $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ on A_ρ. It was shown in [14, Proposition 3.6] that the Morse index of a positive radial solution of

$$\begin{cases} \Delta U + \lambda U + U^p = 0 & \text{in } A_\rho, \\ U = 0 & \text{on } \partial A_\rho \end{cases}$$

diverges as $a \to \infty$. The ratio ρ/R converges to 1 as $a \to \infty$. This divergence result of the Morse index corresponds to Theorem 1.1 (ii). Similar phenomena were also observed in previous researches including [9, 17, 24, 25].

When $(p, \alpha) = (3, N-4)$, the Morse index of positive and negative solutions $U^{\pm}_{1,\rho}$ can be completely determined by the ratio $\rho/R \in (0, 1)$. In particular, the smallness of $\rho > 0$ is not assumed in the following:

Theorem 1.4 *Let $N \geq 3$ and $(p, \alpha) = (3, N-4)$. Let $U^{+}_{1,\rho}$ and $U^{-}_{1,\rho}$ be a positive and negative radial solution of (1.1), respectively. Assume that one of the following* (a) *and* (b) *holds:*

(a) $\mathcal{R}_{\ell,1} < \frac{\rho}{R} \leq \mathcal{R}_{\ell+1,1}$ *for a positive integer* $\ell > \frac{N}{2} - 1$,

(b) $0 < \frac{\rho}{R} \leq \mathcal{R}_{\ell+1,1}$ *for a nonnegative integer* $\ell = \left[\frac{N}{2}\right] - 1$.

Then,

$$\mathsf{m}(U^{\pm}_{1,\rho}) = \sum_{j=0}^{\ell} M_j(N) = M_\ell(N+1).$$

Here,

$$\mathcal{R}_{\ell,n} := \begin{cases} \exp\left(-4n\sqrt{\dfrac{3}{8\nu_\ell - 3(N-2)^2}}\, K\left(\sqrt{\dfrac{4\nu_\ell}{8\nu_\ell - 3(N-2)^2}}\right)\right) & \text{if } \ell > \frac{N}{2} - 1, \\ 0 & \text{if } \ell \leq \frac{N}{2} - 1. \end{cases} \tag{1.6}$$

and K denotes the complete elliptic integral of the first kind whose definition and basic properties are recalled in Sect. 7.1.

Note that if $\ell > (N-2)/2$, then $8\nu_\ell - 3(N-2)^2 > 0$ and $0 < \sqrt{\frac{4\nu_\ell}{8\nu_\ell - 3(N-2)^2}} < 1$. Then $\mathcal{R}_{\ell,1}$ and $\mathcal{R}_{\ell+1,1}$ are well-defined. We can check that $\mathcal{R}_{[\frac{N}{2}],1}$ is also well-defined. Since

$$(0, \mathcal{R}_{[\frac{N}{2}],1}] \cup \left(\bigcup_{\ell > \frac{N}{2} - 1} (\mathcal{R}_{\ell,1}, \mathcal{R}_{\ell+1,1}]\right) = (0, 1),$$

the statements (a) and (b) in Theorem 1.4 cover the whole range (0, 1).

Upper and lower bounds of Morse indices $\mathsf{m}(U^{\pm}_{n,\rho})$, $n \geq 1$, can be obtained as follows:

Theorem 1.5 *Let $N \geq 3$ and $(p, \alpha) = (3, N-4)$. Let $U^{\pm}_{n,\rho}$, $n \geq 1$, denote radial solutions of* (1.1) *with n nodal domains. Then, the following holds:*

(i) *If $\frac{\rho}{R} > \mathcal{R}_{\ell,n}$ for a positive integer $\ell > \frac{N}{2} - 1$, then*

$$\mathsf{m}(U^{\pm}_{n,\rho}) \geq n \sum_{j=0}^{\ell} M_j(N) = n M_\ell(N+1). \tag{1.7}$$

Here, $\mathcal{R}_{\ell,n}$ is defined by (1.6).

(ii) *If $\frac{\rho}{R} \leq \tilde{\mathcal{R}}_{\ell,n}$ for a nonnegative integer $\ell > \frac{N}{2} - 2$, then*

$$\mathsf{m}(U^{\pm}_{n,\rho}) \leq n \sum_{j=0}^{\ell} M_j(N) = n M_\ell(N+1). \tag{1.8}$$

Here,

$$\tilde{\mathcal{R}}_{\ell,n} := \exp\left(-n(N-2)\sqrt{\frac{6}{\nu_{\ell+1}\{2\nu_{\ell+1}-(N-2)^2\}}}K\left(\sqrt{\frac{1}{2}+\frac{3(N-2)^4}{16\nu_{\ell+1}\{2\nu_{\ell+1}-(N-2)^2\}}}\right)\right).$$

Note that if $\ell > \frac{N}{2} - 2$ and $\ell \geq 0$, then

$$2\nu_{\ell+1} - (N-2)^2 > 0 \text{ and } 0 < \sqrt{\frac{1}{2}+\frac{3(N-2)^4}{16\nu_{\ell+1}\{2\nu_{\ell+1}-(N-2)^2\}}} < 1,$$

and hence $\tilde{\mathcal{R}}_{\ell,n}$ is well-defined.

When $(p, \alpha) = (3, N-4)$, all the radial solutions of (1.1) can be written explicitly as follows:

Theorem 1.6 *Let $N \geq 3$ and $(p, \alpha) = (3, N-4)$. Let $U^{\pm}_{n,\rho}$, $n \geq 1$, denote radial solutions of (1.1) with n nodal domains. Then,*

$$U^{\pm}_{n,\rho}(r) = \pm\frac{N-2}{2}\sqrt{\frac{2k^2(1-k^2)}{2k^2-1}}r^{-\frac{N-2}{2}}\operatorname{sd}\left(2nK(k)\frac{\log R - \log r}{\log R - \log \rho}, k\right), \tag{1.9}$$

where $\operatorname{sd}(\xi, k) := \operatorname{sn}(\xi, k)/\operatorname{dn}(\xi, k)$ *and* $k \in (\frac{1}{\sqrt{2}}, 1)$ *is the unique solution of*

$$\frac{4n}{N-2}\sqrt{2k^2-1}K(k) = \log\frac{R}{\rho}. \tag{1.10}$$

In Theorem 1.6 $\operatorname{sn}(\xi, k)$ and $\operatorname{dn}(\xi, k)$ denote Jacobi elliptic functions whose definitions and basic properties are summarized in Sect. 7.

Let us compare our theorems with previous results. Morse indices of radial solutions were studied by Amadori–Gladiali [1–5], by De Marchis–Ianni–Pacella [12, 13], by Gladiali–Grossi–Neves [15] and by Moreira dos Santos–Pacella [20]. As mentioned above, (1.2) does not have nontrivial radial solutions if $p \geq p_c$. Therefore it is appropriate to study differences between (1.1) with $p = p_c$ and (1.2) with $p < p_c$. When $p < p_c$, for each $n \geq 1$, there exist exactly two radial solutions $U^{\pm}_n$ of (1.2) with n nodal domains, which satisfy $U^{-}_n(r) = -U^{+}_n(r)$. The following Morse index formula of $U^{\pm}_n$ was obtained for $\alpha = 0$ in [12] and for $\alpha > 0$ in [2].

Proposition 1.7 *Let $N \geq 3$, $\alpha \geq 0$ and $\ell := \left[\frac{\alpha}{2}\right] + 1$. If $p(< p_c)$ is close to p_c, then*

$$\mathsf{m}(U^{\pm}_n) = \begin{cases} n\sum_{j=0}^{\ell} M_j(N) & \text{if } \alpha \text{ is not an even integer}, \\ n\sum_{j=0}^{\ell} M_j(N) - M_\ell(N) & \text{if } \alpha \text{ is an even integer}. \end{cases}$$

When $\alpha \geq 0$ is not an even integer, we see by Theorem 1.1 that $\mathsf{m}(U^{\pm}_{n,\rho}) = \mathsf{m}(U^{\pm}_n)$ for small $\rho > 0$. On the other hand, when $\alpha \geq 0$ is an even integer, we see that $\mathsf{m}(U^{\pm}_{n,\rho}) > \mathsf{m}(U^{\pm}_n)$ for small $\rho > 0$. Hence, we can say that the critical case $p = p_c$ is more unstable than the subcritical case $p < p_c$ when α is an even integer.

In [3, Theorem 1.1] the following lower bounds of Morse indices for (1.1) and (1.2) were obtained: If $\alpha \geq 0$, then

$$\mathsf{m}(U^{\pm}_{n,\rho}) \geq (n-1)\sum_{j=0}^{\ell} M_j(N) + 1 \quad \text{and} \quad \mathsf{m}(U^{\pm}_{n}) \geq (n-1)\sum_{j=0}^{\ell} M_j(N) + 1, \tag{1.11}$$

where $\ell := \left[\frac{\alpha}{2}\right] + 1$. A simple proof in the case (1.2) with $n \geq 2$ can be found in [12]. By Theorem 1.1 we see that $\mathsf{m}(U^{\pm}_{n,\rho})$ does not attain the lower bound (1.11). On the other hand, if $p(< p_c)$ is close to p_c and $\alpha = 0$, then we see by Proposition 1.7 that $\mathsf{m}(U^{\pm}_{n})$ attains the lower bound (1.11) when $\ell = 1$. We can say that the critical case does not have *the most stable* solution, while the subcritical case has.

Let us mention technical details. In the critical case by Emden's transformation we can transform a radial part of (1.1) into the scalar field equation $u'' - u + |u|^{p-1}u = 0$. As explained in Sect. 2, the Morse index $\mathsf{m}(U^{\pm}_{n,\rho})$ is equal to the number of the negative eigenvalues of the weighted eigenvalue problem

$$\begin{cases} \Delta\tilde{\Phi} + p|x|^{\alpha}|U^{\pm}_{n,\rho}|^{p-1}\tilde{\Phi} = -\frac{\tilde{\lambda}}{|x|^2}\tilde{\Phi} & \text{in } A_\rho, \\ \tilde{\Phi} = 0 & \text{on } \partial A_\rho \end{cases} \tag{1.12}$$

and all the eigenvalue of (1.12), which is denoted by $\tilde{\lambda}_{i,j}$, can be written as follows:

$$\tilde{\lambda}_{i,j} = \tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } i \geq 1 \text{ and } j \geq 0.$$

Here, $\tilde{\lambda}_{\mathrm{rad},i}$, $i = 1, 2, \ldots$, is the i-th radial eigenvalue of (1.12), i.e., the i-th eigenvalue of

$$\begin{cases} \tilde{\Phi}''_{\mathrm{rad}} + \frac{N-1}{r}\tilde{\Phi}'_{\mathrm{rad}} + pr^{\alpha}|U^{\pm}_{n,\rho}|^{p-1}\tilde{\Phi}_{\mathrm{rad}} = -\frac{\tilde{\lambda}_{\mathrm{rad}}}{r^2}\tilde{\Phi}_{\mathrm{rad}} & \text{for } \rho < r < R, \\ \tilde{\Phi}_{\mathrm{rad}}(\rho) = \tilde{\Phi}_{\mathrm{rad}}(R) = 0. \end{cases} \tag{1.13}$$

Since ν_j is explicitly given by (1.5), it becomes important to study all the negative eigenvalues $\tilde{\lambda}_{\mathrm{rad},i}$. In the critical case $\tilde{\lambda}_{\mathrm{rad},i}$, $i = 1, 2, \ldots$, are given as the eigenvalues for the linearization of the scalar field equation as explained in Sect. 4. Let $\ell := [\frac{\alpha}{2}] + 1$. The main part of our analysis is for showing that

$$\tilde{\lambda}_{\mathrm{rad},n} < -\frac{1}{4}(\alpha+2)(\alpha+2N-2) \leq -\nu_\ell < 0 < \tilde{\lambda}_{\mathrm{rad},n+1} \tag{1.14}$$

and that,

$$\text{for } i = 1, \ldots, n,\ \tilde{\lambda}_{\mathrm{rad},i} \to -\frac{1}{4}(\alpha+2)(\alpha+2N-2) \text{ as } \rho \to 0. \tag{1.15}$$

In particular,

$$\tilde{\lambda}_{\mathrm{rad},1} < \cdots < \tilde{\lambda}_{\mathrm{rad},n} < 0 < \tilde{\lambda}_{\mathrm{rad},n+1} < \cdots.$$

The limit (1.15) will be obtained in Remark 5.9. If $\rho > 0$ is small, then all the negative eigenvalues are

$$\tilde{\lambda}_{i,j} = \tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } 1 \leq i \leq n \text{ and } 0 \leq j \leq \ell,$$

which leads to Theorem 1.1. Because of a variational characterization of $\tilde{\lambda}_{\mathrm{rad},n}$, each orthogonal set gives an upper bound of $\tilde{\lambda}_{\mathrm{rad},n}$. We can obtain a sharp upper bound in Remark 5.4. On the other hand, a lower bound of $\tilde{\lambda}_{\mathrm{rad},1}$ is nontrivial. In this paper we use a first Neumann eigenvalue of a linearization problem as a lower bound of $\tilde{\lambda}_{\mathrm{rad},1}$. Then we use a blow-up argument to show that

$$\tilde{\lambda}_{\mathrm{rad},1} \to -\frac{1}{4}(\alpha+2)(\alpha+2N-2) \text{ as } \rho \to 0.$$

This limit implies (1.15), because of (1.14).

Let us again compare our problem with a problem on a ball. It was shown in [3, Proposition 3.3] that, for a certain class of nonlinear terms including $|x|^\alpha |U|^{p-1}U$, radial eigenvalues of a wighted eigenvalue problem on a ball satisfy

$$\tilde{\lambda}_{\mathrm{rad},n-1} < -\frac{1}{4}(\alpha+2)(\alpha+2N-2) < \tilde{\lambda}_{\mathrm{rad},n} < 0. \tag{1.16}$$

It follows from (1.14) and (1.16) that the n-th radial eigenvalue is larger than that of our problem. This causes a difference of lower bounds of Morse indices. A lower bound obtained in [3, Theorem 1.1] is smaller than that in Theorem 1.1 (i).

When $(p,\alpha) = (3, N-4)$, $\tilde{\lambda}_{\mathrm{rad},n}$ can be explicitly written as

$$\tilde{\lambda}_{\mathrm{rad},n} = -\frac{3k^2}{2k^2-1},$$

where k is the solution of (1.10). This explicit eigenvalue relates $\mathsf{m}(U^{\pm}_{n,\rho})$ to the ratio ρ/R, and hence it plays a crucial role in the proof of Theorems 1.4 and 1.5 (i). There are various results about lower bounds of the Morse index, while few results are known for upper bounds. A rather explicit upper bound of the Morse index is obtained in Theorem 1.5 (ii). An upper bound of the Morse index is in general not easy to obtain, because a lower bound of $\tilde{\lambda}_{\mathrm{rad},1}$ is needed. In this paper we use the following explicit lower bound of $\tilde{\lambda}_{\mathrm{rad},1}$:

$$\tilde{\lambda}_{\mathrm{rad},1} \geq -\left(\frac{N-2}{2}\right)^2 \left(1+\sqrt{1+\frac{3}{(2k^2-1)^2}}\right).$$

Recently, an exact expression of all the eigenvalues for the linearization of a Neumann problem $u''-u+u^3=0$ is obtained in [19]. The same method is applicable to the Dirichlet problem. However, we do not use those exact expression in this paper.

In summary, thanks to the critical exponent p_c, we can perform these detailed analysis of eigenvalues $\tilde{\lambda}_{\mathrm{rad},1}, \ldots, \tilde{\lambda}_{\mathrm{rad},n+1}$.

The paper consists of seven sections. In Sect. 2 we recall fundamental results about eigenvalues of (1.3) and (1.12). In Sect. 3 we use Emden's transformation and transform (1.1) into the scalar filed equation on an interval. Then, we prove Theorem 1.6. In Sect. 4 we compare the weighted eigenvalue problem (1.12) with (4.1) which is an eigenvalue problem associated with the scalar field equation. In Sect. 5 we compute the Morse index of $U^{\pm}_{n,\rho}$ and prove Theorem 1.1. In Sect. 6 we consider the case $(p,\alpha) = (3, N-4)$ and prove Theorems 1.4 and 1.5. Section 7 is an appendix. We recall the definition and basic properties of the complete elliptic integral $K(k)$ and Jacobi elliptic functions $\mathrm{sn}(\xi,k)$, $\mathrm{cn}(\xi,k)$, $\mathrm{dn}(\xi,k)$ and $\mathrm{sd}(\xi,k)$.

2 Preliminaries

Let $U^{\pm}_{n,\rho}$ be solutions of (1.1) with n nodal domains. In this paper we mainly study eigenvalues of the wighted eigenvalue problem (1.12). We define the number of the negative eigenvalues of (1.12) counted with their multiplicity by

$$\widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) = \sharp\{\text{negative eigenvalues of (1.12) counted with their multiplicity}\}. \tag{2.1}$$

The following proposition plays a crucial role in the study of the Morse index for radial solutions. It was extensively used in previous researches including [1–5, 12, 13, 15].

Proposition 2.1 *Let* $\mathsf{m}(U^{\pm}_{n,\rho})$ *and* $\widetilde{\mathsf{m}}(U^{\pm}_{n,\rho})$ *defined by* (1.4) *and* (2.1)*, respectively. Then the following holds:*

$$\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}).$$

The proof of Proposition 2.1 is the same as [13, Lemma 4.2 (a)], which proves the case $\alpha = 0$. See also [4, Proposition 1.1]. We omit the proof.

Let $\tilde{L}^{\pm}_n := |x|^2\left(\Delta + p|x|^{\alpha}|U^{\pm}_{n,\rho}|^{p-1}\right)$ defined on $H^1_0(A_\rho)$ and $\tilde{L}^{\pm}_{\mathrm{rad},n} := |x|^2\big(\Delta + p|x|^{\alpha}$ $|U^{\pm}_{n,\rho}|^{p-1}\big)$ defined on $H^1_{0,\mathrm{rad}}(A_\rho)$. The eigenvalue problem $\tilde{L}^{\pm}_{\mathrm{rad},n}\tilde{\Phi}_{\mathrm{rad}} = -\tilde{\lambda}_{\mathrm{rad}}\tilde{\Phi}_{\mathrm{rad}}$ can be also written as (1.13). From now on, $\sigma(\tilde{L}^{\pm}_n)$ and $\sigma(\tilde{L}^{\pm}_{\mathrm{rad},n})$ denote the set of the eigenvalues of (1.12) and (1.13), respectively. Let $\sigma(-\Delta_{\mathbb{S}^{N-1}})$ denote the set of eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$, *i.e.*, $\sigma(-\Delta_{\mathbb{S}^{N-1}}) = \{\nu_j\}_{j=0}^{\infty}$, where ν_j is defined by (1.5). Let $\tilde{\lambda}_{\mathrm{rad},i}$, $i = 1, 2, \ldots$, denote the i-th eigenvalue of (1.13).

Because of Proposition 2.1, we count the number of the negative eigenvalues of (1.12) instead of (1.3). The eigenvalue problem (1.12) is easier to study, since all the eigenvalues of (1.12) can be decomposed into a radial and spherical parts.

Proposition 2.2 *The eigenvalues of* (1.12) *satisfy the following:*

$$\sigma(\tilde{L}^{\pm}_n) = \sigma(\tilde{L}^{\pm}_{\mathrm{rad},n}) + \sigma(-\Delta_{\mathbb{S}^{N-1}}). \tag{2.2}$$

Specifically, each eigenvalue of (1.12)*, which is denoted by* $\tilde{\lambda}_{i,j}$*, can be written as*

$$\tilde{\lambda}_{i,j} = \tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } i \geq 1 \text{ and } j \geq 0, \tag{2.3}$$

where $\tilde{\lambda}_{\mathrm{rad},i} \in \sigma(\tilde{L}^{\pm}_{\mathrm{rad}})$ *and* $\nu_j \in \sigma(-\Delta_{\mathbb{S}^{N-1}})$.

The proof of Proposition 2.2 is the same as [8, Lemma 3.1], which studies (1.2). The relation (2.2) was also extensively used in [1–5, 8, 12–15, 17, 18].

The multiplicity of an eigenvalue of (1.12) can be calculated by the following proposition:

Proposition 2.3 *Let* $\tilde{\lambda} \in \sigma(\tilde{L}^{\pm}_n)$ *be fixed. Let* $m(\tilde{\lambda})$ *denote the multiplicity of* $\tilde{\lambda}$*. Then,*

$$m(\tilde{\lambda}) = \sum_{\substack{(i,j)\\ \tilde{\lambda}_{\mathrm{rad},i}+\nu_j=\tilde{\lambda}}} M_j(N),$$

where the summation takes all pairs (i, j) *satisfying*

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j = \tilde{\lambda},\ \ i \geq 1 \ \text{ and } j \geq 0. \tag{2.4}$$

Moreover, the eigenspace of (1.12) *associated to* $\tilde{\lambda}$ *is spanned by*

$$\tilde{\Phi}_{\mathrm{rad},i}(r)\omega_j(\theta) \text{ for } \rho < r < R \text{ and } \theta \in \mathbb{S}^{N-1},$$

where $\tilde{\Phi}_{\mathrm{rad},i}$ *denotes the* i*-th eigenfunction of* (1.13)*,* $\omega_j(\theta)$ *denotes an eigenfunction of* $-\Delta_{\mathbb{S}^{N-1}}$ *associated to* ν_j *and the pair* (i, j) *satisfies* (2.4).

We call the eigenvalue $\tilde{\lambda}_{\text{rad},i} + \nu_0$ of (1.12) a radial eigenvalue and $\tilde{\lambda}_{\text{rad},i} + \nu_j$, $j \geq 1$, a nonradial eigenvalue.

Since ν_j is explicitly given by (1.5), it is important to study the negative eigenvalues of (1.13). As mentioned in Sect. 1, we show that

$$\tilde{\lambda}_{\text{rad},1} < \cdots < \tilde{\lambda}_{\text{rad},n} < 0 < \tilde{\lambda}_{\text{rad},n+1} < \cdots .$$

3 Exact solutions

Let $N \geq 3$, $\alpha > -2$ and $p = p_c$. The problem

$$\Delta U + |x|^\alpha |U|^{p-1} U = 0 \text{ in } \mathbb{R}^N \tag{3.1}$$

has an exact positive radial singular solution

$$U^*(r) := A r^{-\beta}, \tag{3.2}$$

where

$$\beta := \frac{2+\alpha}{p-1} = \frac{N-2}{2} \quad \text{and} \quad A := \{\beta(N-2-\beta)\}^{\frac{1}{p-1}} = \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}}. \tag{3.3}$$

We use the so-called Emden transformation

$$t := -\frac{1}{m} \log \frac{r}{R} \quad \text{and} \quad u(t) := \frac{U(r)}{U^*(r)}, \tag{3.4}$$

where

$$m := \{\beta(N-2-\beta)\}^{-\frac{1}{2}} = A^{-\frac{p-1}{2}} = \frac{2}{N-2}. \tag{3.5}$$

Then, we see in the following lemma that $u(t)$ is a solution of the problem

$$\begin{cases} u'' + f(u) = 0 & \text{for } 0 < t < t_\rho, \\ u(0) = u(t_\rho) = 0, \end{cases} \tag{3.6}$$

where

$$f(u) := -u + |u|^{p-1} u \quad \text{and} \quad t_\rho := -\frac{1}{m} \log \frac{\rho}{R}. \tag{3.7}$$

Lemma 3.1 *Let $N \geq 3$, $\alpha > -2$ and $p = p_c$. The radial function $U(r) \in C^2((\rho, R)) \cap C([\rho, R])$ is a solution of (1.1) if and only if $u(t)$ is a solution of (3.6).*

Proof By direct calculation we have

$$\begin{aligned} \frac{dU(r)}{dr} &= -\frac{AR^{-\beta-1}}{m} u' e^{(\beta+1)mt} - AR^{-\beta-1} \beta e^{(\beta+1)mt}, \\ \frac{d^2U(r)}{dr^2} &= \frac{AR^{-\beta-2}}{m^2} u'' e^{(\beta+2)mt} + \frac{AR^{-\beta-2}}{m} (2\beta+1) u' e^{(\beta+2)mt} + AR^{-\beta-2} \beta(\beta+1) u e^{(\beta+2)mt}. \end{aligned}$$

Then,

$$0=\frac{d^2U}{dr^2}+\frac{N-1}{r}\frac{dU}{dr}+r^{\alpha}|U|^{p-1}U$$
$$=\frac{AR^{-\beta-2}}{m^2}e^{(\beta+2)mt}\left\{u''+m(2\beta-N+2)u'-m^2\beta(N-2-\beta)u+m^2A^{p-1}|u|^{p-1}u\right\}. \tag{3.8}$$

We see by (3.3) that $2\beta-N+2=0$. By (3.8) we obtain

$$u''-u+|u|^{p-1}u=0.$$

Since $\rho<r<R$, we have that $0<t<t_\rho$ and that $u(t)$ satisfies the Dirichlet boundary condition. Then, $u(t)$ satisfies (3.6). It is clear that the converse is true. The proof is complete. □

It is well known that a solution of (3.6) corresponds to an orbit of the system

$$\begin{cases} u'=v,\\ v'=u-|u|^{p-1}u. \end{cases} \tag{3.9}$$

Since a Dirichlet boundary condition is imposed in (3.6), a corresponding orbit $(u(t),v(t))$ starts from a point on the v-axis and arrives a point on the v-axis. The system (3.9) has three equilibria $(-1,0)$, $(0,0)$ and $(1,0)$. Then, $(\pm1,0)$ are centers and $(0,0)$ is a saddle. Multiplying the equation in (3.6) by u' and integrating it over $[0,x]$, we see that each solution $(u(t),v(t))$ is on a level set

$$\frac{v^2}{2}-\frac{u^2}{2}+\frac{|u|^{p+1}}{p+1}=C.$$

We see that (3.9) has two homoclinic loops connecting $(0,0)$ to itself which are on $v^2-u^2+2|u|^{p+1}/(p+1)=0$. One loop surrounds $(1,0)$ and the other loop surround $(-1,0)$. Hence two loops consist of a *figure eight*. Let

$$\Omega:=\{(u,v);\ v^2\le u^2-2|u|^{p+1}/(p+1)\}. \tag{3.10}$$

Two loops satisfy $v^2=u^2-2|u|^{p+1}/(p+1)$, *i.e.,* the boundary of Ω. It is obvious that there is no orbit in Ω satisfying the boundary condition of (3.6). Therefore, a solution orbit of (3.6) is in $\mathbb{R}^2\setminus\Omega$, and they are periodic orbits. If $(u(t),v(t))$ satisfies (3.9), then $(-u(t),-v(t))$ also satisfies (3.9). This indicates that all times from a point on the v-axis to the next point of the v-axis are equal, and hence the length of each nodal domain of $u(t)$ is equal to each other. Hence, if u has n nodal domains, then the length of each nodal domain is t_ρ/n.

When $(p,\alpha)=(3,N-4)$, the radial solutions of (1.1) can be written explicitly in terms of Jacobi elliptic functions

Proof of Theorem 1.6 Because of Lemma 3.1, it is enough to obtain an exact solution of (3.6) with n nodal domains. Since u satisfies $u''-u+u^3=0$ and u satisfies the Dirichlet boundary condition, a general solution $u(t)$ can be written in terms of elliptic functions as follows:

$$u(t)=\sqrt{\frac{2k^2}{2k^2-1}}\,\mathrm{cn}\left(\frac{t-t_0}{\sqrt{2k^2-1}},k\right),\quad \frac{1}{\sqrt{2}}<k<1 \text{ and } t_0\in\mathbb{R}.$$

See [10, Chapter 7, Section 10] for details of this formula and see Sect. 7 for the definition of $\mathrm{cn}(\xi, k)$. Since $u(0) = u(t_\rho) = 0$ and u has n nodal domains, we see

$$t_0 = \pm\sqrt{2k^2 - 1}K(k) \quad \text{and} \quad \frac{t_\rho}{\sqrt{2k^2 - 1}} = 2nK(k), \tag{3.11}$$

where K denotes the complete elliptic integral of the first kind whose definition and basic properties are recalled in Sect. 7.1. Then,

$$u(t) = \sqrt{\frac{2k^2}{2k^2 - 1}}\mathrm{cn}\left(2nK(k)\frac{t}{t_\rho} \mp K(k), k\right), \quad \frac{1}{\sqrt{2}} < k < 1.$$

By the addition formula

$$\mathrm{cn}(x + y, k) = \frac{\mathrm{cn}(x, k)\mathrm{cn}(y, k) - \mathrm{sn}(x, k)\mathrm{sn}(y, k)\mathrm{dn}(x, k)\mathrm{dn}(y, k)}{1 - k^2\mathrm{sn}^2(x, k)\mathrm{sn}^2(y, k)}$$

we have that $\mathrm{cn}(x \mp K(k), k) = \pm\sqrt{1 - k^2}\mathrm{sd}(x, k)$. We obtain

$$u(t) = \pm\sqrt{\frac{2k^2(1 - k^2)}{2k^2 - 1}}\mathrm{sd}\left(2nK(k)\frac{t}{t_\rho}, k\right), \quad \frac{1}{\sqrt{2}} < k < 1. \tag{3.12}$$

We return to the original variables. Then we obtain (1.9). By the first equality in (3.11) we obtain (1.10). The function $K(k)$ is increasing in $k \in (0, 1)$. See Sect. 7. It is clear that $\sqrt{2k^2 - 1}K(k)$ is strictly increasing in $k \in (\frac{1}{\sqrt{2}}, 1)$,

$$\lim_{k \to \frac{1}{\sqrt{2}}} \sqrt{2k^2 - 1}K(k) = 0 \quad \text{and} \quad \lim_{k \to 1} \sqrt{2k^2 - 1}K(k) = \infty.$$

Thus, (1.10) has a unique solution $k \in \left(\frac{1}{\sqrt{2}}, 1\right)$. □

4 Eigenvalue problem

Let $n > 1$ and let $u(t)$ be a solution of (3.6) with n nodal domains. We consider the linearized eigenvalue problem

$$\begin{cases} \phi'' + f'(u)\phi = -\mu\phi & \text{for } 0 < t < t_\rho, \\ \phi(0) = \phi(t_\rho) = 0. \end{cases} \tag{4.1}$$

Here $f'(u) = -1 + p|u|^{p-1}$. Let μ_i, $i \geq 1$, denote the i-th eigenvalue and let ϕ_i denote an eigenfunction associated with μ_i.

We use the same change of variables as (3.4), i.e., let $t := -\frac{1}{m}\log\frac{r}{R}$ and we define

$$U(r) := u(t)U^*(Re^{-mt}) \quad \text{and} \quad \tilde{\Phi}(r) := \phi(t)U^*(Re^{-mt}),$$

where $U^*(r)$ is the singular solution of (3.1) given by (3.2) and ϕ is an eigenfunction of (4.1). Then $U(r)$ is a solution of (1.1) and $\tilde{\Phi}(r)$ satisfies

$$\begin{cases} \Delta\tilde{\Phi} + p|x|^\alpha|U|^{p-1}\tilde{\Phi} = -\frac{\mu_i}{m^2|x|^2}\tilde{\Phi} & \text{in } A_\rho, \\ \tilde{\Phi} = 0 & \text{on } \partial A_\rho. \end{cases}$$

For $i \geq 1$, let

$$\tilde{\lambda}_{\mathrm{rad},i} = \frac{\mu_i}{m^2} = \left(\frac{N-2}{2}\right)^2 \mu_i. \tag{4.2}$$

Then, the pair $(\tilde{\lambda}_{\mathrm{rad},i}, \tilde{\Phi})$ satisfies (1.12), and hence the radial part of the eigenvalue problem (1.12), which is (1.13), is equivalent to (4.1). All the eigenvalues of (1.12) can be obtained by (2.3). Since all the eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$ are explicitly given by (1.5), it is crucial to study eigenvalues of (4.1).

5 Morse index

5.1 Fundamental results for the scalar field equation

Hereafter, we use a time map. A reader can consult [23, Chapters 1 and 2] for details about relations of a time map and a solution structure of two point boundary value problems.

Let

$$F(v) := \int_0^v f(s)ds = -\frac{v^2}{2} + \frac{|v|^{p+1}}{p+1} \quad \text{and} \quad a_0 := \left(\frac{p+1}{2}\right)^{1/(p-1)}.$$

First, we consider a positive solution of

$$\begin{cases} U'' + \frac{N-1}{r}U' + r^\alpha |U|^{p-1}U = 0 & \text{for } \rho < r < R, \\ U(r) > 0 & \text{for } \rho < r < R, \\ U(\rho) = U(R) = 0. \end{cases} \tag{5.1}$$

We use the change of variables

$$t := -\frac{1}{m}\log\frac{r}{\sqrt{\rho R}} \text{ and } v(t) := \frac{U(r)}{U^*(r)},$$

where m and $U^*(r)$ are defined in (3.5) and (3.2), respectively. Then $v(t)$ satisfies

$$\begin{cases} v'' + f(v) = 0 & \text{for } -T_0 < t < T_0, \\ v(t) > 0 & \text{for } -T_0 < T < T_0, \\ v(-T_0) = v(T_0) = 0, \end{cases} \tag{5.2}$$

where $T_0 := \frac{1}{m}\log\sqrt{\frac{R}{\rho}}$. Note that $T_0 \to 0$ as $\rho \to R$ and $T_0 \to \infty$ as $\rho \to 0$. A solution of (5.1) corresponds to a solution (5.2). A solution of (5.2) corresponds to an orbit of (3.9) in the right half-plane that starts from a point $(0, v'(-T_0))$ and arrives $(0, v'(T_0))$. This orbit goes across the horizontal axis at $(a, 0)$, and a is the maximum value of $v(t)$ for $-T_0 < t < T_0$. The orbit is in $\mathbb{R}^2 \setminus \Omega$, where Ω is defined by (3.10). Hence $a > a_0$. We study the time of this orbit. Multiplying the equation in (5.2) with v' and integrating it over $[T_0, t]$, we have

$$\frac{v'(t)^2}{2} + F(v(t)) = F(a).$$

Integrating $1 = v'(t)/\sqrt{2(F(a) - F(v(t))}$ over $[0, T_0]$, we have

$$T_0 = T_0(a) = \frac{1}{\sqrt{2}}\int_0^a \frac{dv}{\sqrt{F(a) - F(v)}} \quad \text{for } a > a_0.$$

Hence, T_0 can be related to a which is $\max_{-T_0<t<T_0} v(t)$. It was shown in [22] that, for each pair (ρ, R), $0 < \rho < R$, (5.1) has a unique solution. Therefore, (5.2) also has a unique solution for each $T_0 > 0$. Since $T_0(a)$ corresponds to a solution of (5.2) with $T_0 = T_0(a)$, the uniqueness of a solution of (5.2) indicates that $T_0(a)$ is monotone. Since $a \to a_0$, the corresponding orbit converges to a solution corresponding to a homoclinic loop in $C_{loc}(\mathbb{R})$, and hence $T(a) \to \infty$ as $a \to a_0$. This limit, together with the existence of a solution of (5.2) for all $T_0 > 0$, indicates that $T_0(a)$ is decreasing, and $T_0(a) \to 0$ as $a \to \infty$. In summary,

$$T_0(a) \text{ is defined for } a_0 < a < \infty, \text{ it is decreasing, } \lim_{a\to a_0} T_0(a) = \infty \text{ and } \lim_{a\to\infty} T_0(a) = 0.$$

Hence there exists an inverse function $a = a(T_0)$ for $0 < T_0 < \infty$.

We consider the following limit problem of (5.2):

$$\begin{cases} w'' + f(w) = 0 & \text{for } -\infty < t < \infty, \\ w > 0 & \text{for } -\infty < t < \infty, \\ \lim_{t\to\pm\infty} w(t) = 0. \end{cases} \tag{5.3}$$

Then w can be explicitly written as

$$w(t) = a_0 \left(\cosh\left(\frac{p-1}{2}t\right)\right)^{-\frac{2}{p-1}}.$$

In particular, $w(0) = a_0$. We easily see that $\lim_{a\to a_0} T_0 = \infty$ and

$$v(t) \to w(t) \quad \text{in} \quad C_{loc}(\mathbb{R}) \quad \text{as} \quad a \to a_0.$$

Moreover,

$$a \to a_0 \quad \text{as } T_0 \to \infty. \tag{5.4}$$

Lemma 5.1 *Let v be a solution of (5.2). Then there exists $C > 0$ such that, for $a \in (a_0, 2a_0)$,*

$$|v(t)| \le C \exp\left(-\frac{|t|}{\sqrt{2}}\right) \textit{ for } -T_0 \le t \le T_0. \tag{5.5}$$

Proof Since $v(t)$ is even, it is enough to prove (5.5) for $0 \le t \le T_0$. We define

$$\tilde{v}(t) := \begin{cases} v(t) & \text{for } 0 \le t \le T_0, \\ 0 & \text{for } t > T_0. \end{cases}$$

Then $\tilde{v}(t) \to w(t)$ in $C_{loc}([0,\infty))$ as $a \to a_0$, $\tilde{v}(t) \le a$ for $t \ge 0$ and $\tilde{v}(t)$ is nonincreasing in t. There exists $t_0 > 0$ independent of $a \in (a_0, 2a_0)$ such that $0 \le \tilde{v}(t) \le 2^{-1/(p-1)}$ for $t \ge t_0$. Let $\bar{v}(t) := 2a_0 \exp(-(t-t_0)/\sqrt{2})$. Then,

$$-\tilde{v}'' + \frac{1}{2}\tilde{v} \le 0 \le -\bar{v}'' + \frac{1}{2}\bar{v} \quad \text{for } t_0 \le t < T_0.$$

Note that $a_0 = \tilde{v}(t_0) \le \bar{v}(t_0) = 2a_0$ and $0 = \tilde{v}(T_0) \le \bar{v}(T_0)$. We see that $\bar{v}(t) - \tilde{v}(t)$ does not have a negative minimum in $t_0 < t < T_0$. Thus,

$$0 \le \tilde{v}(t) \le \bar{v}(t) \quad \text{for} \quad t_0 \le t \le T_0. \tag{5.6}$$

It is clear that

$$0 \le \tilde{v}(t) \le \bar{v}(t) \quad \text{for} \quad t \in [0, t_0) \cup (T_0, \infty). \tag{5.7}$$

By (5.7) and (5.6) we see that $0 \le \tilde{v}(t) \le \bar{v}(t)$ for $t \ge 0$ and that $\bar{v}(t)$ is independent of $a \in (a_0, 2a_0)$. The proof is complete. □

A linearization problem of (5.3) in $L^2(\mathbb{R})$ becomes

$$\begin{cases} \phi'' + f'(w)\phi = -\mu\phi & \text{for } -\infty < t < \infty, \\ \phi \in L^2(\mathbb{R}). \end{cases} \tag{5.8}$$

The spectra of (5.8) is known as follows:

Proposition 5.2 *Let* $L := \frac{d^2}{dt^2} + f'(w)$*. The problem* (5.8) *has a continuous spectrum* $[1, \infty)$*. Moreover, the first eigenvalue of* (5.8) *is* $-(p-1)(p+3)/4$*, the second eigenvalue is* 0 *and the third eigenvalue is* $(p-1)(5-p)/4$ *if* $1 < p < 3$*. Specifically, the following hold:*

$$\begin{aligned} &\phi_0 := w^{\frac{p+1}{2}}, && L\phi_0 = \frac{1}{4}(p-1)(p+3)\phi_0 && \text{if } p > 1, \\ &\phi_1 := w', && L\phi_1 = 0 && \text{if } p > 1, \\ &\phi_2 := w^{\frac{3-p}{2}} - \frac{p+3}{2(p+1)} w^{\frac{p+1}{2}}, && L\phi_2 = -\frac{1}{4}(p-1)(5-p)\phi_2 && \text{if } 1 < p < 3. \end{aligned}$$

See e.g. [11, p.9] for Proposition 5.2.

5.2 Proof of Theorem 1.1

Let $T_0 > 0$ and $u_1(t)$ be a positive solution of

$$\begin{cases} u'' + f(u) = 0 & \text{for } 0 < t < 2T_0, \\ u(0) = u(2T_0) = 0. \end{cases} \tag{5.9}$$

Let μ^D denote the first eigenvalue of the Dirichlet problem

$$\begin{cases} \phi'' + f'(u_1)\phi = -\mu\phi & \text{for } 0 < t < 2T_0, \\ \phi(0) = \phi(2T_0) = 0. \end{cases} \tag{5.10}$$

Lemma 5.3 *Let* μ^D *be the first Dirichlet eigenvalue of* (5.10)*. Then the following hold:*
(i) *For* $T_0 > 0$*,*

$$\mu^D < -\frac{1}{4}(p-1)(p+3). \tag{5.11}$$

(ii) *As* $T_0 \to 0$*,*

$$\mu^D \to -\infty. \tag{5.12}$$

Proof (i) Let u_1 be a positive solution of (5.9). Let

$$a := \max_{0 \le t \le 2T_0} u_1(t) = u_1(T_0) > a_0 \quad \text{and} \quad b := \sqrt{2F(a)}.$$

Multiplying the equation of (5.9) by u' and integrating it over $[T_0, x]$, we have

$$\frac{u_1'^2}{2} - \frac{u_1^2}{2} + \frac{u^{p+1}}{p+1} = -\frac{a^2}{2} + \frac{a^{p+1}}{p+1}.$$

Since $b = \sqrt{2F(a)}$, we have

$$\frac{u_1'^2}{2} - \frac{u_1^2}{2} + \frac{u_1^{p+1}}{p+1} = -\frac{a^2}{2} + \frac{a^{p+1}}{p+1} = \frac{b^2}{2}. \tag{5.13}$$

Let $I_1 := (0, 2T_0)$ and

$$H(\phi) := \int_{I_1} \phi'^2 - f'(u_1)\phi^2 dt.$$

By the variational characterization we see that

$$\mu^D = \inf_{\varphi \in H_0^1(I_1)\setminus\{0\}} \frac{H(\varphi)}{\|\varphi\|_2^2}. \tag{5.14}$$

We take a test function $\psi := u_1^{\frac{p+1}{2}}$. Using (5.13) and $u_1'' = u_1 - u_1^p$, we have

$$\begin{aligned}
H(\psi) &= \int_{I_1} -\psi''\psi - (-1 + pu_1^{p-1})\psi^2 dt \\
&= \int_{I_1} -\frac{p+1}{2}\left(\frac{p-1}{2}u_1^{\frac{p-3}{2}}u_1'^2 + u_1^{\frac{p-1}{2}}u_1''\right)u_1^{\frac{p+1}{2}} + u_1^{p+1} - pu_1^{2p} dt \\
&= \int_{I_1} -\frac{p^2-1}{4}u_1^{p-1}u_1'^2 - \frac{p-1}{2}u_1^{p+1} - \frac{p-1}{2}u_1^{2p} dt \\
&= -\frac{1}{4}(p-1)(p+3)\int_{I_1} u_1^{p+1} dt - \frac{p^2-1}{4}\left(-a^2 + \frac{2}{p+1}a^{p+1}\right)\int_{I_1} u_1^{p-1} dt.
\end{aligned}$$

By (5.13) we have

$$\frac{H(\psi)}{\|\psi\|_2^2} = -\frac{1}{4}(p-1)(p+3) - \frac{1}{4}(p^2-1)b^2\frac{\int_{I_1} u_1^{p-1} dt}{\int_{I_1} u_1^{p+1} dt}. \tag{5.15}$$

Thus, by (5.15) and (5.14) we have

$$\mu^D = \inf_{\varphi \in H_0^1(I_1)\setminus\{0\}} \frac{H(\varphi)}{\|\varphi\|_2^2} \le \frac{H(\psi)}{\|\psi\|_2^2} < -\frac{1}{4}(p-1)(p+3). \tag{5.16}$$

Note that the largeness of $T_0 > 0$ is not necessary in (5.16). We have shown that (5.11) holds.

(ii) In this case we take a test function u_1. We show that

$$\frac{H(u_1)}{\|u_1\|_2^2} \to -\infty \text{ as } T_0 \to 0. \tag{5.17}$$

If (5.17) holds, then by (5.14) we have

$$\mu^D = \inf_{\varphi \in H_0^1(I_1)\setminus\{0\}} \frac{H(\varphi)}{\|\varphi\|_2^2} \le \frac{H(u_1)}{\|u_1\|_2^2} \to -\infty \text{ as } T_0 \to 0,$$

and hence (5.12) holds.

Hereafter, we prove (5.17). We need an apriori estimate to prove (5.17). Using Hölder's inequality, we have

$$|u_1(t)| \le \left(\int_0^t |u_1'(s)|^2 ds\right)^{1/2}\left(\int_0^t 1^2 ds\right)^{1/2} \le \|u_1'\|_2 t^{1/2}.$$

Using this inequality, we have

$$\|u_1\|_{p+1}^{p+1} = \int_0^{2T_0} |u_1|^{p+1} dt \le \|u_1'\|_2^{p+1} \int_0^{2T_0} t^{\frac{p+1}{2}} dt = \frac{2}{p+3}(2T_0)^{\frac{p+3}{2}} \|u_1'\|_2^{p+1}. \tag{5.18}$$

Multiplying the equation in (5.9) with u_1 and integrating it by parts, we have

$$\int_0^{2T_0} u_1'^2 + u_1^2 dt = \int_0^{2T_0} |u_1|^{p+1} dt. \tag{5.19}$$

By (5.19) and (5.18) we have

$$\|u_1'\|_2^2 \le \|u_1'\|_2^2 + \|u_1\|_2^2 = \|u_1\|_{p+1}^{p+1} \le \frac{2}{p+3}(2T_0)^{\frac{p+3}{2}} \|u_1'\|_2^{p+1}. \tag{5.20}$$

Then, by (5.20) we have

$$\left(\frac{p+3}{2}\right)^{\frac{2}{p-1}} (2T_0)^{-\frac{p+3}{p-1}} \le \|u_1'\|_2^2. \tag{5.21}$$

By (5.21) and (5.19) we have

$$\left(\frac{p+3}{2}\right)^{\frac{2}{p-1}} (2T_0)^{-\frac{p+3}{p-1}} \le \|u_1'\|_2^2 \le \|u_1\|_{p+1}^{p+1}.$$

Therefore,

$$\left(\frac{p+3}{2}\right)^{\frac{2}{p+1}} (2T_0)^{-\frac{p+3}{p+1}} \le \|u_1\|_{p+1}^{p-1}. \tag{5.22}$$

By Hölder's inequality we have

$$\int_0^{2T_0} u_1^2 dt \le \left(\int_0^{2T_0} |u_1|^{p+1} dt\right)^{\frac{2}{p+1}} \left(\int_0^{2T_0} 1^{\frac{p+1}{p-1}}\right)^{\frac{p-1}{p+1}},$$

and hence

$$(2T_0)^{-\frac{p-1}{p+1}} \le \frac{\|u_1\|_{p+1}^2}{\|u_1\|_2^2}. \tag{5.23}$$

By (5.23), (5.22) and (5.19) we have

$$\begin{aligned}
\frac{H(u_1)}{\|u_1\|_2^2} &= \frac{\int_0^{2T_0} u_1'^2 + u_1^2 - p|u_1|^{p+1} dt}{\|u_1\|_2^2} = -(p-1)\frac{\int_0^{2T_0} |u_1|^{p+1} dt}{\|u_1\|_2^2} \\
&= -(p-1)\,\|u_1\|_{p+1}^{p-1} \frac{\|u_1\|_{p+1}^2}{\|u_1\|_2^2} \\
&\le -(p-1)\left(\frac{p+3}{2}\right)^{\frac{2}{p+1}} (2T_0)^{-2} \to -\infty \text{ as } T_0 \to 0.
\end{aligned}$$

We have shown that (5.17) holds. The proof of (ii) is complete. □

Remark 5.4 Let $v(t)$ be a solution of (5.2) and let $a := v(0)$. We can obtain a sharp upper bound of μ^D in the following way: We can prove that

$$\int_{-T_0(a)}^{T_0(a)} |v(t)|^q dt \to \int_{\mathbb{R}} w(t)^q dt \quad \text{as} \quad a \to a_0, \tag{5.24}$$

using the dominated convergence theorem with Lemma 5.1. Here,

$$\int_{\mathbb{R}} w(t)^q dt = \frac{2}{p-1}\left(\frac{p+1}{2}\right)^{\frac{q}{p-1}} B\left(\frac{q}{p-1}, \frac{1}{2}\right) = \frac{2}{p-1}\left(\frac{p+1}{2}\right)^{\frac{q}{p-1}} \frac{\sqrt{\pi}\,\Gamma(\frac{q}{p-1})}{\Gamma(\frac{q}{p-1}+\frac{1}{2})},$$

Γ denotes the Gamma function defined by $\Gamma(z) := \int_0^\infty t^{z-1}e^{-z}dt$ and B denotes the Beta function, which satisfies $B(\xi,\eta) := \Gamma(\xi)\Gamma(\eta)/\Gamma(\xi+\eta)$. Using the same calculation as in the proof of [6, Lemma 3.2], we can obtain

$$2\log b = -2T_0 + 2\log 2a_0 + \frac{4}{p-1}\log 2 + o(1) \text{ as } T_0 \to \infty. \tag{5.25}$$

By (5.25), (5.24) and (5.15) we have

$$\mu^D \le -\frac{1}{4}(p-1)(p+3) - \frac{(p^2-1)2^{\frac{p+3}{p-1}}}{B\left(\frac{p+1}{p-1}, \frac{1}{2}\right)} e^{-2T_0}(1+o(1)) \text{ as } T_0 \to \infty. \tag{5.26}$$

However, we do not use (5.26) in this paper.

Let u_1 be a positive solution of (5.9). Let μ^N denote the first eigenvalue of the Neumann problem

$$\begin{cases} \phi'' + f'(u_1)\phi = -\mu\phi & \text{for } 0 < t < 2T_0, \\ \phi'(0) = \phi'(2T_0) = 0. \end{cases} \tag{5.27}$$

Note that $u_1(0) = u_1(2T_0) = 0$, while $\phi'(0) = \phi'(2T_0) = 0$.

Lemma 5.5 *Let μ^N be the first Neumann eigenvalue of (5.27). Then,*

$$\mu^N \to -\frac{1}{4}(p-1)(p+3) \text{ as } T_0 \to \infty. \tag{5.28}$$

Proof Let $v(t)$ be a solution of (5.2). Then the first eigenvalue of (5.27) is equal to that of

$$\begin{cases} \phi'' + f'(v)\phi = -\mu\phi & \text{for } -T_0 < t < T_0, \\ \phi'(-T_0) = \phi'(T_0) = 0. \end{cases} \tag{5.29}$$

Hereafter, we consider (5.29).

Let $I_{T_0} := (-T_0, T_0)$. Let μ^N be the first eigenvalue of (5.29) and let ϕ be a first eigenfunction of (5.29). We may assume that $\phi > 0$ in $\mathbb{R}$ and $\|\phi\|_{L^2(I_{T_0})} = 1$.

We see that

$$\mu^N = \inf_{\substack{\varphi \in H^1(I_{T_0}) \\ \|\varphi\|_2 = 1}} \int_{I_{T_0}} \varphi'^2 - f'(v)\varphi^2 dt \tag{5.30}$$

and that ϕ attains the infimum of (5.30). Hereafter, we define $\phi = 0$ on $\mathbb{R}\backslash I_{T_0}$ and $\chi_{I_{T_0}}$ denotes the indicator function of I_{T_0}. We also define $v(t) = 0$ on $\mathbb{R} \setminus I_{T_0}$ to extend the domain of $v(t)$. We see that $1 - p\,\|v\|_\infty^{p-1} \le 1 - p|v|^{p-1}$ on I_{T_0}. By Lemma 5.3 (i) we have

$$1 - p\,\|v\|_\infty^{p-1} \le \inf_{\substack{\varphi \in H^1(I_{T_0}) \\ \|\varphi\|_2 = 1}} \int_{I_{T_0}} \varphi'^2 + (1 - p|v|^{p-1})\varphi^2 dt = \mu^N \le \mu^D < -\frac{1}{4}(p-1)(p+3), \tag{5.31}$$

where we used

$$\mu^N = \inf_{\substack{\varphi\in H^1(I_{T_0})\\ \|\varphi\|_2=1}} H(\varphi) \le \inf_{\substack{\varphi\in H_0^1(I_{T_0})\\ \|\varphi\|_2=1}} H(\varphi) = \mu^D. \tag{5.32}$$

Since $\|v\|_\infty$ is bounded for large $T_0 > 0$, by (5.31) we see that μ^N is bounded for large $T_0 > 0$. Hence, $1 + \mu^N + p\,\|v\|_\infty^{p-1}$ is also bounded for large $T_0 > 0$. Since

$$\int_{\mathbb{R}} \phi'^2 \chi_{I_{T_0}} dt \le \left(1 + \mu^N + p\,\|v\|_\infty^{p-1}\right) \int_{\mathbb{R}} \phi^2 \chi_{I_{T_0}} dt \le C_0 \tag{5.33}$$

uniformly for large $T_0 > 0$, we see that, for each compact set K, there are $C_{K_1} > 0$ and a compact set K_1 such that K is in the interior set of K_1 and that $\|\phi\|_{H^1(K_1)} < C_{K_1}$ for large $T_0 > 0$. Since $H^1(K_1) \hookrightarrow C^\gamma(K_1)$, $0 < \gamma < 1/2$, is continuous, $\{\phi\}$ is bounded in $C^\gamma(K_1)$. Note that $\|\phi\|_{L^\infty(\mathbb{R})}$ is bounded uniformly for large $T_0 > 0$ because $\|\phi\|_{H^1(\mathbb{R})}$ is bounded uniformly for large $T_0 > 0$. Since ϕ satisfies the equation in (5.29), by Schauder estimates we see that $\{\phi\}$ is bounded in $C^{2,\gamma}(K)$. It follows from Ascoli-Arzelá theorem with a diagonal argument that there exists $\phi_* \in C^2(\mathbb{R})$ such that $\phi \to \phi_*$ in $C^2_{loc}(\mathbb{R})$ as $T_0 \to \infty$. Moreover, $|\phi_*(t)| \le C_2$ for $t \in \mathbb{R}$, because $H^1(K_1) \hookrightarrow L^\infty(K_1)$ is a continuous inclusion and $\|\phi\|_{H^1(I_{T_0})}$ is bounded uniformly for large $T_0 > 0$. We show that

$$\lim_{T_0\to\infty} \int_{\mathbb{R}} |v|^{p-1}\phi^2 dt = \int_{\mathbb{R}} |w|^{p-1}\phi_*^2 dt, \tag{5.34}$$

where w is a unique solution of (5.3). Since $\|\phi\|_\infty$ is bounded uniformly for large $T_0 > 0$, by Lemma 5.1 we see that $|v|^{p-1}\phi^2$ is dominated by an $L^1(\mathbb{R})$-function which is independent of $T_0 > 0$ large. Note that $T_0 \to \infty$ if and only if $a \to a_0$. The function $|v|^{p-1}\phi^2$ converges pointwise to $w^{p-1}\phi_*^2$ in $\mathbb{R}$ as $T_0 \to \infty$. By the dominated convergence theorem we obtain (5.34). Using (5.34), Fatou's lemma and Lemma 5.3 (i), we have

$$\begin{aligned}
\int_{\mathbb{R}} \phi_*'^2 + \phi_*^2 - p|w|^{p-1}\phi_*^2 dt &\le \liminf_{T_0\to\infty} \int_{\mathbb{R}} \left(\phi'^2 + \phi^2\right) \chi_{I_{T_0}} dt + \liminf_{T_0\to\infty} \int_{\mathbb{R}} -p|v|^{p-1}\phi^2 dt \\
&\le \liminf_{T_0\to\infty} \int_{\mathbb{R}} \left(\phi'^2 + \phi^2 - p|v|^{p-1}\phi^2\right) \chi_{I_{T_0}} dt \le \liminf_{T_0\to\infty} \mu^N \\
&\le \liminf_{T_0\to\infty} \mu^D \le -\frac{1}{4}(p-1)(p+3),
\end{aligned} \tag{5.35}$$

where w is a solution of (5.3) and we used (5.32). Because of (5.35), $\phi_* \not\equiv 0$ in $\mathbb{R}$. Since $\{\mu^N\}$ is bounded, there exists μ_*^N and a subsequence of $\{\mu^N\}$, which is still denoted by $\{\mu^N\}$, such that $\mu^N \to \mu_*^N$ as $T_0 \to \infty$. Applying Fatou's lemma to the LHS of (5.33), we see that $\phi_*' \in L^2(\mathbb{R})$. Since $\|\phi\|_2 = 1$, again by Fatou's lemma we see that $\phi_* \in L^2(\mathbb{R})$, and hence $\phi_* \in H^1(\mathbb{R})$. Since ϕ satisfies (5.29), ϕ_* satisfies the problem

$$\begin{cases} \phi_*'' + f'(w)\phi_* = -\mu_*^N \phi_* & \text{for } -\infty < t < \infty, \\ \phi_* \ge 0. \end{cases}$$

Since $\phi_* \not\equiv 0$ in $\mathbb{R}$, by the strong maximum principle $\phi_* > 0$ in $\mathbb{R}$. Thus, ϕ_* is a first eigenfunction. By Proposition 5.2 we see that $\mu_*^N = -(p-1)(p+3)/4$. This indicates that (5.28) holds. □

The following elementary inequality will be used later.

Lemma 5.6 *Let* $\xi_j \geq 0,\ j = 0, 1, \dots, n$, *and* $\eta_j > 0,\ j = 0, 1, \dots, n$. *If*

$$\frac{\xi_j}{\eta_j} \geq \frac{\xi_0}{\eta_0} \quad \textit{for } j = 1, 2, \dots, n,$$

then

$$\frac{\xi_1 + \cdots + \xi_n}{\eta_1 + \cdots + \eta_n} \geq \frac{\xi_0}{\eta_0}.$$

Proof Without loss of generality we assume that

$$\frac{\xi_1}{\eta_1} = \min\left\{\frac{\xi_1}{\eta_1}, \cdots, \frac{\xi_n}{\eta_n}\right\}.$$

Since $\frac{\xi_j}{\eta_j} \geq \frac{\xi_1}{\eta_1}$ for $j = 1, 2, \cdots, n$, we see that $\xi_j \eta_1 \geq \xi_1 \eta_j$. Then

$$(\xi_1 + \cdots + \xi_n)\eta_1 - \xi_1(\eta_1 + \cdots + \eta_n) \geq \xi_1(\eta_1 + \cdots + \eta_n) - \xi_1(\eta_1 + \cdots + \eta_n) = 0,$$

and hence

$$\frac{\xi_1 + \cdots + \xi_n}{\eta_1 + \cdots + \eta_n} \geq \frac{\xi_1}{\eta_1} \geq \frac{\xi_0}{\eta_0}.$$

□

Let $n \geq 1$. We define

$$T := t_\rho = \frac{1}{m} \log \frac{R}{\rho} \quad \text{and} \quad T_0 := \frac{T}{2n}. \tag{5.36}$$

Then (3.6) can be written as follows:

$$\begin{cases} u'' + f(u) = 0 & \text{for } 0 < t < 2nT_0, \\ u(0) = u(2nT_0) = 0. \end{cases} \tag{5.37}$$

Hereafter, $u_n^\pm$ denotes a solution of (5.37) with n nodal domains such that $(u_n^+)'(0) > 0$ and $(u_n^-)'(0) < 0$. It is easy to see that $u_n^-(t) = -u_n^+(t)$. We do not distinguish u_n^+ and u_n^-. Note that $u_n^\pm(t)$ can be extended to a $2T_0$-antiperiodic and $4T_0$-periodic function and that all the nodal domains of $u_n^\pm(t)$ are

$$(0, 2T_0),\ (2T_0, 4T_0), \dots, (2nT_0 - 2T_0, 2nT_0).$$

The linearization problem of (5.37) becomes

$$\begin{cases} \phi'' + f'(u_n^\pm)\phi = -\mu\phi & \text{for } 0 < t < 2nT_0, \\ \phi(0) = \phi(2nT_0) = 0. \end{cases} \tag{5.38}$$

Let $\{\mu_i\}_{i=1}^\infty$ denote the set of the eigenvalues of (5.38) associated with $u_n^\pm$. Since $2nT_0 = \frac{1}{m} \log \frac{R}{\rho}$, we see that

$$T_0 \to \infty \text{ as } \rho \to 0.$$

Lemma 5.7 *Let $n \geq 1$. Then the following hold:*
(i) *For $\rho < R$,*

$$\mu_n < -\frac{1}{4}(p-1)(p+3). \tag{5.39}$$

(ii) *For each small $\varepsilon > 0$, there is $\rho_\varepsilon > 0$ such that, for $0 < \rho < \rho_\varepsilon$,*

$$-\frac{1}{4}(p-1)(p+3) - \varepsilon < \mu_1. \tag{5.40}$$

Proof (i) First we show that $\mu_n < -(p-1)(p+3)/4$. Recall that $T = 2nT_0$. Let $I := (0, T)$ and let $I_j := (2(j-1)T_0, 2jT_0)$, $j = 1, 2, \ldots, n$. We use the variational characterization of μ_n

$$\mu_n = \inf_{\phi_1,\ldots,\phi_n \in H_0^1(I)} \sup_{\varphi \in \mathrm{span}(\phi_1,\ldots,\phi_n)\setminus\{0\}} \frac{H(\varphi)}{\|\varphi\|_2^2}, \tag{5.41}$$

where

$$H(\varphi) := \int_I \varphi'^2 - f'(u_n^{\pm})\varphi^2 dt.$$

Let $\phi^D(t)$ be the first eigenfunction of (5.10) defined on I_1. For $j = 1, 2, \ldots, n$, let

$$\phi_j(t) := \begin{cases} \phi^D(t - 2(j-1)T_0) & \text{if } 2(j-1)T_0 < t < 2jT_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathrm{supp}\phi_j = \bar{I}_j$. Since $\phi_j(t)$ is continuous on I, we can check that the weak derivative of $\phi_j(t)$ is $\phi_j'(t)$ for almost everywhere in I. The function $\phi_j'(t)$ is bounded on I_j, and it is identically equal to 0 on $I \setminus \bar{I}_j$. Then, the weak derivative of $\phi_j(t)$ is in $L^2(I)$, and hence $\phi_j \in H^1(I)$. Since $\phi_j(0) = \phi_j(2nT_0) = 0$, we see that $\phi_j(t) \in H_0^1(I)$. Moreover, ϕ_j satisfies

$$\begin{aligned} H(\phi_j) &= \int_I \phi_j'^2 - f'(u_n^{\pm})\phi_j^2 dt = \int_{I_j} \phi_j'^2 - f'(u_n^{\pm})\phi_j^2 dt \\ &= \int_{I_j} -\left\{\phi_j'' + f'(u_n^{\pm})\phi_j\right\}\phi_j dt = \mu^D \int_{I_j} \phi_j^2 dt. \end{aligned} \tag{5.42}$$

We take a set of functions $\phi_1, \ldots, \phi_n$ which are orthogonal in $L^2(I)$, and define

$$\psi(t) := \sum_{j=1}^n c_j \phi_j(t),$$

where $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$. Then, $\psi \in \mathrm{span}(\phi_1, \ldots, \phi_n)$. We see that $\psi \in H_0^1(I)$, since ϕ_j, $1 \leq j \leq n$, is in $H_0^1(I)$. Since $\mathrm{supp}\phi_j \cap \mathrm{supp}\phi_k$, $j \neq k$, has zero measure, by (5.42) we have

$$\frac{H(\psi)}{\|\psi\|_2^2} = \frac{\sum_{j=1}^n c_j^2 H(\phi_j)}{\sum_{j=1}^n c_j^2 \|\phi_j\|_2^2} = \frac{\sum_{j=1}^n c_j^2 \mu^D \|\phi_j\|_2^2}{\sum_{j=1}^n c_j^2 \|\phi_j\|_2^2} = \mu^D. \tag{5.43}$$

By (5.43) and (5.41) we have

$$\mu_n = \inf_{\phi_1,\ldots,\phi_n \in H_0^1(I)} \sup_{\varphi \in \mathrm{span}(\phi_1,\ldots,\phi_n)\setminus\{0\}} \frac{H(\varphi)}{\|\varphi\|_2^2} \leq \frac{H(\psi)}{\|\psi\|_2^2} = \mu^D. \tag{5.44}$$

By (5.44) and Lemma 5.3 (i) we have

$$\mu_n \le \mu^D < -\frac{1}{4}(p-1)(p+3).$$

The proof of (i) is complete. Note that the smallness $\rho > 0$ is not used.

(ii) Let $\phi^N(t)$ be the first eigenfunction of (5.27) defined on I_1 and $\tilde{\phi}^N(t)$ be the first eigenfunction of (5.38) defined on I. Let $\xi_0 := H_1(\phi^N)$, $\eta_0 := \|\phi^N\|^2_{L^2(I_1)}$,

$$\xi_j := H_j(\tilde{\phi}^N), \quad \eta_j := \|\tilde{\phi}^N\|^2_{L^2(I_j)}, \quad j = 1, 2, \ldots, n,$$

where, for $j = 1, 2, \ldots, n$,

$$H_j(\phi) := \int_{I_j} \phi'^2 - f'(u_n^{\pm})\phi^2 dt.$$

We restrict a domain of the function $\tilde{\phi}^N$ to I_j for $j = 1, 2, \ldots$. Then, $\tilde{\phi}^N\Big|_{I_j}$ does not necessarily satisfy a Dirichlet boundary condition on ∂I_j, though $\tilde{\phi}^N$ satisfies a Dirichlet boundary condition on ∂I. However, as seen below, a space of test functions is $H^1(I_j)$. Hence, we can use $\tilde{\phi}^N\Big|_{I_j}$ as a test function. By the unique continuation theorem for linear elliptic PDEs that neither ϕ^N nor $\tilde{\phi}^N$ vanishes on an open set, and hence $\eta_j > 0$ for $j = 0, 1, 2, \ldots, n$. Since ϕ^N is the first eigenfunction of (5.27), we see that

$$\frac{\xi_j}{\eta_j} = \frac{H_j(\tilde{\phi}^N)}{\|\tilde{\phi}^N\|^2_{L^2(I_j)}} \ge \inf_{\varphi \in H^1(I_1)} \frac{H_1(\varphi)}{\|\varphi\|^2_{L^1(I_1)}} = \frac{H_1(\phi^N)}{\|\phi^N\|^2_{L^2(I_1)}} = \frac{\xi_0}{\eta_0}.$$

By Lemma 5.6 we see that

$$\frac{H(\tilde{\phi}^N)}{\|\tilde{\phi}^N\|_2^2} = \frac{\xi_1 + \cdots + \xi_n}{\eta_1 + \cdots + \eta_n} \ge \frac{\xi_0}{\eta_0} = \frac{H_1(\phi^N)}{\|\phi^N\|^2_{L^2(I_1)}}. \tag{5.45}$$

Since $\tilde{\phi}^N$ is a first eigenfunction of (5.38), by (5.45) we see that

$$\mu_1 = \inf_{\varphi \in H_0^1(I)} \frac{H(\varphi)}{\|\varphi\|_2^2} = \frac{H(\tilde{\phi}^N)}{\|\tilde{\phi}^N\|_2^2} \ge \frac{H_1(\phi^N)}{\|\phi^N\|^2_{L^2(I_1)}} = \mu^N.$$

By Lemma 5.5 we see that for each $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that

$$\mu_1 > -\frac{1}{4}(p-1)(p+3) - \varepsilon$$

for $0 < \rho < \rho_\varepsilon$. The proof is complete. □

Let $\{\tilde{\lambda}_{\mathrm{rad},i}\}_{i=1}^{\infty}$ denote the set of the eigenvalues of (1.13) associated with the solution $u_n^{\pm}$.

Corollary 5.8 *Let* $\ell := \left[\frac{\alpha}{2}\right] + 1$. *Then the following holds:*

(i) *For* $\rho < R$,

$$\tilde{\lambda}_{\mathrm{rad},n} < -\nu_\ell.$$

(ii) *There exists* $\rho_\varepsilon > 0$ *such that, for* $0 < \rho < \rho_\varepsilon$,

$$-\nu_{\ell+1} < \tilde{\lambda}_{\mathrm{rad},1}.$$

Proof (i) We use (4.2), i.e., $\tilde{\lambda}_{\mathrm{rad},i} = \left(\frac{N-2}{2}\right)^2 \mu_i$. Multiplying (5.39) by $\left(\frac{N-2}{2}\right)^2$, we have

$$\tilde{\lambda}_{\mathrm{rad},n} < -\frac{1}{4}(\alpha+2)(\alpha+2N-2). \tag{5.46}$$

Since $2\ell - 2 \le \alpha < 2\ell$, by elementary calculation we have

$$-\frac{1}{4}(\alpha+2)(\alpha+2N-2) \le -\nu_\ell. \tag{5.47}$$

By (5.47) and (5.46) we see that (i) holds. Note that the smallness of $\rho > 0$ is not used.

(ii) Since $2\ell - 2 \le \alpha < 2\ell$, by elementary calculation we have

$$-\nu_{\ell+1} < -\frac{1}{4}(\alpha+2)(\alpha+2N-2). \tag{5.48}$$

By (5.48) we see that if $\varepsilon > 0$ is small, then

$$-\nu_{\ell+1} < -\frac{1}{4}(\alpha+2)(\alpha+2N-2) - \left(\frac{N-2}{2}\right)^2 \varepsilon. \tag{5.49}$$

Multiplying (5.40) by $\left(\frac{N-2}{2}\right)^2$, we have

$$-\frac{1}{4}(\alpha+2)(\alpha+2N-2) - \left(\frac{N-2}{2}\right)^2 \varepsilon < \tilde{\lambda}_{\mathrm{rad},1} \tag{5.50}$$

when $\rho > 0$ is small. Here, $\varepsilon > 0$ in (5.50) is the same value as $\varepsilon > 0$ in (5.49), since by Lemma 5.7 (ii) we can take an arbitrarily small $\varepsilon > 0$ in (5.50). Thus, by (5.49) and (5.50) we see that the conclusion of (ii) holds. □

Remark 5.9 By (5.50) and (5.46) we see that, for each $1 \le i \le n$,

$$\tilde{\lambda}_{\mathrm{rad},i} \to -\frac{1}{4}(\alpha+2)(\alpha+2N-2) \text{ as } \rho \to 0.$$

Lemma 5.10 *For $\rho < R$, $\mu_{n+1} > 0$ and hence*

$$\tilde{\lambda}_{\mathrm{rad},n+1} > 0.$$

Proof Since $\tilde{\lambda}_{\mathrm{rad},n+1} = \left(\frac{N-2}{2}\right)^2 \mu_{n+1}$, it is enough to show that $\mu_{n+1} > 0$. We prove the lemma by contradiction. Suppose that $\mu_{n+1} \le 0$. Then, the associated eigenfunction ϕ_{n+1} has $n+2$ zeros on $[0, T]$. Let ϕ be a solution of the initial value problem

$$\begin{cases} \phi'' + f'(u_n^{\pm})\phi = 0 & \text{for } t > 0, \\ \phi(0) = 0, \ \phi'(0) = 1. \end{cases}$$

By Sturm's comparison principle ϕ oscillates more rapidly than ϕ_{n+1} or $\phi = c\phi_{n+1}$ for some $c \ne 0$. Hence, ϕ has at least $n+2$ zeros on $[0, T]$. Let $0 = z_0 < z_1 < z_2 < \cdots < z_{n+1}$ denote the first $n+2$ zeros of ϕ. Since $\psi := (u_n^{\pm})'(t)$ satisfies the same equation $\psi'' + (-1 + p|u_n^{\pm}|^{p-1})\psi = 0$, by Sturm separation theorem we see that $\psi(t)$ has one zero in (z_i, z_{i+1}) for $i = 0, 1, 2, \ldots, n$. The function $\psi(t)$ has exactly n zeros in $(0, T)$, while there are at least $n+1$ intervals $\{(z_i, z_{i+1})\}_{i=0}^n$. We obtain a contradiction. Thus, $\mu_{n+1} \le 0$ does not occur, and hence $\mu_{n+1} > 0$. □

Proof of Theorem 1.1 (i) First, we consider the case where $\rho > 0$ is small. When $\rho > 0$ is small, by Corollary 5.8 and Lemma 5.10 we have

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \begin{cases} < 0 & \text{if } 1 \le i \le n \text{ and } 0 \le j \le \ell, \\ > 0 & \text{if } 1 \le i \le n \text{ and } j \ge \ell + 1, \\ > 0 & \text{if } i \ge n+1 \text{ and } j \ge 0. \end{cases} \tag{5.51}$$

Therefore, if $\rho > 0$ is small, then (1.12) has no zero eigenvalue, and hence (1.3) has also no zero eigenvalue. Thus, $U^{\pm}_{n,\rho}$ is nondegenerate for small $\rho > 0$.

By (5.51) we see that all negative eigenvalues are

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \quad \text{for } 1 \le i \le n \text{ and } 0 \le j \le \ell.$$

Moreover, it follows from Proposition 2.3 that a multiplicity of each eigenvalue is $M_j(N)$. Thus,

$$\widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) = n \sum_{j=0}^{\ell} M_j(N) \text{ for small } \rho > 0.$$

By Proposition 2.1 we see that $\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) = n \sum_{j=0}^{\ell} M_j(N)$ for small $\rho > 0$.

We calculate $\sum_{j=0}^{\ell} M_j(N)$. We use the following form of the multiplicity formula of ν_j:

$$M_j(N) = \tilde{M}_j - \tilde{M}_{j-2},$$

where

$$\tilde{M}_j := \begin{cases} \frac{(N+j-1)!}{(N-1)!j!} & \text{if } j \ge 0, \\ 0 & \text{if } j < 0. \end{cases}$$

We assume that ℓ is even. Then

$$\begin{aligned} \sum_{j=0}^{\ell} M_j &= \left\{ (\tilde{M}_\ell - \tilde{M}_{\ell-2}) + \cdots + (\tilde{M}_0 - \tilde{M}_{-2}) \right\} + \left\{ (\tilde{M}_{\ell-1} - \tilde{M}_{\ell-3}) + \cdots + (\tilde{M}_1 - \tilde{M}_{-1}) \right\} \\ &= \tilde{M}_\ell + \tilde{M}_{\ell-1} = \frac{(N+2\ell-1)(N+\ell-2)!}{(N-1)!\ell!}. \end{aligned}$$

We obtain the same formula in the odd case.

Next, we consider the case where $\rho > 0$ is not necessarily small. Even in this case, by Corollary 5.8 (i) we have

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j < 0 \text{ if } 1 \le i \le n \text{ and } 0 \le j \le \ell.$$

Thus, by Proposition 2.1 we have

$$\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) \ge n \sum_{j=0}^{\ell} M_j(N) \text{ for } \rho < R.$$

The proof of (i) is complete.

(ii) Since $2nT_0 = \frac{1}{m} \log \frac{R}{\rho}$, it follows from Lemma 5.3 (ii) that $\mu^D \to -\infty$ as $\rho \to R$. By (5.44) and (4.2) we see that

$$\tilde{\lambda}_{\mathrm{rad},n} = \left(\frac{N-2}{2} \right)^2 \mu_n \le \left(\frac{N-2}{2} \right)^2 \mu^D \to -\infty \quad \text{as } \rho \to R.$$

Therefore, it is clear that, for each large integer $\ell > 0$, there exists $\rho_\ell < R$ such that if $\rho_\ell < \rho < R$, then $\tilde{\lambda}_{\mathrm{rad},n} + \nu_j < 0$ for $0 \le j \le \ell$. The integer ℓ can be arbitrary large if ρ is close to R. This indicates that $\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) \to \infty$ as $\rho \to R$.

(iii) It follows from (5.46) that

$$\tilde{\lambda}_{\mathrm{rad},n} \to -\infty \quad \text{as} \quad \alpha \to \infty.$$

Note that the smallness of $\rho > 0$ is not used in (5.46). By the same argument as in (ii) we see that $\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) \to \infty$ as $\alpha \to \infty$. □

6 The case $(p, \alpha) = (3, N - 4)$

6.1 Proof of Theorem 1.4

We consider the case $(p, \alpha) = (3, N - 4)$. In this section u denotes a solution of (3.6) with n nodal domains for simplicity. Let $a := \max_{0 \le t \le t_\rho} |u(t)|$. Then, $a > a_0 = \sqrt{2}$ and

$$u'^2 - u^2 + \frac{u^4}{2} = -a^2 + \frac{a^4}{2} \tag{6.1}$$

for $0 \le t \le t_\rho$.

The following lemma says that the n-th eigenvalue of (4.1) with respect to a solution u with n nodal domains can be written explicitly.

Lemma 6.1 *Let $p = 3$ and let $a > \sqrt{2}$. Then,*

$$\mu_n := -\frac{3}{2}a^2 \quad \text{and} \quad \phi_n(t) := u(t)\sqrt{u(t)^2 - 2 + a^2}$$

are the n-th eigenvalue of (4.1) *and an associated eigenfunction, respectively. In particular,*

$$\begin{cases} \phi_n'' + f'(u)\phi_n = -\mu_n \phi_n & \text{for } 0 < t < t_\rho, \\ \phi_n(0) = \phi_n(t_\rho) = 0, \end{cases}$$

where t_ρ is defined by (3.7).

Proof Substituting μ_n and ϕ_n into $\phi'' + (-1 + 3u^2 + \mu)\phi$, we have

$$\begin{aligned} &\phi_n'' + (-1 + 3u^2 + \mu_n)\phi_n \\ &= u''\sqrt{u^2 - 2 + a^2} + \frac{3uu'^2 + u^2u''}{\sqrt{u^2 - 2 + a^2}} - \frac{u^3u'^2}{(u^2 - 2 + a^2)^{3/2}} \\ &\quad + \left(-1 + 3u^2 - \frac{3}{2}a^2\right) u\sqrt{u^2 - 2 + a^2}. \end{aligned} \tag{6.2}$$

Using $u'' = u - u^3$ and (6.1), we can check that the RHS of (6.2) is equal to 0. Since $u(t)^2 - 2 + a^2 > 0$ for $0 \le t \le t_\rho$ and $u(t)$ has n nodal domains, we see that $\phi_n(t)$ has $n - 1$ zeros on $(0, t_\rho)$. It follows from Sturm-Liouville theory that ϕ_n is an n-th eigenfunction, and hence μ_n is the n-th eigenvalue. □

Proof of Theorem 1.4 We consider the case $n = 1$. Using Lemma 6.1 with $n = 1$ and Lemma 5.10 with $n = 1$, we see that

$$\mu_1 < 0 < \mu_2.$$

By (4.2) we see that $\tilde{\lambda}_{\mathrm{rad},i} = \left(\frac{N-2}{2}\right)^2 \mu_i$. Therefore, $\tilde{\lambda}_{\mathrm{rad},1}$ is the only negative radial eigenvalue of (1.12). It follows from Propositions 2.1–2.3 that

$$\mathsf{m}(U^{\pm}_{1,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{1,\rho}) = \sum_{j=0}^{\ell} M_j(N)$$
$$\text{if } \tilde{\lambda}_{\mathrm{rad},1} + \nu_\ell < 0 \text{ and } \tilde{\lambda}_{\mathrm{rad},1} + \nu_{\ell+1} \geq 0 \text{ for some } \ell \in \{0, 1, 2, \ldots\}. \tag{6.3}$$

First we consider the case (a). Let ℓ be given in (a). Since $u(t)$ can be written explicitly as (3.12), we see that $a = \sqrt{2k^2/(2k^2-1)}$, and hence

$$\tilde{\lambda}_{\mathrm{rad},1} = -\left(\frac{N-2}{2}\right)^2 \frac{3k^2}{2k^2-1}.$$

Since $\ell > \frac{N}{2} - 1$, we see that $8\nu_\ell - 3(N-2)^2 > 0$ and $8\nu_{\ell+1} - 3(N-2)^2 > 0$, and hence $\mathcal{R}_{\ell,1}$ and $\mathcal{R}_{\ell+1,1}$ are well-defined. Since $\mathcal{R}_{\ell,1} < \frac{\rho}{R} \leq \mathcal{R}_{\ell+1,1}$ for a positive integer $\ell > \frac{N}{2} - 1$, we have

$$\mathcal{R}_{\ell,1} < \exp\left(-\frac{4}{N-2}\sqrt{2k^2-1}K(k)\right) \leq \mathcal{R}_{\ell+1,1}. \tag{6.4}$$

By direct calculation we can check that (6.4) is equivalent to

$$\tilde{\lambda}_{\mathrm{rad},1} + \nu_\ell < 0 \text{ and } \tilde{\lambda}_{\mathrm{rad},1} + \nu_{\ell+1} \geq 0. \tag{6.5}$$

Thus, by (6.3) we see that the conclusion holds.

Second we consider the case (b). Let ℓ be given in (b). Since $\ell = \left[\frac{N}{2}\right] - 1$, we see that $\ell + 1 > (N-2)/2$ and $\ell \leq (N-2)/2$, and hence $8\nu_{\ell+1} - 3(N-2)^2 > 0$. Then, $0 = \mathcal{R}_{\ell,1} < \mathcal{R}_{\ell+1,1}$. Since

$$0 < \frac{\rho}{R} \leq \mathcal{R}_{\ell+1,1}, \tag{6.6}$$

by the same argument as in (a) we can check that (6.6) is equivalent to (6.5). Thus, by (6.3) we see that the conclusion holds. □

6.2 Upper and lower bounds of the Morse index

Proof of Theorem 1.5 (i) It follows from Proposition 2.2 that if $\tilde{\lambda}_{\mathrm{rad},n} + \nu_\ell < 0$, then the following are negative eigenvalues of (1.12):

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \quad \text{for} \quad 1 \leq i \leq n \text{ and } 0 \leq j \leq \ell.$$

By Propositions 2.1–2.3 we see that

$$\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) \geq n \sum_{j=0}^{\ell} M_j(N) \quad \text{if } \tilde{\lambda}_{\mathrm{rad},n} + \nu_\ell < 0.$$

Let $\ell > 0$ be given in Theorem 1.5 (i). In a similar way to the proof of Theorem 1.4 we see that $\tilde{\lambda}_{\mathrm{rad},n} + \nu_\ell < 0$ is equivalent to $\frac{\rho}{R} > \mathcal{R}_{\ell,n}$. Thus, we have shown that (1.7) holds. □

Proof of Theorem 1.5 (ii) Let $u(t)$ be a solution of (3.6) with n nodal domains. Let $a := \|u\|_\infty$. Then $u(t)$ can be written explicitly as (3.12) and

$$a = \sqrt{\frac{2k^2}{2k^2-1}} > a_0 = \sqrt{2}.$$

We define

$$\underline{\mu} := -1 - \sqrt{1 + 3(a^2 - 1)^2} \quad \text{and} \quad \psi(t) := u(t)^2 + \frac{1}{3}\left(\sqrt{1 + 3(a^2 - 1)^2} - 2\right).$$

By direct calculation we see

$$\begin{cases} \psi'' + f'(u)\psi = -\underline{\mu}\psi & \text{for } 0 < t < t_\rho, \\ \psi > 0 & \text{for } 0 \le t \le t_\rho. \end{cases}$$

Let μ_1 be the first eigenvalue of (4.1) and let ϕ_1 be a positive eigenfunction associated to μ_1. We define $\varphi := \psi - c\phi_1$. Here $c > 0$ can be taken such that the following holds: There exists $t_0 \in (0, t_\rho)$ such that $\varphi(t_0) = 0$ and $\varphi(t) \ge 0$ for $0 \le t \le t_\rho$, since $\varphi(0) = \varphi(t_\rho) > 0$.

We prove by contradiction that

$$\underline{\mu} \le \mu_1. \tag{6.7}$$

Suppose the contrary, i.e.,

$$\mu_1 < \underline{\mu}. \tag{6.8}$$

Since $\phi_1(t) > 0$ for $0 < t < t_\rho$, we see that

$$\varphi'' + f'(u)\varphi = -\underline{\mu}\psi + \mu_1 c\phi_1 < -\underline{\mu}\psi + \underline{\mu}c\phi_1 = -\underline{\mu}\varphi$$

for $0 < t < t_\rho$. Since t_0 is a minimum point of φ, we have $\varphi''(t_0) \ge 0$, and hence

$$0 \le \varphi''(t_0) + f'(u(t_0))\varphi(t_0) < -\underline{\mu}\varphi(t_0) = 0.$$

We obtain a contradiction. Thus, (6.8) does not occur, and $\underline{\mu} \le \mu_1$.

We define

$$\underline{\tilde{\lambda}}_{\mathrm{rad},1} := \left(\frac{N-2}{2}\right)^2 \underline{\mu} = -\left(\frac{N-2}{2}\right)^2 \left(1 + \sqrt{1 + \frac{3}{(2k^2 - 1)^2}}\right).$$

Multiplying (6.7) by $\left(\frac{N-2}{2}\right)^2$, by (4.2) we have that $\underline{\tilde{\lambda}}_{\mathrm{rad},1} \le \tilde{\lambda}_{\mathrm{rad},1}$. Hereafter, let $\ell \ge 1$. If $\tilde{\lambda}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$, then the following are nonnegative eigenvalues of (1.12):

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } i \ge 1 \text{ and } j \ge \ell + 1, \quad \text{and}$$
$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } i \ge n + 1 \text{ and } j \ge 0.$$

Hence the following eigenvalues can be negative:

$$\tilde{\lambda}_{\mathrm{rad},i} + \nu_j \text{ for } 1 \le i \le n \text{ and } 0 \le j \le \ell.$$

Therefore, by Propositions 2.1 and 2.3 we see that

$$\mathsf{m}(U^{\pm}_{n,\rho}) = \widetilde{\mathsf{m}}(U^{\pm}_{n,\rho}) \le n \sum_{j=0}^{\ell} M_j(N) \tag{6.9}$$

provided that $\tilde{\lambda}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$. If $\underline{\tilde{\lambda}}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$, then $\tilde{\lambda}_{\mathrm{rad},1} + \nu_{\ell+1} \ge \underline{\tilde{\lambda}}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$, and hence (6.9) holds. Hence $\underline{\tilde{\lambda}}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$ is a sufficient condition for (6.9). Let ℓ be given in Theorem 1.5 (ii). By direct calculation we see that $\underline{\tilde{\lambda}}_{\mathrm{rad},1} + \nu_{\ell+1} \ge 0$ is equivalent to $\frac{\rho}{R} \le \tilde{\mathcal{R}}_{\ell,n}$. Thus, (1.8) holds. □

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7 Appendix: Complete elliptic integral and Jacobi elliptic functions

7.1 Complete elliptic integral

Let $0 \le k < 1$. The complete elliptic integral of the first kind is denoted by

$$K(k) := \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}.$$

One can easily see that $K(k)$ is monotonically increasing in $k \in [0, 1)$,

$$K(0) = \frac{\pi}{2} \quad \text{and} \quad \lim_{k \to 1} K(k) = \infty.$$

7.2 Jacobi elliptic functions

Let $0 < k < 1$. The Jacobi elliptic function $\mathrm{sn}(\xi, k)$ is an odd, periodic and analytic function with period $4K(k)$ as a function for the real domain, and is defined locally by

$$\xi = \int_0^{\mathrm{sn}(\xi,k)} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

for $\xi \in [0, K(k)]$. The function $\mathrm{cn}(\xi, k)$ is an even and $4K(k)$-periodic function defined locally by

$$\mathrm{cn}(\xi, k) := \sqrt{1 - \mathrm{sn}^2(\xi, k)}$$

for $\xi \in [0, K(k)]$ and $\mathrm{dn}(\xi, k)$ is an even and $2K(k)$-periodic function defined by

$$\mathrm{dn}(\xi, k) := \sqrt{1 - k^2\mathrm{sn}^2(\xi, k)}.$$

In particular,

$$\mathrm{sn}^2(\xi, k) + \mathrm{cn}^2(\xi, k) = 1, \quad k^2\mathrm{sn}^2(\xi, k) + \mathrm{dn}^2(\xi, k) = 1$$

for $\xi \in \mathbb{R}$ and $k \in (0, 1)$. The function $\mathrm{sd}(\xi, k)$ is defined by

$$\mathrm{sd}(\xi, k) := \frac{\mathrm{sn}(\xi, k)}{\mathrm{dn}(\xi, k)}.$$

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