



(Quasi-)Hamiltonian manifolds of cohomogeneity one

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Abstract

We classify compact, connected Hamiltonian and quasi-Hamiltonian manifolds of cohomogeneity one (which is the same as being multiplicity free of rank one). The group acting is a compact connected Lie group (simply connected in the quasi-Hamiltonian case). This work is a concretization of a more general classification of multiplicity free manifolds in the special case of rank one. As a result we obtain numerous new concrete examples of multiplicity free quasi-Hamiltonian manifolds or, equivalently, Hamiltonian loop group actions.

Keywords Hamiltonian manifold · Quasi-Hamiltonian manifold · Momentum map · Group valued momentum map · Cohomogeneity · Multiplicity free · Spherical variety

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1 Introduction

Let K be a simply connected, compact Lie group. In order to study Hamiltonian actions of the (infinite dimensional) loop group $\mathcal{L}K$ on (infinite dimensional) manifolds \mathcal{M} , Alekseev–Malkin–Meinrenken [2], introduced the notion of quasi-Hamiltonian K -manifolds. These are finite dimensional K -manifolds, equipped with a 2-form and a momentum map.

To a certain extent, quasi-Hamiltonian manifolds are very similar to classical Hamiltonian manifolds. This means, in particular, that a quasi-Hamiltonian manifold can be locally described by Hamiltonian manifolds. The most striking difference is that the target of the momentum map is the group K itself (instead of its coadjoint representation \mathfrak{k}^*). For that reason, quasi-Hamiltonian manifolds lack functoriality properties, like restriction to a subgroup and are therefore more difficult to construct than Hamiltonian manifolds.

The most basic quasi-Hamiltonian manifolds are the conjugacy classes of K . These are precisely the ones on which K acts transitively. The main purpose of this paper is to classify the next case in difficulty, namely compact manifolds of cohomogeneity one, i.e., where K acts with orbits of codimension one.

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Our approach is based on the papers [15, 16] where the more general class of multiplicity free (quasi-)Hamiltonian manifolds was considered. These are manifolds M for which the momentum map induces an injective map on orbit spaces ($M/K \hookrightarrow \mathfrak{k}^*/K$ or $M/K \hookrightarrow K//K$, respectively). In op. cit. these have been classified in terms of pairs (\mathcal{P}, Γ) where \mathcal{P} is a convex polytope (the momentum polytope) and Γ is a lattice (characterizing the principal isotropy group). The compatibility condition between \mathcal{P} and Γ is expressed in terms of a root system (affine or finite, respectively) and the existence of certain smooth affine spherical varieties. Even though the latter have been determined in [17] (up to coverings, central tori and \mathbb{C}^\times -fibrations), the condition is hard to handle in practice. So the present paper is also meant to be a (successful) test case for the feasibility of using [15, 16] to obtain explicit classification results. More precisely, the case considered here is precisely that of the multiplicity free manifolds which are of rank one, i.e., where $\text{rk } M := \dim \mathcal{P} = \text{rk } \Gamma = 1$. Our classification proceeds in two steps. First, there is an induction procedure from smaller Hamiltonian manifolds. Manifolds which are not induced are called *primitive*. So, in a second step, we determine all primitive manifolds. These can be quasi-Hamiltonian or Hamiltonian. So, as a by-product, but of independent interest *our classification also encompasses (classical) Hamiltonian manifolds of cohomogeneity one*.

Theorem 1.1 *Let M be a primitive, multiplicity free, (quasi-)Hamiltonian manifold of rank one. Then M corresponds to a diagram in Table 8.3. Moreover, to each diagram there corresponds a manifold which is either unique (in the quasi-Hamiltonian case) or unique up to rescaling the symplectic structure (in the Hamiltonian case).*

In Table 8.3 we presented each case by a diagram which is very close to Luna’s [20] for classifying spherical varieties. The classification yields many previously known quasi-Hamiltonian manifolds. For example, the spinning 4-sphere of Hurtubise–Jeffrey [9], Alekseev–Malkin–Woodward [3]) and its generalization, the spinning $2n$ -sphere, by Hurtubise–Jeffrey–Sjamaar [10] are on our list. We also recover the $\text{Sp}(2n)$ -action on $\mathbb{P}_{\mathbb{H}}^n$ discovered by Eshmatov [7] as part of a larger series which seems to be new, namely a quasi-Hamiltonian action of $K = \text{Sp}(2n)$ on the quaternionic Grassmannians $\mathbf{Gr}_k(\mathbb{H}^{n+1})$ (item (cc) for the root system $C_n^{(1)}$ in Table 8.3).

It should be mentioned that there is related work by Lê [18] on more qualitative aspects of Hamiltonian manifolds of cohomogeneity one.

Remark 1.2 Part of this paper is based on part of the second author’s doctoral thesis [25] which was written under the supervision of the first named author.

2 Hamiltonian and quasi-Hamiltonian manifolds

We first recall the most important properties of Hamiltonian and quasi-Hamiltonian manifolds. In the entire paper, K will be a compact connected Lie group with Lie algebra \mathfrak{k} . A *Hamiltonian K -manifold* is a triple (M, w, m) where M is a K -manifold, w is a K -invariant symplectic form on M and $m : M \rightarrow \mathfrak{k}^*$ is a smooth K -equivariant map (the *momentum map*) such that

$$w(\xi x, \eta) = \langle \xi, m_*(\eta) \rangle, \quad \text{for all } \xi \in \mathfrak{k}, x \in M, \eta \in T_x M. \tag{2.1}$$

In [2], Alekseev, Malkin and Meinrenken studied this concept in the context of loop groups. Even though these Loop groups are infinite dimensional, the authors managed to reduce

Hamiltonian loop group action to a finite dimensional concept namely *quasi-Hamiltonian manifolds*. These are very similar to Hamiltonian manifolds.

More precisely, quasi-Hamiltonian manifolds are also triples (M, w, m) where M is a K -manifold, w is a K -invariant 2-form and m is a K -equivariant map. But there are differences.

First of all, the Lie algebra \mathfrak{k} has to be equipped with a K -invariant scalar product. Moreover, a twist $\tau \in \text{Aut } K$ has to be chosen¹ (which may be the identity). The momentum map m has values in K instead of \mathfrak{k}^* and is equivariant with respect to the τ -twisted conjugation action on K , i.e., $g * h := gh\tau(g)^{-1}$. Finally, the closedness and non-degeneracy of w as well as formula (2.1) have to be adapted. For the details one can consult the papers [2, 16, 23]. They are not relevant for the present paper.

In the following, we want to treat the Hamiltonian case and the quasi-Hamiltonian on the same footing. So we talk about U -Hamiltonian manifolds where $U = \mathfrak{k}^*$ in the Hamiltonian and $U = K$ in the quasi-Hamiltonian case.

This momentum map $m : M \rightarrow U$ gives rise to a map between orbits spaces:

$$m/K : M/K \rightarrow U/K. \tag{2.2}$$

By definition, the fibers of this map are the *symplectic reductions* of M . The smooth ones are symplectic manifolds in a natural way. In particular, they are even dimensional. Most important for us are those manifolds for which this dimension is as low as possible, namely 0. These manifolds are called *multiplicity free*.

An important invariant of M is its *momentum image* $(m/K)(M/K) \subseteq U/K$. Its dimension is called the *rank of M* . Multiplicity free manifolds of rank zero are simply the K -orbits in U . In this paper we study the next more difficult case namely multiplicity free manifolds of rank one. These two conditions can be combined into one. For this recall that the dimension of M/K is the *cohomogeneity of M* . Then we have:

Lemma 2.1 *For a U -Hamiltonian manifold M the following are equivalent:*

- (1) *The cohomogeneity of M is 1.*
- (2) *M is multiplicity free of rank one.*

Proof Let $c := \frac{1}{2} \dim ((m/K)^{-1}(a))$ where a is a generic point of the momentum image. As mentioned above, it is an integer. By definition, $c = 0$ is equivalent to multiplicity freeness. Let r be the rank of M . Then we have

$$\dim M/K = 2c + r. \tag{2.3}$$

Hence, $\dim M/K = 1$ if and only if $c = 0$ and $r = 1$. □

3 Affine root systems

Before we go on with explaining the general structure of U -Hamiltonian manifolds we need to set up notation for finite and affine root systems. Here, we largely follow the exposition in [16] which is in turn based on [21, 22].

Let $\bar{\mathfrak{a}}$ be a Euclidean vector space, i.e., a finite dimensional real vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let \mathfrak{a} be an affine space for $\bar{\mathfrak{a}}$, i.e., \mathfrak{a} is equipped with a free and

¹ The original paper [2] deals only with the untwisted case $\tau = \text{id}_K$. The straightforward adaption to the twisted case has been carried out independently in [4], [16], and [23].

transitive $\bar{\alpha}$ -action. We denote the set of affine linear functions on \mathfrak{a} by $A(\mathfrak{a})$. The gradient of a function $\alpha \in A(\mathfrak{a})$ is the element $\bar{\alpha} \in \bar{\mathfrak{a}}$ with

$$\alpha(X + t) = \alpha(X) + \langle \bar{\alpha}, t \rangle, \quad X \in \mathfrak{a}, \quad t \in \bar{\mathfrak{a}}. \tag{3.1}$$

A reflection s is an isometry of \mathfrak{a} whose fixed point set is an affine hyperplane. If that hyperplane is the zero-set of $\alpha \in A(\mathfrak{a})$ then one can express $s = s_\alpha$ as $s_\alpha(X) = X - \alpha(X)\bar{\alpha}^\vee$ with the usual convention $\bar{\alpha}^\vee = \frac{2\bar{\alpha}}{\|\bar{\alpha}\|^2}$.

Definition 3.1 An *affine root system* on \mathfrak{a} is a subset $\Phi \subset A(\mathfrak{a})$ such that:

- (1) $\mathbb{R}1 \cap \Phi = \emptyset$ (in particular $0 \notin \Phi$),
- (2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,
- (3) $\langle \bar{\beta}, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$,
- (4) the Weyl Group $W = \langle s_\alpha, \alpha \in \Phi \rangle$ acts properly discontinuously on \mathfrak{a} ,
- (5) $\mathbb{R}\alpha \cap \Phi = \{+\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

Observe that, with our definition, Φ might be finite or even empty. In that case, the roots have a common zero which we can use as a base point. This way, we can identify \mathfrak{a} with $\bar{\mathfrak{a}}$ and we have $\alpha(X) = \langle \bar{\alpha}, X \rangle$ for all roots α . If $(\mathfrak{a}_1, \Phi_1), \dots, (\mathfrak{a}_s, \Phi_s)$ are affine root systems then

$$(\mathfrak{a}_1, \Phi_1) \times \dots \times (\mathfrak{a}_s, \Phi_s) := (\mathfrak{a}_1 \times \dots \times \mathfrak{a}_s, p_1^* \Phi_1 \cup \dots \cup p_s^* \Phi_s) \tag{3.2}$$

is also one (where the p_i are the projections). Conversely, every affine root system admits such a decomposition such that the Weyl group W_i of Φ_i is either trivial or acts irreducibly on $\bar{\mathfrak{a}}_i$. We say that Φ is *properly affine* if each irreducible factor Φ_i is infinite.

A chamber of Φ is a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Phi} \{\alpha = 0\}$. The closure \mathcal{A} of a chamber is called an *alcove*. If Φ is finite then \mathcal{A} is called a *Weyl chamber*. If Φ is irreducible then \mathcal{A} is either a simplicial cone if Φ is finite or a simplex if Φ is properly affine.

A root $\alpha \in \Phi$ is called *simple* with respect to an alcove \mathcal{A} if $\mathcal{A} \cap \{\alpha = 0\}$ is a wall of \mathcal{A} . The set of simple roots (for a fixed alcove) will be denoted by S .

Put $\bar{\Phi} := \{\bar{\alpha} \mid \alpha \in \Phi\}$ and $\bar{\Phi}^\vee := \{\bar{\alpha}^\vee \mid \alpha \in \Phi\}$. These are possibly non-reduced finite root systems on $\bar{\mathfrak{a}}$. We define:

Definition 3.2 An *integral root system* on \mathfrak{a} is a pair (Φ, Ξ) where $\Phi \subset A(\mathfrak{a})$ is an affine root system and $\Xi \subseteq \bar{\mathfrak{a}}$ is a lattice with $\bar{\Phi} \subseteq \Xi$ and $\langle \Xi, \bar{\Phi}^\vee \rangle \subseteq \mathbb{Z}$. The integral root system is *simply connected* if $\Xi = \{\omega \in \mathfrak{a} \mid \langle \omega, \bar{\Phi}^\vee \rangle \subseteq \mathbb{Z}\}$.

The classification of irreducible (infinite) affine root systems as it can be found, e.g., in [12] is recalled in Table 8.3. In that table, also the *Dynkin label* $k(\alpha)$ of each $\alpha \in S$ is given. These labels are uniquely characterized by being integral, coprime, and having the property that

$$\delta := \sum_{\alpha \in S} k(\alpha)\alpha \tag{3.3}$$

is a positive constant function.

4 Classification of multiplicity free Hamiltonian and quasi-Hamiltonian manifolds

We summarize some known facts about the quotient U/K .

If $U = \mathfrak{k}^*$ then it is classical that U/K is parameterized by a Weyl chamber for the finite root system attached to K .

If $U = K$ we need to assume that K is simply connected which we do from now on.

Then U/K is in bijection with the alcove \mathcal{A} for a properly affine root system which is determined by K and the action of τ in the Dynkin diagram of K , cf. [24] for details. Recall that in this case, \mathfrak{k} is equipped with a scalar product. We use it to identify \mathfrak{k} with \mathfrak{k}^* . Thereby, we obtain a map

$$\psi : \mathfrak{k}^* = \mathfrak{k} \xrightarrow{\text{exp}} K = U. \tag{4.1}$$

In the Hamiltonian case, we put for compatibility reasons $\psi = \text{id}_{\mathfrak{k}^*}$. Likewise, we assume that a scalar product has been selected on \mathfrak{k} even though the results will not depend on it.

Theorem 4.1 *Let K, U be as above. Then there is a subspace $\mathfrak{a} \subseteq \mathfrak{k}^*$ and an integral root system (Φ, Ξ) on \mathfrak{a} such that:*

- (1) *If $\mathcal{A} \subseteq \mathfrak{a}$ is any alcove of Φ , then the map $\psi/K : \mathcal{A} \rightarrow U/K$ is a homeomorphism.*
- (2) *If $X \in \mathcal{A}$ and $a := \psi(X) \in U$, then the isotropy group*

$$K_a = \{k \in K \mid k \cdot a = a\} \tag{4.2}$$

is connected, $\mathfrak{a} \subseteq \mathfrak{k}_a$ is a Cartan subalgebra, the weight lattice of K_a is Ξ , and

$$S(X) := \{\alpha \in S \mid \alpha(X) = 0\}, \tag{4.3}$$

is a set of simple roots of K_a . Here $S \subset \Phi$ is the set of simple roots with respect to \mathcal{A} .

Since K_a depends only on $S(X) \subseteq S$ we also write $K_a = K_{S(X)}$. Let M be a compact, connected U -Hamiltonian manifold. Then the *invariant momentum map* is the composition

$$m_+ : M \xrightarrow{m} U \rightarrow U/K \xrightarrow{\sim} \mathcal{A} \subseteq \mathfrak{a}. \tag{4.4}$$

Its image $\mathcal{P}_M := m_+(M) \subseteq \mathcal{A}$ can be shown to be a convex polytope [13], the so-called *momentum polytope* of M . It is the first main invariant of M .

A second invariant comes from the facts that for generic $a \in \mathcal{P}_M$ the isotropy group K_a acts on the momentum fiber $m_+^{-1}(a)$ via a quotient A_M of K_a which is a torus independent of a . Its character group Γ_M is a subgroup of the weight lattice Ξ .

Theorem 4.2 [15, 16] *Let M_1 and M_2 be two compact, connected multiplicity free U -Hamiltonian manifolds with $\mathcal{P}_{M_1} = \mathcal{P}_{M_2}$ and $\Gamma_{M_1} = \Gamma_{M_2}$. Then M_1 and M_2 are isomorphic as U -Hamiltonian manifolds.*

This begs the question which pairs (\mathcal{P}, Γ) arise this way. The key to the answer lies in the paper [5] of Brion which connects the theory of multiplicity free Hamiltonian manifolds with the theory of complex spherical varieties. In the following we summarize only a simplified version which suffices for our purposes.

We start with a connected, reductive, complex group G . An irreducible algebraic G -variety Z is called *spherical* if a Borel subgroup of G has an open orbit. Now assume also that Z is affine and let $\mathbb{C}[Z]$ be its ring of regular functions. Then the Vinberg-Kimelfeld criterion [29] asserts that Z is spherical if and only if $\mathbb{C}[Z]$ is multiplicity free as a G -module. This means that there is a set (actually a monoid) Λ_Z of dominant integral weights of G such that

$$\mathbb{C}[Z] \cong \bigoplus_{\chi \in \Lambda_Z} V_\chi \tag{4.5}$$

where V_χ is the simple G -module of highest weight χ . A theorem of Losev [19] asserts that in case Z is smooth, the variety Z is in fact uniquely determined by its weight monoid Λ_Z .

Let $K \subseteq G$ be a maximal compact subgroup. Then any smooth affine G -variety can be equipped with the structure of a Hamiltonian K -manifold by embedding Z into a finite dimensional L -module V and using a K -invariant Hermitian scalar product on V to define a momentum map. Then

- (1) Z is spherical as a G -variety if and only if it is multiplicity free as a Hamiltonian K -manifold.
- (2) $\mathcal{P}_Z = \mathbb{R}_{\geq 0}\Lambda_Z$ (the convex cone generated by Λ_Z).
- (3) $\Gamma_Z = \mathbb{Z}\Lambda_Z$ (the group generated by Λ_Z).

The first two items were proved by Brion [5] in the context of projective varieties. The version which we need, namely for affine varieties, was proved by Sjamaar in [27]. For the last item see Losev [19, Prop. 8.6(3)].

Remark 4.3 It follows from the normality of Z that conversely

$$\Lambda_Z = \mathcal{P}_Z \cap \Gamma_Z. \tag{4.6}$$

So Λ_Z and the pair $(\mathcal{P}_Z, \Gamma_Z)$ carry the same information.

Definition 4.4 A pair (\mathcal{P}, Γ) is called G -spherical if there exists a smooth affine spherical G -variety Z with $\mathcal{P} = \mathbb{R}_{\geq 0}\Lambda_Z$ and $\Gamma = \mathbb{Z}\Lambda_Z$. The (unique) smooth variety Z will be called a *model* for (\mathcal{P}, Γ) .

Now we go back to U -Hamiltonian manifolds. For any subset $\mathcal{P} \subseteq \mathcal{A}$ and point $X \in \mathcal{P}$ we define the *tangent cone* of \mathcal{P} at X as

$$T_X\mathcal{P} := \mathbb{R}_{\geq 0}(\mathcal{P} - X). \tag{4.7}$$

Here is a local version of sphericity:

Definition 4.5 Let $\mathcal{P} \subseteq \mathcal{A}$ be a compact convex polytope and $\Gamma \subseteq \Xi$ a subgroup.

- (1) (\mathcal{P}, Γ) is *spherical in* $X \in \mathcal{P}$ if $(T_X\mathcal{P}, \Gamma)$ is L -spherical where $L := K_a^{\mathbb{C}}$ is the Levi subgroup corresponding to $a := \psi(X) \in U$. The model variety for $(T_X\mathcal{P}, \Gamma)$ will be called the *local model* of (\mathcal{P}, Γ) in X .
- (2) The pair (\mathcal{P}, Γ) is *locally spherical* if it is spherical in every vertex of \mathcal{P} .

Remark 4.6 It follows from the definition of G -sphericity that in a locally spherical pair \mathcal{P} and Γ are necessarily parallel in the sense that \mathcal{P} is a polytope of maximal dimension inside the affine subspace $X + \langle \Gamma \rangle_{\mathbb{R}} \subseteq \mathfrak{a}$ for any $X \in \mathcal{P}$.

The classification theorem can now be stated as follows:

Theorem 4.7 [15, 16] *Let K be a connected compact Lie group which is assumed to be simply connected in the quasi-Hamiltonian case. Then the map $M \mapsto (\mathcal{P}_M, \Gamma_M)$ induces a bijection between*

- (1) *isomorphism classes of compact, connected multiplicity free U -Hamiltonian manifolds and*
- (2) *locally spherical pairs (\mathcal{P}, Γ) where $\mathcal{P} \subseteq \mathcal{A}$ is a compact convex polyhedron and $\Gamma \subseteq \Xi$ is a subgroup.*

Remarks 4.8 (1) The relation between pairs and manifolds can be made more precise. Let M be a U -Hamiltonian manifold and $X \in \mathcal{P}_M$. Then there exists a neighborhood \mathcal{P}_0 of X in \mathcal{P} such that

$$M_0 \cong K \times^{K_a} Z_0 \tag{4.8}$$

where $M_0 = m_{\mp}^{-1}(\mathcal{P}_0)$, $a = \psi(X)$, and $Z_0 \subseteq Z$ is a K_a -stable open subset of the local model Z in X .

(2) The construction of locally spherical pairs is quite difficult. Already deciding whether a given pair is locally spherical is intricate. There is an algorithm due to Pezzini-Van Steirteghem [26] for this but we are not going to use it since in our setting it is not necessary.

Because of this theorem, we are going to work from now on exclusively on the “combinatorial side”, i.e., with locally spherical pairs. We start with two reduction steps.

Definition 4.9 Let

$$S(X) := \{\alpha \in S \mid \alpha(X) = 0 \text{ for } X \in \mathcal{P}\} \tag{4.9}$$

be the set of simple roots which are zero for a fixed $X \in \mathcal{P}$ and

$$S(\mathcal{P}) := \bigcup_{X \in \mathcal{P}} S(X) = \{\alpha \in S \mid \alpha(X) = 0 \text{ for some } X \in \mathcal{P}\}. \tag{4.10}$$

Thus, elements of $S(\mathcal{P})$ correspond to walls of \mathcal{A} which contain a point of \mathcal{P} . Let $K_0 := K_{S(\mathcal{P})}$ be the corresponding (twisted) Levi subgroup of K . Then it is immediate that (\mathcal{P}, Γ) is locally spherical for K if and only if it is so for K_0 . This observation reduces classifications largely to pairs with $S(\mathcal{P}) = S$.

Definition 4.10 A polyhedron $\mathcal{P} \subseteq \mathcal{A}$ is called genuine if $S(\mathcal{P}) = S$.

There is another reduction. Assume $S_0 \subseteq S$ is a component of the Dynkin diagram of S . It corresponds to a (locally) direct semisimple factor L_{S_0} of $G = K^{\mathbb{C}}$. Suppose also that $S_0 \subseteq S(X)$ for all $X \in \mathcal{P}$. Then it follows from Remark 4.6 that $(\Gamma, S_0) = 0$. In turn (4.5) implies that L_{S_0} will act trivially on every local model Z of (\mathcal{P}, Γ) . This means that also the roots in S_0 can be ignored for determining the sphericity of (\mathcal{P}, Γ) .

Definition 4.11 A genuine polyhedron $\mathcal{P} \subseteq \mathcal{A}$ is called primitive if S does not contain a component S_0 with $S_0 \subseteq S(X)$ for all $X \in \mathcal{P}$.

The following lemma summarizes our findings:

Lemma 4.12 Let $\mathcal{P} \subseteq \mathcal{A}$ be a compact convex polyhedron and let be $\Gamma \subseteq \Xi$ a subgroup. Let

$$S^c := \{\alpha \in S \mid \alpha(X) \neq 0 \text{ for all } X \in \mathcal{P}\} \tag{4.11}$$

and let S_1 be the union of all components C of $S \setminus S^c$ with $C \subseteq S(X)$ for all $X \in \mathcal{P}$. Let $\overline{\Xi} := \Xi \cap S_1^{\perp}$. Then \mathcal{P} is primitive for $\overline{S} := S \setminus (S^c \cup S_1)$. Moreover, the pair (\mathcal{P}, Γ) is locally spherical for (S, Ξ) if and only if it is so for $(\overline{S}, \overline{\Xi})$.

Now the purpose of this paper is to present a complete classification of primitive locally spherical pairs in the special case when $\text{rk } M = \dim \mathcal{P} = 1$.

In this case the following simplifications occur: the polyhedron \mathcal{P} is a line segment $\mathcal{P} = [X_1, X_2]$ with $X_1, X_2 \in \mathcal{A}$ and $\Gamma = \mathbb{Z}\omega$ with $\omega \in \Xi$. It follows from Remark 4.6 that

$$X_2 = X_1 + c\omega \text{ for some } c \neq 0. \tag{4.12}$$

By replacing ω by $-\omega$ if necessary, we may assume that $c > 0$. Then Theorem 4.7 boils down to:

Corollary 4.13 *The map $M \mapsto (\mathcal{P}_M, \Gamma_M) = ([X_1, X_2], \mathbb{Z}\omega)$ induces a bijection between*

- (1) *isomorphism classes of compact, connected multiplicity free U -Hamiltonian manifolds of rank one and*
- (2) *triples (X_1, X_2, ω) satisfying (4.12) such that $\mathbb{N}\omega$ is the weight monoid of a smooth affine spherical $K_{S(X_1)}^{\mathbb{C}}$ -variety Z_1 and $\mathbb{N}(-\omega)$ is the weight monoid of a smooth affine spherical $K_{S(X_2)}^{\mathbb{C}}$ -variety Z_2 . The triples (X_1, X_2, ω) and $(X_2, X_1, -\omega)$ are considered equal.*

Triples as above will be called *spherical*. A triple is *genuine* or *primitive* if $\mathcal{P} = [X_1, X_2]$ has this property. The varieties Z_i are called the *local models* of the triple.

5 The local models

We proceed by recalling all possible local models, i.e., smooth, affine, spherical L -varieties Z of rank one where L is a connected, reductive, complex, algebraic group. Then

$$\mathbb{C}[Z] = \bigoplus_{n \in \Lambda} V_{n\omega}, \tag{5.1}$$

where ω is a non-zero integral dominant weight, $V_{n\omega}$ is the simple L -module of highest weight $n\omega$, and Λ equals either \mathbb{N} or \mathbb{Z} . The case $\Lambda = \mathbb{Z}$ is actually irrelevant for our purposes since this case only occurs as local model of an interior point of \mathcal{P} (by (4.6)).

In case $\Lambda = \mathbb{N}$, the weight ω is unique.

Theorem 5.1 *Let Z be a smooth, affine, spherical L -variety of rank one. Then one of the following cases holds:*

- (1) *$Z = \mathbb{C}^*$ and L acts via a non-trivial character.*
- (2) *$Z = L_0/H_0$ where (L_0, H_0) appears in the first part of Table 8.2 and L acts via a surjective homomorphism $\varphi : L \rightarrow L_0$.*
- (3) *$Z = V_0$ where (L_0, V_0) appears in the second part of Table 8.2 and L acts via a homomorphism $\varphi : L \rightarrow L_0$ which is surjective modulo scalars (except for case \mathfrak{a}_0 when φ should be surjective).*

Proof Smooth affine spherical varieties have been classified by Knop-Van Steirteghem in [17] and the assertion could be extracted from that paper. A much simpler argument goes as follows. First, a simple application of Luna’s slice theorem (see [17, Cor. 2.2]) yields $Z \cong L \times^H V$ where $H \subseteq L$ is a reductive subgroup and V is a representation of H . As the homogeneous space L/H is the image of $Z = L \times^H V$ under the projection $Z \rightarrow L/H$, the rank of the homogeneous space L/H is at most the rank of Z , so either 0 or 1.

If it is 0, then L/H is projective (see, e.g., [28, prop. 10.1]), but, being also affine, it is a single point, i.e., $L = H$. We deduce $Z = V$, i.e., Z is a spherical module of rank one. The classification of spherical modules (Kac [11], see also [14]), yields the cases in (3).

Assume now that L/H has rank one. This means that Z and L/H have the same rank. Let $F \subseteq L$ be the stabilizer of a point in the open L -orbit of Z such that $F \subseteq H$. By [8, lem. 2.4], the quotient H/F is finite. This implies that the projection $Z \rightarrow H/F$ has finite fibers. Hence $V = 0$ and $Z = L/H$ is homogeneous. The classification of homogeneous spherical varieties of rank one (Akhiezer [1], see also [6], and Wasserman [30]), yields the cases in (1) and (2). Observe, that the non-affine cases of Akhiezer’s list have been left out. \square

Remarks 5.2 Some remarks concerning Table 8.2:

- (1) Observe that items $[\frac{1}{2}]d_2$ and $[\frac{1}{2}]d_3$ could be made part of the series $[\frac{1}{2}]d_n$. Because of their singular behavior we chose not to do so. For example both can be embedded into A_n -diagrams. Moreover, $[\frac{1}{2}]d_2$ are the only cases with a disconnected Dynkin diagram.
- (2) We encode the local models by the diagram given in the last column of Table 8.2. For homogeneous models these diagrams are due to Luna [20]. The inhomogeneous ones are specific to our situation.
- (3) For a homogeneous model the weight ω is a fixed linear combination of simple roots (recorded in the fourth column). Hence it lifts uniquely to a weight of L . On the other hand, for inhomogeneous models the weight of L is only unique up to a character. This is indicated by the notation $\omega \sim \pi_1$ which means that $\langle \omega, \bar{\alpha}^\vee \rangle = 1$ for $\alpha = \alpha_1$ and $= 0$ otherwise.

Let S be the set of simple roots of L_0 , i.e., the set of vertices of a diagram. Then inspection of Table 8.2 shows that the elements of

$$S' := \{\alpha \in S \mid \langle \omega, \bar{\alpha}^\vee \rangle > 0\} \tag{5.2}$$

are exactly those which are decorated. All other simple roots α satisfy $\langle \omega, \bar{\alpha}^\vee \rangle = 0$. Another inspection shows that the diagram of a local model is almost uniquely encoded by the pair (S, S') . What is getting lost is a factor c of $1/2, 1$ or 2 , and the cases a_1 and \mathbf{a}_1 become indistinguishable. So we assign the formal symbol $c = i$ to the inhomogeneous cases. This way, the local model is uniquely encoded by the triple (S, S', c) with $c \in \{1/2, 1, 2, i\}$ which triggers the following

Definition 5.3 A *local diagram* is a triple $\mathcal{D} = (S, S', c)$ associated to a local model in Table 8.2. In the homogeneous case, let $\omega_{\mathcal{D}}$ be the weight given in column 4. If \mathcal{D} is inhomogeneous and S is non-empty then $\alpha_{\mathcal{D}}$ denotes the unique element of S' . Moreover, we put $\bar{\alpha}_{\mathcal{D}}^\vee := \bar{\alpha}^\vee$.

6 The classification

Let (X_1, X_2, ω) be a primitive spherical triple. Then we obtain two local models Z_1, Z_2 which determine two local diagrams $\mathcal{D}_1 = (S_1, S'_1, c_1), \mathcal{D}_2 = (S_2, S'_2, c_2)$ where $S_1, S_2 \subseteq S$. Put

$$S^P(\omega) := \{\alpha \in S \mid \langle \omega, \bar{\alpha}^\vee \rangle = 0\}. \tag{6.1}$$

Lemma 6.1 Let (X_1, X_2, ω) be a primitive spherical triple. Then

$$S(X_1) \cup S(X_2) = S \text{ and } S(X_1) \cap S(X_2) = S^P(\omega). \tag{6.2}$$

Proof The first equality holds because the triple is genuine. The inclusion $S(X_1) \cap S(X_2) \subseteq S^P(\omega)$ follows directly from (4.12). Assume conversely that $\alpha \in S^P(\omega)$. Without loss of generality we may assume that also $\alpha \in S(X_1)$. But then also $\alpha \in S(X_2)$ by (4.12). \square

From now let $i \in \{1, 2\}$ and $j := 3 - i$, so that if Z_i is a local model then Z_j is the other.

Lemma 6.2 *Let (X_1, X_2, ω) be a primitive spherical triple and $\mathcal{D}_1, \mathcal{D}_2$ as above. Then*

$$S = S'_1 \dot{\cup} S^p(\omega) \dot{\cup} S'_2. \tag{6.3}$$

Moreover,

$$S(X_i) = S'_i \dot{\cup} S^p(\omega) = S \setminus S'_j. \tag{6.4}$$

Proof It follows from Theorem 5.1 that every $\alpha \in S(X_i)$ (a simple root of L) is either in S_i (a simple root of L_0) or a simple root of $\ker \varphi$. In the latter case, we have $\langle \omega, \bar{\alpha}^\vee \rangle = 0$.

Now let $\alpha \in S$ and assume first $\langle \omega, \bar{\alpha}^\vee \rangle > 0$. Since the triple is genuine we have $S = S(X_1) \cup S(X_2)$. If $\alpha \in S(X_2)$ then actually $\alpha \in S_2$. This contradicts $\langle -\omega, \bar{\alpha}^\vee \rangle \geq 0$ for all $\alpha \in S_2$. Thus $\alpha \in S(X_1)$. By the same reasoning we have $\alpha \in S_1$. But then $\alpha \in S'_1$ by (5.2).

Analogously, $\langle \omega, \bar{\alpha}^\vee \rangle < 0$ implies $\alpha \in S'_2$. This proves (6.3). The second equality (6.4) now follows from Lemma 6.1. □

Definition 6.3 Let S' be a subset of a graph S . The *connected closure* $C(S', S)$ of S' in S is the union of all connected components of S which meet S' . In other words, $C(S', S)$ is the set of vertices of S for which there exists a path to S' .

The following lemma shows in particular how to recover S_i from the triple (S, S'_1, S'_2) .

Lemma 6.4 *Let (X_1, X_2, ω) be primitive. Then*

- (1) S_i is the connected closure of S'_i in $S \setminus S'_j$.
- (2) S is the connected closure of $S'_1 \cup S'_2$.
- (3) $S = S_1 \cup S_2$.

Proof (1) Recall that $S \setminus S'_j = S(X_i)$ is the disconnected union of S_i and the Dynkin diagram C_i of $\ker \varphi$. Inspection of Table 8.2 shows that S_i is the connected closure of S'_i .

(2) Let $C \subseteq S$ be a component with $C \cap (S'_1 \cup S'_2) = \emptyset$. Then $C \subseteq S^p(\omega) = S(X_1) \cap S(X_2)$ in contradiction to primitivity.

(3) By (1), the connected closure of $S'_1 \cup S'_2$ in S is $S_1 \cup S_2$. □

Definition 6.5 Let \mathcal{D} be the quintuple $\mathcal{D} = (S, S'_1, c_1, S'_2, c_2)$ where S is a (possibly empty) Dynkin diagram, S'_1, S'_2 are disjoint (possibly empty) subsets of S and $c_1, c_2 \in \{\frac{1}{2}, 1, 2\}$.

Let S_i be the connected closure of S'_i in $S \setminus S'_j$. Then \mathcal{D} is a *primitive spherical diagram* if it has following properties:

- (1) $S = S_1 \cup S_2$.
- (2) The triples $\mathcal{D}_i := (S_i, S'_i, c_i)$ are local diagrams.
- (3) a) If both \mathcal{D}_i are homogeneous then $\omega_{\mathcal{D}_1} + \omega_{\mathcal{D}_2} = 0$.
 b) If \mathcal{D}_i is homogeneous and \mathcal{D}_j is inhomogeneous with $S_j \neq \emptyset$ then $\langle \omega_{\mathcal{D}_i}, \bar{\alpha}_{\mathcal{D}_j}^\vee \rangle = -1$.
 c) If both \mathcal{D}_i are inhomogeneous with both $S_i \neq \emptyset$ and S is affine and irreducible then $k(\alpha_{\mathcal{D}_1}^\vee) = k(\alpha_{\mathcal{D}_2}^\vee)$ where $k(\alpha^\vee)$ is the colabel of α , i.e., the label of α^\vee in the dual diagram of S .

A primitive spherical diagram can be represented by the Dynkin diagram of S with decorations as in Table 8.2 which indicate the subsets S'_i and the numbers c_i .

Example 6.6 Consider the following diagram on $F_4^{(1)}$:

$$\begin{array}{c} 1/2 \\ \bullet \rightleftarrows \bullet \end{array} \tag{6.5}$$

It represents the quintuple with $S'_1 = \{\alpha_1\}$, $c_1 = 1/2$, $S'_2 = \{\alpha_3\}$, $c_2 = 1$. Hence $S_1 = \{\alpha_0, \alpha_1, \alpha_2\}$, and $S_2 = \{\alpha_2, \alpha_3, \alpha_4\}$. Comparing with Table 8.2 we see that the local diagrams are of type $\frac{1}{2}d_3$ and c_3 , respectively. The diagram is bihomogeneous so we need to check condition (3)a). Indeed

$$\begin{aligned} \omega_{D_1} + \omega_{D_2} &= \left(\frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{1}{2}\bar{\alpha}_2 \right) + (\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4) \\ &= \frac{1}{2}(\bar{\alpha}_0 + 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 4\bar{\alpha}_3 + 2\bar{\alpha}_4) = 0 \end{aligned} \tag{6.6}$$

(compare with the labels of $F_4^{(1)}$ in Table 8.3). Thus, the above diagram is primitive spherical.

The point of Definition 6.5 is of course:

Corollary 6.7 *Let (X_1, X_2, ω) be a primitive spherical triple with local diagrams (S_1, S'_1, c_1) and (S_2, S'_2, c_2) . Then $(S, S'_1, c_1, S'_2, c_2)$ is a primitive spherical diagram.*

Proof All conditions have been verified except for (3) c). If S is affine then the coroots satisfy the linear dependence relation

$$\sum_{\alpha \in S} k(\alpha^\vee)\bar{\alpha}^\vee = 0. \tag{6.7}$$

We pair this with ω and observe that $\langle \omega, \bar{\alpha}^\vee \rangle = 1, -1, 0$ according to $\alpha = \alpha_{D_1}, \alpha = \alpha_{D_2}$ or otherwise. This implies the claim. \square

The following is our main result. It will be proved in the next section.

Theorem 6.8 *Every primitive spherical diagram is isomorphic to an entry of Table 8.3.*

Explanation of Table 8.3: The first column gives the type of S . The second lists for identification purposes the type of the local models. The diagram is given in the fifth column. If a parameter is involved, its scope is given in the last column. Observe the boundary cases where we used the conventions $b_1 = a_1, 2b_1 = 2a_1, c_2 = b_2$, and $\mathbf{c}_1 = \mathbf{a}_1$. In some cases, besides $(S, S'_1, c_1, S'_2, c_2)$ also $(S, S'_1, c, c_1, S'_2, c, c_2)$ is primitive spherical where c is the factor in the column ‘‘factor’’. An entry of the form $[c]_{n=a}$ indicates that the factor applies only to the case $n = a$.

Finally, the weight ω can be read off from the third column. More precisely, if \mathcal{D}_i is homogeneous then ω_i indicates the unique lift of ω or $-\omega$ to an affine linear function with $\omega_i(X_i) = 0$. If both local models $\mathcal{D}_1, \mathcal{D}_2$ are inhomogeneous then ω is only unique up to a character of G . Thus, the notation $\omega \sim \omega_0$ means $\langle \omega, \alpha^\vee \rangle = \langle \omega_0, \alpha^\vee \rangle$ for all $\alpha \in S$. Here, $\pi_i \in \mathbb{E} \otimes \mathbb{Q}$ denotes the i -th fundamental weight.

Example 6.9 The primitive diagrams for $A_1^{(1)}$ and $A_2^{(2)}$ are

$$\begin{array}{cccccc} \bullet \rightleftarrows \bullet & \circ \rightleftarrows \circ & \circ \rightleftarrows \circ & \circ \rightleftarrows \bullet & \circ \rightleftarrows \circ & \circ \rightleftarrows \circ \end{array} \tag{6.8}$$

In these cases, we have $\mathcal{P} = \mathcal{A}$ and $\omega = \frac{1}{2}\bar{\alpha}_1, \bar{\alpha}_1, 2\bar{\alpha}_1, \frac{1}{2}\bar{\alpha}_1, \bar{\alpha}_1$, respectively (where $S = \{\alpha_0, \alpha_1\}$).

The conditions defining a primitive spherical diagram \mathcal{D} have been shown to be necessary but it is not clear whether each of them can be realized by a (quasi-)Hamiltonian manifold M . And if so, how unique is M ? We answer these questions in Theorem 6.12 below. To state it we need more notation.

- Definition 6.10** (1) For a finite root system Φ let π_α be the fundamental weight corresponding to $\alpha \in S$.
 (2) If Φ is affine and irreducible (hence \mathcal{A} is a simplex) let $P_\alpha \in \mathcal{A}$ be the vertex of \mathcal{A} with $\alpha(P_\alpha) > 0$.

Let \mathcal{D} be a local diagram $\neq \mathbf{a}_0$. An inspection of Table 8.2 shows that the pairing $\langle \omega, \bar{\alpha}^\vee \rangle$ is in fact independent of $\alpha \in S'$ (actually only $\alpha_{n \geq 2}$ and $[\frac{1}{2}]d_{n \geq 2}$ have to be checked). The common value will be denoted by $n_{\mathcal{D}}$. Here is a list:

Table 6.11 The numbers $n_{\mathcal{D}}$

\mathcal{D}	a_1	$2a_1$	$a_{n \geq 2}$	$b_{n \geq 2}$	$2b_{n \geq 2}$	$c_{n \geq 3}$	$d_{n \geq 2}$	$\frac{1}{2}d_{n \geq 2}$	f_4	g_2	$2g_2$	b'_3	$\frac{1}{2}b'_3$	$\mathbf{a}_{n \geq 1}$	$\mathbf{c}_{n \geq 2}$
$n_{\mathcal{D}}$	2	4	1	1	2	1	2	1	1	1	2	2	1	1	1

Theorem 6.12 Let K be simply connected (also in the Hamiltonian case) and let $\mathcal{D} \neq (\emptyset)$ be a primitive diagram for $(\mathfrak{a}, \Phi, \Xi)$.

- (1) If Φ is finite then \mathcal{D} can be realized by a multiplicity free Hamiltonian manifold of rank one. This manifold is unique up to a positive factor of the symplectic structure. The momentum polytope is given by

$$X_i = c n_{\mathcal{D}_j} \sum_{\alpha \in S'_j} \pi_\alpha \tag{6.9}$$

(see Table 6.11 for $n_{\mathcal{D}}$) if both S'_i are non-empty. If $S'_1 \neq \emptyset$ and $S'_2 = \emptyset$ then

$$X_1 = 0 \text{ and } X_2 = c\omega. \tag{6.10}$$

In both cases, c is some arbitrary positive factor.

- (2) If Φ is infinite and irreducible then \mathcal{D} can be realized by a unique multiplicity free quasi-Hamiltonian manifold of rank one. The momentum polytope is given by

$$X_i = \begin{cases} P_\alpha & \text{if } S'_j = \{\alpha\} \\ \frac{k(\alpha^\vee)}{k(\alpha^\vee)+k(\beta^\vee)} P_\alpha + \frac{k(\beta^\vee)}{k(\alpha^\vee)+k(\beta^\vee)} P_\beta & \text{if } S'_j = \{\alpha, \beta\}. \end{cases} \tag{6.11}$$

- (3) If Φ is infinite and reducible (cases $A_1^{(1)} \times A_1^{(1)}$ and $A_2^{(2)} \times A_2^{(2)}$) then \mathcal{D} can be realized by a multiplicity free quasi-Hamiltonian manifold of rank one if and only if the scalar product is chosen to be the same on both factors of K , i.e., if the alcove \mathcal{A} is a metric square. This manifold is then unique.

Proof Let $L_i \subseteq K^\mathbb{C}$ be the (twisted) Levi subgroup having the simple roots $S(i) := S \setminus S'_j$. We have to construct (X_1, X_2, ω) such that $S(X_i) = S(i)$, $X_2 - X_1 \in \mathbb{R}_{>0}\omega$, and ω or $-\omega$ generates the weight monoid of a smooth affine spherical L_1 -variety or L_2 -variety, respectively.

If both \mathcal{D}_1 and \mathcal{D}_2 are homogeneous then there are exactly two choices for ω namely $\omega_{\mathcal{D}_1}$ and $\omega_{\mathcal{D}_2}$ which are related by $\omega_{\mathcal{D}_1} + \omega_{\mathcal{D}_2} = 0$. We claim that $\langle \omega_{\mathcal{D}_i}, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in S$. This follows by inspection for $\alpha \in S_i$. From $\omega_{\mathcal{D}_1} = -\omega_{\mathcal{D}_2}$ we get it also for $\alpha \in S_j$. Since K is simply-connected, the weights $\omega_{\mathcal{D}_i}$ are integral, i.e., $\omega \in \Xi$.

If \mathcal{D}_1 is homogeneous and \mathcal{D}_2 is inhomogeneous we must put $\omega = \omega_{\mathcal{D}_1}$. By condition (2)b) of Definition 6.5 we have $\langle -\omega, \bar{\alpha}_{\mathcal{D}_2}^\vee \rangle = 1$. Let $\beta \in S_2 \setminus \{\alpha_{\mathcal{D}_2}\}$. Then β is not connected to any $\alpha \in S_1$ by Lemma 6.4(1). From $\omega \in \mathbb{Q}S_1$ we get $\langle \omega, \bar{\beta}^\vee \rangle = 0$. Hence $\omega \in \Xi$ and both $\mathbb{N}\omega$ and $\mathbb{N}(-\omega)$ form the weight monoid of a smooth affine spherical L_1 - or L_2 -variety, respectively.

If both \mathcal{D}_i are inhomogeneous then we need a weight ω with $\langle \omega, \bar{\alpha}_{\mathcal{D}_1}^\vee \rangle = 1$, $\langle \omega, \bar{\alpha}_{\mathcal{D}_2}^\vee \rangle = -1$, and $\langle \omega, \bar{\alpha}^\vee \rangle = 0$ otherwise. If Φ is finite then ω exists and is unique since S is a basis of $\Xi \otimes \mathbb{Q}$. If S is affine and irreducible then ω exists and is unique because of condition (3)c) of Definition 6.5. In both cases ω is integral. The case of reducible affine root systems will be discussed at the end.

This settles the reconstruction of ω . It remains to construct points $X_1, X_2 \in \mathcal{A}$ with $S(X_1) = S(1)$, $S(X_2) = S(2)$ and $X_2 - X_1 \in \mathbb{R}_{>0}\omega$. These boil down to the following set of linear (in-)equalities (where the last column just records the known behavior of ω):

$$\begin{array}{c|cc|c}
 \alpha & X_1 & X_2 & \omega \\
 \hline
 \alpha \in S'_1 & \alpha(X_1) = 0 & \alpha(X_2) > 0 & \langle \omega, \bar{\alpha} \rangle > 0 \\
 \alpha \in S'_2 & \alpha(X_1) > 0 & \alpha(X_2) = 0 & \langle \omega, \bar{\alpha} \rangle < 0 \\
 \alpha \notin S'_1 \cup S'_2 & \alpha(X_1) = 0 & \alpha(X_2) = 0 & \langle \omega, \bar{\alpha} \rangle = 0
 \end{array} \tag{6.12}$$

$$X_2 = X_1 + c\omega \text{ with } c > 0. \tag{6.13}$$

The inequalities (6.12) for X_1 define the relative interior of a face of the alcove \mathcal{A} (observe that $S'_i \neq \emptyset$ if Φ is affine). The first and the third set of inequalities for X_2 then follow from (6.13). Inserting (6.13) into the second set we get equalities for X_1 and c :

$$\alpha(X_1) = c\langle -\omega, \bar{\alpha} \rangle > 0 \text{ for all } \alpha \in S'_2. \tag{6.14}$$

Define the affine linear function $\alpha^\vee := \frac{2}{\|\bar{\alpha}\|}\alpha$. Then (6.14) is equivalent to

$$\alpha^\vee(X_1) = cn_{\mathcal{D}_2} > 0 \text{ for all } \alpha \in S'_2. \tag{6.15}$$

This already shows assertion (1) of the theorem. Now assume that Φ is affine and irreducible. Then there is the additional relation

$$\sum_{\beta \in S} k(\beta^\vee)\beta^\vee(X) = \varepsilon \equiv \text{const.} > 0, \quad X \in \mathfrak{a}. \tag{6.16}$$

Setting $X = X_1$, we get

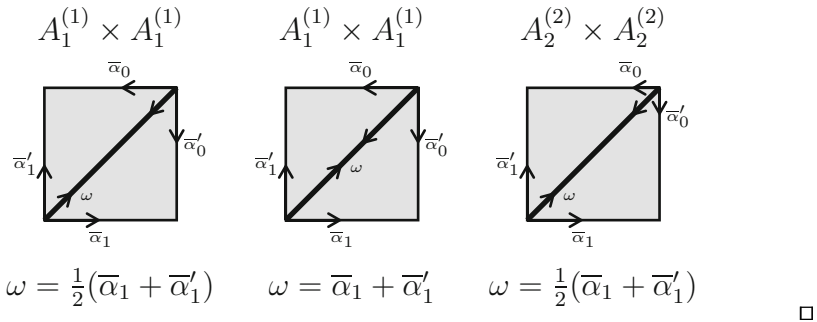
$$cn_{\mathcal{D}_2} \sum_{\beta \in S'_2} k(\beta^\vee) = \varepsilon. \tag{6.17}$$

This means that c is unique and positive. From (6.15) we get

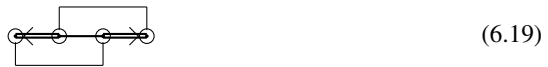
$$\alpha^\vee(X_1) = \left[\sum_{\beta \in S'_2} k(\beta^\vee) \right]^{-1} \varepsilon. \tag{6.18}$$

Evaluation of (6.16) at $X = P_\alpha$ yields $\alpha^\vee(P_\alpha) = \frac{\varepsilon}{k(\alpha^\vee)}$. To obtain (6.11) just observe that S'_2 has either one or two elements.

Finally, assume Φ is reducible. The mixed types “finite times infinite” do not appear in our context. For the two other cases, the existence of $([X_1, X_2], \omega)$ is clear from the following graphics. In particular, they show why \mathcal{A} must be a metric square.



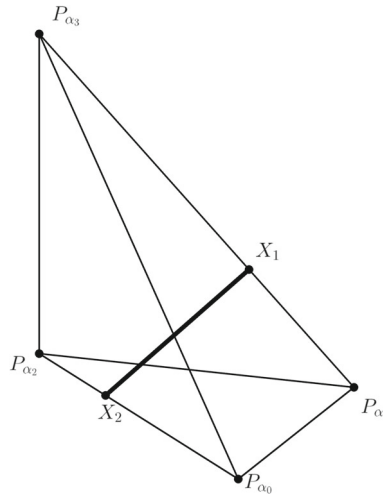
Example 6.13 Consider the diagram $D_4^{(2)}(dd)$



Then $k(\alpha_0^\vee) = k(\alpha_3^\vee) = 1$ and $k(\alpha_1^\vee) = k(\alpha_2^\vee) = 2$, and $\omega_1 = \bar{\alpha}_0 + \bar{\alpha}_2$ and $\omega_2 = \bar{\alpha}_1 + \bar{\alpha}_3$ and

$$X_1 = \frac{2}{3}P_{\alpha_1} + \frac{1}{3}P_{\alpha_3}, \quad X_2 = \frac{1}{3}P_{\alpha_0} + \frac{2}{3}P_{\alpha_2}. \tag{6.20}$$

Here is a picture of \mathcal{P} inside \mathcal{A}



Remark 6.14 (1) The three primitive diagrams for $A_1^{(1)}$ (see Example 6.9) correspond to the manifolds S^4 (the so-called “spinning 4-sphere”, [3, 9]), $S^2 \times S^2$, and $\mathbb{P}^2(\mathbb{C})$, respectively (see [16, §2.7] for details).
 (2) Generalizing (1), the diagram $A_{n-1}^{(1)}(\mathbf{aa})$ with $n \geq 2$ corresponds to the “spinning $2n$ -sphere” S^{2n} discovered by Hurtubise–Jeffrey–Sjamaar [10].

- (3) The cases $C_{n \geq 2}^{(1)}(\mathbb{C}\mathbb{C})$ are realized by $\text{Sp}(2n)$ acting on the quaternionic Grassmannians $M = \text{Gr}_d(\mathbb{H}^{n+1})$ (see [16, Thm. 2.7.2]). This is a generalization of a result by Eshmatov [7] for $d = 1$.

One can combine the classification of primitive diagrams with the Reduction Lemma 4.12. For the formulation of the lemma, we define

$$k^\vee(S^c) := \text{gcd}\{k(\alpha^\vee) \mid \alpha \in S^c\}. \tag{6.21}$$

in case Φ is an irreducible affine root system.

Definition 6.15 Assume Φ is finite or irreducible. A *spherical diagram* is a 6-tuple $(S, S^c, S'_1, c_1, S'_2, c_2)$ with:

- (1) $S'_1, S'_2, S^c \subseteq S$ are pairwise disjoint
- (2) $(S_{12}, S'_1, c_1, S'_2, c_2)$ is a primitive diagram where S_{12} is the connected closure of $S'_1 \cup S'_2$ in $S \setminus S^c$. Set $\mathcal{D}_i = (S_i, S'_i, c_i)$ where S_i is the connected closure of S'_i in $S_{12} \setminus S'_j$.
- (3) If \mathcal{D}_i is homogeneous then $\langle w_{\mathcal{D}_i}, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in S^c$.
- (4) Assume \mathcal{D}_1 and \mathcal{D}_2 are both inhomogeneous with $\alpha_i := \alpha_{\mathcal{D}_i}$. Assume also that Φ is affine and irreducible. Then $k^\vee(S^c)$ divides $k(\alpha_1^\vee) - k(\alpha_2^\vee)$.

Remark 6.16 The condition (3) is only relevant if \mathcal{D}_i is of type $\frac{1}{2}d_{n \geq 2}$ or $\frac{1}{2}b'_3$.

Again, the point of the definition is:

Lemma 6.17 Let (X_1, X_2, ω) be a spherical triple. Put $S^c := S \setminus (S(X_1) \cup S(X_2))$ and let (S_i, S'_i, c_i) be the local diagram at X_i . Then $(S, S^c, S'_1, c_1, S'_2, c_2)$ is a spherical diagram.

Proof Only (4) needs an argument. The weight ω satisfies $\langle \omega, \bar{\alpha}_{\mathcal{D}_1}^\vee \rangle = 1, \langle \omega, \bar{\alpha}_{\mathcal{D}_2}^\vee \rangle = -1, \langle \omega, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for $\alpha \in S^c$, and $\langle \omega, \bar{\alpha}^\vee \rangle = 0$ for all other $\alpha \in S$. Hence (4) follows from the linear dependence relation (6.7). □

Theorem 6.18 Let (Φ, Ξ) be simply connected and \mathcal{D} a non-empty spherical diagram on Φ . Let Φ be finite or affine, irreducible. Then \mathcal{D} is realized by a spherical triple (X_1, X_2, ω) .

Proof The two last conditions (3) and (4) of Definition 6.15 make sure that there exists $\omega \in \Xi$ which gives rise to the appropriate local model at X_i (for (4), see the argument in the proof of Lemma 6.17). We show the existence of a matching polytope first in the finite case. In this case, we may assume that all roots $\alpha = \bar{\alpha}$ are linear and S is linearly independent. Additionally to the inequalities (6.12) for $\alpha \notin S^c$ we get the inequalities $\alpha(X_1), \alpha(X_2) > 0$ for $\alpha \in S^c$. Because of linear independence, the values $\alpha(X_1)$ with $\alpha \in S$ can be prescribed arbitrarily. By Theorem 6.12, we can do that in such a way that all inequalities for $\alpha \notin S^c$ are satisfied. Now we choose $\alpha(X_1) \gg 0$ for all $\alpha \in S^c$. Since c in (6.13) is not affected by this choice, this yields $\alpha(X_2) > 0$ for all $\alpha \in S^c$, as well.

Now let Φ be affine, irreducible. If $S^c = \emptyset$ then the spherical diagram would be in fact primitive. This case has been already dealt with. So let $S^c \neq \emptyset$ and fix $\alpha_0 \in S^c$. Then $S^f := S \setminus \{\alpha_0\}$ generates a finite root system. We may assume that all $\alpha \in S^f$ are linear. Then the existence of a solution (X_1, X_2) satisfying all inequalities for all $\alpha \in S^f$ has been shown above. Now observe that the set of these solutions form a cone. If we choose a solution sufficiently close to the origin we get $\alpha_0(X_1), \alpha_0(X_2) > 0$. □

A spherical diagram is drawn like a primitive diagram where the roots $\alpha \in S^c$ are indicated by circling them.

Example 6.19 Consider the diagram on D_7 :



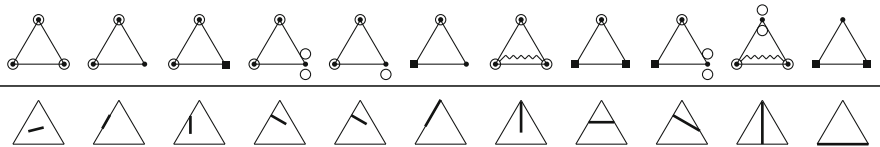
Then $S^c = \{\alpha_3\}$, $S_1 = \{\alpha_5, \alpha_6, \alpha_7\}$ and $S_2 = \{\alpha_4, \alpha_5\}$.

Example 6.20 In addition to the primitive diagrams of Example 6.9, the root system $A_1^{(1)}$ supports the following spherical diagrams:



If one identifies the alcove \mathcal{A} with the interval $[0, 1]$ then $\mathcal{P} = [t, 1]$ in the first three cases, $\mathcal{P} = [0, t]$ in cases 4 through 6 and $\mathcal{P} = [t_1, t_2]$ in the last case where $0 < t < 1$ and $0 < t_1 < t_2 < 1$ are arbitrary.

Example 6.21 Up to automorphisms, all spherical diagrams supported on $A_2^{(1)}$ are listed in the top row of



The bottom row lists the corresponding momentum polytopes. Observe that each simple root of a Dynkin diagram in the first row corresponds to the opposite edge of the alcove below it.

Remark 6.22 If Φ is a product of more than one affine root system then there are problems with the metric of \mathcal{A} as we have already seen for the primitive case where \mathcal{A} must be a metric square. We have not explored this case in full generality.

7 Verification of Theorem 6.8

Recall $i \in \{1, 2\}$ and $j := 3 - i$. Let $(S, S'_1, c_1, S'_2, c_2)$ be a primitive diagram. Recall that S_i is the connected closure of S'_i in $S \setminus S'_j$.

Put

$$S_0^p := S_1 \cap S_2. \tag{7.1}$$

Observe that these are exactly the simple roots that are not decorated in the diagram. Our strategy is to construct S by gluing S_1 and S_2 along S_0^p . For this we have to make sure that S_i remains the connected closure of S'_i .

Lemma 7.1 *Let S be a graph with subsets S'_1, S_1, S'_2, S_2 such that $S'_i \subseteq S_i \subseteq S \setminus S'_j$. Assume that $S = S_1 \cup S_2$ and $S_i = C(S'_i, S_i)$. Then the following are equivalent:*

- (1) S_i is the connected closure of S'_i in $S \setminus S_j$.
- (2) The following two conditions hold:

- a) $S_1 \cap S_2$ is the union of connected components of $S_i \setminus S'_i$.
- b) If $\alpha_1 \in S_1 \setminus S_2$ is connected to $\alpha_2 \in S_2 \setminus S_1$ in S then $\alpha_1 \in S'_1$ and $\alpha_2 \in S'_2$.

Proof Because of $S_i = C(S'_i, S_i)$, the assertion $S_i = C(S'_i, S \setminus S'_j)$ means that there are no edges between S_i and

$$S \setminus (S_i \cup S'_j) = (S_j \setminus S'_j) \setminus (S_i \cap S_j) = (S_j \setminus S_i) \setminus S'_j. \tag{7.2}$$

This statement breaks up into two parts: There are no edges between $S_i \cap S_j$ and $(S_j \setminus S'_j) \setminus (S_i \cap S_j)$ which is just condition (2)a). And there are no edges between $S_i \setminus S_j$ and $(S_j \setminus S_i) \setminus S'_j$ which is just condition (2)b). □

7.1 The case $S_0^p \neq \emptyset$

We start our classification with:

Lemma 7.2 *Let \mathcal{D} be a primitive spherical diagram with $S_0^p \neq \emptyset$ such that there is at least one edge between S'_1 and S'_2 . Then $\mathcal{D} \cong A_{n \geq 2}^{(1)}(\mathbf{aa})$:*



Proof Let $\alpha \in S_0^p$ and let there be an edge between $\alpha_1 \in S'_1$ and $\alpha_2 \in S'_2$. Inspection of Table 8.2 implies that there are paths from α_1 and α_2 to α respectively. Together with the edge they form a cycle in S which implies that S is of type $A_{n \geq 2}^{(1)}$ with α_1, α_2 being adjacent. Therefore the local diagrams are either of type a_m or \mathbf{a}_m . Since only diagrams of the latter type can be glued such that $S_0^p \neq \emptyset$ and S is a cycle we get $A_{n \geq 2}^{(1)}(\mathbf{aa})$ as the only possibility. □

Thus, we may assume from now on that S is the union of S_1 and S_2 minimally glued along S_0^p , i.e., with no further edges added. To classify these diagrams we proceed by the type of S_0^p . Helpful is the following table which lists for all isomorphy types of S_0^p the possible candidates for S_1 and S_2 . The factor c is suppressed.

Table 7.3 Gluing data

S_0^p	Candidates for S_1 and S_2							
A_1	a_3	\mathbf{a}_2	b_2	$c_{n \geq 3}$	c_3	d_3	g_2	\mathbf{c}_2
A_2	a_4	\mathbf{a}_3	b'_3					
A_3	a_5	\mathbf{a}_4	d_4					
$A_{n \geq 4}$	a_{n+2}	\mathbf{a}_{n+1}						
B_2	b_3	c_4	\mathbf{c}_3					
B_3	b_4	f_4						
$B_{n \geq 4}$	b_{n+1}							
$C_{n \geq 3}$	c_{n+2}	\mathbf{c}_{n+1}						
D_4	d_5							
$D_{n \geq 5}$	d_{n+1}							
$A_1 A_1$	c_3	d_3						
$A_1 C_{n \geq 2}$	c_{n+2}							

Remark 7.4 For c_3 , the graph $S \setminus S'$ consists of two A_1 -components. Therefore c_3 is listed twice in the A_1 -row. Also the case $S_0^p = D_4$ is listed separately since D_4 has automorphisms which don't extend to D_5 .

Using the table, it easy to find all primitive triples with $S_0^p \neq \emptyset$. Since the case-by-case considerations are lengthy we just give an instructive example namely where $S_0^p = A_2$. Here the following nine combinations have to be considered:

$$a_4 \cup a_4, a_4 \cup a_3, a_3 \cup a_3(2\times), a_4 \cup b'_3, a_3 \cup b'_3(2\times), b'_3 \cup b'_3(2\times). \tag{7.4}$$

We omitted the possibility of a factor of $1/2$ for b'_3 . Observe that in three cases two different gluings are possible. It turns out that all cases lead to a valid spherical diagram (namely $D_5^{(1)}(aa)$, $D_5(aa)$, $D_4(aa)$, $A_4(aa')$, $B_4(b'a)$ and $D_3^{(2)}(b'b')$ in the notation of Table 8.3) except for $a_4 \cup b'_3$ and one of the gluings of $a_3 \cup b'_3$ which do not lead to affine Dynkin diagrams.

7.2 The case $S_0^p = \emptyset$

Here, according to Lemma 7.1, S is the disjoint union of S_1 and S_2 stitched together with edges between S'_1 and S'_2 .

A rather trivial subcase is when $S_2 = S'_2 = \emptyset$. Then \mathcal{D} is just a local diagram all of which appear in Table 8.3.

Therefore, assume from now on that $S_1, S_2 \neq \emptyset$. Since then $1 \leq |S'_i| \leq 2$, the number N of edges between S'_1 and S'_2 is at most 4. This yields 5 subcases.

7.2.1 $N = 0$

In this case, S is the disconnected union of S_1 and S_2 . If the triple were bihomogeneous we cannot have $\omega_{\mathcal{D}_1} + \omega_{\mathcal{D}_2} = 0$. If \mathcal{D}_1 were homogeneous and \mathcal{D}_2 inhomogeneous then $\langle \omega_{\mathcal{D}_1}, \alpha_{\mathcal{D}_2}^\vee \rangle = 0 \neq -1$. Therefore, the triple is bi-inhomogeneous and we get the three items $A_{m \geq 1} \times A_{n \geq 1}, A_{m \geq 1} \times C_{n \geq 2}$, and $C_{m \geq 2} \times C_{n \geq 2}$ near the end of the table.

7.2.2 $N = 1$

In this case the diagram \mathcal{D} is the disjoint union of two local diagrams connected by one edge between some $\alpha_1 \in S'_1$ and $\alpha_2 \in S'_2$. One can now go through all pairs of local diagrams and all possibilities for the connecting edge. This works well if one or both local diagrams are of type (d_2) . Otherwise, it is easier to go through all possible connected Dynkin diagrams for S and omit one of its edges. The remaining diagram admits very few possibilities for \mathcal{D}_1 and \mathcal{D}_2 . This way one can check easily that the table is complete with respect to this subcase. Let's look, e.g., at $S = F_4^{(1)}$

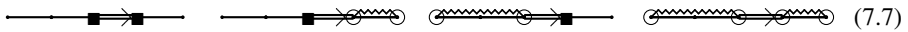
$$\overset{1}{\circ} \overset{2}{\circ} \overset{3}{\circ} \overset{4}{\circ} \overset{2}{\circ} \tag{7.5}$$

Omitting one edge yields

$$\circ \circ \circ \circ \circ \circ \circ \circ \tag{7.6}$$

Each component has to support a local diagram such that the circled vertex is in S' . This rules out all cases except the third one where the local diagrams could be of type a_n or \mathbf{a}_n .

This yields



The first diagram violates Definition 6.5(3)c), the second is primitive and is contained in Table 8.3, the third violates (3)b), and the fourth violates (3)a).

7.2.3 N = 2

In this case, at least one of the S'_i , say S'_1 , has two different elements α_1, α'_1 which are connected to elements $\alpha_2, \alpha'_2 \in S'_2$, respectively.

Assume first that $\alpha_2 \neq \alpha'_2$. Then both local models are either of type $a_{n \geq 2}$ or d_2 . One checks easily that this yields the cases

$$A_{n \geq 3}^{(1)}(aa), C_{n \geq 3}^{(1)}(ad), D_{n+1}^{(2)}(ad), A_1^{(1)} \times A_1^{(1)}(dd), \text{ or } A_2^{(2)} \times A_2^{(2)}(dd). \tag{7.8}$$

The second subcase is $\alpha_2 = \alpha'_2$. If S_1 is of type $a_{n \geq 2}$ one ends up with $A_{n \geq 2}^{(1)}(aa)$, $d = 1$. Otherwise S_1 is of type d_2 . Now one can go through all local diagrams for S_2 and all possible edges between α_1, α'_1 and $\alpha_2 \in S'_2$.

7.2.4 N = 3

In this case, there are distinct elements $\alpha_1, \alpha'_1 \in S'_1$ and $\alpha_2, \alpha'_2 \in S'_2$ which form a string $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$. Both local diagrams are of type d_2 which leaves only the case $D_4^{(2)}(dd)$.

7.2.5 N = 4

Here $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$ form a cycle, so S is of type $A_3^{(1)}$. The local diagrams are both d_2 . This yields $A_3^{(1)}(dd)$.

This concludes the verification of the table.

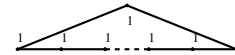
8 Tables

Table 8.1 Affine Dynkin diagrams and their labels

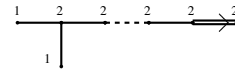
$A_1^{(1)}$



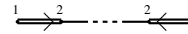
$A_n^{(1)}, n \geq 2$



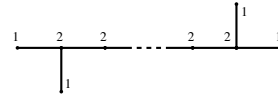
$B_n^{(1)}, n \geq 3$



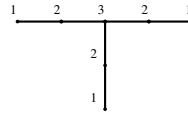
$C_n^{(1)}, n \geq 2$



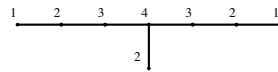
$D_n^{(1)}, n \geq 4$



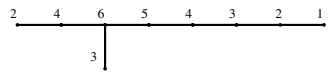
$E_6^{(1)}$



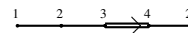
$E_7^{(1)}$



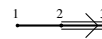
$E_8^{(1)}$



$F_4^{(1)}$



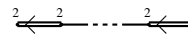
$G_2^{(1)}$



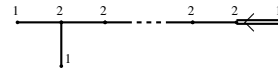
$A_2^{(2)}$



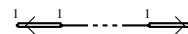
$A_{2n}^{(2)}, n \geq 2$



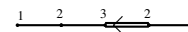
$A_{2n-1}^{(2)}, n \geq 3$



$D_{n+1}^{(2)}, n \geq 2$



$E_6^{(2)}$



$D_4^{(3)}$

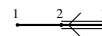


Table 8.2 Primitive local models

Homogeneous models				
	L_0	H_0	ω	Diagram
a_1	PGL(2)	GL(1)	α_1	
$2a_1$	PGL(2)	$N(\mathbb{C}^*)$	$2\alpha_1$	
$a_{n \geq 2}$	PGL(n+1)	GL(n)/μ_{n+1}	$\alpha_1 + \dots + \alpha_n$	
$b_{n \geq 2}$	SO(2n+1)	SO(2n)	$\alpha_1 + \dots + \alpha_n$	
$2b_{n \geq 2}$	SO(2n+1)	O(2n)	$2\alpha_1 + \dots + 2\alpha_n$	
$c_{n \geq 3}$	PSp(2n)	Sp(2) × μ^2 Sp(2n-2)	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$	
$\frac{1}{2}d_{n \geq 4}$	SO(2n)	SO(2n-1)	$\alpha_1 + \dots + \alpha_{n-2} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n$	
$d_{n \geq 4}$	PSO(2n)	SO(2n-1)	$2\alpha_1 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	
$\frac{1}{2}d_2$	SO(4)	SO(3)	$\alpha + \alpha'$	
d_2	SO(3) × SO(3)	SO(3)	$\frac{1}{2}\alpha + \frac{1}{2}\alpha'$	
$\frac{1}{2}d_3$	SO(6)	SO(5)	$\frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3$	
d_3	PSO(6)	SO(5)	$\alpha_1 + 2\alpha_2 + \alpha_3$	
f_4	F_4	Spin(9)	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	
g_2	G_2	SL(3)	$2\alpha_1 + \alpha_2$	
$2g_2$	G_2	N(SL(3))	$4\alpha_1 + 2\alpha_2$	
$\frac{1}{2}b'_3$	Spin(7)	G_2	$\frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3$	
b'_3	SO(7)	G_2	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	

Inhomogeneous models				
	L_0	V_0	ω	Diagram
\mathfrak{a}_0	$GL(1)$	\mathbb{C}	~ 0	\emptyset
$\mathfrak{a}_{n \geq 1}$	$GL(n+1)$	\mathbb{C}^{n+1}	$\sim \pi_1$	
$\mathfrak{c}_{n \geq 2}$	$GSp(2n)$	\mathbb{C}^{2n}	$\sim \pi_1$	

Table 8.3 Affine Dynkin diagrams and their labels

Φ	case	$\omega = \bar{\omega}_1 = -\bar{\omega}_2$	factor	diagram	scope
The empty case					
\emptyset	(\emptyset)	$\omega \neq 0$		\emptyset	
The affine simple cases					
$A_{n \geq 1}^{(1)}$	(aa)	$\omega_1 = \alpha_0 + \dots + \alpha_{d-1}$ $\omega_2 = \alpha_d + \dots + \alpha_n$	$[2]_{n=1}$		$1 \leq d \leq n$
	(dd ₂)	$\omega_1 = \omega_2 = \alpha_0 + \alpha_2$ $\omega_2 = \omega_2 = \alpha_1 + \alpha_3$	$\frac{1}{2}$		
	(dd ₃)	$\omega_1 = \alpha_0 + 2\alpha_1 + \alpha_2$ $\omega_2 = \alpha_2 + 2\alpha_3 + \alpha_0$	$\frac{1}{2}$		
	(aa)	$\omega \sim \pi_0 - \pi_n$			
$B_{n \geq 3}^{(1)}$	(bd)	$\omega_1 = \alpha_0 + \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{d-1}$ $\omega_2 = 2\alpha_d + \dots + 2\alpha_n$	$\frac{1}{2}$		$2 \leq d \leq n$
	(bb)	$\omega_1 = \alpha_1 + \alpha_2 + \dots + \alpha_n$ $\omega_2 = \alpha_0 + \alpha_2 + \dots + \alpha_n$	2		
	(b'a)	$\omega_1 = \text{dir}(\alpha_1 + 2\alpha_2 + 3\alpha_3)$			
	(da)	$\omega_1 = \text{dir}(\alpha_1 + \alpha_3)$			
$C_{n \geq 2}^{(1)}$	(cc)	$\omega_1 = \alpha_0 + 2\alpha_1 + \dots + 2\alpha_{d-1} + \alpha_d$ $\omega_2 = \alpha_d + 2\alpha_{d+1} + \dots + 2\alpha_{n-1} + \alpha_n$	$[2]_{n=2}$		$1 \leq d < n$
	(ad)	$\omega_1 = \text{dir}(\alpha_0 + \alpha_n)$ $\omega_2 = \alpha_1 + \dots + \alpha_{n-1}$	$[2]_{n=2}$		
	(cc)	$\omega \sim \pi_{d-1} - \pi_d$			$1 \leq d \leq n$
$D_{n \geq 4}^{(1)}$	(dd)	$\omega_1 = \alpha_0 + \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{d-1}$ $\omega_2 = 2\alpha_d + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\frac{1}{2}$		$2 \leq d \leq n-1$

Φ	case	$\omega = \bar{\omega}_1 = -\bar{\omega}_2$	factor	diagram	scope
	(dd')	$\omega_1 = 2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ $\omega_2 = 2\alpha_0 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\frac{1}{2}$		
	(aa)	$\omega_1 = \alpha_1 + \alpha_2 + \dots + \alpha_{n-2} + \alpha_{n-1}$ $\omega_2 = \alpha_0 + \alpha_2 + \dots + \alpha_{n-2} + \alpha_n$			
$F_4^{(1)}$	(bf)	$\omega_1 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ $\omega_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$			
	(cd)	$\omega_1 = \frac{1}{2}\alpha_0 + \alpha_1 + \frac{1}{2}\alpha_2$ $\omega_2 = \alpha_2 + 2\alpha_3 + \alpha_4$			
	(aa)	$\omega_1 = \alpha_3 + \alpha_4$			
$G_2^{(1)}$	(ga)	$\omega_1 = \alpha_2 + 2\alpha_1$			
	(aa)	$\omega_1 = \alpha_1$			
	(da)	$\omega_1 = \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_1$			
$A_{2n}^{(2)}$ $n \geq 1$	(ab)	$\omega_1 = 2\alpha_0 + 2\alpha_1 + \dots + 2\alpha_{n-1}$ $\omega_2 = \alpha_n$			
	(bc)	$\omega_1 = \alpha_0 + \alpha_1 + \dots + \alpha_{d-1}$			$1 \leq d \leq n$
$A_{2n-1}^{(2)}$ $n \geq 3$	$(a1c)$	$\omega_1 = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$ $\omega_2 = \alpha_0$			
	$(a3c)$	$\omega_1 = \alpha_0 + \alpha_1 + \alpha_2$ $\omega_2 = \alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-1} + \alpha_n$			
	(ad)	$\omega_1 = \alpha_0 + \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1}$ $\omega_2 = \alpha_n$			
	(dc)	$\omega_1 = \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_1 + \alpha_2 + \dots + \alpha_{d-1}$			$2 \leq d \leq n$
	(cc)	$\omega \sim \pi_0 - \pi_1$			
$D_{n+1}^{(2)}$ $n \geq 2$	(bb)	$\omega_1 = \alpha_0 + \dots + \alpha_{d-1}$ $\omega_2 = \alpha_d + \dots + \alpha_n$	2		$1 \leq d \leq n$
	(ad)	$\omega_1 = \alpha_1 + \dots + \alpha_{n-1}$ $\omega_2 = \alpha_0 + \alpha_n$	$[\frac{1}{2}]_{n=2}$		
	(dd)	$\omega_1 = \alpha_0 + \alpha_2$ $\omega_2 = \alpha_1 + \alpha_3$	$\frac{1}{2}$		
	$(b'b')$	$\omega_1 = 3\alpha_0 + 2\alpha_1 + \alpha_2$ $\omega_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3$	$\frac{1}{2}$		
	(da)	$\omega_1 = \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_2$			

Φ	case	$\omega = \bar{\omega}_1 = -\bar{\omega}_2$	factor	diagram	scope
	(cc)	$\omega \sim \pi_0 - \pi_2$			
$E_6^{(2)}$	(af)	$\omega_1 = \alpha_0$ $\omega_2 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$			
	(bc)	$\omega_1 = \alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3$ $\omega_2 = \alpha_2 + \alpha_3 + \alpha_4$			
	(aa)	$\omega_1 = \alpha_0 + \alpha_1 + \alpha_2$			
$D_4^{(3)}$	(ag)	$\omega_1 = \alpha_0$ $\omega_2 = 2\alpha_1 + \alpha_2$	2		
	(ad)	$\omega_1 = \alpha_0 + \alpha_2$ $\omega_2 = 2\alpha_1$	$\frac{1}{2}$		
	(aa)	$\omega_1 = \alpha_0 + \alpha_1$			

The finite simple cases

$A_{n \geq 1}$	(aa)	$\omega_1 = \alpha_1 + \dots + \alpha_{d-1}$			$2 \leq d \leq n$
	(d2a)	$\omega_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3$			
	(d3a)	$\omega_1 = \alpha_1 + 2\alpha_2 + \alpha_3$			
	(aa)	$\omega \sim \pi_{d-1} - \pi_d$			$2 \leq d \leq n$
	(aa')	$\omega \sim \pi_1 - \pi_n$			$n \geq 2$
	(a)	$\omega_1 = \alpha_1 + \dots + \alpha_n$	$[2]_{n=1}$		
	(d)	$\omega_1 = \alpha_1 + 2\alpha_2 + \alpha_3$	$\frac{1}{2}$		
(a)	$\omega \sim \pi_1$				
$B_{n \geq 2}$	(ba)	$\omega_1 = \alpha_d + \dots + \alpha_n$			$2 \leq d \leq n$
	(da)	$\omega_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3$			
	(dc)	$\omega_1 = \alpha_1 + 2\alpha_2 + \alpha_3$			
	(b'a)	$\omega_1 = \alpha_2 + 2\alpha_3 + 3\alpha_4$			
	(aa)	$\omega \sim \pi_{n-1} - \pi_n$			
	(ac)	$\omega \sim \pi_1 - \pi_3$			
	(b)	$\omega_1 = \alpha_1 + \dots + \alpha_n$	2		
	(b')	$\omega_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3$	$\frac{1}{2}$		
$C_{n \geq 3}$	(ac)	$\omega_1 = \alpha_1 + \dots + \alpha_{d-1}$			$2 \leq d \leq n$
	(ca)	$\omega_1 = \alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-1} + \alpha_n$			
	(ac)	$\omega \sim \pi_{d-1} - \pi_d$			$2 \leq d \leq n$
	(c)	$\omega_1 = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$			
	(e)	$\omega \sim \pi_1$			

Φ	case	$\omega = \bar{\omega}_1 = -\bar{\omega}_2$	factor	diagram	scope
$D_{n \geq 4}$	(da)	$\omega_1 = \alpha_d + \dots + \alpha_{n-2} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n$			$2 \leq d < n$
	(aa)	$\omega_1 = \alpha_1 + \dots + \alpha_{n-1}$			
	(d4a)	$\omega_1 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$			
	(aa)	$\omega \sim \pi_{n-1} - \pi_n$			
	(d)	$\omega_1 = 2\alpha_1 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\frac{1}{2}$		
F_4	(ca)	$\omega_1 = \alpha_2 + 2\alpha_3 + \alpha_3$			
	(aa)	$\omega_1 = \alpha_3 + \alpha_4$			
	(bc)	$\omega_1 = \alpha_1 + \alpha_2 + \alpha_3$			
	(aa)	$\omega \sim \pi_2 - \pi_3$			
	(f)	$\omega_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$			
G_2	(aa)	$\omega_1 = \alpha_1$			
	(aa)	$\omega \sim \pi_1 - \pi_2$			
	(g)	$\omega_1 = 2\alpha_1 + \alpha_2$	2		

The reducible cases

$A_1^{(1)} \times A_1^{(1)}$	(dd)	$\omega_1 = \alpha_0 + \alpha'_0$ $\omega_2 = \alpha_1 + \alpha'_1$	$\frac{1}{2}$		$\frac{\delta}{\ \alpha_i\ } = \frac{\delta'}{\ \alpha'_i\ }$
$A_2^{(2)} \times A_2^{(2)}$	(dd)	$\omega_1 = \alpha_0 + \alpha'_0$ $\omega_2 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha'_1$			$\frac{\delta}{\ \alpha_i\ } = \frac{\delta'}{\ \alpha'_i\ }$
$A_1 \times A_1^{(1)}$	(da)	$\omega_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha'_0$			
$A_1 \times A_{2n}^{(2)}$ $n \geq 1$	(dc)	$\omega_1 = \alpha_1 + \alpha'_0$			
$A_1 \times C_{n \geq 2}^{(1)}$	(dc)	$\omega_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha'_0$			
$A_1 \times G_2^{(1)}$	(da)	$\omega_1 = \alpha_1 + \alpha'_0$			
$A_1 \times A_{n \geq 2}$	(da)	$\omega_1 = \alpha_1 + \alpha'_1$			
$A_1 \times C_{n \geq 3}$	(dc)	$\omega_1 = \alpha_1 + \alpha'_1$			
$A_1 \times B_{n \geq 2}$	(da)	$\omega_1 = \alpha_1 + \alpha'_1$			
$A_1 \times C_{n \geq 2}$	(da)	$\omega_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha'_1$			

Φ	case	$\omega = \bar{\omega}_1 = -\bar{\omega}_2$	factor	diagram	scope
$A_1 \times G_2$	(da)	$\omega_1 = \alpha_1 + \alpha'_1$			
$A_{m \geq 1} \times A_{n \geq 1}$	(aa)	$\omega \sim \pi_1 - \pi'_1$			
$A_{m \geq 1} \times C_{n \geq 2}$	(ac)	$\omega \sim \pi_1 - \pi'_1$			
$C_{m \geq 2} \times C_{n \geq 2}$	(cc)	$\omega \sim \pi_1 - \pi'_1$			
$A_1 \times A_1$	(d)	$\omega_1 = \alpha_1 + \alpha'_1$	$\frac{1}{2}$		

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