



# Bounds for invariants of numerical semigroups and Wilf's conjecture

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## Abstract

Given coprime positive integers  $g_1 < \dots < g_e$ , the Frobenius number  $F = F(g_1, \dots, g_e)$  is the largest integer not representable as a linear combination of  $g_1, \dots, g_e$  with non-negative integer coefficients. Let  $n$  denote the number of all representable non-negative integers less than  $F$ ; Wilf conjectured that  $F + 1 \leq en$ . We provide bounds for  $g_1$  and for the type of the numerical semigroup  $S = \langle g_1, \dots, g_e \rangle$  in function of  $e$  and  $n$ , and use these bounds to prove that  $F + 1 \leq qen$ , where  $q = \left\lceil \frac{F+1}{g_1} \right\rceil$ , and  $F + 1 \leq en^2$ . Finally, we give an alternative, simpler proof for the Wilf conjecture if the numerical semigroup  $S = \langle g_1, \dots, g_e \rangle$  is almost-symmetric.

**Keywords** Wilf conjecture · Numerical semigroups · Multiplicity · Embedding dimension · Type · Almost symmetric numerical semigroup

**Mathematics Subject Classification** 05A99 · 11B75 · 20M14

## 1 Introduction

The classical money-changing problem consists of finding what sums of money can be changed, using  $e$  different denominations of coins  $2 \leq g_1 < \dots < g_e$ . Assuming, without loss of generality, that  $\gcd(g_1, \dots, g_e) = 1$ , it is well-known that only a finite number of sums cannot be changed, and there exists a maximum integer  $F = F(g_1, \dots, g_e)$  which cannot be represented as a linear combination of the *generators*  $g_1, \dots, g_e$ , with coefficients in the set of natural numbers  $\mathbb{N}$ .

Determining this maximum integer  $F$ , called the Frobenius number, is the subject of the Diophantine Frobenius Problem. This challenging problem has been extensively studied

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over the past decades, and presents applications in several areas of mathematics, including Commutative Algebra, Combinatorics and Coding Theory (see [9] for a monograph on this problem). Nonetheless, as of today there is an exact solution only for the special case  $e = 2$ , where Sylvester showed that  $F = g_1g_2 - g_1 - g_2$ . In the general case, it is known that no polynomial formula for  $F$  in function of  $g_1, \dots, g_e$  can exist (cf. [3]), and presently the literature is mostly focused on finding algorithms and bounds for  $F$ .

In 1978, H.S. Wilf proposed an upper bound for the Frobenius number  $F$  (cf. [11]), namely

$$F + 1 \leq en, \tag{1}$$

where  $n$  is the number of solutions of the money-changing problem for  $g_1, \dots, g_e$  less than  $F$  (actually, Wilf’s original question was a lower bound for  $e$ , but we choose this equivalent and simpler formulation of his conjecture).

This problem, now known as the Wilf Conjecture, has been considered by several authors; however, only special cases have been solved. For instance, it is known that the Conjecture is true in the following cases:  $e \leq 3$  (cf. [7]),  $|\mathbb{N} \setminus S| \leq 65$  (where  $S$  denotes the numerical semigroup generated by  $g_1, \dots, g_e$ ; cf. [2]),  $e \geq \frac{g_1}{3}$  (cf. [6]), when  $g_1$  is large enough and its prime factors are not smaller than  $\left\lceil \frac{g_1}{e} \right\rceil$  (cf. [8]), if  $F + 1 \leq 3g_1$  (cf. [5]). Most notably, the last case, due to Eliahou, coupled with a previous result by Zhai (cf. [12]), infers that the Wilf Conjecture is, in a sense, asymptotically true; the survey [4] describes the state of the research on the Wilf Conjecture.

Despite this vibrant literature, the general case is still very elusive, and in fact, no bound for  $F$  in function of  $e$  and  $n$ , that holds true for all numerical semigroups, is known. In this work, we provide such a bound, by virtue of a bound for the smallest generator  $g_1$  in function of  $e$  and  $n$ .

**Theorem 1** *Let  $g_1 < \dots < g_e$  be coprime positive integers larger than 1, let  $F$  be the Frobenius number,  $n$  be the number of integers less than  $F$  which are representable as a linear combination with coefficients in  $\mathbb{N}$  of  $g_1, \dots, g_e$ , and  $q = \left\lceil \frac{F + 1}{g_1} \right\rceil$ . Then*

- (1)  $F + 1 \leq qen$ ;
- (2)  $F + 1 \leq en^2$ .

Then, we provide a bound for the *type* of the numerical semigroup  $S = \langle g_1, \dots, g_e \rangle$  in function of  $e$  and  $n$ , and use this bound to give an alternative proof of the Wilf Conjecture when the numerical semigroup  $S$  is almost-symmetric.

## 2 Main result

Let  $\mathbb{Z}$  denote the set of integers, and  $\mathbb{N}$  the set of non-negative integers. Given  $e \geq 2$  and  $g_1, \dots, g_e \in \mathbb{N}$  such that  $\gcd(g_1, \dots, g_e) = 1$ , it is well-known that the set

$$S = \langle g_1, \dots, g_e \rangle = \{a_1g_1 + \dots + a_eg_e \mid a_i \in \mathbb{N}\}$$

is a submonoid of  $(\mathbb{N}, +)$  such that the set  $\mathbb{N} \setminus S$  is finite; a monoid  $S$  satisfying this property is called a *numerical semigroup* (see [10] for a detailed monograph on this algebraic structure). With the notation  $S = \langle g_1, \dots, g_e \rangle$  we will assume that  $\{g_1, \dots, g_e\}$  is a minimal generating system (which is unique for any numerical semigroup) for  $S$ , and we will thus say that  $e$

is the *embedding dimension* of  $S$ . We also denote by  $F$  the *Frobenius number* of  $S$ , that is,  $F = \max \mathbb{Z} \setminus S$ . Denote by  $N(S) = S \cap [0, F]$  the set of elements of  $S$  less than  $F$  (called *small elements*), and let  $n = |N(S)|$ .

Given an element  $m \in S$ , define the *Apéry set* of  $S$  with respect to  $s$  as

$$Ap(S, m) = \{\omega \in S \mid \omega - m \notin S\}.$$

Clearly  $Ap(S, m)$  consists of the smallest elements of  $S$  in every residual class modulo  $m$ , therefore  $0 \in Ap(S, m)$ ,  $\max Ap(S, m) = F + m$  and  $|Ap(S, m)| = m$ .

Our first result is a bound for the smallest generator  $g_1$  (often called the *multiplicity*) of  $S$ , in function of  $e$  and  $n$ . The main result is a direct corollary of this bound.

**Theorem 2** *Let  $2 \leq g_1 < \dots < g_e$  be coprime positive integers, and let  $S = \langle g_1, \dots, g_e \rangle$ . Then*

$$g_1 \leq (e - 1)n + 1.$$

**Proof** Define the map

$$\varphi : Ap(S, g_1) \setminus \{0\} \rightarrow \mathcal{P}(N(S) \times \{g_2, \dots, g_e\}), \quad \varphi(\omega) = \{(\omega - g_i, g_i) \mid \omega - g_i \in S\}.$$

This map is well defined since, if  $\omega \in Ap(S, g_1) \setminus \{0\}$ , then  $\omega \leq F + g_1$ , therefore  $\omega - g_i \in S$  implies  $\omega - g_i \in N(S)$ . Moreover, for every  $\omega \in Ap(S, g_1) \setminus \{0\}$ , there exists a generator  $g_i$  such that  $\omega - g_i \in S$ , and therefore  $\varphi(\omega) \neq \emptyset$ . Finally, for  $\omega_1, \omega_2 \in Ap(S, g_1) \setminus \{0\}$ , if  $(s, g_i) \in \varphi(\omega_1) \cap \varphi(\omega_2)$  then  $s = \omega_1 - g_i = \omega_2 - g_i$  and thus  $\omega_1 = \omega_2$ : hence the sets  $\varphi(\omega)$  are pairwise disjoint. Therefore the collection  $\{\varphi(\omega)\}_{\omega \in Ap(S, g_1) \setminus \{0\}}$  is a partition of a subset of  $N(S) \times \{g_2, \dots, g_e\}$ , and thus we conclude that

$$g_1 - 1 = |Ap(S, g_1) \setminus \{0\}| \leq \sum_{\omega \in Ap(S, g_1) \setminus \{0\}} |\varphi(\omega)| \leq |N(S) \times \{g_2, \dots, g_e\}| = n(e - 1).$$

□

**Proof of Theorem 1** By Theorem 2, we know that  $g_1 \leq (e - 1)n + 1$ , thus multiplying by  $q$  and remembering that  $n \geq 1$ , we obtain

$$F + 1 \leq g_1 q \leq q(e - 1)n + q = qen - qn + q \leq qen.$$

Finally, since by definition of  $q$  we have  $\{0, g_1, 2g_1, \dots, (q - 1)g_1\} \subseteq S \cap [0, F] = N(S)$ , we have  $q \leq n$ , therefore

$$F + 1 \leq qen \leq en^2.$$

□

For a numerical semigroup  $S = \langle g_1, \dots, g_e \rangle$ , define the set of *pseudo-Frobenius numbers* of  $S$  as the set

$$PF(S) = \{\omega \notin S \mid \omega + s \in S \text{ for every } s \in S \setminus \{0\}\}.$$

The cardinality of  $PF(S)$  is called the *type* of  $S$ , denoted by  $t$ . Since for every  $\omega \in PF(S)$  and  $m \in S \setminus \{0\}$ ,  $\omega + m \in Ap(S, m) \setminus \{0\}$ , we have  $t \leq g_1 - 1$ . Our next result is a bound for  $t$  in function of  $e$  and  $n$ .

**Theorem 3** Let  $2 \leq g_1 < \dots < g_e$  be positive coprime integers, let  $S = \langle g_1, \dots, g_e \rangle$ , and define  $q = \left\lceil \frac{F + 1}{g_1} \right\rceil \geq 1$ . Then

$$t \leq (e - 2)[n - q + 1] + 2 \leq (e - 2)n + 2.$$

**Proof** Assume that there are two elements  $f_1, f_2 \in PF(S)$  such that  $f_1 = \lambda_1 g_2 - g_1$  and  $f_2 = \lambda_2 g_2 - g_1$ , with  $\lambda_1, \lambda_2 \in \mathbb{N}$  and  $\lambda_1 > \lambda_2$ ; then  $s = f_1 - f_2 = (\lambda_1 - \lambda_2)g_2 \in S$ , yielding  $f_2 + s = f_1 \in S$ , a contradiction. Then there is at most one element of the form  $\lambda g_2 - g_1$  in the set  $PF(S)$ ; let  $f_2$  be such an element (if it exists), and let  $PF'(S) = PF(S) \setminus \{F, f_2\}$  (if  $f_2$  does not exist, then take  $PF'(S) = PF(S)$ ).

Define the function  $\varphi : PF'(S) \rightarrow \mathcal{P}(N(S) \times \{g_3, \dots, g_e\})$  as  $\varphi(f) = \{(s, g_i) \mid s = f + g_1 - g_i \in S\}$ . This function is well-defined since  $(s, g_i) \in \varphi(f)$  is such that  $s = f + g_1 - g_i < f \leq F$ ,  $\varphi(f) \neq \emptyset$  (because, being  $f \neq f_2$ ,  $f + g_1$  cannot be of the form  $Kg_2$ , for some integer  $K$ ), and clearly  $\varphi(f) \cap \varphi(f') = \emptyset$  if  $f \neq f'$ , since  $(s, g_i) \in \varphi(f) \cap \varphi(f')$  would imply  $f = s + g_i - g_1 = f'$ . Therefore the collection  $\{\varphi(f)\}_{f \in PF'(S)}$  is a partition of a subset of  $N(S) \times \{g_3, \dots, g_e\}$ . Moreover, our choice of  $q$  means that for  $i = 1, \dots, q - 1$ ,  $ig_1 \in N(S)$ , but if  $(ig_1, g_i) \in \varphi(f)$  for some  $f$  and  $g_i$ , then  $f = ig_1 + g_i - g_1 \in S$ , which is impossible. Therefore for every  $i = 1, \dots, q - 1$  and  $j = 3, \dots, e$ ,  $(ig_1, g_j)$  cannot belong to any set  $\varphi(f)$ . Combining these facts, and remembering that  $q \geq 1$ , we obtain

$$t - 2 \leq |PF'(S)| \leq \sum_{f \in PF'(S)} |\varphi(f)| = \left| \bigcup_{f \in PF'(S)} \varphi(f) \right| \leq (e - 2)[n - q + 1] \leq (e - 2)n.$$

□

Let  $S = \langle g_1, \dots, g_e \rangle$  be a numerical semigroup. We say that  $S$  is *almost-symmetric* if, for every  $x \notin S$ , either  $F - x \in S$  or  $\{x, F - x\} \subseteq PF(S)$ . Partitioning the interval  $[0, F]$  in couples  $\{x, F - x\}$ , it is simple to see that, for an almost-symmetric numerical semigroup,  $2n + t = F + 2$ . Then Theorem 3 can be used to provide an alternative proof of Wilf’s Conjecture for almost-symmetric numerical semigroups (see [1] for the original proof).

**Corollary 4** *Almost symmetric numerical semigroups satisfy Wilf’s conjecture.*

**Proof** Let  $S$  be an almost symmetric numerical semigroup,  $S \neq \{0, g_1, \rightarrow\}$  (in this case it is immediate to check that the Wilf Conjecture still holds). Then in Theorem 3 we have  $q \geq 2$ , and thus

$$t \leq [e - 2][n - 1] + 2 = [e - 2]n - e + 4.$$

By definition of almost symmetric numerical semigroup we have  $2n + t = F + 2$ , hence assuming that  $e \geq 4$  (we recall that Wilf’s Conjecture holds in case  $e \leq 3$ ) we have

$$F + 1 \leq en - e + 3 < en.$$

□

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## Declarations

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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