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Geometric amenability in totally disconnected locally compact groups

Romain Tessera[1] · **Matthew Tointon**[2]

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Abstract
We give a short geometric proof of a result of Soardi and Woess and Salvatori that a quasitransitive graph is amenable if and only if its automorphism group is amenable and unimodular. We also strengthen one direction of that result by showing that if a compactly generated totally disconnected locally compact group admits a proper Lipschitz action on a bounded degree amenable graph then that group is amenable and unimodular. We pass via the notion of *geometric amenability* of a locally compact group, which has previously been studied by the second author and is defined by analogy with amenability, only using right Følner sets instead of left Følner sets. We also introduce a notion of *uniform geometric non-amenability* of a locally compact group, and relate this notion in various ways to actions of that group on graphs and to its modular homomorphism.

Keywords Totally disconnected locally compact group · Amenable group · Unimodular locally compact group · Graph automorphism

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✉ Matthew Tointon
m.tointon@bristol.ac.uk

Romain Tessera
romain.tessera@imj-prg.fr

[1] Institut de Mathématiques de Jussieu-Paris Rive Gauche, Paris, France

[2] School of Mathematics, University of Bristol, Bristol, UK

1 Introduction

A well-known result of Soardi and Woess [16, Corollary 1] states that a vertex-transitive graph is amenable if and only if its automorphism group is amenable and unimodular. Salvatori [15, Theorem 1] generalised this result to quasitransitive graphs. Benjamini, Lyons, Peres and Schramm [4, Remarks 3.11 and 6.3] later gave an alternative proof. Each of these proofs emerged as a corollary of broader work: Soardi and Woess's and Salvatori's proofs came out of work on random walks (see also [14]), whilst one direction of Benjamini, Lyons, Peres and Schramm's proof came out of work on percolation. Lyons and Peres subsequently gave a geometric proof of this direction [11, Proposition 8.14]. The initial purpose of this paper is to offer a quick, geometric proof of the Soardi–Woess–Salvatori theorem, and strengthen one direction of it by showing that the presence of a proper action on a bounded-degree amenable graph that is merely Lipschitz, and not necessarily quasitransitive, is enough to imply that a compactly generated totally disconnected locally compact group is amenable and unimodular.

We start by presenting the necessary definitions. Let Γ be a graph. Given a set $A \subseteq \Gamma$ of vertices, we define the *boundary* ∂A to consist of those vertices that do not belong to A but have a neighbour in A. The *(vertex) Cheeger constant* $\mathrm{h}(\Gamma)$ of Γ is defined via

$$\mathrm{h}(\Gamma) = \inf_{\substack{A \subset V \\ |A| < \infty}} \frac{|\partial A|}{|A|},$$

and Γ is then called *amenable* if $\mathrm{h}(\Gamma) = 0$.

The group $\mathrm{Aut}\,(\Gamma)$ of automorphisms of Γ is a locally compact group with respect to the topology of pointwise convergence, which is metrisable. Every closed subgroup of $\mathrm{Aut}\,(\Gamma)$ is then a compactly generated, totally disconnected, locally compact group in which vertex stabilisers are compact and open. This is classical, and treated in detail in [19–21] for instance.

Now let G be an arbitrary compactly generated locally compact group. An action of G by automorphisms on Γ is called *continuous* if the homomorphism $G \to \mathrm{Aut}\,(\Gamma)$ it induces is continuous. In this paper, we assume by definition that all actions of topological groups on graphs are continuous. An action is called *proper* if vertex stabilisers are compact in G; in particular, every closed subgroup of $\mathrm{Aut}\,(\Gamma)$ acts properly on Γ. It is called *transitive* if there is a unique orbit of vertices, and *quasitransitive* if there are only finitely many orbits of vertices. A theorem of Abels [1] (see [5, Proposition 2.E.9.] or [10, Theorem 2.2^{+}]) states that every compactly generated, totally disconnected, locally compact group admits a transitive proper continuous action on some connected, locally finite graph.

Suppose G is a compactly generated locally compact group acting properly continuously by automorphisms on a locally finite graph Γ. Given $C > 0$, we say that this action is C-*Lipschitz* with respect to a given compact symmetric generating set S if in each orbit there exists some vertex x such that the orbit map

$$\begin{array}{ccc}(G,S) & \to & G\cdot x\\ g & \mapsto & g\cdot x\end{array}$$

is C-Lipschitz (here, (G, S) means the group G endowed with the word metric with respect to S). We will say that the action is *C-Lipschitz* to mean that there exists some S with respect to which it is C-Lipschitz, and simply *Lipschitz* to mean that there exists some $C > 0$ such that the action is C-Lipschitz.

Every quasitransitive action is Lipschitz. Indeed, if vertices $x_1, \ldots, x_k$ are representatives of the orbits, then the action is C-Lipschitz where C is the maximal integer n such that $S \cdot x_i \subset B(x_i, n)$ for every $i = 1, \ldots, k$.

A locally compact group G admits a *(left) Haar measure* μ, the properties of which include that

(i) $\mu(K) < \infty$ if K is compact,
(ii) $\mu(U) > 0$ if U is open and nonempty,
(iii) $\mu(gA) = \mu(A)$ for every Borel set $A \subset G$ and every $g \in G$, and
(iv) if μ' is another Haar measure on G then there exists $\lambda > 0$ such that $\mu' = \lambda \cdot \mu$.

See [8, §15] for a detailed introduction to Haar measures. Note that since a right translate of a Haar measure is again a Haar measure, by property (iv) there exists a homomorphism $\Delta_G : G \to \mathbb{R}^+$, called the *modular homomorphism*, such that

$$\mu(Ag) = \Delta_G(g)\mu(A)$$

for every Borel set A. Property (iv) also implies that Δ_G depends only on G, and not on μ. The group G is called *unimodular* if $\Delta_G \equiv 1$, in which case $\mu(gA) = \mu(Ag) = \mu(A)$ for every Borel set $A \subset G$ and every $g \in G$. Note that if G is unimodular then we may define another Haar measure μ' by $\mu'(A) = \mu(A^{-1})$, and then by property (iv) we have $\mu = \mu'$ so that μ is symmetric.

A locally compact group G with Haar measure μ is called *amenable* if for every compact subset $K \subseteq G$ and every $\varepsilon > 0$ there exists a compact set $U \subseteq G$ of positive measure such that

$$\frac{\mu(KU)}{\mu(U)} \leqslant 1 + \varepsilon.$$

The group G is called *geometrically amenable* if for every compact subset $K \subseteq G$ and every $\varepsilon > 0$ there exists a compact set $U \subseteq G$ of positive measure such that

$$\frac{\mu(UK)}{\mu(U)} \leqslant 1 + \varepsilon.$$

In a unimodular group these notions coincide by the symmetry of μ. In a non-unimodular group G, there exists $k \in G$ such that $\Delta_G(k) > 1$, and then since $\mu(Uk) = \Delta_G(k)\mu(U)$ for all compact sets U of positive measure, G is not geometrically amenable. Thus, we have the following lemma, previously noted by the first author [18, §11].

Lemma 1.1 *A locally compact group is geometrically amenable if and only if it is amenable and unimodular.*

This opens a new avenue to understanding which groups are both amenable and unimodular, which we exploit to prove the following result, which in some sense shows that simultaneous amenability and unimodularity of an *arbitrary* compactly generated, totally disconnected, locally compact group necessarily reflects an action of that group on an amenable graph.

Theorem 1.2 *Let G be a compactly generated, totally disconnected, locally compact group. Then the following are equivalent:*

(i) *G is amenable and unimodular;*
(ii) *G is geometrically amenable;*
(iii) *there exists a bounded-degree amenable graph admitting a proper Lipschitz action of G;*
(iv) *there exists an amenable graph admitting a proper quasitransitive action of G;*
(v) *there exists an amenable graph admitting a proper transitive action of G;*
(vi) *every graph admitting a proper quasitransitive action of G is amenable.*

The implication (v) $\implies$ (iv) is trivial, and we discussed the implication (iv) $\implies$ (iii) above. The implication (vi) $\implies$ (v) follows from Abels's theorem, whilst of course the equivalence (i) $\iff$ (ii) is a special case of Lemma 1.1.

The equivalence of (i), (iv) and (vi) recovers the Soardi–Woess–Salvatori theorem. However, we prove the implication (i) $\implies$ (vi) via the following version of the Soardi–Woess–Salvatori theorem, in particular giving a more direct proof of that result than any previous reference we are aware of.

Theorem 1.3 *Suppose Γ is a connected, locally finite graph, and that G is locally compact group admitting a proper quasitransitive action on Γ. Then Γ is amenable if and only if G is amenable and unimodular.*

We prove the one outstanding implication of Theorem 1.2, (iii) $\implies$ (ii), via the following result, which we believe to be completely new.

Theorem 1.4 *Suppose G is a compactly generated, totally disconnected, locally compact group admitting a proper Lipschitz action on a bounded-degree amenable graph. Then G is geometrically amenable.*

If one is willing to replace Lipschitz by 1-Lipschitz, then we can drop the assumption that the graph must have bounded degree (see the first part of Theorem 1.9). Note that, thanks to the absence of any quasitransitivity assumption, Theorem 1.3 is a significant strengthening of one direction of the Soardi–Woess–Salvatori theorem.

We prove Theorem 1.3 in Sect. 2. Theorem 1.4 follows from the first part of Theorem 1.9, which itself results from Proposition 4.1.

Remark 1.5 The properness of the actions in statements (iii)–(vi) of Theorem 1.2 cannot be removed. On the one hand, if Γ is Cayley graph of degree d on some group H, and G is a free group of rank greater than d, then one may define a transitive (and hence quasitransitive and Lipschitz) action of the free group G on Γ by projecting $F_r \to H$ and letting H act on Γ by translations. In particular, amenability of Γ in this instance does not imply amenability of G. Conversely, defining them as HNN-extensions, one can let the lamplighter and solvable Baumslag–Solitar groups act faithfully and transitively on regular trees (see for instance [12]), so that amenability and unimodularity of G does not preclude the existence of improper transitive actions of G on nonamenable graphs.

Remark 1.6 Statement (vi). of Theorem 1.2 cannot be strengthened to say that every bounded-degree graph admitting a proper *Lipschitz* action of G is amenable. For example, if G is a finitely generated group and T is a tree then the obvious action of G on $G \times T$ is both proper and Lipschitz.

In light of the lamplighter and Baumslag–Solitar examples described in Remark 1.5, one might reasonably wonder whether Sol admits a faithful, transitive action on a regular tree. In Proposition 7.1 we show that it does not, in fact, admit any quasitransitive action on any non-amenable locally finite graph.

Uniform non-amenability

Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura [2] define a uniform notion of non-amenability for finitely generated groups (Osin [13] considers a related notion called *weak amenabilty*). In this paper we extend this definition to locally compact groups, and to geometric amenability. First, given a compactly generated locally compact group G, we follow Arzhantseva et. al. in defining

$$\operatorname{Føl} G = \inf_S \inf_U \frac{\mu(SU \backslash U)}{\mu(U)},$$

where the infima are over all compact symmetric generating sets S for G and all compact subsets $U \subseteq G$. (In fact, this differs slightly from Arzhantseva et al.'s definition in that they consider the interior boundary, where we consider the exterior boundary.) If G is amenable then $\operatorname{Føl} G = 0$ by definition; we call a group G satisfying $\operatorname{Føl} G > 0$ *uniformly non-amenable*.

In the context of the present work it is natural to define analogously uniform *geometric* non-amenability. Given a compactly generated locally compact group G, we therefore set

$$\operatorname{Føl}^* G = \inf_S \inf_U \frac{\mu(US \backslash U)}{\mu(U)},$$

where again the infima are over all compact symmetric generating sets S for G and all compact subsets $U \subseteq G$. If G is geometrically amenable then $\operatorname{Føl}^* G = 0$ by definition, and we call a group G *uniformly geometrically non-amenable* if $\operatorname{Føl}^* G > 0$.

Recall that a connected Lie group is generated by any neighbourhood of the identity. It follows that if G is such a group then $\operatorname{Føl} G = 0$ and $\operatorname{Føl}^* G = 0$; indeed, if U is any compact subset of positive measure, and $(S_n)_{n=1}^{\infty}$ is a sequence of compact symmetric neighbourhoods of the identity converging to the identity, then

$$\frac{\mu(S_n U \backslash U)}{\mu(U)} \to 0, \qquad \frac{\mu(U S_n \backslash U)}{\mu(U)} \to 0.$$

These notions are therefore more appropriately studied in the setting of totally disconnected locally compact groups.

It turns out that, in that setting, these notions relate to unimodularity and the presence of certain actions on graphs in a number of ways that are strongly analogous to Theorem 1.2. For example, the following statement (which we prove in a more detailed form in Proposition 6.3) shows that in an amenable group G, uniform geometric non-amenability can be characterised in terms of the modular homomorphism, just as geometric amenability can be by the equivalence (i) $\iff$ (ii).

Theorem 1.7 *Suppose G is a compactly generated totally disconnected locally compact group. Suppose further that G is amenable and non-unimodular. Then* $\operatorname{Føl}^* G = 0$ *if and only if the image of G under the modular homomorphism is dense in* $\mathbb{R}_+^*$.

The following result, on the other hand, is directly analogous to the equivalence (ii) $\iff$ (v).

Theorem 1.8 *Let G be a compactly generated locally compact totally disconnected group. Then* $\text{Føl}^* G = 0$ *if and only if there exists a sequence of G-transitive proper locally finite graphs Γ_n which are asymptotically amenable in the sense that there exists a sequence $A_n \subseteq \Gamma_n$ such that $|\partial A_n|/|A_n| \to 0$.*

Theorem 1.8 actually follows from the more refined Theorem 1.12, below.

Finally, we have an analogue of the equivalence (ii) $\iff$ (iii). To state it requires a further definition. Suppose G is a compactly generated locally compact group acting properly on a locally finite graph Γ. Given $C > 0$, we say that this action is *contingently C-Lipschitz* if for every $x \in \Gamma$ there exists a compact symmetric generating subset $S_x \subset G$ such that the orbit map

$$\begin{aligned} (G, S_x) &\to G \cdot x \\ g &\mapsto g \cdot x \end{aligned}$$

is C-Lipschitz. We will say that the action is *contingently Lipschitz* to mean that there exists some $C > 0$ such that the action is contingently C-Lipschitz.

Theorem 1.9 *Suppose G is a compactly generated totally disconnected locally compact group. Then*

- *G is geometrically amenable if and only if it admits a* 1*-Lipschitz proper action on a locally finite amenable graph; and*
- $\text{Føl}^* G = 0$ *if and only if G admits a contingently* 1*-Lipschitz proper action on a locally finite amenable graph.*

We actually prove a slightly more detailed result than Theorem 1.9, which we state below as Theorem 6.5.

Remark 1.10 Lemma 1.1 implies that geometric amenabillity is stronger than amenability. However, it is not clear whether $\text{Føl}^* G = 0$ is a stronger property than $\text{Føl}\, G = 0$. Clearly the two conditions coincide when G is unimodular. If G is non-unimodular and amenable then $\text{Føl}\, G = 0$. On the other hand, it is easy to see that $\text{Føl}^* G$ is not 0 if the modular homomorphism has discrete image; this is the case, for example, in the affine group over $\mathbb{Q}_p$, where the image of the modular homomorphism is the powers of p. Note that Theorem 1.7 shows that if G is amenable, then the converse holds as well. We do not know what happens if G is neither unimodular nor amenable.

Qustion 1.11 If $\text{Føl}\, G = 0$ and Δ_G has dense image in $\mathbb{R}_+^*$, must it be the case that $\text{Føl}^* G = 0$?

The space of *G*-transitive graphs

Let $\mathfrak{G}$ be the set of isomorphism classes of locally finite vertex transitive graphs. Given a compactly generated totally disconnected locally compact group G, we define $\mathfrak{G}(G)$ to be the subset of $\mathfrak{G}$ consisting of graphs admitting a proper transitive action of G. We then define $\mathrm{h}_G = \inf_{\Gamma \in \mathfrak{G}(G)} \mathrm{h}(\Gamma)$, where $\mathrm{h}(\Gamma)$ is the Cheeger constant of Γ as above. This allows us to formulate the following refinement of Theorem 1.8.

Theorem 1.12 *Suppose G is a compactly generated totally disconnected locally compact group. Then $h_G =$* $\text{Føl}^* G$.

We recall that $\mathfrak{G}$ comes with a natural topology, obtained from the following distance: we say that two graphs $\Gamma, \Gamma' \in \mathfrak{G}$ are at distance at most 2^{-n} if their balls of radius n are isomorphic. We observe that two graphs with different degrees are at distance 1 apart. Moreover a standard compactness argument shows that two graphs at distance 0 must be isomorphic, so that this indeed defines a distance on $\mathfrak{G}$. Write $\overline{\mathfrak{G}(G)}$ the closure of $\mathfrak{G}(G)$ in $\mathfrak{G}$ for this topology.

Theorem 1.13 *The map $\Gamma \mapsto h_\Gamma$ is upper semicontinuous on $\mathfrak{G}$. In particular, if G is a compactly generated totally disconnected locally compact group such that $\overline{\mathfrak{G}(G)}$ contains an amenable graph, then $h_G = 0$.*

We prove Theorems 1.12 and 1.13 in Sect. 5.

For every $k \in \mathbb{N}$, let $\mathfrak{G}_k$ be the set of isomorphism classes of vertex transitive graphs of degree at most k, and let $\mathfrak{G}_k(G) = \mathfrak{G}(G) \cap \mathfrak{G}_k$. It is natural to consider the quantity $\mathrm{h}_{G,k} = \inf_{\Gamma \in \mathfrak{G}_k(G)} \mathrm{h}(\Gamma)$.

Qustion 1.14 Can we have $\mathrm{h}_{G,k} > 0$ for all k but $\mathrm{h}_G = 0$?

We strongly expect the answer to be positive although we do not currently have an example.

Qustion 1.15 Does $\mathrm{h}_{G,k} = 0$ for some k imply that $\overline{\mathfrak{G}(G)}$ contains an amenable graph?

2 The Soardi–Woess–Salvatori theorem

In this section we prove Theorem 1.3. Our proof consists of combining Lemma 1.1 with the following two results, the second of which is similar to a reduction appearing in [15] and [4, Lemma 3.10].

Proposition 2.1 *Suppose Γ is a connected, locally finite vertex-transitive graph, and G is a locally compact group admitting a proper transitive action on Γ. Then Γ is amenable if and only if G is geometrically amenable.*

Lemma 2.2 *Suppose Γ is a connected, locally finite quasitransitive graph, and G is a locally compact group admitting a proper quasitransitive action on Γ. Then there exists a connected, locally finite vertex-transitive graph Γ' quasi-isometric to Γ, and a compact normal subgroup $H \lhd G$ such that G/H acts properly transitively on Γ'.*

Given these results, it is straightforward to deduce Theorem 1.3, as follows.

Proof of Theorem 1.3 Let Γ' be the graph and $H \lhd G$ be the compact normal subgroup given by Lemma 2.2. Since Γ and Γ' are quasi-isometric, either both are amenable or neither is [6, Theorem 18.13]. Moreover, since H is compact, G is amenable if and only if G/H is amenable, and unimodular if and only if G/H is unimodular, and hence, by Lemma 1.1, geometrically amenable if and only if G' is geometrically amenable. The theorem therefore follows from applying Proposition 2.1 to Γ' and G/H. □

All that remains, then, is to prove Proposition 2.1 and Lemma 2.2. We start with the following result, which is basically the key reason why geometric amenability of a group relates to amenability of a graph it acts on transitively. Here, and throughout this paper, given a group G acting on a graph Γ, and a vertex $o \in \Gamma$ and a subset $X \subseteq \Gamma$ of vertices, we write G_o for the stabiliser of o in G, and

$$G_{o\to X} = \{g \in G : g \cdot o \in X\}.$$

Moreover, given a subset X of a graph Γ and a natural number r, we write $[X]_r = \{y \in \Gamma : d(y, X) \leqslant r\}$ for the r-neighbourhood of X, and $\partial_r X = [X]_r \backslash X$ for the r-exterior boundary of X.

Proposition 2.3 *Suppose Γ is a connected, locally finite vertex-transitive graph, and G is a locally compact group admitting a proper transitive action on Γ. Let $o \in \Gamma$, and let $S = \{g \in G : d(g \cdot o, o) \leqslant 1\}$. Then S is a symmetric compact open generating set for G, and for every subset $X \subseteq \Gamma$ the set $G_{o \to X}$ is compact and open and satisfies*

$$\mu(G_{o\to X}) = |X| \cdot \mu(G_o), \tag{2.1}$$

and more generally

$$\mu(G_{o\to X} S^r) = |[X]_r| \cdot \mu(G_o) \tag{2.2}$$

for every $r \in \mathbb{N}$.

Proof The first part is essentially [21, Lemma 3]. To see that S is symmetric, note that

$$d(g^{-1} \cdot o, o) = d(g^{-1} \cdot o, g^{-1} g \cdot o) = d(g \cdot o, o).$$

We will prove by induction on n that $d(g \cdot o, o) \leqslant n$ for a given $n \geqslant 1$ if and only if $g \in S^n$, which implies in particular that S generates G. The base case $n = 1$ is true by definition, whilst for $n \geqslant 2$ we have

$$\begin{aligned} d(g\cdot o, o) \leqslant n &\iff d(g \cdot o, x) \leqslant 1 \text{ for some } x \text{ with } d(x, o) \leqslant n-1 \\ &\iff d(g \cdot o, h \cdot o) \leqslant 1 \text{ for some } h \in S^{n-1} && \text{(by induction)} \\ &\iff h^{-1} g \in S \text{ for some } h \in S^{n-1} && \text{(by the } n = 1 \text{ case)} \\ &\iff g \in S^n, \end{aligned}$$

as claimed.

By transitivity of the action, we may pick, for each $x \in \Gamma$, an automorphism $g_x \in G$ such that $g_x \cdot o = x$. Note then that

$$G_{o\to X} = \bigcup_{x \in X} g_x G_o$$

for an arbitrary subset $X \subseteq \Gamma$, which immediately implies (2.1). Furthermore, G_o is compact by properness and open by continuity, so this also means that $G_{o\to X}$ is compact and open whenever X is finite, and in particular that S is compact and open, as required.

Finally, for every $g \in G$ we have

$$\begin{aligned} g \in G_{o\to X} S^r &\iff \text{there exists } q \in G_{o\to X} \text{ such that } d(q^{-1} g \cdot o, o) \leqslant r \\ &\iff \text{there exists } q \in G_{o\to X} \text{ such that } d(g \cdot o, q \cdot o) \leqslant r \\ &\iff g \cdot o \in [X]_r \\ &\iff g \in G_{o\to [X]_r}, \end{aligned}$$

and so (2.2) follows from (2.1). □

Proof of Proposition 2.1 First, suppose that Γ is amenable, and let $(A_n)_{n=1}^{\infty}$ be a sequence of finite subsets of Γ such that $|\partial A_n|/|A_n| \to 0$. Since Γ is transitive, and hence has uniformly bounded degrees, we in fact have that $|\partial_r A_n|/|A_n| \to 0$ for all r. Let K be a compact subset of G. Proposition 2.3 says that S is a symmetric open generating set for G, so we have $K \subseteq S^r$ for some r. This in turn implies that $G_{o\to A_n}K \subseteq G_{o\to A_n}S^r$ for each n, and hence, by Proposition 2.3, that

$$\frac{\mu(G_{o\to A_n}K)}{\mu(G_{o\to A_n})} \leqslant \frac{\mu(G_{o\to A_n}S^r)}{\mu(G_{o\to A_n})} = \frac{|[A_n]_r|}{|A_n|} \to 1.$$

Conversely, if G is geometrically amenable then since S is compact there exists a sequence $(U_n)_{n=1}^{\infty}$ of compact subsets of positive measure in G such that $\mu(U_nS)/\mu(S) \to 1$. Since $G_o \subseteq S$, this implies in particular that $\mu(U_nS)/\mu(U_nG_o) \to 1$. Using the fact that $SG_o = S$ and $S = S^{-1}$ we have $G_oS = (SG_o)^{-1} = S$, and so we deduce further that $\mu(U_nG_oS)/\mu(U_nG_o) \to 1$. Since $U_nG_o = G_{o\to U_n\cdot o}$, this combines with Proposition 2.3 to show that $|\partial(U_n\cdot o)|/|U_n\cdot o| \to 0$, and so Γ is amenable. □

Proof of Lemma 2.2 It is well known that if G has n orbits, then if we fix one of these orbits V, and define $E = \{(x, y) \in V \times V : 1 \leqslant d(x, y) \leqslant 2n\}$, the resulting graph $\Gamma' = (V, E)$ is connected, locally finite and quasi-isometric to Γ (see e.g. the proof of [9, Proposition 2.13]). We claim we may take H to be the kernel of the homomorphism $G \to \mathrm{Aut}\,(\Gamma')$ given by the restriction to V of the G-action on Γ. Indeed, the action of G/H on Γ' induced by this homomorphism is transitive by definition, whilst $H = \{g \in G : g\cdot v = v \text{ for all } v \in V\}$ is a closed subset of a vertex stabiliser, and hence compact. □

3 Equivalent formulations of amenability and geometric amenability

It is well known that in order to decide whether a locally compact group is amenable it suffices to consider the individual elements of a single compact generating set, as follows.

Lemma 3.1 *Suppose G is a locally compact group, and $S \subseteq G$ is a compact set generating G as a semigroup. Then G is amenable if and only if for each $\varepsilon > 0$ there exists a compact set $F \subseteq G$ of positive measure such that* $\sup_{s\in S}\mu(sF \bigtriangleup F)/\mu(F) \leqslant \varepsilon$.

Although this is well known, we have not been able to locate a convenient self-contained reference, so we provide a proof.

Proof If G is amenable then by definition there exist compact sets $(F_n)_{n=1}^{\infty}$ of positive measure such that $\mu((S \cup S^{-1})F_n\backslash F_n)/\mu(F_n) \to 0$. In particular, for each $s \in S$ we have $\mu(sF_n\backslash F_n)/\mu(F_n) \to 0$ and $\mu(F_n\backslash sF_n)/\mu(F_n) = \mu(s(s^{-1}F_n\backslash F_n))/\mu(F_n) = \mu(s^{-1}F_n\backslash F_n)/\mu(F_n) \to 0$.

Conversely, suppose $(F_n)_{n=1}^{\infty}$ is a sequence of compact subsets of positive measure in G such that $\mu(sF_n \bigtriangleup F_n)/\mu(F_n) \leqslant \frac{1}{n}$ for all $s \in S$. We claim more generally that $\mu(gF_n \bigtriangleup F_n)/\mu(F_n) \to 0$ for all $g \in G$. Indeed, since S generates G as a semigroup, an arbitrary element $g \in G$ can be written in the form $g = s_1 \cdots s_m$ with $s_i \in S$, and then using the well-known and easily verified fact that $\mu(A \bigtriangleup B)$ satisfies the triangle inequality we obtain

$$\begin{aligned}&\mu(gF_n \bigtriangleup F_n)\\ &\leqslant \mu(s_1\cdots s_{m-1}(s_mF_n \bigtriangleup F_n)) + \mu(s_1\cdots s_{m-2}(s_{m-1}F_n \bigtriangleup F_n)) + \cdots + \mu(s_1F_n \bigtriangleup F_n)\\ &\leqslant \frac{m}{n}\mu(F_n) \to 0.\end{aligned}$$

The implication (iv) $\implies$ (v) of [3, Theorem G.3.1] then implies that $L^{\infty}(G)$ admits an invariant mean, so that G is *amenable* in the sense of [7], and then the implication (amenable)

$\Longrightarrow$ (A) proved in [7, Sect. 1.2] implies that G is amenable in our sense by the remarks at the end of [7, Sect. 1.2]. □

In the present work we need the following analogous result for geometric amenability.

Lemma 3.2 *Suppose G is a locally compact group, and $S \subseteq G$ is a compact set generating G as a semigroup. Then G is geometrically amenable if and only if for each $\varepsilon > 0$ there exists a compact set $F \subseteq G$ of positive measure such that* $\sup_{s\in S} \mu(Fs \vartriangle F)/\mu(F) \leqslant \varepsilon$.

Proof We first show that if G is not unimodular then neither condition holds. It is convenient to prove the contrapositive. If G is geometrically amenable this is immediate from Lemma 1.1. On the other hand, if $(F_n)_{n=1}^{\infty}$ is a sequence of compact subsets of positive measure in G such that $\mu(F_n s \vartriangle F_n)/\mu(F_n) \leqslant \frac{1}{n}$ for all $s \in S$, then $\mu(F_n s) \leqslant (1+\frac{1}{n})\mu(F_n)$ for every $s \in S$ and $n \in \mathbb{N}$, hence $\Delta_G(s) \leqslant 1 + \frac{1}{n}$ for every $s \in S$ and $n \in \mathbb{N}$, and hence $\Delta_G(s) \leqslant 1$ for every $s \in S$. Since S generates G as a semigroup, this implies that $\Delta_G \equiv 1$ on G as claimed.

We may therefore assume that G is unimodular, and in particular that μ is symmetric. By Lemma 1.1, G is then geometrically amenable if and only if it is amenable; by Lemma 3.1, G is amenable if and only if for each $\varepsilon > 0$ there exists a compact set $F \subseteq G$ of positive measure such that $\sup_{t\in S^{-1}} \mu(sF \vartriangle F)/\mu(F) \leqslant \varepsilon$; and by symmetry of μ, this occurs if and only if for each $\varepsilon > 0$ there exists a compact set $F^{-1} \subseteq G$ of positive measure such that $\sup_{s\in S} \mu(F^{-1}s \vartriangle F^{-1})/\mu(F^{-1}) \leqslant \varepsilon$. □

4 Lipschitz proper actions and geometric amenability

In this section we generalise Theorem 1.4 to graphs of unbounded degree. This generalisation necessitates a further definition: we will say that a locally finite graph is r-amenable for $r \geqslant 1$, and for all $\varepsilon > 0$, there exists a finite set of vertices F such that $|\partial_r F|/|F| \leqslant \varepsilon$. When $r = 1$, we simply recover the usual notion of amenability. Note that if the graph has uniformly bounded degrees then amenability implies r-amenability for all r.

Proposition 4.1 *Let $r \in \mathbb{N}$. Suppose G is a compactly generated, totally disconnected, locally compact group acting r-Lipschitz properly on a locally finite r-amenable graph Γ. Then G is geometrically amenable.*

Before proving Proposition 4.1 we present two lemmas.

Lemma 4.2 *Suppose G is a compactly generated locally compact group acting properly on a locally finite graph Γ. Then the action of G on Γ is contingently 1-Lipschitz if and only if every subgraph induced by an orbit of G on Γ is connected.*

Proof Suppose first that the action is contingently 1-Lipschitz. Given $x \in \Gamma$, there therefore exists a generating set S_x such that $g \mapsto g \cdot x$ is a 1-Lipschitz map from the Cayley graph (G, S_x) to X. The fact that (G, S_x) is connected implies that the range of this map is connected as well, hence the orbit of x is connected.

Conversely, suppose that the orbit of x is connected. Then the subgraph Γ_x induced by $G \cdot x$ is a connected locally finite vertex-transitive graph, and by Proposition 2.3 the set $S_x = \{g \in G : d_{\Gamma_x}(g \cdot x, x) \leqslant 1\}$ is a symmetric compact open generating set for G, with respect to which $g \mapsto g \cdot x$ is trivially 1-Lipschitz. □

Lemma 4.3 *Suppose G is a locally compact group with Haar measure μ acting transitively on a locally finite graph Γ. Suppose further that this action is 1-Lipschitz with respect to some compact symmetric generating set S for G. Then for every finite subset $F \subseteq \Gamma$ there exists a compact open subset $A \subseteq G$ such that*

$$\frac{\mu(AS\backslash A)}{\mu(A)} \leqslant \frac{|\partial F|}{|F|}.$$

Proof Let $x \in \Gamma$ such that the orbit map $(G, S) \to \Gamma$, $g \mapsto g \cdot x$ is 1-Lipschitz. Let $\hat{S} = \{g \in G : d(x, g \cdot x) \leqslant 1\}$. Proposition 2.3 implies that $G_{x \to F}$ is a compact open set satisfying

$$\frac{\mu(G_{x\to F}\hat{S}\backslash G_{x\to F})}{\mu(G_{x\to F})} = \frac{|\partial F|}{|F|}.$$

Moreover, the fact that the orbit map is 1-Lipschitz implies that $d(x, gx) \leqslant |g|_S$ for all $g \in G$. Applying this to $g \in S$ implies that $S \subseteq \hat{S}$, hence that $G_{x\to F}S \subseteq G_{x\to F}\hat{S}$, and hence that $\mu(G_{x\to F}S\backslash G_{x\to F}) \leqslant \mu(G_{x\to F}\hat{S}\backslash G_{x\to F})$, so that we may take $A = G_{x\to F}$. □

Proof of Proposition 4.1 Upon adding edges between all pairs of vertices at distance at most r, we may assume that $r = 1$. By Lemma 4.2, the subgraph induced by each orbit is connected. Pick a compact symmetric generating set S, and a vertex z in each orbit such that the orbit map $g \to g \cdot z$ is 1-Lipschitz with respect to S, and write Z for the set of such z. For every $z \in Z$, denote by Y_z the graph induced by the orbit of z, and let $Y = \bigsqcup_{z\in Z} Y_z$. In other words, Y is obtained from Γ by removing all edges joining different orbits. Note that each Y_z is a vertex-transitive graph such that the orbit map $(G, S) \to Y_z$, $g \mapsto g \cdot z$ is 1-Lipschitz.

Let $\varepsilon > 0$. By amenability of Γ there exists $F \subseteq \Gamma$ be such that $|\partial_F|/|F| \leqslant \varepsilon$. Write $\partial^Y F$ for the external boundary of F in Y, noting that $\partial^Y F \subseteq \partial F$ and that $\partial^Y F = \bigsqcup_{z\in Z} \partial^Y F_z$, where $F_z = F \cap G\cdot z$. By the pigeonhole principle, there exists $z \in Z$, such that $|\partial_1^Y F_z|/|F_z| \leqslant |\partial^Y F|/|F| \leqslant |\partial F|/|F| \leqslant \varepsilon$. Applying Lemma 4.3 to the action of G on the vertex-transitive graph Y_z, we therefore conclude that there exists a compact open set $A \subseteq G$ such that $\mu(AS\backslash A)/\mu(A) \leqslant \varepsilon$. In particular, this implies that

$$\begin{aligned}\sup_{s\in S}\frac{\mu(As \vartriangle A)}{\mu(A)} &= \sup_{s\in S}\frac{\mu(As\backslash A) + \mu((As^{-1}\backslash A)s)}{\mu(A)} \\ &\leqslant \big(1 + \sup_{s\in S}\Delta_G(s)\big)\frac{\mu(AS\backslash A)}{\mu(A)} \leqslant \big(1 + \sup_{s\in S}\Delta_G(s)\big)\varepsilon,\end{aligned}$$

so that G is geometrically amenable by Lemma 3.2. □

5 The space of G-transitive graphs

In this section we prove Theorems 1.12 and 1.13.

Proof of Theorem 1.12 Let us start proving that $\text{Føl}^* G \leqslant \text{h}_G$. Assume the existence of a sequence Γ_n of proper G-transitive graphs, and of finite subsets A_n such that $|\partial A_n|/|A_n| \to \text{Føl}^* G$. Let o_n be some vertex in Γ_n, denote by K_n the stabilizer of o_n in G, and let $S_n = \{g \in G : d(g \cdot o_n, o_n) \leqslant 1\}$. By Proposition 2.3, S_n is a compact open generating subset of G, and we have that

$$\mu(G_{o_n\to A_n}S_n)/\mu(G_{o_n\to A_n}) = |[A_n]_1|/|A_n|.$$

Hence we deduce that $\text{Føl}^* G \leqslant \text{h}_G$.

Observe that compact subsets of the form $\{g \in G : d(g \cdot o, o) \leqslant 1\}$ for a proper G-transitive pointed graph (Γ, o) satisfy $S = KSK$ for some compact open subgroup K. The main point of the converse inequality is to show that in the definition of $\text{Føl}^* G$, we only need to take the infimum over such generating subsets. Precisely: given $\varepsilon > 0$, two compact subsets S and F such that $\mu(F) > 0$, we claim that there exists a compact open subgroup K such that $\mu(FKSK \backslash FS) \leqslant \varepsilon\mu(FS)$. Since the Haar measure is regular, one can find an open neighbourhood of the identity W of G such that $\mu(FSW \backslash FS) \leqslant \varepsilon\mu(FS)$. We now have to find a compact open subgroup K such that $FKSK \subseteq FSW$. Using that the multiplication is continuous, we see that there exist neighbourhoods S' and F' of S and F such that $F'S' \subseteq FSW$. Now because S and F are compact, such neighbourhoods can be taken to be of the form $S' = SU$ and $F' = FU$ for some neighbourhood U of the neutral element in G. But since G is totally disconnected, U contains some compact open subgroup K. So we finally have that $FKSK \subseteq FSW$, and so the claim follows.

We are now ready to prove that $\text{h}_G \leqslant \text{Føl}^* G$. Consider a sequence of compact symmetric generating subsets S_n and a sequence of compact subsets of positive measure F_n such that $\mu(F_n S_n \backslash F_n S_n)/\mu(F_n)$ tends to $\text{Føl}^* G$. By our claim, we deduce the existence of a sequence of compact open subgroups K_n such that $\mu(F_n K_n S_n K_n \backslash F_n S_n)/\mu(F_n)$ tends to zero. Hence, $\mu(F_n K_n S_n K_n \backslash F_n)/\mu(F_n)$ tends to $\text{Føl}^* G$. Letting $S_n' = K_n S_n K_n$, we deduce that

$$\liminf \mu(F_n K_n S_n' \backslash F_n K_n)/\mu(F_n K_n) \leqslant \text{Føl}^* G.$$

Consider now the Cayley–Abels graph Γ_n obtained as right quotient of (G_n, S_n') by K_n (see [5, Proposition 2.E.9.] for the definition of a Cayley–Abels graph, originally due to Abels [1]). Denote by π_n the projection modulo K_n. Let $A_n = \pi_n(F_n)$. Since $F_n K_n$ and $F_n K_n S_n' \backslash F_n K_n$ are unions of K_n left cosets, we have

$$\partial A_n = \pi(F_n K_n S_n') \backslash \pi(F_n K_n) = \pi(F_n K_n S_n' \backslash F_n K_n).$$

Now, because μ is left-invariant, we deduce that $|\partial A_n|\mu(K_n) = \mu(F_n K_n S_n' \backslash F_n K_n)$, and $|A_n|\mu(K_n) = \mu(F_n K_n)$. Hence $\liminf |\partial A_n|/|A_n| \leqslant \text{Føl}^* G$ as required. □

We now turn to the proof of Theorem 1.13. Recall that the isoperimetric profile of a graph Γ is defined via

$$j_\Gamma(n) = \inf_{|A| \leqslant n} \left\{ \frac{|\partial A|}{|A|} \right\}.$$

Proposition 5.1 *For every $n \in \mathbb{N}$, the map $\Gamma \mapsto j_\Gamma(n)$ is continuous on $\mathfrak{G}$.*

Proof Let A be a finite subset of a graph Γ, and assume that $A = A_1 \sqcup A_2$ are such that $d(A_1, A_2) \geqslant 3$. Then we have $\partial A = \partial A_1 \sqcup \partial A_2$. We therefore deduce that $|\partial A| = |\partial A_1| + |\partial A_2|$. Hence we deduce that

$$\frac{|\partial A|}{|A|} \leqslant \min \left\{ \frac{|\partial A_1|}{|A_1|}, \frac{|\partial A_2|}{|A_2|} \right\}.$$

Hence $j_\Gamma(n)$ is attained on subsets A that are 2-connected: meaning that every pair of vertices x, y can be joined by a chain of vertices $x = x_0, \ldots, x_k = y$ such that $d(x_i, x_{i+1}) \leqslant 2$. Since such sets are contained in a ball of radius $2n$, we have that $j_\Gamma(n) = j_{\Gamma'}(n)$ as soon as $d(\Gamma, \Gamma') \leqslant 2^{-2n-1}$, meaning that the balls of radius $n+1$ of these two graphs coincide. This proves the proposition. □

Proof of Theorem 1.13 Note that $h_\Gamma = \inf_n j_\Gamma(n)$, so $\Gamma \mapsto h_\Gamma$ is an infimum of continuous functions by Proposition 5.1, hence is upper semicontinuous. □

6 Føl* *G*

In this section we prove our various results about Føl* G, which recall we defined via

$$\text{Føl}^* G = \inf_S \inf_U \frac{\mu(US \backslash U)}{\mu(U)},$$

where the infima are over all compact symmetric generating sets S for G and all compact subsets $U \subseteq G$. We start by observing that one direction of Theorem 1.7 does not need the amenability assumption.

Proposition 6.1 *Suppose G is a non-unimodular locally compact group, and that $\Delta_G(G)$ is discrete. Then G is uniformly geometrically non-amenable, i.e.* Føl* $G > 0$.

Note that $\Delta_G(G)$ is either discrete or dense, so that this really does prove one direction of Theorem 1.7.

Proof of Proposition 6.1 The fact that $\Delta_G(G)$ is non-trivial and discrete implies that it is cyclic, so that we may fix a generator $t > 1$. Note, then, that every symmetric generating set S for G must contain an element s such that $\Delta_G(s) \geqslant t$, so that for every compact subset F of positive measure we have

$$\mu(FS \backslash F) \geqslant \mu(Fs \backslash F) \geqslant \mu(Fs) - \mu(F) = (t-1)\mu(F).$$

□

Example 6.2 Here is an example of a group with discrete image of the modular homomorphism: the affine group $\mathcal{A}(\mathbb{Q}_p) = \mathbb{Q}_p \rtimes \mathbb{Z}$ (for which $t = p$ in the proof of Proposition 6.1). On the other hand, modular homomorphism of the direct product $\mathcal{A}(\mathbb{Q}_p) \times \mathcal{A}(\mathbb{Q}_q)$ has dense image whenever p and q are not powers of a common integer. It turns out that this group is not uniformly geometrically non-amenable, as shown by the following proposition.

Proposition 6.3 *Suppose that G is an amenable, non-unimodular compactly generated totally disconnected locally compact group. Then* Føl* $G = 0$ *if and only if the image of the modular homomorphism is dense in $\mathbb{R}_+^*$. Moreover, if Δ_G is split and* Føl* $G = 0$, *then there exists a sequence $(S_n)_{n=1}^\infty$ of generating subsets of the form $S_n = K \sqcup T_n$, where K is a fixed compact subset of* $\ker \Delta_G$ *and $(T_n)_{n=1}^\infty$ is a sequence of finite subsets of bounded cardinality satisfying*

$$\frac{\mu(F_n S_n \backslash F_n)}{\mu(F_n)} \to 0$$

for some sequence $(F_n)_{n=1}^\infty$ of compact subsets of G.

We start with a lemma to help us construct the required generating sets S_n.

Lemma 6.4 *Suppose that G is a compactly generated locally compact group, that $1 \to N \to G \to Q \to 1$ is a short exact sequence of locally compact groups. Then given any compact generating set U for Q, and any relatively compact lift V of U in G, there exists a compact symmetric subset R of N such that $R \cup V$ generates G. Moreover, if the sequence is split then there exists a fixed compact symmetric subset $K \subseteq N$ such that given any compact generating set U for Q, the set $K \cup U$ generates G.*

Proof Write $\pi : G \to Q$ for the quotient homomorphism. Let S be a compact symmetric generating set for G, and fix $n \in \mathbb{N}$ so that $\pi(S) \subseteq U^n$. We claim that we may take R to be the (compact) closure of $(SV^{-n} \cup V^n S) \cap N$. Indeed, this set is symmetric by definition, and given s in S, there exists g in V^n such that $sg^{-1} \in N$, hence $sg^{-1} \in R$, and hence $s = sg^{-1}g \in RV^n$.

If the sequence is split then let $V = \pi(S)$, and let K be the (compact) closure of $(SV^{-1} \cup VS) \cap N$, noting that $K \cup V$ generates G by the previous paragraph. Then if U is a compact generating set for Q, we have $V \subseteq U^m$ for some $m \in \mathbb{N}$, so that $K \cup U$ generates G as required. □

Proof of Proposition 6.3 As noted above, $\Delta_G(G)$ is either discrete or dense in $\mathbb{R}^*_+$, so Proposition 6.1 implies that it is dense if $\text{Føl}^* G = 0$. It therefore remains to prove that $\text{Føl}^* G = 0$ assuming that $\Delta_G(G)$ is dense.

Being totally disconnected, G has a compact open subgroup [8, Theorem 7.7]. The image of this subgroup under Δ_G is a compact subrgoup of $\mathbb{R}$, and hence trivial, so Δ_G factors through a discrete, and hence finitely generated, quotient. Since $\mathbb{R}^*_+$ is abelian and torsion-free, Δ_G factors through a finitely generated torsion-free abelian quotient $G \to A \cong \mathbb{Z}^d$. Write $\pi : G \to A$ for the quotient homomorphism, and let $\Delta' : A \to \mathbb{R}^*_+$ be an injective homomorphism such that $\Delta_G = \Delta' \circ \pi$. If Δ_G is split, let K be the symmetric compact set given by Lemma 6.4.

We claim that $V \cap \Delta'(A)$ spans $\Delta'(A)$ for every neighbourhood V of $1 \in \mathbb{R}^*_+$. It is more convenient to work additively: consider $\delta' = \log \circ \Delta' : A \to (\mathbb{R}, +)$, and take $W = \log V$ of the form $W = (-t, t)$ for some $t > 0$. Let $a \in \delta'(A)$. By density of $\Delta'(A) = \Delta_G(G)$, there exists a sequence $0 = a_1, a_2, \ldots, a_m = a$ of elements of $\delta'(A)$ such that $|a_{i+1} - a_i| < t$ for each i, and hence $a_{i+1} - a_i \in W \cap \delta'(A)$ for each i, so that $\delta'(A) \cap W$ generates $\delta'(A)$ as claimed. In particular, for all such V we can find a basis $(x_1, \ldots, x_d)$ of $A \cong \mathbb{Z}^d$ whose image under Δ' lies in V.

Let $n \in \mathbb{N}$. Note that for any $t > 0$ close enough to 1, we have $t^{n+1} \leqslant \frac{1}{n}(\sum_{i=-n}^{n} t^i)$ (as for $t = 1$ we have strict inequality). We may therefore pick $V_n \subseteq (\frac{1}{2}, 2)$ small enough such that

$$v^{n+1} \leqslant \frac{1}{n}\left(\sum_{i=-n}^{n} v^i\right) \qquad \forall\, v \in V_n, \tag{6.1}$$

and fix some basis $(x_1, \ldots, x_d)$ for A whose image under Δ' lies in V_n. From now on, we identify A with $\mathbb{Z}^d$ via this basis.

Let $j : A \to G$ be a cross section of π satisfying $j(0) = 1$, chosen to be the natural embedding $A \hookrightarrow \ker \pi \rtimes A$ if Δ_G is split, and set $T_n = \{j(x_1)^{\pm 1}, \ldots, j(x_d)^{\pm 1}\}$. Lemma 6.4 then implies that there exists a compact subset $R_n \subseteq \ker \pi$ such that $S_n = R_n \cup T_n$ forms a symmetric compact generating subset of G. Moreover, if Δ_G is split then by definition of K we may take $S_n = K \sqcup T_n$ for each n, as required.

Let $K_n = j([-n, n]^d \cap \mathbb{Z}^d) \subseteq G$. Define $X_n = \bigcup_{k \in K_n} k R_n k^{-1}$, which is a compact subset of $\ker \pi$. Finally, for all $a \in K_n$, we have $\pi(a) = (a_1, \ldots, a_d)$ with $a_i \in [-n, n]$, and for each permutation $\sigma \in \mathfrak{S}(d)$ the elements a and $j(x_{\sigma(1)})^{a_{\sigma(1)}} \ldots j(x_{\sigma(d)})^{a_{\sigma(d)}}$ differ by an element of $\ker \pi$. Let $L_n \subseteq \ker \pi$ be the (finite) subset of all such elements and their inverses.

We now form a compact subset $M_n \subseteq \ker \pi$ by setting $M_n = L_n' \cup L_n \cup X_n$. Note that since $\ker \pi$ is open in G, the restriction of μ to $\ker \pi$ is a Haar measure on $\ker \pi$. Moreover, since $\ker \pi$ is the kernel of Δ_G, it is unimodular, and since it is a closed subgroup of G, it is also amenable. Hence $\ker\pi$ is geometrically amenable by Lemma 1.1, and so there exists a compact subset $Q_n \subseteq \ker\pi$ such that

$$\mu(Q_n M_n \backslash Q_n) \leqslant \tfrac{1}{n}\mu(Q_n).$$

Let $F_n = Q_n K_n$.

We claim that

$$\mu(F_n S_n \backslash F_n) \leqslant O_d(\tfrac{1}{n})\mu(F_n), \tag{6.2}$$

which will immediately finish the proof of the proposition. To see this, first note that $F_n = \bigsqcup_{k\in K_n} Qk$, and hence

$$\mu(F_n) = \left(\sum_{k\in K_n} \Delta_G(k)\right)\mu(Q_n). \tag{6.3}$$

Note also that by definition of X_n we have

$$F_n R_n = Q_n K_n R_n \subseteq Q_n X_n K_n \subseteq Q_n M_n K_n = \bigsqcup_{k\in K_n} Q_n M_n k,$$

hence

$$F_n R_n \backslash F_n \subseteq \bigsqcup_{k\in K_n} (Q_n M_n \backslash Q_n)k,$$

and hence

$$\mu(F_n R_n \backslash F_n) \leqslant \left(\sum_{k\in K_n} \Delta_G(k)\right)\mu(Q_n M_n \backslash Q_n).$$

It then follows from (6.3) and the definition of Q_n that

$$\mu(F_n R_n \backslash F_n) \leqslant \tfrac{1}{n}\mu(F_n). \tag{6.4}$$

We now turn to T_n. By symmetry in $x_1, \ldots, x_d$ and in x_d, x_d^{-1}, it is enough to show that

$$\mu(F_n j(x_d) \backslash F_n) \leqslant O(\tfrac{1}{n})\mu(F_n); \tag{6.5}$$

this will combine with (6.4) to prove (6.2), as claimed. To that end, let

$$K_n' = \{j(x_1)^{a_1} \ldots j(x_d)^{a_d} : (a_1, \ldots, a_d) \in [-n, n]^d \cap \mathbb{Z}^d\},$$

and let $F_n' = Q_n K_n'$. By definition of L_n', we have $F_n \subseteq Q_n L_n K_n'$ and $F_n' \subseteq Q_n L_n K_n$. We claim that

$$\mu(F_n \mathbin{\triangle} F_n') \leqslant O(\tfrac{1}{n})\mu(F_n). \tag{6.6}$$

To see this, first note that by definition of L_n' we have $F_n' \subseteq Q_n L_n K_n \subseteq Q_n M_n K_n$, and hence

$$F_n' \backslash F_n \subseteq Q_n M_n K_n \backslash Q_n K_n = \bigsqcup_{k\in K_n} (Q_n M_n \backslash Q_n)k.$$

In particular, this implies that

$$\mu(F_n'\backslash F_n) \leqslant \left(\sum_{k\in K_n} \Delta_G(k)\right) \mu(Q_n M_n \backslash Q_n),$$

so that by (6.3) and definition of Q_n we have $\mu(F_n'\backslash F_n) \leqslant \frac{1}{n}\mu(F_n)$, as required. On the other hand, we similarly have $F_n \subseteq Q_n L_n K_n'$, and a similar argument then shows that $\mu(F_n\backslash F_n') \leqslant \frac{1}{n}\mu(F_n') \leqslant \frac{2}{n}\mu(F_n)$, giving (6.6) as claimed. Since $\Delta'(x_d) < 2$, (6.6) in turn implies that

$$\mu(F_n j(x_d)\backslash F_n' j(x_d)) = \mu((F_n\backslash F_n')j(x_d)) = \Delta'(x_d)\mu((F_n\backslash F_n') < \tfrac{2}{n}\mu(F_n).$$

which then combines with (6.6) to show that in order to prove (6.5)—and hence the proposition—it is enough to prove that

$$\mu(F_n' j(x_d)\backslash F_n') \leqslant \tfrac{1}{n}\mu(F_n'). \tag{6.7}$$

Let $K_n'' = \{j(x_1)^{a_1} \dots j(x_{d-1})^{a_{d-1}} : (a_1, \dots, a_{d-1}) \in [-n, n]^{d-1} \cap \mathbb{Z}^{d-1}\}$. We have

$$F_n' = \bigsqcup_{i=-n}^{n} Q_n K_n'' j(x_d)^i,$$

so that

$$\mu(F_n') = \left(\sum_{i=-n}^{n} \Delta_G(j(x_d))^i\right) \mu(Q_n K_n'') = \left(\sum_{i=-n}^{n} \Delta'(x_d)^i\right) \mu(Q_n K_n'')$$

and

$$F_n' j(x_d)\backslash F_n' \subseteq Q_n K_n'' j(x_d)^{n+1},$$

and hence

$$\mu(F' j(x_d)\backslash F_n') \leqslant \Delta_G(j(x_d))^{n+1}\mu(Q_n K_n'') = \Delta'(x_d)^{n+1}\mu(Q_n K_n'').$$

Applying (6.1) with $v = \Delta'(x_d)$, we deduce that (6.7) holds as required. □

Finally, we prove the following slight refinement of Theorem 1.9.

Theorem 6.5 *Suppose G is a compactly generated totally disconnected locally compact group. Then*

- *G is geometrically amenable if and only if it admits a 1-Lipschitz proper action on a locally finite amenable graph; and*
- $\mathrm{Føl}^* G = 0$ *if and only if G admits a contingently 1-Lipschitz proper action on a locally finite amenable graph, if and only if there exists $r \in \mathbb{N}$ such that G admits a contingently r-Lipschitz proper action on a locally finite r-amenable graph for some $r \geqslant 1$.*

Proof If G is geometrically amenable then by Theorem 1.2 there exists $r \in \mathbb{N}$ such that G admits an r-Lipschitz proper action on a bounded-degree amenable graph Γ_0. Then G acts 1-Lipschitz properly on the bounded-degree amenable graph obtained from Γ_0 by adding edges between all pairs of vertices at distance at most r. The converse is given by Proposition 4.1.

Now suppose that $\mathrm{Føl}^* G = 0$. By Theorems 1.8 (which recall followed from Theorem 1.12), there exists a sequence Γ_n of G-vertex transitive proper locally finite graphs and a sequence of finite subsets $A_n \subset \Gamma_n$ such that $|\partial A_n|/|A_n| \to 0$. Consider the disjoint union $\Gamma = \bigsqcup \Gamma_n$: since the graphs induced by the orbits are connected, we deduce from Lemma 4.2 that the action is contingently 1-Lipschitz. Besides, Γ is obviously amenable, and the G-action on it is proper.

Conversely, assume that G admits a contingently r-Lipschitz proper action on a locally finite r-amenable graph for some $r \geqslant 1$. On adding edges between all pairs of points at distance at most r, we reduce to the case $r = 1$. Hence we can assume that G admits a 1-Lipschitz proper action on a locally finite amenable graph Γ. Removing edges from Γ does not change the fact that it is amenable, so we may assume that Γ is a disjoint union of vertex-transitive proper locally finite graphs Γ_n. Now if $F \subseteq \Gamma$, we have $F = \bigsqcup_n F_n$, where $F_n \subseteq \Gamma_n$, and $\partial F = \bigsqcup_n \partial F_n$. By the pigeonhole principle, there exists n such that $|\partial F_n|/|F_n| \leqslant |\partial F|/|F|$. We deduce that if (A_k) is a sequence of subsets of Γ such that $|\partial A_k|/|A_k| \to 0$, there exists $A'_k \subseteq A_k$ such that $|\partial A'_k|/|A'_k| \to 0$ and such that A'_k is contained in some Γ_{n_k} for each k. Hence the sequence Γ_{n_k} is asymptotically amenable, which implies that $\text{Føl}^* G = 0$ by Theorem 1.8. □

7 Sol

Recall that the group Sol is defined as the semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{Z}^2$ via the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Proposition 7.1 *Sol does not admit a continuous quasitransitive action on a locally finite non-amenable graph.*

Proof Assume by contradiction that such action exists on a graph X. Then since the image of Sol is cocompact in the automorphism group of X, its closure G is quasi-isometric to X. But because Sol is amenable, then so is G. Since X is non amenable though, G is not geometrically amenable, and so the only way this can happen is if G is non-unimodular. Hence we are reduced to proving that Sol does not admit any continuous morphism with dense image $\pi : Sol \to G$, where G is totally disconnected and non-unimodular.

We start observing that since Sol is metabelian, then so is G, we deduce that $A := \overline{[G, G]}$ is abelian. Recall that Sol is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where $\mathbb{Z}^2$ coincides with the derived subgroup. We therefore have that $\mathbb{Z}^2 \subset \overline{[G, G]}$. On the other hand, we have $\mathbb{Z} \cap A = \{1\}$. Indeed, since $\overline{[G, G]}$ is abelian, this would imply that the image of Sol is virtually abelian, which would be incompatible with the fact that G is non-unimodular. This implies that $Sol \cap A = \mathbb{Z}^2$, and therefore that $\mathbb{Z}^2$ is dense in A.

Moreover, the image of G in $G^{ab} := G/A$ coincides with the image of $\mathbb{Z}$, which is dense. We have two possibilities: either G^{ab} is compact, which again would be at odd with the fact that G is non-unimodular, or it is discrete and therefore isomorphic to $\mathbb{Z}$. Hence $G = A \rtimes \mathbb{Z}$. Now let K be an open compact subgroup of A. Since it is open and $\mathbb{Z}^2$ is dense in A, $H_K = \mathbb{Z}^2 \cap K$ is dense in K. Since G is not discrete, H_K must be non-trival. Let $u \in H_K \backslash \{1_G\}$. Since the conjugation by the generator t in $\mathbb{Z}$ is continuous, we have that $t^{-1}ut$ must also be contained in a compact subgroup of A. But since A is abelian, this implies that $t^{-1}ut$ and u are contained in a compact subgroup of A. But since these are linearly independent vectors in $\mathbb{Q}^2$, they generate a finite index subgroup of $\mathbb{Z}^2$. Hence since $\mathbb{Z}^2$ is dense in A, this would imply that A is compact, again not compatible with G being non-unimodular. So we are done. □

8 Edge boundaries

In this paper, we chose to focus on the exterior boundary. The important aspect of this choice is that this boundary is a set of vertices, not of edges. The main reason for this choice comes from the fact that it is well suited for Cayley graphs of locally compact compactly generated groups: indeed the measure of the exterior boundary is well-behaved despite the fact that the graph may have infinite degree. Moreover, as seen in Lemma 4.3, there is a nice connection between Følner sets in the group, and Følner sets in the graph on which G acts transitively.

Say that a graph Γ is *edge-amenable* if there exists a sequence of finite subset F_n such that $|\partial_e F_n|/|F_n|$ tends to zero, where the edge-boundary $\partial_e F_n$ is the set of edges joining a vertex of F_n to a vertex of its complement. Since amenability and edge-amenability are obviously equivalent for bounded degree graphs, this discussion is only relevant for the statements involving locally finite graphs with unbounded degree. Let us focus our discussion here on Theorem 1.9. The first statement remains true for the edge boundary: one implication is immediate by the previous discussion, and the other one follows from the simple observation that the size of the edge boundary is always larger than than that of the exterior boundary.

However, the second statement of Theorem 1.9 does not have an obvious analogue, motivating the following question.

Qustion 8.1 Characterise those locally compact groups G admitting a contingently 1-Lipschitz proper action on a locally finite edge-amenable graph.

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