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Bogomolov–Sommese type vanishing theorem for holomorphic vector bundles equipped with positive singular Hermitian metrics

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Abstract
In this article, we obtain the Bogomolov–Sommese type vanishing theorem involving multiplier ideal sheaves for big line bundles. We define a dual Nakano semi-positivity of singular Hermitian metrics with L^2-estimates and prove a vanishing theorem which is a generalization of the Bogomolov–Sommese type vanishing theorem to holomorphic vector bundles.

Keywords L^2-estimates · Singular Hermitian metrics · Cohomology vanishing · Nakano positivity

Mathematics Subject Classification 32L20 · 32L10 · 14F17 · 14F18

1 Introduction

Positivity notions for holomorphic vector bundles and multiplier ideal sheaves play an important role in several complex variables and complex algebraic geometry. For holomorphic vector bundles, singular Hermitian metrics and its positivity are very interesting subjects. On holomorphic line bundles, positivity of a singular Hermitian metric corresponds to plurisubharmonicity of the local weight and the multiplier ideal sheaf is an invariant of the singularities of the plurisubharmonic functions.

Let X be a complex manifold and φ be a plurisubharmonic function. Let $\mathscr{I}(\varphi)$ be the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable which is called the multiplier ideal sheaf. Let h be a singular Hermitian metric on a holomorphic line bundle L over X and φ be the local weight of h, i.e. $h = e^{-\varphi}$. Then we define the multiplier ideal sheaf for h by $\mathscr{I}(h) := \mathscr{I}(\varphi)$.

For a holomorphic line bundle L over a projective manifold X of $\dim X = n$, the famous Bogomolov–Sommese vanishing theorem [3] asserts that $H^0(X, \Omega^p_X \otimes L^*) = 0$ for $p < \kappa(L)$. In particular, if L is big then we have that

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$$H^n(X, \Omega_X^p \otimes L) = 0$$

for $p > 0$ by taking the dual. The Bogomolov–Sommese type vanishing theorem have been studied in the direction of weakening the positivity (cf. [20, 30]).

For big line bundles, we first obtain the following Bogomolov–Sommese type vanishing theorem which involves a multiplier ideal sheaf as in the Demailly–Nadel vanishing theorem (cf. [8, 21, 22]) and which is an extension of the Demailly–Nadel vanishing theorem to (p, n)-forms.

Theorem 1.1 *Let X be a projective manifold of dimension n equipped with a Kähler metric ω on X. Let L be a holomorphic line bundle on X equipped with a singular Hermitian metric h. We assume that*

$$i\Theta_{L,h} \geq \varepsilon\omega$$

in the sense of currents for some $\varepsilon > 0$. Then we have that

$$H^n(X, \Omega_X^p \otimes L \otimes \mathscr{I}(h)) = 0$$

for $p > 0$.

Theorem 1.1 is shown using the L^2-estimate theorem (see Theorem 2.1) for (p, n)-forms and a fine resolution of $\Omega_X^p \otimes L \otimes \mathscr{I}(h)$.

Notions of singular Hermitian metrics for holomorphic vector bundles were introduced and investigated (cf. [4, 5]). However, it is known that we cannot always define the curvature currents with measure coefficients (see [27]). Hence, Griffiths semi-negativity or semi-positivity ([4, 27], see Definition 4.3) and Nakano semi-negativity ([27], see Definition 4.4) is defined without using the curvature currents by using the properties of plurisubharmonic functions. Here, Griffiths semi-positivity can be returned to Griffiths semi-negativity by considering the duality, but this method cannot be used for Nakano semi-positivity because the dual of a Nakano negative bundle in general is not Nakano positive.

After that, Nakano semi-positivity for singular Hermitian metrics (see Definition 4.6) was defined in [18], who establishes the singular-type Nakano vanishing theorem, i.e. the Demailly–Nadel type vanishing theorem extended to holomorphic vector bundles. This definition is based on characterization of Nakano positivity using the so called "optimal L^2-estimate condition" for $(n, 1)$-forms by Deng–Ning–Wang–Zhou [13], and does not require the use of curvature currents. In [29], these characterizations of positivity using L^2-estimates for $(n, 1)$-forms are extended to (n, q) and (p, n)-forms.

Throughout this paper, we let X be an n-dimensional complex manifold and $E \to X$ be a holomorphic vector bundle of finite rank r. From the definition of Nakano semi-negativity ([27], see Definition 4.4), we naturally define dual Nakano semi-positive singular Hermitian metrics (see Definition 4.5) with characterization using L^2-estimates (see Proposition 4.10). Then, by using the method of the proof of Theorem 1.1, we obtain the following vanishing theorem which is a generalization of dual Nakano vanishing theorem to singular Hermitian metrics and of the Bogomolov–Sommese vanishing theorem to holomorphic vector bundles.

Theorem 1.2 *Let X be a projective manifold equipped with a Hodge metric ω_X on X. We assume that (E, h) is strictly dual Nakano δ_{ω_X}-positive in the sense of Definition 4.11 on X and* $\det h$ *is bounded on X. Then we have the following vanishing:*

$$H^n(X, \Omega_X^p \otimes E) = 0,$$

for $p > 0$.

We get the following result which is a generalization of the Griffiths vanishing theorem (cf. [10, Chapter VII, Corollary 9.4], [19]) to singular Hermitian metrics and which can also be considered as a generalization of the Demailly–Nadel vanishing theorem and Theorem 1.1 to holomorphic vector bundles. Here, the generalization up to (n, q)-forms for singular Hermitian metrics is already known in [18].

Theorem 1.3 *Let X be a projective manifold equipped with a Kähler metric ω_X on X. We assume that (E, h) is strictly Griffiths δ_{ω_X}-positive in the sense of Definition 4.7 on X. Then we have the following vanishing:*

$$H^q(X, K_X \otimes \mathscr{E}(h \otimes \det h)) = 0$$
$$H^n(X, \Omega_X^p \otimes \mathscr{E}(h \otimes \det h)) = 0,$$

for $p, q > 0$.

2 Proof of Theorem 1.1

In this section, we first prove Theorem 2.1 and then use it to show Theorem 1.1.

Theorem 2.1 *Let X be a projective manifold of dimension n and ω be a Kähler metric on X. Let L be a holomorphic line bundle equipped with a singular Hermitian metric h whose local weights are denoted $\varphi \in L^1_{loc}$, i.e. $h = e^{-\varphi}$. We assume that*

$$i\Theta_{L,h} = i\partial\overline{\partial}\varphi \geq \varepsilon\omega$$

in the sense of currents for some $\varepsilon > 0$. Then for any $f \in L^2_{p,n}(X, L, h, \omega)$ satisfying $\overline{\partial} f = 0$, there exists $u \in L^2_{p,n-1}(X, L, h, \omega)$ such that $\overline{\partial} u = f$ and

$$\int_X |u|^2_{h,\omega} dV_\omega \leq \frac{1}{p\varepsilon} \int_X |f|^2_{h,\omega} dV_\omega.$$

First, we consider Theorem 2.1 on a Stein manifold (= Proposition 2.6) and consider Lemma 2.2 to show this. Here, the claim of the type of Theorem 2.1 and Lemma 2.2 for (n, q)-forms rather than (p, n)-forms is already known (see [6, 8, 10, Chapter VIII]).

Let (X, ω) be a Hermitian manifold and (E, h) be a holomorphic Hermitian vector bundle over X. We denote the curvature operator $[i\Theta_{E,h}, \Lambda_\omega]$ on $\Lambda^{p,q} T_X^* \otimes E$ by $A^{p,q}_{E,h,\omega}$. And the fact that the curvature operator $[i\Theta_{E,h}, \Lambda_\omega]$ is positive (resp. semi-positive) definite on $\Lambda^{p,q} T_X^* \otimes E$ is simply written as $A^{p,q}_{E,h,\omega} > 0$ (resp. ≥ 0).

Lemma 2.2 *Let (E, h) be a holomorphic Hermitian vector bundle over X and ω, γ be Hermitian metrics on X such that $\gamma \geq \omega$. For any $u \in \Lambda^{p,n} T_X^* \otimes E$, $p \geq 1$, we have that $|u|^2_{h,\gamma} dV_\gamma \leq |u|^2_{h,\omega} dV_\omega$ and that if $A^{p,n}_{E,h,\omega} > 0$ (resp. ≥ 0) then*

$$A^{p,n}_{E,h,\gamma} > 0 \ (resp. \ \geq 0), \quad \langle (A^{p,n}_{E,h,\gamma})^{-1} u, u \rangle_{h,\gamma} dV_\gamma \leq \langle (A^{p,n}_{E,h,\omega})^{-1} u, u \rangle_{h,\omega} dV_\omega.$$

To show Lemma 2.2, we use the following symbolic definition and lemma which is the calculation results.

Definition 2.3 (cf. [29, Definition 2.1]) Let (M, g) be an oriented Riemannian $\mathcal{C}^\infty$-manifold with $\dim_{\mathbb{R}} M = m$ and $(\xi_1, \ldots, \xi_m)$ be an orthonormal basis of (T_M, g) at $x_0 \in M$. For any ordered multi-index I, we define $\varepsilon(s, I) \in \{-1, 0, 1\}$ as the number that satisfies $\xi_s \lrcorner \, \xi_I^* =$

$\varepsilon(s, I)\xi^*_{I\setminus s}$, where if $s \notin I$ then $\varepsilon(s, I) = 0$ and if $s \in I$ then $\varepsilon(s, I) \in \{-1, 1\}$. Here, the symbol $\bullet \lrcorner \bullet$ represents the interior product, i.e. $\xi_s \lrcorner \xi^*_I = \iota_{\xi_s}\xi^*_I$.

Let (X, ω) be a Hermitian manifold of $\dim_{\mathbb{C}} X = n$. If $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ is an orthonormal basis of (T_X, ω) at x_0 then we define $\varepsilon(s, I)$ in the same way as follows $\frac{\partial}{\partial z_s} \lrcorner\, dz_I = \varepsilon(s, I)dz_{I\setminus s}$. In particular, we have that $\frac{\partial}{\partial \overline{z}_s} \lrcorner\, d\overline{z}_I = \varepsilon(s, I)d\overline{z}_{I\setminus s}$.

Lemma 2.4 (cf. [29, Proposition 2.2]) *Let (X, ω) be a Hermitian manifold and (E, h) be a holomorphic Hermitian vector bundle over X. Let $x_0 \in X$ and $(z_1, \dots, z_n)$ be local coordinates such that $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ is an orthonormal basis of (T_X, ω) at x_0. Let $(e_1, \dots, e_r)$ be an orthonormal basis of E_{x_0}. We can write*

$$\omega_{x_0} = i \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j, \quad i\Theta_{E,h,x_0} = i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu.$$

Let J, K, L and M be ordered multi-indices with $|J| = |L| = p$ and $|K| = |M| = q$. For any (p, q)-form $u = \sum_{|J|=p,|K|=q,\lambda} u_{J,K,\lambda} dz_J \wedge d\overline{z}_K \otimes e_\lambda \in \Lambda^{p,q}T^*_{X,x_0} \otimes E_{x_0}$, we have the following calculation results:

$$\begin{aligned}\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle_\omega = &\left(\sum_{j\in J} + \sum_{j\in K} - \sum_{1\le j\le n}\right) c_{jj\lambda\mu} u_{J,K,\lambda}\overline{u}_{J,K,\mu} \\ &+ \sum_{j\ne k, K\setminus j = M\setminus k} c_{jk\lambda\mu} u_{J,K,\lambda}\overline{u}_{J,M,\mu}\varepsilon(j, K)\varepsilon(k, M) \\ &+ \sum_{j\ne k, L\setminus j = J\setminus k} c_{jk\lambda\mu} u_{L,K,\lambda}\overline{u}_{J,K,\mu}\varepsilon(k, J)\varepsilon(j, L).\end{aligned}$$

Proof of Lemma 2.2 For any $x_0 \in X$, after a linearly transformation, we may assume $\omega = i\sum_{j=1}^n dz_j \wedge d\overline{z}_j$ and $\gamma = i\sum_{j=1}^n \gamma_j^2 dz_j \wedge d\overline{z}_j$ at x_0 with $\gamma_j \ge 1$. Let $w_j = \gamma_j z_j$ for $j = 1, 2, \dots, n$ and $(e_1, \dots, e_r)$ be an orthonormal basis of E_{x_0}. Then we can write

$$\begin{aligned}\gamma &= i \sum_{1\le j\le n} dw_j \wedge d\overline{w}_j, \\ i\Theta_{E,h} &= i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu = i \sum_{j,k,\lambda,\mu} c'_{jk\lambda\mu} dw_j \wedge d\overline{w}_k \otimes e^*_\lambda \otimes e_\mu\end{aligned}$$

with $c'_{jk\lambda\mu} = c_{jk\lambda\mu}/\gamma_j\gamma_k$. For any ordered multi-indices J we denote $\gamma_J = \Pi_{j\in J}\gamma_j$. For any $u \in \Lambda^{p,n}T^*_{X,x_0} \otimes E_{x_0}$ we can write

$$u = \sum u_{J\lambda} dz_J \wedge d\overline{z}_N \otimes e_\lambda = \sum u'_{J\lambda} dw_J \wedge d\overline{w}_N \otimes e_\lambda$$

with $u'_{J\lambda} = u_{J\lambda}/\gamma_J\gamma_N$ where $N = \{1, \dots, n\}$.

Then we obtain that

$$\begin{aligned}|u|^2_{h,\gamma} &= \sum |u'_{J\lambda}|^2 = \sum \gamma_J^{-2}\gamma_N^{-2}|u_{J\lambda}|^2, \qquad dV_\gamma = \gamma_N^2 dV_\omega, \\ |u|^2_{h,\gamma} dV_\gamma &= \sum \gamma_J^{-2}|u_{J\lambda}|^2 dV_\omega \le |u|^2_{h,\omega} dV_\omega.\end{aligned}$$

From Lemma 2.4, we have that

$$
\begin{aligned}
\langle A^{p,n}_{E,h,\gamma}u,u\rangle_\gamma &= \sum_{j\in J} c'_{jj\lambda\mu}u'_{J\lambda}\overline{u}'_{J\mu} + \sum_{j\neq k, L\setminus j=J\setminus k} c'_{jk\lambda\mu}u'_{L\lambda}\overline{u}'_{J\mu}\varepsilon(k,J)\varepsilon(j,L)\\
&= \sum_{L\setminus j=J\setminus k} c'_{jk\lambda\mu}u'_{L\lambda}\overline{u}'_{J\mu}\varepsilon(k,J)\varepsilon(j,L)\\
&= \gamma_N^{-2}\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_j\gamma_k\gamma_L\gamma_J)\\
&= \gamma_N^{-2}\sum_I\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_j\gamma_k\gamma_I)^2 \quad (I:=L\setminus j=J\setminus k)\\
&= \gamma_N^{-2}\sum_I\gamma_I^2\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_j\gamma_k\gamma_I^2)^2\\
&= \gamma_N^{-2}\sum_I\gamma_I^2\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_L^2\gamma_J^2)\\
&\geq \gamma_N^{-2}\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_L^2\gamma_J^2)\\
&= \gamma_N^{2}\sum_{L\setminus j=J\setminus k} c_{jk\lambda\mu}u_{L\lambda}\overline{u}_{J\mu}\varepsilon(k,J)\varepsilon(j,L)/(\gamma_L^2\gamma_J^2\gamma_N^4)\\
&= \gamma_N^2\langle A^{p,n}_{E,h,\omega}S_\gamma u, S_\gamma u\rangle_\omega
\end{aligned}
$$

where S_γ is the operator defined by

$$S_\gamma u = \sum u_{J\lambda}\gamma_J^{-2}\gamma_N^{-2}dz_J\wedge d\overline{z}_N\otimes e_\lambda \in \Lambda^{p,n}T^*_{X,x_0}\otimes E_{x_0}.$$

Therefore we obtain that $A^{p,n}_{E,h,\omega}>0 \implies A^{p,n}_{E,h,\gamma}>0$.

Hence for any $u,v\in\Lambda^{p,n}T^*_{X,x_0}\otimes E_{x_0}$ we have that

$$
\begin{aligned}
|\langle u,v\rangle_\gamma|^2 = |\langle u,S_\gamma v\rangle_\omega|^2 &\leq \langle (A^{p,n}_{E,h,\omega})^{-1}u,u\rangle_\omega\langle A^{p,n}_{E,h,\omega}S_\gamma v,S_\gamma v\rangle_\omega\\
&\leq \gamma_N^{-2}\langle (A^{p,n}_{E,h,\omega})^{-1}u,u\rangle_\omega\langle A^{p,n}_{E,h,\gamma}v,v\rangle_\gamma,
\end{aligned}
$$

and the choice $v=(A^{p,n}_{E,h,\gamma})^{-1}u$ implies

$$\langle (A^{p,n}_{E,h,\gamma})^{-1}u,u\rangle_{h,\gamma}\gamma_N^2 \leq \langle (A^{p,n}_{E,h,\omega})^{-1}u,u\rangle_{h,\omega}.$$

From the above and $dV_\gamma=\gamma_N^2 dV_\omega$, this proof is completed. □

Lemma 2.5 *Let X be a complex manifold and (E,h) be a holomorphic Hermitian vector bundle over X. Let ω,γ be Hermitian metrics on X such that $\gamma\geq\omega$. For any $u\in\Lambda^{p,q}T^*_X\otimes E$, we have that $|u|^2_{h,\gamma}\leq|u|^2_{h,\omega}$.*

Proof Let notation be the same as one in the proof of Lemma 2.2.

Then for any $u\in\Lambda^{p,q}T^*_{X,x_0}\otimes E_{x_0}$, we can write

$$u=\sum_{J,K,\lambda}u_{JK\lambda}dz_J\wedge d\overline{z}_K\otimes e_\lambda = \sum_{J,K,\lambda}u'_{JK\lambda}dw_J\wedge d\overline{w}_K\otimes e_\lambda$$

with $u'_{JK\lambda}=u_{JK\lambda}/\gamma_J\gamma_K$. Hence, we have that

$$|u|^2_{h,\gamma}=\sum|u'_{JK\lambda}|^2=\sum|u_{JK\lambda}|^2/(\gamma_J\gamma_K)^2\leq\sum|u_{JK\lambda}|^2=|u|^2_{h,\omega}.$$

From the above, this proof is completed. □

Using Lemmas 2.2 and 2.5 , we obtain the following proposition.

Proposition 2.6 *Let S be a Stein manifold of dimension n and ω be a Kähler metric on S. Let φ be a strictly plurisubharmonic function on S. We assume that*

$$i\partial\overline{\partial}\varphi \geq \varepsilon\omega$$

in the sense of currents for some $\varepsilon > 0$. Then for any $f \in L^2_{p,n}(S, \varphi, \omega)$ satisfying $\overline{\partial} f = 0$, there exists $u \in L^2_{p,n-1}(S, \varphi, \omega)$ such that $\overline{\partial} u = f$ and

$$\int_S |u|^2_\omega e^{-\varphi} dV_\omega \leq \frac{1}{p\varepsilon} \int_S |f|^2_\omega e^{-\varphi} dV_\omega.$$

Proof We may assume that S is a submanifold of $\mathbb{C}^N$. By the theorem of Docquier and Grauert, there exists an open neighborhood $W \subset \mathbb{C}^N$ of S and a holomorphic retraction $\mu : W \to S$ (cf. Chapter V of [14]). Let $\rho : \mathbb{C}^N \to \mathbb{R}_{\geq 0}$ be a smooth function depending only on $|z|$ such that $\operatorname{supp} \rho \subset \mathbb{B}^N$ and that $\int_{\mathbb{C}^N} \rho(z) dV = 1$, where $\mathbb{B}^N$ is the unit ball. Define $\rho_\varepsilon(z) = (1/\varepsilon^{2n}) h(z/\varepsilon)$ for $\varepsilon > 0$. Let $S^\nu := \{z \in S \mid d_N(z, S^c) > 1/\nu\}$ be a subset of $S \subset \mathbb{C}^N$. For any plurisubharmonic function α on S we define the function $\alpha_\nu := (\alpha \circ \mu) * \rho_{1/\nu}$. Then α_ν is a smooth plurisubharmonic function on S^ν.

Let U be a open subset and Ω be a local Kähler potential of ω on U, i.e. Ω satisfies $i\partial\overline{\partial}\Omega = \omega$. By the assumption, we get $i\partial\overline{\partial}(\varphi - \varepsilon\Omega) = i\Theta_{L,h} - \varepsilon\omega \geq 0$ in the sense of currents. Then the function $(\varphi - \varepsilon\Omega)_\nu = \varphi_\nu - \varepsilon\Omega_\nu$ is a smooth plurisubharmonic function defined on U^ν. Since Ω_ν is strictly plurisubharmonic, φ_ν also is a smooth strictly plurisubharmonic function on S^ν and satisfies the following condition

$$i\partial\overline{\partial}\varphi_\nu \geq \varepsilon i\partial\overline{\partial}\Omega_\nu \geq \varepsilon_\nu \omega,$$

where $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ is a positive number sequence such that $0 < \varepsilon/2 < \varepsilon_\nu \nearrow \varepsilon, \ (\nu \to +\infty)$. Let $\varphi_\infty := \lim_{\nu\to+\infty} \varphi_\nu$ then φ_∞ is a plurisubharmonic function on S such that $\varphi_\infty = \varphi \ a.e.$ and a smooth functions sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ is decreasing to φ_∞.

By the Stein-ness of S, there exists a smooth exhaustive plurisubharmonic function ψ on S. We can assume that $\sup_S \psi = +\infty$. For any number $c < \sup_S \psi = +\infty$, we define the sublevel sets $S_c := \{z \in S \mid \psi(z) < c\}$ which is Stein. Fixed $j \in \mathbb{N}$. There exists $\nu_0 \in \mathbb{N}$ such that for any integer $\nu \geq \nu_0$, $S_j \subset\subset S^{\nu_0} \subset\subset S^\nu$. From Stein-ness of S_j, there exists a complete Kähler metric $\widehat{\omega}_j$ on S_j. Then we define the complete Kähler metric $\omega_\delta := \omega + \delta\widehat{\omega}_j > \omega$ on S_j for $\delta > 0$.

For any $\nu \geq \nu_0$ and any $v \in \Lambda^{p,n} T^*_{S_j}$, we obtain

$$\langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_\omega] v, v\rangle_\omega \geq \langle [\varepsilon_\nu\omega, \Lambda_\omega] v, v\rangle_\omega = p\varepsilon_\nu |v|^2 \ \text{ and } \ A^{p,n}_{e^{-\varphi_\nu},\omega} = [i\partial\overline{\partial}\varphi_\nu, \Lambda_\omega] > 0.$$

From this and Lemma 2.2, we have that $A^{p,n}_{e^{-\varphi_\nu},\omega_\delta} = [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega_\delta}] > 0$ and

$$\begin{aligned}
\int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega_\delta}]^{-1} f, f\rangle_{\omega_\delta} e^{-\varphi_\nu} dV_{\omega_\delta} &\leq \int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_\omega]^{-1} f, f\rangle_\omega e^{-\varphi_\nu} dV_\omega \\
&\leq \frac{1}{p\varepsilon_\nu} \int_{S_j} |f|^2_\omega e^{-\varphi_\nu} dV_\omega \\
&\leq \frac{1}{p\varepsilon_\nu} \int_{S_j} |f|^2_\omega e^{-\varphi} dV_\omega
\end{aligned}$$

$$\leq \frac{2}{p\varepsilon}\int_S |f|^2_\omega e^{-\varphi} dV_\omega < +\infty.$$

For any two Hermitian metrics γ_1, γ_2 and any locally integrable function $\Phi \in \mathcal{L}_{loc}$, we define the Hilbert space $L^2_{p,q}(S, \Phi, \gamma_1, \gamma_2)$ of (p, q)-forms g on S with measurable coefficients such that

$$\int_S |g|^2_{\gamma_1} e^{-\Phi} dV_{\gamma_2} < +\infty.$$

Here there exists a positive smooth function $\tilde{\gamma} \in \mathcal{E}(S, \mathbb{R}_{>0})$ such that $dV_{\gamma_2} = \tilde{\gamma} dV_{\gamma_1}$ then we have that $L^2_{p,q}(S, \Phi, \gamma_1, \gamma_2) = L^2_{p,q}(S, \Phi - \log\tilde{\gamma}, \gamma_1)$.

Thanks to Hörmander's L^2-estimate for smooth Hermitian metric with weight φ_ν and complete Kähler metric ω_δ, we get a solution $u_{j,\nu,\delta} \in L^2_{p,n-1}(S_j, \varphi_\nu, \omega_\delta) \subset L^2_{p,n-1}(S_j, \varphi_\nu, \omega_\delta, \omega)$ of $\overline{\partial} u_{j,\nu,\delta} = f$ on S_j such that

$$\begin{aligned}\int_{S_j} |u_{j,\nu,\delta}|^2_{\omega_\delta} e^{-\varphi_\nu} dV_\omega \leq \int_{S_j} |u_{j,\nu,\delta}|^2_{\omega_\delta} e^{-\varphi_\nu} dV_{\omega_\delta} &\leq \int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega_\delta}]^{-1} f, f\rangle_{\omega_\delta} e^{-\varphi_\nu} dV_{\omega_\delta} \\ &\leq \int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega}]^{-1} f, f\rangle_{\omega} e^{-\varphi_\nu} dV_{\omega} \\ &\leq \frac{2}{p\varepsilon}\int_S |f|^2_\omega e^{-\varphi} dV_\omega < +\infty.\end{aligned}$$

For fixed integer $\lambda_1 \geq 1$, $(u_{j,\nu,1/\lambda})_{\lambda_1 \leq \lambda \in \mathbb{N}}$ forms a bounded sequence in $L^2_{p,n-1}(S_j, \varphi_\nu, \omega_{1/\lambda_1}, \omega)$ due to the monotonicity of $|\bullet|^2_{\omega_{1/\lambda}}$, i.e. Lemma 2.5. Therefore we can obtain a weakly convergent subsequence in $L^2_{p,n-1}(S_j, \varphi_\nu, \omega_{1/\lambda_1}, \omega)$. By using a diagonal argument, we get a subsequence $(u_{j,\nu,\lambda_k})_{k\in\mathbb{N}}$ of $(u_{j,\nu,1/\lambda})_{\lambda\geq\lambda_1}$ converging weakly in $L^2_{p,n-1}(S_j, \varphi_\nu, \omega_{1/\lambda_1}, \omega)$ for any λ_1, where $u_{j,\nu,\lambda_k} \in L^2_{p,n-1}(S_j, \varphi_\nu, \omega_{1/\lambda_k}, \omega) \subset L^2_{p,n-1}(S_j, \varphi_\nu, \omega_{1/\lambda_1}, \omega)$. We denote by $u_{j,\nu}$ the weak limit of $(u_{j,\nu,\lambda_k})_{k\in\mathbb{N}}$. Then $u_{j,\nu}$ satisfies $\overline{\partial} u_{j,\nu} = f$ on S_j and

$$\int_{S_j} |u_{j,\nu}|^2_{\omega_{\lambda_k}} e^{-\varphi_\nu} dV_\omega \leq \int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega}]^{-1} f, f\rangle_{\omega} e^{-\varphi_\nu} dV_{\omega}$$

for each $k \in \mathbb{N}$. Taking weak limit $k \to +\infty$ and using the monotone convergence theorem, we have the following estimate

$$\begin{aligned}\int_{S_j} |u_{j,\nu}|^2_{\omega} e^{-\varphi_\nu} dV_\omega &\leq \int_{S_j} \langle [i\partial\overline{\partial}\varphi_\nu, \Lambda_{\omega}]^{-1} f, f\rangle_{\omega} e^{-\varphi_\nu} dV_{\omega} \\ &\leq \frac{1}{p\varepsilon_\nu}\int_{S_j} |f|^2_\omega e^{-\varphi} dV_\omega \leq \frac{2}{p\varepsilon}\int_S |f|^2_\omega e^{-\varphi} dV_\omega < +\infty,\end{aligned}$$

i.e. $u_{j,\nu} \in L^2_{p,n-1}(S_j, \varphi_\nu, \omega)$. For fixed $\nu_1 \geq \nu_0$, $(u_{j,\nu})_{\nu\geq\nu_1}$ forms a bounded sequence in $L^2_{p,n-1}(S_j, \varphi_{\nu_1}, \omega)$ due to the monotonicity of $(\varphi_\nu)_{\nu\in\mathbb{N}}$. Repeating the above argument and taking the weak limit $\nu \to +\infty$, we get a solution $u_j \in L^2_{p,n-1}(S_j, \varphi, \omega)$ of $\overline{\partial} u_j = f$ on S_j such that

$$p\varepsilon_\nu \int_{S_j} |u_j|^2_\omega e^{-\varphi_\nu} dV_\omega \leq \int_S |f|^2_\omega e^{-\varphi} dV_\omega,$$

for each $\nu \in \mathbb{N}$. Taking weak limit $\nu \to +\infty$ and using the monotone convergence theorem, we have the following estimate

$$p\varepsilon \int_{S_j} |u_j|^2_\omega e^{-\varphi} dV_\omega \leq \int_S |f|^2_\omega e^{-\varphi} dV_\omega.$$

Finally, repeating the above argument and taking the weak limit $j \to +\infty$, we get a solution $u \in L^2_{p,n-1}(S, \varphi, \omega)$ of $\overline{\partial} u = f$ on S such that

$$\int_S |u|^2_\omega e^{-\varphi} dV_\omega \leq \frac{1}{p\varepsilon} \int_S |f|^2_\omega e^{-\varphi} dV_\omega.$$

From the above, this proof is completed. □

Then, use Proposition 2.6 to prove Theorem 2.1.

Proof of Theorem 2.1 By Serre's GAGA, there exists a hypersurface $H \subset X$ such that $X \setminus H$ is Stein and L is trivial over $X \setminus H$. From Proposition 2.6, for any $\overline{\partial}$-closed $f \in L^2_{p,n}(X, L, h, \omega)$ there exists $u \in L^2_{p,n-1}(X \setminus H, -\log\det h, \omega) = L^2_{p,n-1}(X \setminus H, L, h, \omega)$ such that $\overline{\partial} u = f$ and

$$\begin{aligned}\int_{X\setminus H} |u|^2_{h,\omega} dV_\omega &\leq \frac{1}{p\varepsilon} \int_{X\setminus H} |f|^2_{h,\omega} dV_\omega \\ &\leq \frac{1}{p\varepsilon} \int_X |f|^2_{h,\omega} dV_\omega < +\infty.\end{aligned}$$

Letting $u = 0$ on H, we have that $u \in L^2_{p,n-1}(X, L, h, \omega)$, $\overline{\partial} u = f$ on X and

$$\int_X |u|^2_{h,\omega} dV_\omega \leq \frac{1}{p\varepsilon} \int_X |f|^2_{h,\omega} dV_\omega,$$

from the following lemma. □

Lemma 2.7 (cf. [2, Lemma 5.1.3]) *Let X be a complex manifold and H be a hypersurface in X. Let u and f be (possibly bundle valued) forms in L^2_{loc} of X satisfying $\overline{\partial} u = f$ on $X \setminus H$. Then the same equation holds on X (in the sense of distributions).*

Finally, we prove Theorem 1.1 using Theorem 2.1 and the following lemma and theorem.

Lemma 2.8 (Dolbeault–Grothendieck lemma, cf. [10, ChapterI]) *Let T be a current of type $(p, 0)$ on some open subset $U \subset \mathbb{C}^n$. If T is $\overline{\partial}$-closed then it is a holomorphic differential form, i.e. a smooth differential form with holomorphic coefficients.*

Theorem 2.9 (cf. [14, Theorem 4.4.2]) *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ be any plurisubharmonic function in Ω. For any $f \in L^2_{p,q+1}(\Omega, \varphi)$ with $\overline{\partial} f = 0$ there exists a solution $u \in L^2_{p,q}(\Omega, \mathrm{loc})$ of the equation $\overline{\partial} u = f$ such that*

$$\int_\Omega |u|^2 e^{-\varphi}(1 + |z|^2)^{-2} dV_\Omega \leq \int_\Omega |f|^2 e^{-\varphi} dV_\Omega.$$

Proof of Theorem 1.1 We define the subsheaf $\mathscr{L}^{p,q}_{L,h}$ of germs of (p, q)-forms u with values in L and with measurable coefficients such that both $|u|^2_h$ and $|\overline{\partial} u|^2_h$ are locally integrable. And we consider the following sheaves sequence:

$$0 \longrightarrow \ker \overline{\partial}_0 \hookrightarrow \mathscr{L}^{p,0}_{L,h} \xrightarrow{\overline{\partial}_0} \mathscr{L}^{p,1}_{L,h} \xrightarrow{\overline{\partial}_1} \cdots \xrightarrow{\overline{\partial}_{n-1}} \mathscr{L}^{p,n}_{L,h} \longrightarrow 0.$$

For any $x_0 \in X$, there exists a bounded Stein open neighborhood Ω of x_0 such that $L|_\Omega$ is trivial. Then $-\log h$ is strictly plurisubharmonic function on Ω and $L^2_{p,q}(\Omega, L, h, \omega) = L^2_{p,q}(\Omega, -\log h, \omega)$. From Theorem 2.9, for any $f \in L^2_{p,q}(\Omega, -\log h, \omega)$ with $\overline{\partial} f = 0$ there exists a solution $u \in L^2_{p,q-1}(\Omega, \mathrm{loc})$ of the equation $\overline{\partial} u = f$ such that

$$\begin{aligned}\inf_{z\in\Omega} \frac{1}{(1+|z|^2)^2} \int_\Omega |u|^2 e^{\log h} dV_\omega &\leq \int_\Omega |u|^2 e^{\log h}(1+|z|^2)^{-2} dV_\omega \\ &\leq \int_\Omega |f|^2 e^{\log h} dV_\omega < +\infty.\end{aligned}$$

By the boundedness of Ω, we get $0 < \inf_{z\in\Omega}(1+|z|^2)^{-2}$ and $u \in L^2_{p,q-1}(\Omega, -\log h, \omega) = L^2_{p,q-1}(\Omega, L, h, \omega)$. Then we have that the above sheaves sequence is exact.

From Lemma 2.8, the kernel of $\overline{\partial}_0$ consists of all germs of holomorphic $(p, 0)$-forms with values in L which satisfy the integrability condition and we have that $\ker \overline{\partial}_0 = \Omega^p_X \otimes L \otimes \mathscr{I}(h)$. In fact, for any locally open subset $U \subset \mathbb{C}^n$ we obtain

$$\begin{aligned}f \in \ker \overline{\partial}_0(U) \iff f = \sum f_I dz_I \in H^0(U, \Omega^p_X \otimes L) \ \text{ such that} \\ \int_U |f|^2_{h,\omega} dV_\omega = \int_U |f|^2 e^{\log h} dV_\omega = \sum \int_U |f_I|^2 e^{\log h} dV_\omega < +\infty.\end{aligned}$$

Therefore any $f_I \in H^0(U, \mathbb{C})$ satisfy the condition $f_I \in \mathscr{I}(h)(U)$.

From the acyclicity of each $\mathscr{L}^{p,q}_{L,h}$, we obtain that

$$H^q(X, \Omega^p_X \otimes L \otimes \mathscr{I}(h)) \cong H^q(\Gamma(X, \mathscr{L}^{p,\bullet}_{L,h})).$$

By Theorem 2.1, we conclude that $H^n(\Gamma(X, \mathscr{L}^{p,\bullet}_{L,h})) = 0$. □

From the Demailly–Nadel vanishing theorem and Theorem 1.1, we get the following results (= extension of the Demailly–Nadel vanishing theorem) immediately:

Let X be a projective manifold of dimension n equipped with a Kähler metric ω on X. Let L be a holomorphic line bundle on X equipped with a singular Hermitian metric h. We assume that

$$i\Theta_{L,h} \geq \varepsilon\omega$$

in the sense of currents for some $\varepsilon > 0$. Then we have that

$$\begin{aligned}H^p(X, K_X \otimes L \otimes \mathscr{I}(h)) = 0, \\ H^n(X, \Omega^p_X \otimes L \otimes \mathscr{I}(h)) = 0\end{aligned}$$

for $p > 0$.

Remark 2.10 The above extension of the Demailly–Nadel vanishing theorem cannot be extended to the same bidegree (p, q) with $p + q > n$ as the Kodaira–Akizuki–Nakano type vanishing theorem.

In fact, Ramanujam has given in the following counterexample to the extension of the Kodaira–Akizuki–Nakano type vanishing theorem to nef and big line bundles.

Counterexample. (cf. [10, ChapterVII], [26]) Let X be a blown up of one point in $\mathbb{P}^n$ and $\pi : X \to \mathbb{P}^n$ be the natural morphism. Clearly the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ is nef and big. Then we have the following non-vanishing cohomologies:

$$H^{p,p}(X, \pi^*\mathcal{O}_{\mathbb{P}^n}(1)) \neq 0 \quad \text{for} \quad 0 \leq p \leq n-1.$$

And, from the analytical characterization of nef and big line bundles (see [7, 9, Chapter 6]), there exist a singular Hermitian metric $h_{\pi^*O(1)}$ on $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ such that $\mathscr{I}(h_{\pi^*O(1)}) = \mathscr{O}_X$ and $i\Theta_{\pi^*\mathcal{O}_{\mathbb{P}^n}(1),h_{\pi^*O(1)}} \geq \varepsilon\omega$ in the sense of currents for some $\varepsilon > 0$, where ω is a Kähler metric on X. Then we get the following counterexample:

$$H^p(X, \Omega_X^p \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathscr{I}(h_{\pi^*O(1)})) \cong H^{p,p}(X, \pi^*\mathcal{O}_{\mathbb{P}^n}(1)) \neq 0 \quad \text{for} \quad 0 \leq p \leq n-1.$$

3 Smooth Hermitian metrics and dual Nakano positivity

Let (X, ω) be a complex manifold of complex dimension n equipped with a Hermitian metric ω on X and (E, h) be a holomorphic Hermitian vector bundle of rank r over X. Let $D = D' + \overline{\partial}$ be the Chern connection of (E, h), and $\Theta_{E,h} = [D', \overline{\partial}] = D'\overline{\partial} + \overline{\partial}D'$ be the Chern curvature tensor. Let $(U, (z_1, \ldots, z_n))$ be local coordinates. Denote by $(e_1, \ldots, e_r)$ an orthonormal frame of E over $U \subset X$, and

$$i\Theta_{E,h,x_0} = i\sum_{j,k} \Theta_{jk} dz_j \wedge d\overline{z}_k = i\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu, \quad \overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}.$$

To $i\Theta_{E,h}$ corresponds a natural Hermitian form $\theta_{E,h}$ on $T_X \otimes E$ defined by

$$\theta_{E,h}(u) := \theta_{E,h}(u, u) = \sum c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu}, \quad u = \sum u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_{X,x_0} \otimes E_x,$$

$$\text{i.e.} \quad \theta_{E,h} = \sum c_{jk\lambda\mu} (dz_j \otimes e_\lambda^*) \otimes \overline{(dz_k \otimes e_\mu^*)}.$$

Definition 3.1 Let X be a complex manifold and (E, h) be a holomorphic Hermitian vector bundle over X.

- (E, h) is said to be *Griffiths positive* (resp. *Griffiths semi-positive*) if for any $\xi \in T_{X,x}$, $\xi \neq 0$ and $s \in E_x$, $s \neq 0$, we have

$$\theta_{E,h}(\xi \otimes s, \xi \otimes s) > 0 \quad (\text{resp.} \geq 0).$$

  We write $(E, h) >_{Grif} 0$, i.e. $i\Theta_{E,h} >_{Grif} 0$ (resp. $\geq_{Grif} 0$) for Griffiths positivity (resp. semi-positivity).
- (E, h) is said to be *Nakano positive* (resp. *Nakano semi-positive*) if $\theta_{E,h}$ is positive (resp. semi-positive) definite as a Hermitian form on $T_X \otimes E$, i.e. for any $u \in T_X \otimes E$, $u \neq 0$, we have

$$\theta_{E,h}(u, u) > 0 \quad (\text{resp.} \geq 0).$$

  We write $(E, h) >_{Nak} 0$, i.e. $i\Theta_{E,h} >_{Nak} 0$ (resp. $\geq_{Nak} 0$) for Nakano positivity (resp. semi-positivity).

We introduce another notion about Nakano-type positivity.

Definition 3.2 (cf. [11, Section 1], [19, Definition 2.1]) Let X be a complex manifold of dimension n and (E, h) be a holomorphic Hermitian vector bundle of rank r over X. (E, h) is said to be *dual Nakano positive* (resp. *dual Nakano semi-positive*) if (E^*, h^*) is Nakano negative (resp. Nakano semi-negative).

From definitions, we see immediately that if (E, h) is Nakano positive or dual Nakano positive then (E, h) is Griffiths positive. And there is an example of dual Nakano positive as follows. Let h_{FS} be the Fubini-Study metric on $T_{\mathbb{P}^n}$, then $(T_{\mathbb{P}^n}, h_{FS})$ is dual Nakano positive and Nakano semi-positive (cf. [19, Corollary 7.3]). $(T_{\mathbb{P}^n}, h_{FS})$ is easyly shown to be ample, but it is not Nakano positive. In fact, if $(T_{\mathbb{P}^n}, h_{FS})$ is Nakano positive then from the Nakano vanishing theorem (see [23]), we have that

$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0.$$

However, this contradicts $H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = \mathbb{C}$.

Here, the following theorem is known, which expresses the relationship for the three positivity, i.e. Griffiths, Nakano and dual Nakano.

Theorem 3.3 (cf. [12, Theorem 1], [19, Theorem 7.2]) *Let h be a smooth Hermitian metric on E. If (E, h) is Griffiths semi-positive then $(E \otimes \det E, h \otimes \det h)$ is Nakano semi-positive and dual Nakano semi-positive.*

Let $\mathcal{E}^{p,q}(E)$ be the sheaf of germs of $\mathcal{C}^\infty$ sections of $\Lambda^{p,q}T^*_X \otimes E$ and $\mathcal{D}^{p,q}(E)$ be the space of $\mathcal{C}^\infty$ sections of $\Lambda^{p,q}T^*_X \otimes E$ with compact support on X.

Deng, Ning, Wang and Zhou introduced a positive notion of Hörmander type in [13], which is named as *the optimal L^p-estimate condition* and characterizes Nakano semi-positivity, i.e. $A^{n,1}_{E,h} \geq 0$, for holomorphic vector bundles (E, h). Then we introduced the following positive notion of Hörmander type in [29], which is an extension of the optimal L^2-estimate condition from $(n, 1)$-forms to (p, n)-forms and which characterizes the condition $A^{p,n}_{E,h} \geq 0$ (see Theorem 3.5).

Definition 3.4 (cf. [29, Definition 1.4]) Let (X, ω) be a Kähler manifold of dimension n which admits a positive holomorphic Hermitian line bundle and E be a holomorphic vector bundle over X equipped with a (singular) Hermitian metric h. (E, h) satisfies *the (p, n)-L^2_ω-estimate condition* on X, if for any positive holomorphic Hermitian line bundle (A, h_A) on X and for any $f \in \mathcal{D}^{p,n}(X, E \otimes A)$ with $\overline{\partial} f = 0$, there is $u \in L^2_{p,n-1}(X, E \otimes A)$ satisfying $\overline{\partial} u = f$ and

$$\int_X |u|^2_{h\otimes h_A,\omega} dV_\omega \leq \int_X \langle [i\Theta_{A,h_A} \otimes \mathrm{id}_E, \Lambda_\omega]^{-1} f, f\rangle_{h\otimes h_A,\omega} dV_\omega,$$

provided that the right hand side is finite.

And (E, h) satisfies *the (p, n)-L^2-estimate condition* on X if for any Kähler metric $\tilde{\omega}$, (E, h) satisfies *the (p, n)-$L^2_{\tilde{\omega}}$-estimate condition* on X

Theorem 3.5 (cf. [29, Theorem 1.6]) *Let (X, ω) be a Kähler manifold of dimension n which admits a positive holomorphic Hermitian line bundle and (E, h) be a holomorphic Hermitian vector bundle over X and p be a nonnegative integer. If (E, h) satisfies the (p, n)-L^2_ω-estimate condition on X then we have that $A^{p,n}_{E,h,\omega} \geq 0$.*

Here, as is well known, we know the following two facts about smooth Hermitian metrics h on E: Let (X, ω) be a Kähler manifold.

$$A^{n,1}_{E,h,\omega} \geq 0 \ (\text{resp.} \leq 0) \Longrightarrow A^{n,q}_{E,h,\omega} \geq 0 \ (\text{resp.} \leq 0) \ \text{ for all } q \geq 1 \quad (\text{see [6]}),$$

$$(E, h) \geq_{Nak} 0 \ (\text{resp.} \leq_{Nak} 0) \iff A^{n,1}_{E,h,\omega} \geq 0 \ (\text{resp.} \leq 0) \qquad (\text{see [12]}).$$

Therefore, from these two facts, Lemma 3.6 and the definition of Nakano semi-positivity, we obtain the following characterizations:

$$(a) \qquad (E, h) \textit{ is Nakano semi-positive } \iff A^{n,q}_{E,h,\omega} \geq 0 \textit{ for all } q \geq 1,$$

$$(b) \qquad (E, h) \textit{ is dual Nakano semi-positive } \iff A^{p,n}_{E,h,\omega} \geq 0 \textit{ for all } p \geq 1.$$

Lemma 3.6 (cf. [29, Theorem 2.3 and 2.5]) *Let (X,ω) be a Hermitian manifold and (E,h) be a holomorphic vector bundle over X. We have that*

$$A^{n-p,n-q}_{E^*,h^*,\omega}\geq 0 \ (resp. \leq 0) \iff A^{p,q}_{E,h,\omega}\geq 0 \ (resp. \leq 0) \iff A^{n-q,n-p}_{E,h,\omega}\leq 0 \ (resp. \geq 0).$$

By using the second condition (b), we show the following theorem which is already known as (n,q)-forms in the case of Nakano semi-positive.

Theorem 3.7 *Let $(X,\widehat{\omega})$ be a complete Kähler manifold, ω be another Kähler metric which is not necessarily complete and (E,h) be a dual Nakano semi-positive vector bundle. Then for any $\overline{\partial}$-closed $f\in L^2_{p,n}(X,E,h,\omega)$ there exists $u\in L^2_{p,n-1}(X,E,h,\omega)$ satisfies $\overline{\partial}u=f$ and*

$$\int_X |u|^2_{h,\omega}dV_\omega \leq \int_X \langle (A^{p,n}_{E,h,\omega})^{-1}f,f\rangle_{h,\omega}dV_\omega,$$

where we assume that the right-hand side is finite.

Furthermore, from the condition (b), Theorems 3.5 and 3.7, we obtain the following characterization of dual Nakano semi-positivity by using L^2-estimates.

Theorem 3.8 (cf. [29, Corollary 4.5]) *Let X be a complex manifold of dimension n which admits a complete Kähler metric and a positive holomorphic Hermitian line bundle. Let (E,h) be a holomorphic Hermitian vector bundle over X. Then (E,h) satisfies the (p,n)-L^2-estimate condition for all $p\geq 1$ if and only if (E,h) is dual Nakano semi-positive.*

Proof of Theorem 3.7 For any two Hermitian metrics γ_1,γ_2, we define the Hilbert space $L^2_{p,q}(X,E,h,\gamma_1,\gamma_2)$ of (p,q)-forms g on X with measurable coefficients such that

$$\int_X |g|^2_{h,\gamma_1}dV_{\gamma_2}<+\infty.$$

Here there exists a positive smooth function $\tilde{\gamma}\in\mathcal{E}(X,\mathbb{R}_{>0})$ such that $dV_{\gamma_2}=\tilde{\gamma}dV_{\gamma_1}$ then we have that $L^2_{p,q}(X,E,h,\gamma_1,\gamma_2)=L^2_{p,q}(X,E,\tilde{\gamma}h,\gamma_1)$.

For every $\varepsilon>0$, the Kähler metric $\omega_\varepsilon=\omega+\varepsilon\widehat{\omega}$ is complete. The idea of the proof is to apply the L^2-estimates to ω_ε and to let ε tend to zero. It follows from Lemma 2.2 and the equivalence condition of dual Nakano semi-positivity, i.e. $A^{p,n}_{E,h,\omega}\geq 0$ for $p\geq 1$ that

$$\langle (A^{p,n}_{E,h,\omega_\varepsilon})^{-1}g,g\rangle_{h,\omega_\varepsilon}dV_{\omega_\varepsilon}\leq \langle (A^{p,n}_{E,h,\omega})^{-1}g,g\rangle_{h,\omega}dV_\omega$$

for any $g\in\Lambda^{p,n}T^*_X\otimes E$. Thanks to Hörmander's L^2-estimate, we get the solution $u_\varepsilon\in L^2_{p,n-1}(X,E,h,\omega_\varepsilon)\subset L^2_{p,n-1}(X,E,h,\omega_\varepsilon,\omega)$ of $\overline{\partial}u_\varepsilon=f$ such that

$$\begin{aligned}\int_X |u|^2_{h,\omega_\varepsilon}dV_\omega &\leq \int_X |u|^2_{h,\omega_\varepsilon}dV_{\omega_\varepsilon}\leq \int_X \langle (A^{p,n}_{E,h,\omega_\varepsilon})^{-1}f,f\rangle_{h,\omega_\varepsilon}dV_{\omega_\varepsilon}\\ &\leq \int_X \langle (A^{p,n}_{E,h,\omega})^{-1}f,f\rangle_{h,\omega}dV_\omega,\end{aligned}$$

where $dV_\omega\leq dV_{\omega_\varepsilon}$.

For fixed integer $j_0\geq 1$, $(u_{1/j})_{j\in\mathbb{N}_{\geq j_0}}$ forms a bounded sequence in $L^2_{p,n-1}(X,E,h,\omega_{1/j_0},\omega)$ due to the monotonicity of $|\bullet|^2_{\omega_{1/j}}$, i.e. Lemma 2.5. Therefore we can obtain a weakly convergent subsequence in $L^2_{p,n-1}(X,E,h,\omega_{1/j_0},\omega)$. By using a diagonal argument, we get a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ of $(u_{1/j})_{j\in\mathbb{N}_{\geq j_0}}$ converging weakly in $L^2_{p,n-1}(X,E,h,\omega_{1/j_0},\omega)$ for any j_0, where $u_{j_k} \in L^2_{p,n-1}(X, E, h, \omega_{1/j_k}, \omega) \subset L^2_{p,n-1}(X, E, h, \omega_{1/j_0}, \omega)$. We denote by u the weak limit of $(u_{j_k})_{k\in\mathbb{N}}$. Then u satisfies $\overline{\partial} u = f$ and

$$\int_X |u|^2_{h,\omega_{1/j_k}} dV_\omega \le \int_X \langle (A^{p,n}_{E,h,\omega})^{-1} f, f\rangle_{h,\omega} dV_\omega,$$

for each $k \in \mathbb{N}$. Taking weak limit $k \to +\infty$ and using the monotone convergence theorem, we have the following estimate

$$\int_X |u|^2_{h,\omega} dV_\omega \le \int_X \langle (A^{p,n}_{E,h,\omega})^{-1} f, f\rangle_{h,\omega} dV_\omega < +\infty,$$

i.e. $u \in L^2_{p,n-1}(X, E, h, \omega)$. □

Finally, we get the following proposition by applying and modifying Theorem 3.8.

Proposition 3.9 *Let h be a smooth Hermitian metric on E. We consider the following conditions:*

(1) *h is dual Nakano semi-positive.*
(2) *For any Stein coordinate S such that $E|_S$ is trivial on S, any Kähler metric ω_S on S, any smooth strictly plurisubharmonic function ψ on S, any integer $p \in \{1, \dots, n\}$ and any $\overline{\partial}$-closed $f \in L^2_{p,n}(S, E, he^{-\psi}, \omega_S)$, there exists $u \in L^2_{p,n-1}(S, E, he^{-\psi}, \omega_S)$ satisfying $\overline{\partial} u = f$ and*

$$\int_S |u|^2_{h,\omega_S} e^{-\psi} dV_{\omega_S} \le \int_S \langle B^{-1}_{\psi,\omega_S} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S},$$

provided the right-hand side is finite, where $B_{\psi,\omega_S} = [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$.
(3) *(E, h) satisfies the (p, n)-L^2-estimate condition for all $p \ge 1$.*

Then two conditions (1) *and* (2) *are equivalent. If X admits a complete Kähler metric ω and a positive holomorphic line bundle on X, the above three conditions are equivalent.*

Proof First, we consider (1) $\Longrightarrow$ (2). We have that $i\Theta_{E,he^{-\psi}} = i\Theta_{E,h} + i\partial\overline{\partial}\psi \otimes \mathrm{id}_E$ is dual Nakano positive on S and

$$A^{p,n}_{E,he^{-\psi},\omega_S} = [i\Theta_{E,h}, \Lambda_{\omega_S}] + [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_{\omega_S}] = A^{p,n}_{E,h,\omega_S} + B_{\psi,\omega_S} \ge B_{\psi,\omega_S} > 0 \text{ on } S.$$

By Theorem 3.7, for any $p \ge 1$ and for any $\overline{\partial}$-closed $f \in L^2_{p,n}(S, E, he^{-\psi}, \omega_S)$ there exists $u \in L^2_{p,n-1}(S, E, he^{-\psi}, \omega_S)$ such that $\overline{\partial} u = f$ and

$$\begin{aligned}\int_S |u|^2_{h,\omega} e^{-\psi} dV_{\omega_S} &\le \int_S \langle (A^{p,n}_{E,he^{-\psi},\omega_S})^{-1} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S} \\ &\le \int_S \langle B^{-1}_{\psi,\omega_S} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S}.\end{aligned}$$

Next, we consider (2) $\Longrightarrow$ (1). From the condition (2), for any very small Stein coordinate S, (E, h) satisfies the (p, n)-$L^2_{\omega_S}$-estimate condition on S. By Theorem 3.8, we have that $A^{p,n}_{E,h,\omega_S} \ge 0$ which is equivalent to dual Nakano semi-positive on S. Since dual Nakano semi-positive is a local property, we get the condition (1).

Finally, we assume that X admits a complete Kähler metric ω and a positive holomorphic line bundle on X. From Theorems 3.7 and 3.8, we have that (3) $\Longleftrightarrow$ (1). □

4 Singular Hermitian metrics and characterization of dual Nakano positivity

In this section, we consider the case where a Hermitian metric of a holomorphic vector bundle has singularities. First, for holomorphic vector bundles, we introduce the definition of singular Hermitian metrics h and the multiplier submodule sheaf $\mathscr{E}(h)$ of $\mathscr{O}(E)$ with respect to h that is analogous to the multiplier ideal sheaf.

Definition 4.1 (cf. [4, Section 3], [25, Definition 2.2.1] and [27, Definition 1.1]) We say that h is a *singular Hermitian metric* on E if h is a measurable map from the base manifold X to the space of non-negative Hermitian forms on the fibers satisfying $0 < \det h < +\infty$ almost everywhere.

Definition 4.2 (cf. [5, Definition 2.3.1]) Let h be a singular Hermitian metric on E. We define the L^2-subsheaf $\mathscr{E}(h)$ of germs of local holomorphic sections of E by

$$\mathscr{E}(h)_x := \{s_x \in \mathscr{O}(E)_x \mid |s_x|_h^2 \text{ is locally integrable around } x\}.$$

Moreover, we introduce the definitions of positivity and negativity, such as Griffiths and Nakano, for singular Hermitian metrics.

Definition 4.3 (cf. [4, Definition 3.1], [25, Definition 2.2.2] and [27, Definition 1.2]) We say that a singular Hermitian metric h is

(1) *Griffiths semi-negative* if $|u|_h$ is plurisubharmonic for any local holomorphic section $u \in \mathscr{O}(E)$ of E.
(2) *Griffiths semi-positive* if the dual metric h^* on E^* is Griffiths semi-negative.

Let h be a smooth Hermitian metric on E and $u = (u_1, \dots, u_n)$ be an n-tuple of local holomorphic sections of E. We define T_u^h, an $(n-1, n-1)$-form through

$$T_u^h = \sum_{j,k=1}^{n} (u_j, u_k)_h \widehat{dz_j \wedge d\overline{z}_k}$$

where $(z_1, \dots, z_n)$ are local coordinates on X, and $\widehat{dz_j \wedge d\overline{z}_k}$ denotes the wedge product of all dz_i and $d\overline{z}_i$ expect dz_j and $d\overline{z}_k$, multiplied by a constant of absolute value 1, chosen so that T_u is a positive form. Then a short computation yields that (E, h) is Nakano semi-negative if and only if T_u^h is plurisubharmonic in the sense that $i\partial\overline{\partial}T_u^h \geq 0$ (see [1, 27]). In the case of $u_j = u_k = u$, (E, h) is Griffiths semi-negative.

From the above, we introduce the definition of Nakano semi-negativity for singular Hermitian metrics.

Definition 4.4 (cf. [27, Section 1]) We say that a singular Hermitian metric h on E is Nakano semi-negative if the $(n-1, n-1)$-form T_u^h is plurisubharmonic for any n-tuple of local holomorphic sections $u = (u_1, \dots, u_n)$.

Here, since the dual of a Nakano negative bundle in general is not Nakano positive, we cannot define Nakano semi-positivity for singular Hermitian metrics as in the case of Griffiths semi-positive, but we naturally define dual Nakano semi-positivity (see [28]) for singular Hermitian metrics as follows.

Definition 4.5 We say that a singular Hermitian metric h on E is dual Nakano semi-positive if the dual metric h^* on E^* is Nakano semi-negative.

For Nakano semi-positivity of singular Hermitian metrics, we already know one definition in [18], which is based on the optimal L^2-estimate condition in [13] and is equivalent to the usual definition for the smooth case by Deng–Ning–Wang–Zhou's characterization of Nakano semi-positivity in terms of optimal L^2-estimate condition in [13].

Definition 4.6 (cf. [18, Definition 1.1]) Assume that h is a Griffiths semi-positive singular Hermitian metric. We say that h is *Nakano semi-positive* if for any Stein coordinate S such that $E|_S$ is trivial, any Kähler metric ω_S on S, any smooth strictly plurisubharmonic function ψ on S, any positive integer $q \in \{1, \dots, n\}$ and any $\overline{\partial}$-closed $f \in L^2_{n,q}(S, E, he^{-\psi}, \omega_S)$ there exists $u \in L^2_{n,q-1}(S, E, he^{-\psi}, \omega_S)$ satisfying $\overline{\partial} u = f$ and

$$\int_S |u|^2_{h,\omega_S} e^{-\psi} dV_{\omega_S} \leq \int_S \langle B^{-1}_{\psi,\omega_S} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S},$$

where $B_{\psi,\omega_S} = [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$. Here we assume that the right-hand side is finite.

In [21, 22], Nadel proved that $\mathscr{I}(h)$ is coherent by using the Hörmander L^2-estimate. After that, as holomorphic vector bundles case, Hosono and Inayama proved that $\mathscr{E}(h)$ is coherent if h is Nakano semi-positive in the sense of singular as in Definition 4.6 in [15] and [18].

For singular Hermitian metrics, we cannot always define the curvature currents with measure coefficients (see [27]). However, the above Definition 4.6 can be defined by not using the curvature currents of a singular Hermitian metric directly. Therefore, by using these definitions, the following definition of strictly positivity for Griffiths and Nakano is known.

Definition 4.7 (cf. [16, Definition 2.6], [18, Definition 2.16]) Let (X, ω_X) be a Kähler manifold and h be a singular Hermitian metric on E.

- We say that h is *strictly Griffiths δ_{ω_X}-positive* if for any open subset U and any Kähler potential φ of ω_X on U, $he^{\delta\varphi}$ is Griffiths semi-positive on U.
- We say that h is *strictly Nakano δ_{ω_X}-positive* if for any open subset U and any Kähler potential φ of ω_X on U, $he^{\delta\varphi}$ is Nakano semi-positive on U in the sense of Definition 4.6.

This definition for Nakano gives the following L^2-estimate theorem and establishes the singular-type Nakano vanishing theorem (Theorem 4.9) by using this L^2-estimate theorem.

Theorem 4.8 (cf. [18, Theorem 1.4]) *Let (X, ω_X) be a projective manifold and a Hodge metric on X and q be a positive integer. We assume that (E, h) is strictly Nakano δ_{ω_X}-positive in the sense of Definition 4.7 on X. Then for any $\overline{\partial}$-closed $f \in L^2_{n,q}(X, E, h, \omega_X)$ there exists $u \in L^2_{n,q-1}(X, E, h, \omega_X)$ satisfies $\overline{\partial} u = f$ and*

$$\int_X |u|^2_{h,\omega_X} dV_{\omega_X} \leq \frac{1}{\delta q} \int_X |f|^2_{h,\omega_X} dV_{\omega_X}.$$

Theorem 4.9 (cf. [18, Theorem 1.5]) *Let (X, ω_X) be a projective manifold and a Hodge metric on X. We assume that (E, h) is strictly Nakano δ_{ω_X}-positive in the sense of Definition 4.7 on X. Then the q-th cohomology group of X with coefficients in the sheaf of germs of holomorphic sections of $K_X \otimes \mathscr{E}(h)$ vanishes for $q > 0$:*

$$H^q(X, K_X \otimes \mathscr{E}(h)) = 0,$$

where $\mathscr{E}(h)$ is the sheaf of germs of locally square integrable holomorphic sections of E with respect to h.

Here, from the above discussion we consider dual Nakano positivity using L^2-estimates and its relation to Definition 4.5. For convenience, we say that h is *L^2-type dual Nakano semi-positive* if h is a Griffiths semi-positive singular Hermitian metric and has the following L^2-estimates condition (= the singular case of condition (2) in Proposition 3.9):

For any Stein coordinate S such that $E|_S$ is trivial, any Kähler metric ω_S on S, any smooth strictly plurisubharmonic function ψ on S, any positive integer $p \in \{1, \dots, n\}$ and any $\overline{\partial}$-closed $f \in L^2_{p,n}(S, E, he^{-\psi}, \omega_S)$ there exists $u \in L^2_{p,n-1}(S, E, he^{-\psi}, \omega_S)$ satisfying $\overline{\partial} u = f$ and

$$\int_S |u|^2_{h,\omega_S} e^{-\psi} dV_{\omega_S} \leq \int_S \langle B^{-1}_{\psi,\omega_S} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S},$$

where $B_{\psi,\omega_S} = [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$. Here we assume that the right-hand side is finite.

Then we obtain the following proposition that the natural Definition 4.5 satisfies the above condition analogous to Definition 4.6.

Proposition 4.10 *Assume that a singular Hermitian metric h on E is dual Nakano semi-positive. Then h is L^2-type dual Nakano semi-positive.*

This proof is in the next section. The above discussion of Nakano semi-positivity and this proposition show the usefulness of Definition 4.5. From Proposition 3.9, this L^2-estimates condition can be considered a natural extension of dual Nakano semi-positivity to singular Hermitian metrics and coincides with the usual definition if h is smooth.

Using Definition 4.7 as a reference, we introduce a definition of strictly positivity for dual Nakano as follows.

Definition 4.11 Let (X, ω_X) be a Kähler manifold and h be a singular Hermitian metric on E. We say that h is *strictly dual Nakano δ_{ω_X}-positive* if for any open subset U and any Kähler potential φ of ω_X on U, $he^{\delta\varphi}$ is dual Nakano semi-positive on U.

By using this definition, we get the following L^2-estimate theorem which is an extension of Theorem 2.1 to holomorphic vector bundles.

Theorem 4.12 *Let (X, ω_X) be a projective manifold and a Hodge metric on X and p be a positive integer. We assume that (E, h) is strictly dual Nakano δ_{ω_X}-positive on X. Then for any $\overline{\partial}$-closed $f \in L^2_{p,n}(X, E, h, \omega_X)$ there exists $u \in L^2_{p,n-1}(X, E, h, \omega_X)$ satisfies $\overline{\partial} u = f$ and*

$$\int_X |u|^2_{h,\omega_X} dV_{\omega_X} \leq \frac{1}{\delta p} \int_X |f|^2_{h,\omega_X} dV_{\omega_X}.$$

Proof There exists an ample divisor D on X with respect to the Hodge class $\{\omega_X\}$. Then $S_D := X \setminus D$ is Stein and ω_X has a Kähler potential φ on S_D. And there exists a hypersurface H on S_D such that $S := S_D \setminus H$ is also Stein and that $E|_S$ is trivial by Stein-ness of S_D.

Then we have that

$$\langle B_{\delta\varphi,\omega_X} f, f\rangle_{h,\omega_X} = \delta p |f|^2_{h,\omega_X}, \quad \langle B^{-1}_{\delta\varphi,\omega_X} f, f\rangle_{h,\omega_X} = \frac{1}{\delta p} |f|^2_{h,\omega_X}.$$

From Definition 4.11 and Proposition 4.10, for any smooth strictly plurisubharmonic function ψ on S, there exists $u \in L^2_{p,n-1}(S, E, he^{\delta\varphi-\psi}, \omega_X)$ such that $\overline{\partial} u = f$ and

$$\int_S |u|^2_{h,\omega_X} e^{\delta\varphi-\psi} dV_{\omega_X} \leq \int_S \langle B^{-1}_{\delta\varphi,\omega_X} f, f\rangle_{h,\omega_X} e^{\delta\varphi-\psi} dV_{\omega_X}$$

if the right-hand side is finite. Taking $\psi = \delta\varphi$, we get a solution $u \in L^2_{p,n-1}(S, E, h, \omega_X)$ of $\overline{\partial} u = f$ such that

$$\int_S |u|^2_{h,\omega_X} dV_{\omega_X} \leq \int_S \langle B^{-1}_{\delta\varphi,\omega_X} f, f\rangle_{h,\omega_X} dV_{\omega_X} = \frac{1}{\delta p}\int_S |f|^2_{h,\omega_X} dV_{\omega_X}$$
$$\leq \frac{1}{\delta p}\int_X |f|^2_{h,\omega_X} dV_{\omega_X} < +\infty.$$

Letting $u = 0$ on $X \setminus S$, we have that $u \in L^2_{p,n-1}(X, E, h, \omega_X)$, $\overline{\partial} u = f$ on X and

$$\int_X |u|^2_{h,\omega_X} dV_{\omega_X} \leq \frac{1}{\delta p}\int_X |f|^2_{h,\omega_X} dV_{\omega_X},$$

from Lemma 2.7. □

5 Applications and proof of Proposition 4.10

In this section, as applications of Theorem 3.8, we introduce a property necessary for proofs of Proposition 4.10 and Theorem 6.1 that the (n, q) and (p, n)-L^2-estimate condition is preserved with respect to increasing sequences and we prove Proposition 4.10. This phenomenon is first mentioned in [17] as an extension of the properties seen in plurisubharmonic functions. After that, it is extended to the case of singular Nakano semi-positivity in [18], and then to the case of the (n, q) and (p, n)-L^2-estimate condition in [29].

Proposition 5.1 (cf. [18, Proposition 6.1]) *Let h be a singular Hermitian metric on E. Assume that there exists a sequence of smooth Nakano semi-positive metrics $(h_\nu)_{\nu\in\mathbb{N}}$ increasing to h pointwise. Then h is Nakano semi-positive in the sense of Definition 4.6, (i.e. L^2-type).*

Here, more general versions of Proposition 5.1 are obtained in [24].

Proposition 5.2 (cf. [29, Corollary 5.7]) *Let h be a singular Hermitian metric on E. Assume that there exists a sequence of smooth dual Nakano semi-positive metrics $(h_\nu)_{\nu\in\mathbb{N}}$ increasing to h pointwise. Then h is L^2-type dual Nakano semi-positive.*

By using this proposition, we prove Proposition 4.10.

Proof of Proposition 4.10 Let S be a Stein coordinate such that $E|_S$ is trivial. From Proposition 5.2, it is sufficient to show that there exists a sequence of smooth dual Nakano semi-positive metrics $(h_\nu)_{\nu\in\mathbb{N}}$ over any relatively compact subset of S increasing to h pointwise.

Here, h^* is Nakano semi-negative singular Hermitian metric on E^* over S. We define a sequence of smooth Hermitian metrics $(h^*_\nu)_{\nu\in\mathbb{N}}$ approximating h^* by a convolution of h^* with an approximate identity. In other words, let $h^*_\nu := h^* * \rho_\nu$ where ρ_ν is an approximate identity, i.e. $\rho \in \mathcal{D}(S)$ with $\rho \geq 0$, $\rho(z) = \rho(|z|)$, $\int \rho = 1$ and $\rho_\nu(z) = \nu^n \rho(\nu z)$. From Griffiths semi-negativity of h^*, each h^*_ν is Griffiths semi-negative and $(h^*_\nu)_{\nu\in\mathbb{N}}$ is decreasing to h^* pointwise (cf. [4, Proposition 3.1], [27, Proposition 1.3]).

Finally, we show that h^*_ν is Nakano semi-negative (cf. [27]). For any n-tuple of holomorphic sections $u = (u_1, \ldots, u_n)$ of E, we have locally that

$$(u_j, u_k)_{h^*_\nu}(z) = \int (u_j, u_k)_{h^*_w}(z)\rho_\nu(w)dV_w,$$

where $h_w^*(z) := h^*(z-w)$ and that

$$T_u^{h_\nu^*}(z) = \int T_u^{h_w^*}(z)\rho_\nu(w)dV_w.$$

By Nakano semi-negativity of h_w^*, for any test form $\phi \in \mathcal{D}(S)$ we have that

$$\begin{aligned} i\partial\overline{\partial}T_u^{h_\nu^*}(\phi) &= \int \phi i\partial\overline{\partial}T_u^{h_\nu^*} = \int T_u^{h_\nu^*} \wedge i\partial\overline{\partial}\phi \\ &= \int_z \Big\{\int_w T_u^{h_w^*}(z)\rho_\nu(w)dV_w\Big\} \wedge i\partial\overline{\partial}\phi \\ &= \int_w \Big\{\int_z T_u^{h_w^*}(z) \wedge i\partial\overline{\partial}\phi\Big\}\rho_\nu(w)dV_w \\ &= \int i\partial\overline{\partial}T_u^{h_w^*}(\phi)\rho_\nu(w)dV_w \geq 0, \end{aligned}$$

where $i\partial\overline{\partial}T_u^{h_w^*}(\phi) \geq 0$. Hence, $T_u^{h_\nu^*}$ is plurisubharmonic i.e. $h_\nu^* \leq_{Nak} 0$ and we let $h_\nu := (h_\nu^*)^*$ then $(h_\nu)_{\nu\in\mathbb{N}}$ is a sequence of smooth dual Nakano semi-positive metrics satisfying the necessary conditions. □

Here, for convenience, we also introduce the following notion for strictly dual Nakano positivity using L^2-estimates. Let (X, ω_X) be a Kähler manifold, we say that h is *L^2-type strictly dual Nakano δ_ω-positive* if for any open subset U and any Kähler potential φ of ω_X on U, $he^{\delta\varphi}$ is L^2-type dual Nakano semi-positive on U.

From Proposition 4.10, we immediately obtain the following two facts:

- Let ω_X be a Kähler metric on X and h be a singular Hermitian metric on E. If h is strictly dual Nakano δ_{ω_X}-positive then h is L^2-type strictly dual Nakano δ_{ω_X}-positive.
- Theorem 4.12 holds under the weaker assumption that h is L^2-type strictly dual Nakano δ_{ω_X}-positive from its proof.

Finally, using these two propositions, we obtain the following two theorems which is a generalization of Demailly–Skoda type theorem (see [12], [19, Theorem 3.3]). These theorems were shown in [18] up to the Nakano (semi-)positive case, and can be shown for the dual Nakano (semi-)positive case in almost the same way using Proposition 5.2.

Theorem 5.3 *Let h be a singular Hermitian metric on E. If h is Griffiths semi-positive then $(E \otimes \det E, h \otimes \det h)$ is Nakano semi-positive in the sense of Definition 4.6 (i.e. L^2-type) and L^2-type dual Nakano semi-positive.*

Theorem 5.4 *Let ω_X be a Kähler metric on X and h be a singular Hermitian metric on E. If h is strictly Griffiths δ_{ω_X}-positive then $(E \otimes \det E, h \otimes \det h)$ is strictly Nakano $(r+1)\delta_{\omega_X}$-positive in the sense of Definition 4.7 (i.e. L^2-type) and L^2-type strictly dual Nakano $(r+1)\delta_{\omega_X}$-positive.*

6 Proofs of Theorem 1.3 and 1.2

In this section, we get the proofs of Theorem 1.2 and 1.3. First, we prove the following theorem and corollary, which is an extension of Theorem 2.9 to holomorphic vector bundles, to show these theorems.

Theorem 6.1 *Let X be a complex manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that h is Griffiths semi-positive on X. Then for any $x_0 \in X$, there exist an open neighborhood U of x_0 and a Kähler metric ω on U satisfying that for any $\overline{\partial}$-closed $f \in L^2_{p,q}(U, E \otimes \det E, h \otimes \det h, \omega)$, there exists $u \in L^2_{p,q-1}(U, E \otimes \det E, h \otimes \det h, \omega)$ such that $\overline{\partial} u = f$.*

Proof For any $x_0 \in X$, there exist a bounded Stein neighborhood U of x_0 such that $E|_U$ and T_U are trivial and a sequence of smooth Griffiths positive metrics $(h_\nu)_{\nu\in\mathbb{N}}$ on U increasing to h pointwise (see [4, Proposition 3.1], [27]). Here, $(E \otimes \det E, h_\nu \otimes \det h_\nu)$ is Nakano positive and $(E \otimes \det E, h \otimes \det h)$ is Nakano semi-positive in the sense of Definition 4.6 by Proposition 5.1. We fix a bounded Kähler potential ψ of ω on U and define the trivial vector bundle $F := E \otimes \det E \otimes \Lambda^{n-p} T_U$ over U, where $\Lambda^{p,q} T^*_U \otimes E \otimes \det E \cong \Lambda^{n,q} T^*_U \otimes F$. Let $I_{T^p_U}$ be a trivial Hermitian metric and $h_{T^p_U} := I_{T^p_U} e^{-\psi}$ be a smooth Nakano positive metric on a trivial vector bundle $\Lambda^{n-p} T_U$.

Define a singular Hermitian metric $h_F := h \otimes \det h \otimes h_{T^p_U}$ on trivial vector bundle F over U. Then we have that (F, h_F) is strictly Nakano 1_ω-positive on U in the sense of Definition 4.7. This is enough to show that for any Kähler potential φ of ω, $h_F e^{\varphi}$ is Nakano semi-positive on U in the sense of Definition 4.6. Let $h_{F,\nu} := h_\nu \otimes \det h_\nu \otimes h_{T^p_U}$ be a smooth Hermitian metric on F then $h_{F,\nu} e^{\varphi}$ is Nakano semi-positive. In fact, from $h_{F,\nu} e^{\varphi} = h_\nu \otimes \det h_\nu \otimes I_{T^p_U} e^{-\psi+\varphi}$ and Nakano positivity of $h_\nu \otimes \det h_\nu$, we get

$$\begin{aligned} i\Theta_{F,h_{F,\nu}e^{\varphi}} &= i\Theta_{E\otimes\det E, h_\nu\otimes\det h_\nu} \otimes \mathrm{id}_{\Lambda^{n-p}T_U} + i\partial\overline{\partial}(\psi - \varphi) \otimes \mathrm{id}_F \\ &= i\Theta_{E\otimes\det E, h_\nu\otimes\det h_\nu} \otimes \mathrm{id}_{\Lambda^{n-p}T_U} \geq_{Nak} 0. \end{aligned}$$

Therefore $(h_{F,\nu} e^{\varphi})_{\nu\in\mathbb{N}}$ is a sequence of smooth Nakano semi-positive metric on U increasing to $h_F e^{\varphi}$. From Proposition 5.1, we have that $h_F e^{\varphi}$ is Nakano semi-positive on U in the sense of Definition 4.6.

For any (p, q)-form v with values in $E \otimes \det E$, the form v is considered (n, q)-form with values in F and we have that $|v|^2_{h_F,\omega} = |v|^2_{h\otimes\det h,\omega} e^{-\psi}$. In fact, we can write

$$\begin{aligned} \omega &= \sum dz_j \wedge d\overline{z}_j, \\ v &= \sum v_{IJ\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda \in \Lambda^{p,q} T^*_U \otimes E \otimes \det E \\ &= \sum v_{IJ\lambda} dz_N \wedge d\overline{z}_J \otimes \frac{\partial}{\partial z_{N\setminus I}} \otimes e_\lambda \in \Lambda^{n,q} T^*_U \otimes \Lambda^{n-p} T_U \otimes E \otimes \det E \\ &\cong \Lambda^{n,q} T^*_U \otimes F, \end{aligned}$$

at any fixed point, where $(e_j)_{1\leq j\leq r}$ is an orthonormal basis of $E \otimes \det E$. Then we get

$$\begin{aligned} |v|^2_{h_F,\omega} = |v|^2_{h\otimes\det h\otimes I_{T^p_U},\omega} e^{-\psi} &= \sum v_{IJ\lambda} \overline{v}_{KJ\mu} (h \otimes \det h)_{\lambda\mu} \delta_{N\setminus I, N\setminus K} e^{-\psi} \\ &= \sum v_{IJ\lambda} \overline{v}_{IJ\mu} (h \otimes \det h)_{\lambda\mu} e^{-\psi} = |v|^2_{h\otimes\det h,\omega} e^{-\psi}, \end{aligned}$$

where $\delta_{IK} = \Pi_j \delta_{i_j k_j}$ is multi-Kronecker's delta.

By using the boundedness of ψ on U, for any $\overline{\partial}$-closed $f \in L^2_{p,q}(U, E \otimes \det E, h \otimes \det h, \omega)$, we have that

$$\int_U |f|^2_{h_F,\omega} dV_\omega = \int_U |f|^2_{h\otimes\det h,\omega} e^{-\psi} dV_\omega \leq \sup_U e^{-\psi} \int_U |f|^2_{h\otimes\det h,\omega} dV_\omega < +\infty,$$

i.e. $f \in L^2_{n,q}(U, F, h_F, \omega)$, where $\sup_U e^{-\psi} < +\infty$. Therefore, from strictly Nakano 1_ω-positivity of (F, h_F) in the sense of Definition 4.7 and Theorem 4.8, there exists $u \in L^2_{n,q-1}(U, F, h_F, \omega)$ such that $\overline{\partial} u = f$ and

$$\int_U |u|^2_{h_F,\omega} dV_\omega \leq \frac{1}{q} \int_U |f|^2_{h_F,\omega} dV_\omega < +\infty.$$

Repeating the above argument, we have that

$$+\infty > \int_U |u|^2_{h_F,\omega} dV_\omega = \int_U |u|^2_{h\otimes\det h,\omega} e^{-\psi} dV_\omega \geq \inf_U e^{-\psi} \int_U |u|^2_{h\otimes\det h,\omega} dV_\omega,$$

i.e. $u \in L^2_{p,q-1}(U, E \otimes \det E, h \otimes \det h, \omega)$ and the following L^2-estimate

$$\int_U |u|^2_{h\otimes\det h,\omega} dV_\omega \leq \frac{1}{q} \frac{\sup_U e^{-\psi}}{\inf_U e^{-\psi}} \int_U |f|^2_{h\otimes\det h,\omega} dV_\omega,$$

where the right-hand side is finite. □

Corollary 6.2 *Let X be a complex manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that h is Griffiths semi-positive on X and that $\det h$ is bounded on X. Then for any $x_0 \in X$, there exist an open neighborhood U of x_0 and a Kähler metric ω on U satisfying that for any $\overline{\partial}$-closed $f \in L^2_{p,q}(U, E, h, \omega)$, there exists $u \in L^2_{p,q-1}(U, E, h, \omega)$ such that $\overline{\partial} u = f$.*

Proof For any x_0, there exists a bounded Stein neighborhood U of x_0 such that $E|_U$ is trivial then $E \otimes \det E|_U \cong E|_U$. By the boundedness of $\det h$ on U, for any $\overline{\partial}$-closed $f \in L^2_{p,q}(U, E, h, \omega)$ we have that

$$\int_U |f|^2_{h\otimes\det h,\omega} dV_\omega = \int_U |f|^2_{h,\omega} \det h \, dV_\omega \leq \sup_U \det h \int_U |f|^2_{h,\omega} dV_\omega < +\infty,$$

i.e. $f \in L^2_{p,q}(U, E \otimes \det E, h \otimes \det h, \omega)$.

From Theorem 6.1, there exists $u \in L^2_{p,q-1}(U, E \otimes \det E, h \otimes \det h, \omega)$ satisfies $\overline{\partial} u = f$. Then we have that $\inf_U \det h > 0$ and that

$$\inf_U \det h \int_U |u|^2_{h,\omega} dV_\omega \leq \int_U |u|^2_{h,\omega} \det h \, dV_\omega = \int_U |u|^2_{h\otimes\det h,\omega} dV_\omega < +\infty,$$

i.e. $u \in L^2_{p,q-1}(U, E, h, \omega)$. □

Finally, we prove Theorems 1.2 and 1.3 by using the above result and the following proposition. For singular Hermitian metrics h on E, we define the subsheaf $\mathscr{L}^{p,q}_{E,h}$ of germs of (p, q)-forms u with values in E and with measurable coefficients such that both $|u|^2_h$ and $|\overline{\partial} u|^2_h$ are locally integrable.

Proof of Theorem 1.3 There exist a Hodge metric γ_X and a positive number $c > 0$ such that $\omega_X \geq c\gamma_X$ on X. Since Griffiths positivity is local property, h is strictly Griffiths $c\delta_{\gamma_X}$-positive. For any open subset U and any Kähler potential ψ of γ_X on U, we prove that $he^{c\delta\psi}$ is Griffiths semi-positive on U. It is known that this is equivalent to $\log |u|^2_{h^* e^{-c\delta\psi}}$ being plurisubharmonic for any local holomorphic section $u \in \mathscr{O}(E^*)$ (see [27]). Here, for any $x_0 \in U$ and any ball neighborhood B of x_0 such that $B \subset U$, there is a Kähler potential

φ of ω_X on B. Then $\varphi - c\psi$ is plurisubharmonic on B. By the assumption, we have that $\log |u|^2_{h^* e^{-\delta\varphi}}$ is plurisubharmonic for any $u \in \mathscr{O}(E^*)_{x_0}$. From the inequality

$$i\partial\overline{\partial}\log |u|^2_{h^* e^{-c\delta\psi}} = i\partial\overline{\partial}\log |u|^2_{h^* e^{-\delta\varphi+\delta(\varphi-c\psi)}} = i\partial\overline{\partial}\log |u|^2_{h^* e^{-\delta\varphi}} + \delta i\partial\overline{\partial}(\varphi - c\psi) \geq 0,$$

the function $\log |u|^2_{h^* e^{-c\delta\psi}}$ is plurisubharmonic.

By Theorem 4.8 and 5.4, we obtain the cohomology vanishing for (n, q)-forms, i.e. $H^q(X, K_X \otimes \mathscr{E}(h \otimes \det h)) = 0$ for $q > 0$, already known (see [18, Theorem 1.6]).

We consider the following sheaves sequence:

$$0 \longrightarrow \ker \overline{\partial}_0 \hookrightarrow \mathscr{L}^{p,0}_{E\otimes\det E, h\otimes\det h} \xrightarrow{\overline{\partial}_0} \mathscr{L}^{p,1}_{E\otimes\det E, h\otimes\det h} \xrightarrow{\overline{\partial}_1} \cdots \xrightarrow{\overline{\partial}_{n-1}} \mathscr{L}^{p,n}_{E\otimes\det E, h\otimes\det h} \longrightarrow 0.$$

By Theorem 6.1, we have that the above sheaves sequence is exact.

From Lemma 2.8, the kernel of $\overline{\partial}_0$ consists of all germs of holomorphic $(p, 0)$-forms with values in $E \otimes \det E$ which satisfy the integrability condition and we have that $\ker \overline{\partial}_0 = \Omega^p_X \otimes \mathscr{E}(h \otimes \det h)$. In fact, for any locally open subset $U \subset \mathbb{C}^n$ we obtain

$$f \in \ker \overline{\partial}_0(U) \iff f = \sum f_I dz_I \in H^0(U, \Omega^p_X \otimes E \otimes \det E) \ \text{ such that}$$
$$\int_U |f|^2_{h\otimes\det h,\omega} dV_\omega = \sum_I \int_U |f_I|^2_{h\otimes\det h} dV_\omega < +\infty.$$

Therefore any $f_I \in H^0(U, E \otimes \det E)$ satisfy the condition $f_I \in \mathscr{E}(h \otimes \det h)(U)$.

From the acyclicity of each $\mathscr{L}^{p,q}_{E\otimes\det E, h\otimes\det h}$, we have that

$$H^q(X, \Omega^p_X \otimes \mathscr{E}(h \otimes \det h)) \cong H^q(\Gamma(X, \mathscr{L}^{p,\bullet}_{E\otimes\det E, h\otimes\det h})).$$

By Theorem 4.12 and 5.4, we get $H^n(\Gamma(X, \mathscr{L}^{p,\bullet}_{E\otimes\det E, h\otimes\det h})) = 0$. □

Proof of Theorem 1.2 We consider the following sheaves sequence:

$$0 \longrightarrow \ker \overline{\partial}_0 \hookrightarrow \mathscr{L}^{p,0}_{E,h} \xrightarrow{\overline{\partial}_0} \mathscr{L}^{p,1}_{E,h} \xrightarrow{\overline{\partial}_1} \cdots \xrightarrow{\overline{\partial}_{n-1}} \mathscr{L}^{p,n}_{E,h} \longrightarrow 0.$$

By Corollary 6.2, we have that the above sheaves sequence is exact.

Locally, we see $h = \det h \cdot \widehat{h^*}$ where $\widehat{h^*}$ is the adjugate matrix of h^*. From Griffiths semi-negativity of h^*, each element of $\widehat{h^*}$ is locally bounded [25, Lemma 2.2.4]. From the assumption $\det h$ is bounded, we get $\mathscr{E}(h) = \mathscr{O}(E)$.

Repeating the above argument, we have that $\ker \overline{\partial}_0 = \Omega^p_X \otimes \mathscr{E}(h) = \Omega^p_X \otimes \mathscr{O}(E)$. From the acyclicity of each $\mathscr{L}^{p,q}_{E,h}$, we have that $H^q(X, \Omega^p_X \otimes E) \cong H^q(\Gamma(X, \mathscr{L}^{p,\bullet}_{E,h}))$. By Theorem 4.12, we have that $H^n(\Gamma(X, \mathscr{L}^{p,\bullet}_{E,h})) = 0$. □

Remark 6.3 Since Theorem 4.12 holds, Theorem 1.2 also holds under the weaker assumption that h is L^2-type strictly dual Nakano δ_{ω_X}-positive.

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