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Tensor triangular geometry of filtered objects and sheaves

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Abstract

We compute the Balmer spectra of compact objects of tensor triangulated categories whose objects are filtered or graded objects of (or sheaves valued in) another tensor triangulated category. Notable examples include the filtered derived category of a scheme as well as the homotopy category of filtered spectra. We use an ∞-categorical method to properly formulate and deal with the problem. Our computations are based on a point-free approach, so that distributive lattices and semilattices are used as key tools. In the Appendix, we prove that the ∞-topos of hypercomplete sheaves on an ∞-site is recovered from a basis, which may be of independent interest.

Contents

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1 Introduction

In the subject called tensor triangular geometry, a basic object to study is a tt-category, which is a triangulated category equipped with a compatible symmetric monoidal structure. Balmer introduced a way to associate to each tt-category $\mathcal{T}^{\otimes}$ a topological space $\mathrm{Spc}(\mathcal{T})$, which we call the Balmer spectrum. We refer the reader to [3] for a survey on tensor triangular geometry.

In this paper, we compute the Balmer spectra of mainly two families of tt-categories, whose objects are diagrams in another tt-category. Although those two computations are logically independent, many techniques used in them are similar.

To state the main results, we introduce some terminology. See Remark 3.2 for the comparison with the common setting of tensor triangular geometry.

Definition 1.1 A *big tt-∞-category* is a compactly generated stable ∞-category equipped with an $\mathbb{E}_2$-monoidal structure whose tensor product preserves (small) colimits separately in each variable and restrict to compact objects.

We here state only a main consequence of the first computation because it requires some notions to state it in full generality.

Theorem I *Suppose that $\mathcal{C}^{\otimes}$ is a big tt-∞-category.*

(1) *For a nonzero Archimedean group A (for example, $\mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{R}$, equipped with their usual orderings), there is a canonical homeomorphism*

$$\mathrm{Spc}(\mathrm{Fun}(A, \mathcal{C})^{\omega}) \simeq S \times \mathrm{Spc}(\mathcal{C}^{\omega}),$$

where S denotes the Sierpiński space (that is, the Zariski spectrum of a discrete valuation ring); see Fig. 1.

(2) *For an abelian group A, considered as a discrete symmetric monoidal poset, there is a canonical homeomorphism*

$$\mathrm{Spc}(\mathrm{Fun}(A, \mathcal{C})^{\omega}) \simeq \mathrm{Spc}(\mathcal{C}^{\omega}).$$

In each statement, we consider the Day convolution $\mathbb{E}_2$-monoidal structure on the ∞-category $\mathrm{Fun}(A, \mathcal{C})$.

Specializing to the case $A = \mathbf{Z}$, this theorem has several consequences:

Example 1.2 Let X be a quasicompact quasiseparated scheme. The ∞-category $\mathrm{QCoh}(X)$, whose homotopy category is the derived category of discrete quasicoherent sheaves, can be equipped with a symmetric monoidal structure by considering the (derived) tensor product.

In some situations, it is useful to consider also the ∞-category $\mathrm{Fun}(\mathbf{Z}, \mathrm{QCoh}(X))$, whose compact objects are perfect filtered complexes on X, which was first introduced and studied by Illusie in [7, Section V.3]. We know that $\mathrm{QCoh}(X)^{\otimes}$ is a tt-∞-category and that $\mathrm{Spc}(\mathrm{QCoh}(X)^{\omega})$ is the underlying topological space of X according to the reconstruction theorem (see [3, Theorem 54]). We can apply (1) of Theorem I to see that the Balmer spectrum of perfect filtered complexes on X is the product of the Sierpiński space and the underlying topological space of X. In the affine case, this result is obtained by Gallauer in [5] using a different method.

If we regard $\mathbf{Z}$ as a discrete abelian group, $\mathrm{Fun}(\mathbf{Z}, \mathrm{QCoh}(X))$ is the ∞-category of graded quasicoherent sheaves. Then (1) of Theorem I says that the Balmer spectrum of perfect graded complexes on X is the underlying topological space of X.

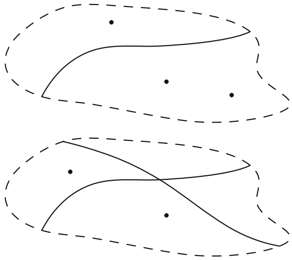

Fig. 1 As a set, $\mathrm{Spc}(\mathrm{Fun}(\mathbf{Z}, \mathcal{C})^\omega)$ consists of two copies of $\mathrm{Spc}(\mathcal{C}^\omega)$. Its closed subsets correspond to inclusions between two closed subsets of $\mathrm{Spc}(\mathcal{C}^\omega)$

Example 1.3 The space $\mathrm{Spc}(\mathsf{Sp}^\omega)$ is already calculated; its points are indexed by the Morava K-theories (see [3, Theorem 51]). Applying Theorem I to $\mathcal{C}^\otimes = \mathsf{Sp}^\otimes$, we get the Balmer spectrum of compact filtered (or graded) spectra.

Example 1.4 One advantage of the generality of the statement of Theorem I is that it can be applied iteratively. For example, it can be used when objects are $\mathbf{Z}$-filtered in several directions. Actually, what we prove in Sect. 5 is so general that we can determine the Balmer spectrum of $\mathbf{Z}^\kappa$-filtered objects for any cardinal κ.

We note that this theorem has a geometric interpretation:

Example 1.5 For an $\mathbb{E}_\infty$-ring R, Moulinos proved in [15] that there exist the following equivalences of symmetric monoidal ∞-categories:

$$\mathrm{Fun}(\mathbf{Z}, \mathsf{Mod}_R)^\otimes \simeq \mathrm{QCoh}(\mathbb{A}^1_R/\mathbb{G}_{\mathrm{m},R})^\otimes, \quad \mathrm{Fun}(\mathbf{Z}^{\mathrm{disc}}, \mathsf{Mod}_R)^\otimes \simeq \mathrm{QCoh}(\mathrm{B}\mathbb{G}_{\mathrm{m},R})^\otimes,$$

where we write $\mathbf{Z}^{\mathrm{disc}}$ for the abelian group of integers without a poset structure. Applying our result to the case when $A = \mathbf{Z}$, $\mathbf{Z}^{\mathrm{disc}}$ and $\mathcal{C}^\otimes = \mathsf{Mod}_R^\otimes$, we get the Balmer spectra of perfect complexes on these geometric stacks (but note that the determination of $\mathrm{Spc}(\mathsf{Mod}_R^\omega)$ for a general $\mathbb{E}_\infty$-ring R is a difficult problem). At least when R is a field, our computations reflect a naive intuition on how these stacks look like.

The second result is the following:

Theorem II *Suppose that $\mathcal{C}^\otimes$ is a big tt-∞-category and X is a coherent topological space (that is, a space which arises as the underlying topological space of a quasicompact quasiseparated scheme). Let $\mathrm{Shv}_\mathcal{C}(X)$ denote the ∞-category of $\mathcal{C}$-valued sheaves on X. Considering the pointwise $\mathbb{E}_2$-monoidal structure on it, we have a canonical homeomorphism*

$$\mathrm{Spc}(\mathrm{Shv}_\mathcal{C}(X)^\omega) \simeq X_{\mathrm{cons}} \times \mathrm{Spc}(\mathcal{C}^\omega),$$

where X_{cons} denotes the set X endowed with the constructible topology of X.

One big problem in tensor triangular geometry is to compute the Balmer spectrum of compact objects of the stable motivic homotopy theory $\mathrm{SH}(X)^\otimes$ associated to a quasicompact quasiseparated scheme X. The first step would be the determination of that of spectrum-valued sheaves on the smooth-Nisnevich site of X. Theorem II can be considered as an easy variant of that calculation. Nevertheless, it is an interesting fact in its own right.

In this paper, we use distributive lattices to deal with the Balmer spectra. More precisely, we introduce the notion of the Zariski lattice of a tt-(∞-)category, which turns out to be just the opposite of the distributive lattice of quasicompact open sets of the Balmer spectrum. Thus it contains the same information as the Balmer spectrum, but it has a more algebraic nature, which makes our computations possible. We can find a similar approach in [9], where Kock and Pitsch used the “Zariski frame” of a tt-category as a central notion, which in our terminology is the ideal frame of the Zariski lattice. We note that we also use (upper) semilattices to consider a “tensorless” variant of tensor triangular geometry.

Outline

In Sect. 2, we study ∞-categorical machinery related to functor categories. In Sect. 3, we review basic notions in tensor triangular geometry using the language of distributive lattices. Its “tensorless” variant is also introduced there. Sections 4 and 5 are devoted to the proofs of Theorem II and a general version of Theorem I, respectively. These two sections are logically independent, but we arrange them in this way because the former is much simpler. In Appendix A, we develop some technical material on ∞-toposes which we need in Sect. 4.

Conventions

Concerning ∞-categories, we will follow terminology and notation used in [11–13] with minor exceptions, which we will explicitly mention. For example, S and Cat_∞ denote the large ∞-categories of spaces and ∞-categories, respectively.

2 Functor categories

In this section, we study general properties of the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ for a small ∞-category K and a presentable ∞-category $\mathcal{C}$.

2.1 Tensor products of presentable ∞-categories

Let Pr denote the large ∞-category of presentable ∞-categories whose morphisms are those functors that preserve colimits. This is what is denoted by Pr^{L} in [11]. Similarly, for an infinite regular cardinal κ, we let Pr_κ denote the large ∞-category of κ-compactly generated ∞-categories whose morphisms are those functors that preserve colimits and κ-compact objects; see [11, Section 5.5.7].

Here we list basic properties of the symmetric monoidal structure on Pr as one theorem; see [12, Section 4.8.1] for the proofs.

Theorem 2.1 *There exists a symmetric monoidal structure on* Pr *which satisfies the following properties*:

(1) *For* $\mathcal{C}, \mathcal{D} \in \mathsf{Pr}$, *the tensor product* $\mathcal{C} \otimes \mathcal{D}$ *is canonically equivalent to the full subcategory of* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ *spanned by those functors preserving limits.*
(2) *For* $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{D} \in \mathsf{Pr}$, *the full subcategory of* $\mathrm{Fun}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n, \mathcal{D})$ *spanned by those functors preserving colimits is equivalent to that of* $\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D})$ *spanned by those functors preserving colimits in each variable.*
(3) *The functor* $\mathrm{PShv}\colon \mathsf{Cat}_\infty \to \mathsf{Pr}$ *has a symmetric monoidal refinement*, *where we consider the cartesian symmetric monoidal structure* (*see* [12, Remark 2.4.2.6]) *on* Cat_∞.
(4) *The tensor product operations preserve small colimits in each variable.*
(5) *For an infinite regular cardinal* κ *and* $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{D} \in \mathsf{Pr}_\kappa$, *we have* $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \in \mathsf{Pr}_\kappa$. *Moreover*, *the full subcategory of* $\mathrm{Fun}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n, \mathcal{D})$ *spanned by those functors*

preserving colimits and κ-compact objects is canonically equivalent to that of $\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D})$ *spanned by those functors that preserve colimits in each variable and restrict to determine functors from* $(\mathcal{C}_1)^\kappa \times \cdots \times (\mathcal{C}_n)^\kappa$ *to* $\mathcal{D}^\kappa$.

The useful consequence for us is the following:

Corollary 2.2 *For a presentable ∞-category $\mathcal{C}$ and a small ∞-category K, we have a canonical equivalence* $\mathrm{Fun}(K, \mathsf{S}) \otimes \mathcal{C} \simeq \mathrm{Fun}(K, \mathcal{C})$.

Proof According to (1) of Theorem 2.1, the left hand side can be regarded as the full subcategory of $\mathrm{Fun}(\mathrm{Fun}(K, \mathsf{S})^{\mathrm{op}}, \mathcal{C})$ spanned by those functors preserving limits. Hence the equivalence follows from [11, Theorem 5.1.5.6]. □

We have the following result from this description, although it can be proven more concretely:

Corollary 2.3 *Let κ be an infinite regular cardinal. If $\mathcal{C}$ is a κ-compactly generated ∞-category, so is* $\mathrm{Fun}(K, \mathcal{C})$ *for any small ∞-category K.*

2.2 Compact objects in a functor category

In this subsection, we fix an infinite regular cardinal κ.

For a κ-compactly generated ∞-category $\mathcal{C}$, according to Corollary 2.3, the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is also κ-compactly generated for any small ∞-category K. The aim of this subsection is to determine κ-compact objects of $\mathrm{Fun}(K, \mathcal{C})$ under some assumptions on K.

Definition 2.4 Let K be a small ∞-category. We say that K is *κ-small* if there exist a simplicial set K' with $< \kappa$ nondegenerate simplices and a Joyal equivalence $K' \to K$. We say that K is *locally κ-compact* if the mapping space functor $\mathrm{Map}\colon K^{\mathrm{op}} \times K \to \mathsf{S}$ factors through the full subcategory S^κ spanned by κ-compact spaces.

Remark 2.5 When κ is uncountable, the notion of κ-smallness introduced here coincides with that of essential κ-smallness given in [11, Definition 5.4.1.3]. Furthermore, according to [11, Proposition 5.4.1.2], every κ-small category is locally κ-compact. However in the case $\kappa = \omega$, the analogous result does not hold: For example, the 1-sphere S^1, regarded as an ∞-category, is ω-finite but not locally ω-compact; see also Remark 2.9.

Lemma 2.6 *Let $\mathcal{C}$ be a κ-compactly generated ∞-category and K a κ-small ∞-category. Then every functor $K \to \mathcal{C}$ that factors through $\mathcal{C}^\kappa$ is a κ-compact object of* $\mathrm{Fun}(K, \mathcal{C})$.

Proof This is a corollary of [11, Proposition 5.3.4.13]. □

Lemma 2.7 *Let $\mathcal{C}$ be a κ-compactly generated ∞-category and K a locally κ-compact ∞-category. Then every κ-compact object $K \to \mathcal{C}$ of* $\mathrm{Fun}(K, \mathcal{C})$ *factors through $\mathcal{C}^\kappa$.*

Proof Let k be an object of K. The inclusion $i\colon * \hookrightarrow K$ corresponding to k induces a functor $i^*\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$ by composition. Since $F(k) \simeq i^*F$ holds for every functor $F\colon K \to \mathcal{C}$, it suffices to show that the functor i^* preserves κ-compact objects. By [11, Proposition 5.5.7.2], this is equivalent to the assertion that its right adjoint i_* preserves κ-filtered colimits. The functor i_* can be concretely described as the assignment $C \mapsto \left(l \mapsto C^{\mathrm{Map}_K(l,k)}\right)$ using the cotensor structure on $\mathcal{C}$. Since $\mathcal{C}$ is κ-compactly generated and $\mathrm{Map}_K(l, k)$ is κ-compact for every object $l \in K$, we can see that i_* preserves κ-filtered colimits. □

Combining these two lemmas, we get the following result:

Proposition 2.8 *Let $\mathcal{C}$ be a κ-compactly generated ∞-category and K a κ-small locally κ-compact ∞-category. Then an object $F \in \mathrm{Fun}(K, \mathcal{C})$ is κ-compact if and only if it takes values in κ-compact objects.*

Remark 2.9 Proposition 2.8 does not hold without the local condition: Take $K = S^1$ and consider the object X of $\mathrm{Fun}(S^1, \mathrm{S})$ corresponding to the universal covering $* \to S^1$ by the Grothendieck construction. By [11, Lemma 5.1.6.7] the object X is compact, but $X(*) \simeq \mathbf{Z}$ is not compact.

Remark 2.10 In the case $\kappa = \omega$, the assumption on K in the statement of Proposition 2.8 is relatively restrictive. For example, if K is also assumed to be equivalent to a space, K must be equivalent to a finite set: When K is simply connected, this can be deduced by considering the homological Serre spectral sequence associated to the fiber sequence $\Omega K \to * \to K$. The general case follows by taking the universal cover of each connected component of K and using the fact that the classifying space of a nontrivial finite group is not finite.

Later in this paper, we consider a slightly more general situation than that of Proposition 2.8. The following result is useful in that case:

Corollary 2.11 *Let K be a locally κ-compact ∞-category and $\mathcal{C}$ a κ-compactly generated ∞-category. Suppose that there is a κ-directed family $(K_j)_{j \in J}$ of full subcategories of K such that K_j is κ-small for $j \in J$ and $\bigcup_j K_j = K$. We let $(i_j)_!$ denote the left Kan extension functor along the inclusion $i_j \colon K_j \hookrightarrow K$. Then we have an equality*

$$\mathrm{Fun}(K, \mathcal{C})^\kappa = \bigcup_{j \in J} (i_j)_!(\mathrm{Fun}(K_j, \mathcal{C})^\kappa)$$

of full subcategories of $\mathrm{Fun}(K, \mathcal{C})$.

Proof Since the right adjoint of the left Kan extension functor preserves colimits, we obtain one inclusion by applying [11, Proposition 5.5.7.2].

Conversely, let $F \in \mathrm{Fun}(K, \mathcal{C})$ be a κ-compact object. By assumption, F is the colimit of the κ-filtered diagram $j \mapsto (i_j)_!(F|_{K_j})$. Hence we can take $j \in J$ such that F is a retract of $(i_j)_!(F|_{K_j})$. Since the essential image of $(i_j)_!$ is closed under retracts, F is in fact equivalent to $(i_j)_!(F|_{K_j})$. From Lemma 2.7 and Proposition 2.8, we observe that $F|_{K_j}$ is κ-compact, which completes the proof of the other inclusion. □

2.3 Recollement

We refer the reader to [12, Section A.8] for the theory of recollements for ∞-categories.

Definition 2.12 Suppose that $\mathcal{C}$ is an ∞-category and $i \colon \mathcal{C}_0 \to \mathcal{C}$ and $j \colon \mathcal{C}_1 \to \mathcal{C}$ are fully faithful functors. We say that $\mathcal{C}$ is a *recollement* of i and j if $\mathcal{C}$ is a recollement of the essential images of i and j in the sense of [12, Definition A.8.1].

There are many ways to write a functor category as a recollement due to the following observation:

Proposition 2.13 *Let $\mathcal{C}$ be a presentable ∞-category and $K_1 \subset K$ a full inclusion of ∞-categories. Suppose that K_1 is a cosieve on K; that is, $k \in K_1$ implies $l \in K_1$ if there exists a morphism $k \to l$ in K. Let K_0 denote the full subcategory of K spanned by objects not in K_1. Let $i_*\colon \mathrm{Fun}(K_0, \mathcal{C}) \hookrightarrow \mathrm{Fun}(K, \mathcal{C})$ and $j_*\colon \mathrm{Fun}(K_1, \mathcal{C}) \hookrightarrow \mathrm{Fun}(K, \mathcal{C})$ denote the functors defined by right Kan extensions. Then $\mathrm{Fun}(K, \mathcal{C})$ is a recollement of i_* and j_*.*

Proof The only nontrivial point is to verify that j^*i_* sends every object to the final object, where we write j^* for the functor given by restriction along the inclusion $K_1 \subset K$. For $k \in K_1$, the cosieve condition implies $(K_0)_{k/} = \emptyset$. Hence for any $F \in \mathrm{Fun}(K_0, \mathcal{C})$ and any $k \in K_1$, we have $(j^*i_*F)(k) = (i_*F)(k) \simeq \varprojlim_{l\in(K_0)_{k/}} F(l) \simeq *$, which completes the proof. $\square$

We state a lemma on recollements in the stable setting.

Lemma 2.14 *Let $\mathcal{C}$ be a stable ∞-category, which is a recollement of i_* and j_*. We write i^*, j^* for the left adjoints of i_*, j_*, respectively, and $j_!$ for that of j^*. Then for an object $C \in \mathcal{C}$, the cofiber sequence $j_!j^*C \to C \to i_*i^*C$ splits if and only if the map $i^*C \to i^*j_*j^*C$ is zero.*

Proof We write $i^!$ for the right adjoint of i_*. The "only if" direction follows from the fact that $i^*(j_!j^*C)$ and $i^*j_*j^*(i_*i^*C)$ are both zero. We wish to prove the converse. By applying i^* to the cofiber sequence $i_*i^!C \to C \to j_*j^*C$ and shifting, we obtain a cofiber sequence $\Sigma^{-1}i^*j_*j^*C \to i^!C \to i^*C$, which splits by assumption. Then the map $i_*i^*C \to C$ corresponding to the section $i^*C \to i^!C$ by adjunction induces the desired splitting. $\square$

2.4 The two monoidal structures on a functor category

In this subsection, we fix an ∞-operad $\mathbb{E}_k^{\otimes}$, where k is a positive integer or the symbol ∞.

Recall that an $\mathbb{E}_k$-monoidal ∞-category can be regarded as an $\mathbb{E}_k$-algebra object of the symmetric monoidal category $\mathsf{Cat}_\infty^{\times}$. In the following discussion, we often use this identification implicitly.

The following is a special case of [12, Proposition 3.2.4.4]:

Lemma 2.15 *There exists a symmetric monoidal structure on $\mathrm{Alg}_{\mathbb{E}_k}(\mathsf{Pr})$ such that the forgetful functor $\mathrm{Alg}_{\mathbb{E}_k}(\mathsf{Pr}) \to \mathsf{Pr}$ has a symmetric monoidal refinement. Also, the same holds for $\mathsf{Pr}_\kappa^{\otimes}$, where κ is an infinite regular cardinal.*

First we use this to construct the pointwise $\mathbb{E}_k$-monoidal structure.

Definition 2.16 Let K be a small ∞-category and $\mathcal{C}^{\otimes}$ a presentable $\mathbb{E}_k$-monoidal ∞-category whose tensor product preserve colimits in each variable. Then combining with the cartesian $\mathbb{E}_k$-monoidal structure on $\mathrm{Fun}(K, \mathsf{S})$, we obtain an $\mathbb{E}_k$-monoidal structure on $\mathrm{Fun}(K, \mathcal{C}) \simeq \mathrm{Fun}(K, \mathsf{S}) \otimes \mathcal{C}$ by using Lemma 2.15. We call this the *pointwise $\mathbb{E}_k$- monoidal structure*.

The pointwise tensor product operations can be computed pointwise, as the name suggests.

We consider a condition under which this construction is compatible with the compact generation property. See also Corollary 2.22 for another result in this direction.

Proposition 2.17 *Let κ be an infinite regular cardinal. In the situation of Definition 2.16, suppose furthermore that K is κ-small and locally κ-compact, $\mathcal{C}$ is κ-compactly generated, and the tensor product on $\mathcal{C}$ restricts to $\mathcal{C}^{\kappa}$. Then the pointwise tensor product on $\mathrm{Fun}(K, \mathcal{C})$ also restricts to $\mathrm{Fun}(K, \mathcal{C})^{\kappa}$.*

Proof Without loss of generality we may assume that $\mathcal{C} = \mathsf{S}$. Since finite products of κ-compact spaces are again κ-compact, the desired result follows from Proposition 2.8. □

We then consider the Day convolution $\mathbb{E}_k$-monoidal structure. We first note that for an $\mathbb{E}_k$-monoidal ∞-category $K^{\otimes}$, we can equip a canonical $\mathbb{E}_k$-monoidal structure on the opposite K^{op}; see [12, Remark 2.4.2.7]. Therefore, for such $K^{\otimes}$, we have an $\mathbb{E}_k$-monoidal structure on $\mathrm{Fun}(K, \mathsf{S}) \simeq \mathrm{PShv}(K^{\mathrm{op}})$ by (3) of Theorem 2.1.

Definition 2.18 Let $K^{\otimes}$ be a small $\mathbb{E}_k$-monoidal ∞-category and $\mathcal{C}^{\otimes}$ a presentable $\mathbb{E}_k$-monoidal ∞-category whose tensor product preserves colimits in each variable. Then considering the $\mathbb{E}_k$-monoidal structure on $\mathrm{Fun}(K, \mathsf{S})$ explained above, we obtain an $\mathbb{E}_k$-monoidal structure on $\mathrm{Fun}(K, \mathcal{C}) \simeq \mathrm{Fun}(K, \mathsf{S}) \otimes \mathcal{C}$ by using Lemma 2.15. We call this the *Day convolution $\mathbb{E}_k$-monoidal structure*.

Concretely, the Day convolution tensor product can be computed as follows:

Lemma 2.19 *In the situation of Definition 2.18, the Day convolution tensor product of $F_1, \ldots, F_n \in \mathrm{Fun}(K, \mathcal{C})$ is equivalent to the left Kan extension of the composite*

$$K \times \cdots \times K \xrightarrow{F_1 \times \cdots \times F_n} \mathcal{C} \times \cdots \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

along the tensor product $K \times \cdots \times K \to K$. Hence for $k \in K$ we have

$$(F_1 \otimes \cdots \otimes F_n)(k) \simeq \varinjlim_{k_1 \otimes \cdots \otimes k_n \to k} F_1(k_1) \otimes \cdots \otimes F_n(k_n).$$

We note that in the case $\mathcal{C} = \mathsf{S}$, this is claimed in [12, Remark 4.8.1.13].

Proof By universality, we have a canonical map from the functor constructed in the statement to the tensor product. Since both constructions are compatible with colimits in each variable, we can assume that $\mathcal{C} = \mathsf{S}$ and that $F_1, \ldots, F_n$ are in the image of the Yoneda embedding $K^{\mathrm{op}} \hookrightarrow \mathrm{PShv}(K^{\mathrm{op}})$. In this case, the desired claim is trivial. □

The author learned the following fact from Jacob Lurie, which says that the Day convolution counterpart of Proposition 2.17 does not need any assumption on $K^{\otimes}$:

Lemma 2.20 *Let κ be an infinite regular cardinal. In the situation of Definition 2.18, suppose furthermore that $\mathcal{C}$ is κ-compactly generated and the tensor product on $\mathcal{C}$ restricts to $\mathcal{C}^{\kappa}$. Then the Day convolution tensor product on $\mathrm{Fun}(K, \mathcal{C})$ also restricts to $\mathrm{Fun}(K, \mathcal{C})^{\kappa}$.*

Proof Without loss of generality we may assume that $\mathcal{C} = \mathsf{S}$. According to [11, Proposition 5.3.4.17], we observe that $\mathrm{PShv}(K^{\mathrm{op}})^{\kappa}$ is the smallest full subcategory of $\mathrm{PShv}(K^{\mathrm{op}})$ that contains the image of the Yoneda embedding and is closed under κ-small colimits and retracts. Since the Yoneda embedding has an $\mathbb{E}_k$-monoidal refinement, the desired result follows. □

Now we give a comparison result of these two $\mathbb{E}_k$-monoidal structures.

Proposition 2.21 *In the situation of Definition 2.18, suppose furthermore that the $\mathbb{E}_k$-monoidal structure on K is cocartesian. Then on $\mathrm{Fun}(K, \mathcal{C})$ the pointwise and Day convolution $\mathbb{E}_k$-monoidal structures are equivalent.*

Proof Without loss of generality we may assume that $\mathcal{C} = \mathsf{S}$ and $k = \infty$. Since both tensor product preserves colimits in each variable and restrict the image of the Yoneda embedding, it suffices to show that they are equivalent on the image. Hence the result follows from the uniqueness of cartesian symmetric monoidal structures on K^{op}. □

Combining this with Lemma 2.20, we have the following:

Corollary 2.22 *Let κ be an infinite regular cardinal. In the situation of Definition 2.16, suppose furthermore that K has finite coproducts and $\mathcal{C}$ is κ-compactly generated. Then the conclusion of Proposition 2.17 holds.*

Remark 2.23 We cannot completely remove the assumptions on K: When $K = \mathbf{Z}$, the final object of $\mathrm{Fun}(\mathbf{Z}, \mathsf{S})$ is not compact. See also Example A.13.

3 Latticial approach to tensor triangular geometry

In this section, first we review the notion of the Balmer spectrum using distributive lattices. In Sect. 3.4 we introduce a tensorless variant of tensor triangular geometry.

3.1 Our setting

Let $\mathsf{Pr}^{\mathrm{st}}_{\omega}$ denote the full subcategory of Pr_{ω} spanned by compactly generated stable ∞-categories, to which the symmetric monoidal structure on Pr_{ω} explained in Theorem 2.1 restricts. A big tt-∞-category, which is defined in Definition 1.1, can be seen as an $\mathbb{E}_2$-algebra object of $(\mathsf{Pr}^{\mathrm{st}}_{\omega})^{\otimes}$.

We let $\mathsf{Cat}^{\mathrm{perf}}_{\infty}$ denote the large ∞-category of idempotent complete stable ∞-categories whose morphisms are exact functors. The equivalence $\mathrm{Ind}\colon \mathsf{Cat}^{\mathrm{perf}}_{\infty} \to \mathsf{Pr}^{\mathrm{st}}_{\omega}$ induces a symmetric monoidal structure on it.

Definition 3.1 A *tt-∞-category* is an $\mathbb{E}_2$-algebra object of $(\mathsf{Cat}^{\mathrm{perf}}_{\infty})^{\otimes}$; in concrete terms, a tt-∞-category is an idempotent complete stable ∞-category equipped with an $\mathbb{E}_2$-monoidal structure whose tensor product is exact in each variable.

By definition, the large ∞-category $\mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Cat}^{\mathrm{perf}}_{\infty})$ of tt-∞-categories and the large ∞-category $\mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Pr}^{\mathrm{st}}_{\omega})$ of big tt-∞-categories are equivalent.

Remark 3.2 There are several differences between our setting and that of the usual theory of tensor triangular geometry, as found in [3, Hypothesis 21]:

(1) We use an ∞-categorical enhancement. We note that by [12, Lemma 1.2.4.6] the idempotent completeness assumptions in both settings are equivalent.
(2) We consider an $\mathbb{E}_2$-monoidal structure, so that the induced monoidal structure on the underlying triangulated category is not necessarily symmetric, but braided. Actually, the arguments of this paper work with slight modifications even if we call an ($\mathbb{E}_1$-)algebra object of $(\mathsf{Cat}^{\mathrm{perf}}_{\infty})^{\otimes}$ a tt-∞-category, mainly since many notions including the Balmer spectrum only depend on the underlying ($\mathbb{E}_1$-)monoidal structure. However, the author does not know if such a generalization is useful.
(3) We do not impose any rigidity condition. This is because we do not need it for our computations.

3.2 Stone duality

We review Stone duality for distributive lattices. For the basic theory, we refer the reader to [8, Section II.3].

We write DLat, Loc, $\mathsf{Loc}^{\mathrm{coh}}$, Top and $\mathsf{Top}^{\mathrm{coh}}$ for the category of distributive lattices, locales, coherent locales, topological spaces, and coherent topological spaces (also called spectral spaces), respectively. Note that both inclusions $\mathsf{Loc}^{\mathrm{coh}} \subset \mathsf{Loc}$ and $\mathsf{Top}^{\mathrm{coh}} \subset \mathsf{Top}$ are not full since only quasicompact maps are considered as morphisms in them. The Stone duality theorem for distributive lattices states that the ideal frame functor $\mathrm{Idl}\colon \mathsf{DLat} \to (\mathsf{Loc}^{\mathrm{coh}})^{\mathrm{op}}$ and the spectrum functor $\mathrm{Spec}\colon \mathsf{DLat}^{\mathrm{op}} \to \mathsf{Top}^{\mathrm{coh}}$ are equivalences, and these two are compatible with the functor $\mathrm{pt}\colon \mathsf{Loc} \to \mathsf{Top}$ that sends a locale to its space of points.

We have the following consequences:

Lemma 3.3 *The (nonfull) inclusion* $\mathsf{Loc}^{\mathrm{coh}} \hookrightarrow \mathsf{Loc}$ *preserves (small) limits.*

Proof This follows from [8, Corollary II.2.11] and Stone duality. □

Proposition 3.4 *The spectrum functor* $\mathsf{DLat}^{\mathrm{op}} \to \mathsf{Top}$ *preserves (small) limits.*

Proof Since the functor $\mathrm{pt}\colon \mathsf{Loc} \to \mathsf{Top}$ has a left adjoint, which sends a topological space to its frame of open sets, it preserves limits. From Lemma 3.3 and the fact that coherent locales are spatial, we obtain the result. □

3.3 The Zariski lattice and the Balmer spectrum

In this subsection, we introduce the notion of the Zariski lattice of a tt-∞-category.

Definition 3.5 Let $\mathcal{T}^{\otimes}$ be a tt-∞-category. A *radical ideal* of $\mathcal{T}^{\otimes}$ is a stable full replete subcategory $I \subset \mathcal{T}$ that satisfies the following conditions:

(1) If $C \oplus D \in I$ for some $C, D \in \mathcal{T}$, we have $C, D \in I$.
(2) For any $C \in \mathcal{T}$ and $D \in I$, we have $C \otimes D \in I$.
(3) If $C \in \mathcal{T}$ satisfies $C^{\otimes k} \in I$ for some $k \geq 0$, we have $C \in I$.

We denote the smallest radical ideal containing an object $C \in \mathcal{T}$ by $\sqrt{C}$.

We note that the notion of radical ideal of a tt-∞-category $\mathcal{T}^{\otimes}$ only depends on the underlying tensor triangulated category $(\mathrm{h}\mathcal{T})^{\otimes}$.

Definition 3.6 Let $\mathcal{T}^{\otimes}$ be a tt-∞-category. A *support* for $\mathcal{T}^{\otimes}$ is a pair (L, s) of a distributive lattice L and a function $s\colon \mathcal{T} \to L$ satisfying the following:

(0) The function s takes the same values on equivalent objects. Hence we can evaluate $s(C)$ even if C is only determined up to equivalence.
(1) For $C_1, \dots, C_n \in \mathcal{T}$, we have $s(C_1 \oplus \cdots \oplus C_n) = s(C_1) \vee \cdots \vee s(C_n)$. In particular, we have $s(0) = 0$.
(2) For any cofiber sequence $C' \to C \to C''$ in $\mathcal{T}$, we have $s(C') \vee s(C) = s(C) \vee s(C'') = s(C'') \vee s(C')$.
(3) For $C_1, \dots, C_n \in \mathcal{T}$, we have $s(C_1 \otimes \cdots \otimes C_n) = s(C_1) \wedge \cdots \wedge s(C_n)$. In particular, we have $s(\mathbf{1}) = 1$, where $\mathbf{1}$ denotes the unit.

Note in particular that for $C \in \mathcal{T}$, by $s(0) = 0$ and the cofiber sequence $C \to 0 \to \Sigma C$, we have $s(\Sigma C) = s(C)$. They form a category, with morphisms $(L, s) \to (L', s')$ defined to be morphisms of distributive lattices $f\colon L \to L'$ satisfying $f \circ s = s'$.

Remark 3.7 The notion of support introduced here is different from what is called "support on $\mathcal{T}$" in [9, Definition 3.2.1], which values in a frame.

Definition 3.8 The *Zariski lattice* $\mathrm{Zar}(\mathcal{T})$ of a tt-∞-category $\mathcal{T}^{\otimes}$ is the partially ordered set $\left\{\sqrt{C} \mid C \in \mathcal{T}\right\}$ ordered by inclusion.

Proposition 3.9 *For a tt-∞-category $\mathcal{T}^{\otimes}$, the following assertions hold*:

(1) *The Zariski lattice* $\mathrm{Zar}(\mathcal{T})$ *is a distributive lattice.*
(2) *The pair* $\left(\mathrm{Zar}(\mathcal{T}), C \mapsto \sqrt{C}\right)$ *is a support for* $\mathcal{T}^{\otimes}$.
(3) *It is an initial support; that is, an initial object of the category described in Definition 3.6.*

Although this can be proven directly, here we give a proof using several results obtained in [9, Section 3].

Proof According to [9, Theorem 3.1.9], all radical ideals of $\mathcal{T}^{\otimes}$ form a coherent frame by inclusion and its compact objects are precisely the elements of $\mathrm{Zar}(\mathcal{T})$, so (1) holds. Also, (2) follows from this observation, together with [9, Lemma 3.2.2]. Assertion (3) follows from essentially the same argument as that given in the proof of [9, Theorem 3.2.3]. □

Remark 3.10 This construction determines a functor $\mathrm{Zar}\colon \mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Cat}_{\infty}^{\mathrm{perf}}) \to \mathsf{DLat}$.

Now we can give a definition of the Balmer spectrum in this paper; this is equivalent to the original definition by [9, Theorem 3.1.9 and Corollary 3.4.2].

Definition 3.11 For a tt-∞-category $\mathcal{T}^{\otimes}$, we let $\mathrm{Spc}(\mathcal{T})$ denote the coherent topological space $\mathrm{Spec}(\mathrm{Zar}(\mathcal{T})^{\mathrm{op}})$ and call it the *Balmer spectrum* of $\mathcal{T}^{\otimes}$.

We conclude this subsection by proving a property of the functor Zar .

Lemma 3.12 *The functor* $\mathrm{Zar}\colon \mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Cat}_{\infty}^{\mathrm{perf}}) \to \mathsf{DLat}$ *preserves filtered colimits.*

The classical version of this result is [5, Proposition 8.2], which Gallauer proved as a corollary of a more general result there. This lemma might be seen as a consequence of its variants, but we here give a different proof using supports.

Proof Suppose that I is a directed poset and that $\mathcal{T}^{\otimes}$ is the colimit of a diagram $I \to \mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Cat}_{\infty}^{\mathrm{perf}})$, which maps i to $\mathcal{T}_i^{\otimes}$. We wish to show that the morphism $\varinjlim_i \mathrm{Zar}(\mathcal{T}_i) \to \mathrm{Zar}(\mathcal{T})$ is an equivalence. By [12, Corollary 3.2.2.5] and the fact that the (nonfull) inclusion $\mathsf{Cat}_{\infty}^{\mathrm{perf}} \to \mathsf{Cat}_{\infty}$ preserves filtered colimits, $\mathcal{T}$ is the colimit of the composite $I \to \mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Cat}_{\infty}^{\mathrm{perf}}) \to \mathsf{Cat}_{\infty}$. Hence it suffices to prove that a function from $\mathcal{T}$ to a distributive lattice L is a support if the composite $\mathcal{T}_i \to \mathcal{T} \to L$ is a support for each i. This follows from the definition of a support. □

3.4 Tensorless tensor triangular geometry

In this subsection, we develop the "tensorless" counterpart of the theory described in the previous section. This is used in Sect. 5.

First recall that an upper semilattice is a poset that has finite joins. A morphism between upper semilattices is defined to be a function that preserves finite joins. We let SLat denote the category of upper semilattices.

Definition 3.13 Suppose that $\mathcal{T}$ is an idempotent complete stable ∞-category.

(1) A *semisupport* for $\mathcal{T}$ is a pair (U, s) of an upper semilattice U and a function $s : \mathcal{T} \to U$ satisfying conditions (0), (1), (2) of Definition 3.6, which make sense in this situation.
(2) A *thick subcategory* of $\mathcal{T}$ is an idempotent complete stable full replete subcategory of $\mathcal{T}$. It is called *principal* if it is generated, as a thick subcategory, by one object.

The following is the counterpart of Proposition 3.9 for semisupports:

Proposition 3.14 *For an idempotent complete stable ∞-category $\mathcal{T}$, the set of principal thick subcategories of $\mathcal{T}$ ordered by inclusion is a semilattice, which is (the target of) the initial semisupport.*

Lemma 3.15 *For any semisupport (U, s) and any object $C \in \mathcal{T}$, the full subcategory $I \subset \mathcal{C}$ spanned by objects D satisfying $s(D) \leq s(C)$ is a thick subcategory of $\mathcal{T}$.*

Proof In this proof, we refer to the conditions given in Definition 3.6. From (0) we see that I is a full replete subcategory. Condition (1) implies $0 \in I$ and (2) implies that I is closed under shifts and (co)fibers. Hence I is a stable subcategory. Also, from (1) we see that I is idempotent complete, which completes the proof. □

Proof of Proposition 3.14 For $C_1, \dots, C_n \in \mathcal{T}$, it is easy to see that the join of $\langle C_1 \rangle, \dots, \langle C_n \rangle$ can be computed as $\langle C_1 \oplus \cdots \oplus C_n \rangle$, where $\langle C \rangle$ denotes the thick subcategory of $\mathcal{T}$ generated by an object $C \in \mathcal{T}$. Hence it suffices to show that if objects $C, D \in \mathcal{T}$ satisfy $s(C) = s(D)$ for some semisupport s, they generate the same thick subcategory. This follows from Lemma 3.15. □

Remark 3.16 We can also prove the tensorless counterpart of Lemma 3.12 by the same argument.

We state a well-known concrete description of the free functor $\mathrm{Free} : \mathsf{SLat} \to \mathsf{DLat}$, which is defined as the left adjoint of the forgetful functor.

Lemma 3.17 *For an upper semilattice U, we let $\mathrm{P}(U)$ denote the power set of U ordered by inclusion. Then the morphism $U \to \mathrm{P}(U)$ that maps u to $\{v \in U \mid u \not\leq v\}$ induces a monomorphism of distributive lattices $\mathrm{Free}(U) \hookrightarrow \mathrm{P}(U)$.*

4 Tensor triangular geometry of sheaves

The main aim of this section is to prove Theorem II, which is stated in Sect. 1. It is a computation of the Balmer spectrum of $\mathrm{Shv}_{\mathcal{C}}(X)^{\omega}$ for a big tt-∞-category $\mathcal{C}$ and a coherent topological space X. Our main technique is a limiting argument: It is well known that a coherent topological space can be regarded as a Pro-object of the category of finite posets. The case of finite posets is treated in Sect. 4.1 with additional generality. The limiting argument is done in Sect. 4.2. Note that there is a subtlety in identifying the case of finite posets and finite coherent topological spaces, and we need a technical input which we prove in Appendix A.

4.1 Tensor triangular geometry of the pointwise monoidal structure

First we define a class of categories. Beware that there are other usages of the word "acyclic category" in the literature.

Definition 4.1 An (ordinary) category is called *acyclic* if only identity morphisms are isomorphisms or endomorphisms in it.

Example 4.2 Any poset, considered as a category, is an acyclic category.

Note that every finite acyclic category is ω-finite and locally ω-compact as an ∞-category, so that we can apply Propositions 2.8 and 2.17.

Theorem 4.3 *Let $\mathcal{C}^{\otimes}$ be a big tt-∞-category and K a finite acyclic category. Then we have a canonical isomorphism*

$$\mathrm{Zar}(\mathrm{Fun}(K,\mathcal{C})^{\omega}) \simeq \mathrm{Zar}(\mathcal{C}^{\omega})^{K_0},$$

where the right hand side denotes the power of $\mathrm{Zar}(\mathcal{C}^{\omega})$ *indexed by the set of objects of K, computed in the category* DLat.

Remark 4.4 In the language of usual tensor triangular geometry, the conclusion of Theorem 4.3 just says that the Balmer spectrum of $\mathrm{Fun}(K,\mathcal{C})^{\omega}$ is homeomorphic to that of $\mathcal{C}^{\omega}$ to the power of the cardinality of objects of K.

Example 4.5 In the case $\mathcal{C}^{\otimes} = \mathsf{Mod}_k^{\otimes}$ for some field k, this result is the special case of [10, Theorem 2.1.5.1] when the quiver is not equipped with relations.

To give the proof of Theorem 4.3, we introduce some notation.

Definition 4.6 (*Used only in this subsection*) In the situation of Theorem 4.3, suppose that k is an object of K. We let K' denote the cosieve generated by k. Let $X(k) \in \mathrm{Fun}(K,\mathcal{C})$ denote the left Kan extension of the object of $\mathrm{Fun}(K',\mathcal{C})$ which is obtained as the right Kan extension of the unit $\mathbf{1} \in \mathcal{C} \simeq \mathrm{Fun}(\{k\},\mathcal{C})$. This object satisfies $X(k)(k) \simeq \mathbf{1}$ and $X(k)(l) \simeq 0$ for $l \neq k$.

Lemma 4.7 *In the situation of Theorem 4.3, suppose that (L,s) is a support for* $(\mathrm{Fun}(K,\mathcal{C})^{\omega})^{\otimes}$. *Then we have $\bigvee_{k\in K} s(X(k)) = 1$. In other words, the object $\bigoplus_{k\in K} X(k)$ generates* $\mathrm{Fun}(K,\mathcal{C})^{\omega}$ *as a radical ideal.*

Proof First, we name objects of K as $k_1,\dots,k_n$ so that we have $\mathrm{Hom}_K(k_j,k_i)=\emptyset$ for any $i<j$. This is possible since K is acyclic. For $0\le i\le n$, let K_i denote the full subcategory of K whose set of objects is $\{k_j \mid j\le i\}$. We write $F_i \in \mathrm{Fun}(K,\mathcal{C})^{\omega}$ for the right Kan extension of $\mathbf{1}|_{K_i}$, where $\mathbf{1}$ denotes the unit of $\mathrm{Fun}(K,\mathcal{C})^{\otimes}$.

We wish to prove $s(F_i) = s(F_{i-1}) \vee s(X(k_i))$ for $1\le i\le n$, which completes the proof since $s(F_n)=s(\mathbf{1})=1$ and $s(F_0)=s(0)=0$ holds. Now since $K\backslash K_{i-1}$ is a cosieve, we get a cofiber sequence $X(k_i)\to F_i\to F_{i-1}$ by applying Proposition 2.13. Combining this with an equivalence $X(k_i)\simeq F_i\otimes X(k_i)$, we obtain the desired equality. □

Proof of Theorem 4.3 Let $\mathrm{P}(K_0)$ denote the power set of the set of objects of K ordered by inclusion. Then there exists a canonical isomorphism $\mathrm{Zar}(\mathcal{C}^{\omega})\otimes \mathrm{P}(K_0) \simeq \mathrm{Zar}(\mathcal{C}^{\omega})^{K_0}$ of distributive lattices.

First, we claim that there exists a (unique) morphism of distributive lattice $\mathrm{P}(K_0)\to \mathrm{Zar}(\mathrm{Fun}(K,\mathcal{C})^{\omega})$ that maps $\{k\}$ to $\sqrt{X(k)}$ for $k\in K$. This follows from the following two observations:

(1) For $k\neq l$, we have $X(k)\otimes X(l)\simeq 0$; this can be checked pointwise.

(2) The object $\bigoplus_{k\in K} X(k)$ generates $\mathrm{Fun}(K, \mathcal{C})^{\omega}$ as a radical ideal; this is the content of Lemma 4.7.

Combining this morphism with the one $\mathrm{Zar}(\mathcal{C}^{\omega}) \to \mathrm{Zar}(\mathrm{Fun}(K, \mathcal{C})^{\omega})$ induced by the functor $K \to *$, we obtain a morphism $f\colon \mathrm{Zar}(\mathcal{C}^{\omega}) \otimes \mathrm{P}(K_0) \to \mathrm{Zar}(\mathrm{Fun}(K, \mathcal{C})^{\omega})$.

For each $k \in K$, the inclusion $\{k\} \hookrightarrow K$ induces a morphism $\mathrm{Zar}(\mathrm{Fun}(K, \mathcal{C})^{\omega}) \to \mathrm{Zar}(\mathrm{Fun}(\{k\}, \mathcal{C})^{\omega}) \simeq \mathrm{Zar}(\mathcal{C}^{\omega})$. From them, we get a morphism $g\colon \mathrm{Zar}(\mathrm{Fun}(K, \mathcal{C})^{\omega}) \to \mathrm{Zar}(\mathcal{C}^{\omega})^{K_0} \simeq \mathrm{Zar}(\mathcal{C}^{\omega}) \otimes \mathrm{P}(K_0)$, where we use the identification described above.

To complete the proof, we wish to prove that $g \circ f$ and $f \circ g$ are identities. By construction, it is easy to see that $g \circ f$ is the identity, so it remains to prove the other claim. We take an object $F \in \mathrm{Fun}(K, \mathcal{C})^{\omega}$ and for $k \in K$ let F_k denote the object of $\mathrm{Fun}(K, \mathcal{C})^{\omega}$ obtained by precomposing the functor $K \to \{k\} \hookrightarrow K$ with F. Unwinding the definitions, the assertion $(g \circ f)\big(\sqrt{F}\big) = \sqrt{F}$ is equivalent to the assertion that F and $\bigoplus_{k\in K} F_k \otimes X(k)$ generate the same radical ideal. Since we have an equivalence $F \otimes X(k) \simeq F_k \otimes X(k)$ for each $k \in K$, this follows from Lemma 4.7. □

4.2 Main result

In order for Theorem II to make sense, we need to clarify what the pointwise $\mathbb{E}_2$-monoidal structure on $\mathrm{Shv}_{\mathcal{C}}(X)$ is.

Proposition 4.8 *For a coherent topological space X, the ∞-topos $\mathrm{Shv}(X)$ is compactly generated. Moreover, finite products of compact objects are again compact.*

Proof The first assertion is the content of [11, Proposition 6.5.4.4]. Let L denote the distributive lattice of quasicompact open subsets of X. We have $\mathrm{Shv}(L) \simeq \mathrm{Shv}(X)$, where L is equipped with the induced Grothendieck topology (see Definition A.2). We let $L'\colon \mathrm{PShv}(L) \to \mathrm{Shv}(L)$ denote the sheafification functor. It follows from the proof of [11, Proposition 6.5.4.4] that $\mathrm{Shv}(L)^{\omega}$ is the smallest full subcategory that contains the image of $\mathrm{PShv}(L)^{\omega}$ under L' and is closed under finite colimits and retracts. Since finite products preserve (finite) colimits in each variable in $\mathrm{Shv}(L)$ and the functor L' preserves finite products, it suffices to show that finite products of compact objects are again compact in $\mathrm{PShv}(L)$. This follows from Corollary 2.22 since L has finite products. □

Using the equivalence $\mathrm{Shv}_{\mathcal{C}}(X) \simeq \mathrm{Shv}(X) \otimes \mathcal{C}$ and Lemma 2.15, we can equip an $\mathbb{E}_2$-monoidal structure on $\mathrm{Shv}_{\mathcal{C}}(X)$.

Corollary 4.9 *For a big tt-∞-category $\mathcal{C}$ and a coherent topological space X, the $\mathbb{E}_2$-monoidal ∞-category $\mathrm{Shv}_{\mathcal{C}}(X)^{\otimes}$, defined as above, is a big tt-∞-category.*

To state the main theorem, we recall basic facts on Boolean algebras. See [8, Section II.4] for details. First recall that Boolean algebras form a reflective subcategory of DLat, which we denote by BAlg. The left adjoint of the inclusion $\mathsf{BAlg} \hookrightarrow \mathsf{DLat}$ is called the Booleanization functor, which we denote by B. For a coherent topological space X, the spectrum of its Booleanization of the distributive lattice of quasicompact open subsets of X is the Stone space whose topology is the constructible topology (also referred to as the patch topology) of X. Hence using Proposition 3.4, Theorem II can be rephrased as follows:

Theorem 4.10 *Let $\mathcal{C}^{\otimes}$ be a big tt-∞-category and L a distributive lattice. Then we have a canonical isomorphism*

$$\mathrm{Zar}(\mathrm{Shv}_{\mathcal{C}}(\mathrm{Spec}(L))^{\omega}) \simeq \mathrm{Zar}(\mathcal{C}^{\omega}) \otimes \mathrm{B}(L).$$

The proof uses the following notion:

Definition 4.11 For a poset P, the *Alexandroff topology* is a topology on the underlying set of P whose open sets are cosieves (or equivalently, upward closed subsets). Let $\mathrm{Alex}(P)$ denote the set P equipped with this topology.

Lemma 4.12 *For a big tt-∞-category $\mathcal{C}^{\otimes}$ and a finite poset P, there exists a canonical equivalence between big tt-∞-categories*

$$\mathrm{Fun}(P, \mathcal{C})^{\otimes} \simeq \mathrm{Shv}_{\mathcal{C}}(\mathrm{Alex}(P))^{\otimes}.$$

The proof relies on a result obtained in Appendix A.

Proof We have the desired equivalence by taking the tensor product of (the cartesian $\mathbb{E}_2$-monoidal refinement of) the equivalence $\mathrm{PShv}(P^{\mathrm{op}}) \to \mathrm{Shv}(\mathrm{Alex}(P))$ obtained in Example A.12 with $\mathcal{C}^{\otimes}$. □

Lemma 4.13 *For a big tt-∞-category $\mathcal{C}^{\otimes}$, the functor from* DLat *to* $\mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Pr}^{\mathrm{st}}_{\omega})$ *that maps L to* $(\mathrm{Shv}_{\mathcal{C}}(\mathrm{Spec}(L)))^{\otimes}$ *preserves filtered colimits.*

Proof We first note that the composite of forgetful functors $\mathrm{Alg}_{\mathbb{E}_2}(\mathsf{Pr}^{\mathrm{st}}_{\omega}) \to \mathsf{Pr}^{\mathrm{st}}_{\omega} \to \mathsf{Pr}_{\omega}$ preserves sifted colimits and is conservative. Since limits in the large ∞-categories $\mathsf{Pr}^{\mathrm{op}}$ and $\mathsf{Pr}^{\mathrm{op}}_{\omega}$ are both computed in the very large ∞-category of large ∞-categories, the forgetful functor $\mathsf{Pr}_{\omega} \to \mathsf{Pr}$ preserves colimits and is obviously conservative. Hence we are reduced to showing that the following composite preserves filtered colimits:

$$\mathsf{DLat} \xrightarrow{\mathrm{Idl}} \mathsf{Loc}^{\mathrm{op}} \xrightarrow{\mathrm{Shv}^{\mathrm{op}}} \mathsf{Top}^{\mathrm{op}}_{\infty} \xrightarrow{\text{forgetful}} \mathsf{Pr} \xrightarrow{-\otimes\mathcal{C}} \mathsf{Pr},$$

where Top_{∞} denotes the large ∞-category of ∞-toposes whose morphisms are geometric morphisms. Now we can check that each functor preserves filtered colimits as follows:

(1) The first functor preserves colimits by Lemma 3.3.
(2) The second functor preserves colimits since 0-localic ∞-toposes form a reflective subcategory of Top_{∞}.
(3) The third functor preserves filtered colimits since cofiltered limits in $\mathsf{Top}^{\mathrm{op}}_{\infty}$ can be computed in the very large ∞-category of large ∞-categories.
(4) The fourth functor preserves colimits by (5) of Theorem 2.1.

Hence the composite also preserves filtered colimits. □

Proof of Theorem 4.10 Since every finitely generated distributive lattice has a finite number of objects, we can write L as a filtered colimit of finite distributive lattices. Hence by Lemmas 4.13 and 3.12, it suffices to consider the case when L is finite. Let P be a poset of points of $\mathrm{Spec}(L)$ with the specialization order; that is, the partial order in which $p \le q$ if and only if the point p is contained in the closure of the singleton $\{q\}$. Since $\mathrm{Spec}(L)$ is finite, there is a canonical homeomorphism $\mathrm{Spec}(L) \simeq \mathrm{Alex}(P)$. Booleanizing their associated distributive lattices, we observe that $\mathrm{B}(L)$ is canonically isomorphic to the power set of P ordered by inclusion. Then applying Lemma 4.12, we obtain the desired equivalence as a corollary of Theorem 4.3. □

5 Tensor triangular geometry of the Day convolution

The main result of this section is Theorem 5.17, which is a generalization of Theorem I. It is a computation of the Balmer spectrum of $\mathrm{Fun}(A, \mathcal{C})^{\omega}$ for a partially ordered abelian group A and a big tt-∞-category $\mathcal{C}^{\otimes}$. We first recall basic notions about partially ordered abelian groups in Sect. 5.1 and then in Sect. 5.2 we introduce the Archimedean semilattice $\mathrm{Arch}(A)$, which appears in the statement. We construct a map between the Balmer spectrum of $\mathrm{Fun}(A, \mathcal{C})^{\omega}$ and a certain space obtained from $\mathrm{Arch}(A)$ and the Balmer spectrum of $\mathcal{C}^{\omega}$ in Sect. 5.3. We show that it is an equivalence under a suitable assumption in Sect. 5.4. Section 5.5 is devoted for technical and intricate proofs of some statements we use in Sects. 5.3 and 5.4.

5.1 Partially ordered abelian groups

We begin with reviewing some notions used in the theory of partially ordered abelian groups.

Definition 5.1 A *partially ordered abelian group* is an abelian group object of the category of posets; that is, an abelian group A equipped with a partial order in which the map $a + -$ is order preserving for any $a \in A$.

We can regard a partially ordered abelian group as a symmetric monoidal poset.

Definition 5.2 Let A be a partially ordered abelian group.

(1) The submonoid $A_{\geq 0} = \{a \in A \mid a \geq 0\}$ is called the *positive cone* of A.
(2) The subgroup $A^{\circ} = \{a - b \mid a, b \in A_{\geq 0}\}$ is called the *identity component* of A. As the name suggests, this is the connected component containing 0 when A is regarded as a category by its partial order.
(3) If $A^{\circ} = A$ holds and $A_{\geq 0}$ has finite joins, A is called *lattice ordered*. This is equivalent to the condition that A has binary joins (but beware that A does not have the nullary join unless A is trivial). Note that in this case A also has binary meets, which are computed as $a \wedge b = a + b - (a \vee b)$ for $a, b \in A$.

Example 5.3 For a cardinal κ, the (categorical) product ordering on $\mathbf{Z}^{\kappa}$ defines a lattice ordered abelian group.

The assignment $f : (x_1, x_2) \mapsto (x_1, x_2, x_1 + x_2)$ gives a morphism $\mathbf{Z}^2 \to \mathbf{Z}^3$ of partially ordered abelian groups, but does not define a morphism of unbounded lattices: Indeed, we have $f((1, 0)) \vee f((0, 1)) = (1, 1, 1) \neq (1, 1, 2) = f((1, 0) \vee (0, 1))$.

Example 5.4 There are many partially ordered abelian groups that are connected and not lattice ordered. We here give two relatively simple examples. The first is $\mathbf{Z}$ with the ordering that makes its positive cone the submonoid generated by 2 and 3. The second is $\mathbf{Z} \times \mathbf{Z}/2$ with the ordering that makes its positive cone the submonoid generated by $(1, 0)$ and $(1, 1)$.

5.2 The Archimedes semilattice

In this subsection, we introduce the notion of the Archimedes semilattice of a partially ordered abelian group.

Definition 5.5 For a partially ordered abelian group A, a submonoid $J \subset A_{\geq 0}$ is called an *ideal* of $A_{\geq 0}$ if it is downward closed; that is, if $a, b \in A_{\geq 0}$ satisfies $a \leq b$ and $b \in J$, we have $a \in J$. An ideal J is called *principal* if it is generated as an ideal by an element of $A_{\geq 0}$.

Proposition 5.6 *For a partially ordered abelian group A, the set of principal ideals of $A_{\geq 0}$ ordered by inclusion is an upper semilattice.*

Proof It is easy to see that $\langle a_1 + \cdots + a_n \rangle$ is a join of $\langle a_1 \rangle, \ldots, \langle a_n \rangle$ for $a_1, \ldots, a_n \in A_{\geq 0}$, where $\langle a \rangle$ denotes the ideal of $A_{\geq 0}$ generated by an element $a \in A_{\geq 0}$. □

Definition 5.7 For a partially ordered abelian group A, the upper semilattice of principal ideals of $A_{\geq 0}$ is denoted by $\mathrm{Arch}(A)$. We call this the *Archimedes semilattice* of A. Note that this only depends on its positive cone $A_{\geq 0}$, regarded as a partially ordered abelian monoid.

Remark 5.8 There is a characterization of the Archimedes semilattice similar to that of the Zariski lattice given in Proposition 3.9: Namely, the Archimedes semilattice $\mathrm{Arch}(A)$ is initial among pairs (U, s) where U is an upper semilattice and $s\colon A_{\geq 0} \to U$ is a function satisfying the following conditions:

(1) For $a_1, \ldots, a_n \in A_{\geq 0}$, we have $s(a_1 + \cdots + a_n) = s(a_1) \vee \cdots \vee s(a_n)$.
(2) If $a, b \in A_{\geq 0}$ satisfies $a \leq b$, we have $s(a) \leq s(b)$.

Example 5.9 If A is a totally ordered abelian group, its Archimedes semilattice $\mathrm{Arch}(A)$ consists of all Archimedean classes of A and the singleton $\{0\}$. This observation justifies the name. In particular, if A is nonzero Archimedean, we have $\mathrm{Arch}(A) \simeq \{0 < 1\}$.

Example 5.10 Any Riesz space R can be regarded as a lattice ordered abelian group in a trivial way. There is a bijective (and order preserving) correspondence between (principal) ideals of $R_{\geq 0}$ and those of R in the usual sense.

5.3 Tensorless tensor triangular geometry with actions

In this subsection, we construct the comparison map, which we prove to be an isomorphism under some mild assumptions.

Proposition 5.11 *Let $\mathcal{C}^{\otimes}$ be a big tt-∞-category and A a partially ordered abelian group. Then we have a canonical morphism of distributive lattices*

$$f\colon \mathrm{Zar}(\mathcal{C}^{\omega}) \otimes \mathrm{Free}(\mathrm{Arch}(A)) \to \mathrm{Zar}(\mathrm{Fun}(A, \mathcal{C})^{\omega}),$$

where $\mathrm{Free}\colon \mathsf{SLat} \to \mathsf{DLat}$ *denotes the left adjoint to the forgetful functor.*

It is convenient to regard the ∞-category $\mathrm{Fun}(A, \mathcal{C})$ as equipped with the action of A, which is described in the following definition:

Definition 5.12 Suppose that $\mathcal{C}$ is a compactly generated stable ∞-category and A is a partially ordered abelian group.

Then precomposition with the map $(a, b) \mapsto b - a$ induces a functor $\mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}(A^{\mathrm{op}} \times A, \mathcal{C})$, which can be seen as a functor from $\mathrm{Fun}(A, \mathcal{C}) \times A^{\mathrm{op}}$ to $\mathrm{Fun}(A, \mathcal{C})$. We write $F\{a\}$ for the value of this functor at (F, a) and $F\{a/b\}$ for the cofiber of the map $F\{b\} \to F\{a\}$, which is only defined when $a \leq b$; concretely, $F\{a\}$ is an object satisfying $F\{a\}(b) \simeq F(b - a)$ for $b \in A$.

We call a semisupport s for $\mathrm{Fun}(A, \mathcal{C})^{\omega}$ an *A-semisupport* if for $a \in A$ and $F \in \mathrm{Fun}(A, \mathcal{C})^{\omega}$, we have $s(F\{a\}) = s(F)$. Similarly, we call a thick subcategory of $\mathrm{Fun}(A, \mathcal{C})^{\omega}$ a *thick A-subcategory* if it is stable under the operation $F \mapsto F\{a\}$ for any $a \in A$.

From now on, we abuse notation by identifying the object $C \in \mathcal{C}^\omega$ with its left Kan extension along the inclusion $0 \hookrightarrow A$ of partially ordered abelian groups. Especially, we do not distinguish the units of $\mathcal{C}^\otimes$ and $\mathrm{Fun}(A, \mathcal{C})^\otimes$, which we denote by $\mathbf{1}$.

Example 5.13 Suppose that $\mathcal{C}^\otimes$ is a big tt-∞-category. For any object $F \in \mathrm{Fun}(A, \mathcal{C})^\omega$ and any element $a \in A$ we have $F\{a\} \simeq F \otimes \mathbf{1}\{a\}$ and that $\mathbf{1}\{a\}$ is invertible with inverse $\mathbf{1}\{-a\}$. Thus any support for $(\mathrm{Fun}(A, \mathcal{C})^\omega)^\otimes$ can be regarded as an A-semisupport. More generally, for any object $G \in \mathrm{Fun}(A, \mathcal{C})^\omega$ and any support s, the assignment $F \mapsto s(G \otimes F)$ defines an A-semisupport for $\mathrm{Fun}(A, \mathcal{C})^\omega$.

Lemma 5.14 *In the situation of Definition 5.12, suppose that s is an A-semisupport for* $\mathrm{Fun}(A, \mathcal{C})^\omega$ *and F is an object of* $\mathrm{Fun}(A, \mathcal{C})^\omega$. *Then the following assertions hold*:

(1) *For $a_1, \ldots, a_n \in A_{\geq 0}$, we have $s(F\{0/(a_1 + \cdots + a_n)\}) = s(F\{0/a_1\}) \vee \cdots \vee s(F\{0/a_n\})$.*
(2) *For $a, b \in A_{\geq 0}$ satisfying $a \leq b$, we have $s(F\{0/a\}) \leq s(F\{0/b\})$.*

Proof We first prove (2). Consider the following diagram, in which all the rows and columns are cofiber sequences:

$$\begin{tikzcd}
F\{a+b\} \arrow[r] \arrow[d] & F\{a\} \arrow[r] \arrow[d] & F\{a/(a+b)\} \arrow[d] \\
F\{b\} \arrow[r] \arrow[d] & F\{0\} \arrow[r] \arrow[d] & F\{0/b\} \arrow[d] \\
F\{b/(a+b)\} \arrow[r, "f"] & F\{0/a\} \arrow[r] & C.
\end{tikzcd}$$

Since there exists an equivalence $F\{a/(a+b)\} \simeq F\{0/b\}\{a\}$, we have $s(F\{a/(a+b)\}) = s(F\{0/b\})$. Similarly we have $s(F\{b/(a+b)\}) = s(F\{0/a\})$. Using the right cofiber sequence, we have $s(C) \leq s(F\{0/b\})$. We complete the proof by showing $s(C) = s(F\{0/a\})$. To prove this, it suffices to show that the morphism f in the diagram is zero. This follows from the fact that the left top square can be decomposed as follows:

$$\begin{tikzcd}
F\{a+b\} \arrow[r] \arrow[d] & F\{a\} \arrow[r, equal] \arrow[d, equal] & F\{a\} \arrow[d] \\
F\{b\} \arrow[r] & F\{a\} \arrow[r] & F\{0\}.
\end{tikzcd}$$

We then prove (1). The case $n = 0$ is trivial. Hence it suffices to consider the case $n = 2$. Consider the following diagram:

$$\begin{tikzcd}
F\{a_1+a_2\} \arrow[r, equal] \arrow[d] & F\{a_1+a_2\} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F\{a_2\} \arrow[r] \arrow[d] & F\{0\} \arrow[r] \arrow[d] & F\{0/a_2\} \arrow[d, equal] \\
F\{a_2/(a_1+a_2)\} \arrow[r] & F\{0/(a_1+a_2)\} \arrow[r] & F\{0/a_2\}.
\end{tikzcd}$$

Since all the other rows and columns are cofiber sequences, so is the bottom row. Hence we have $s(F\{0/(a_1 + a_2)\}) \leq s(F\{a_2/(a_1 + a_2)\}) \vee s(F\{0/a_2\}) = s(F\{0/a_1\}) \vee s(F\{0/a_2\})$.

On the other hand, applying (2), we have $s(F\{0/a_1\}) \vee s(F\{0/a_2\}) \leq s(F\{0/(a_1+a_2)\})$. Therefore, the desired equality follows. □

Proof of Proposition 5.11 The left Kan extension along the inclusion $0 \hookrightarrow A$ defines a morphism $\mathrm{Zar}(\mathcal{C}^{\omega}) \to \mathrm{Zar}(\mathrm{Fun}(A,\mathcal{C})^{\omega})$. By Lemma 5.14, the assignment $a \mapsto \sqrt{\mathbf{1}\{0/a\}}$ satisfies the conditions given in Remark 5.8, so we have a morphism $\mathrm{Free}(\mathrm{Arch}(A)) \to \mathrm{Zar}(\mathrm{Fun}(A,\mathcal{C})^{\omega})$. Combining these two, we obtain the desired morphism. □

Now we study A-semisupports in more detail. First, by mimicking the proof of Proposition 3.14, we can deduce the following:

Proposition 5.15 *In the situation of Definition 5.12, principal thick A-subcategories form an upper semilattice by inclusion and the assignment that takes an object of* $\mathrm{Fun}(A,\mathcal{C})^{\omega}$ *to the thick A-subcategory generated by it defines an A-semisupport. Furthermore, it is an initial A-semisupport.*

If A is lattice ordered, the initial A-semisupport has simple generators.

Proposition 5.16 *In the situation of Definition 5.12, we furthermore assume that A is lattice ordered. Then the target of the initial A-semisupport described in Proposition 5.15 is generated as an upper semilattice by thick A-subcategories generated by an object of the form $C\{0/a_1\}\cdots\{0/a_n\}$ with $C \in \mathcal{C}^{\omega}$ and $a_1,\dots,a_n \in A_{\geq 0}$ satisfying $a_i \wedge a_j = 0$ if $i \neq j$.*

We prove this in Sect. 5.5. Note that the conclusion also holds if we only require the positive cone to have binary joins; see the proof of Proposition 5.20.

5.4 Main theorem

We now state the main result of this section.

Theorem 5.17 *In the situation of Proposition 5.11, suppose furthermore that $A_{\geq 0}$ has finite joins. Then the morphism f is an isomorphism.*

Question 5.18 *In Theorem 5.17, what happens if A does not satisfy the hypothesis? By Proposition 5.20, we may assume that A is connected. To consider the case A is one of the examples given in Example 5.4 would be a starting point.*

Example 5.19 Claim (2) of Theorem I is a direct consequence of Theorem 5.17.

Since $\mathrm{Free}(\{0 < 1\})$ is the linearly ordered set consisting of three elements, $\mathrm{Spec}(\mathrm{Free}(\{0 < 1\}))$ is homeomorphic to the Sierpiński space. Hence by using Proposition 3.4 and Example 5.9, we can deduce (1) of Theorem I.

Since the inclusion $A^{\circ} \hookrightarrow A$ induces an isomorphism $\mathrm{Arch}(A^{\circ}) \simeq \mathrm{Arch}(A)$, Theorem 5.17 is a consequence of the following two results:

Proposition 5.20 *Let $\mathcal{C}^{\otimes}$ be a big tt-∞-category and A a partially ordered abelian group. Then the morphism $i\colon \mathrm{Zar}(\mathrm{Fun}(A^{\circ},\mathcal{C})^{\omega}) \to \mathrm{Zar}(\mathrm{Fun}(A,\mathcal{C})^{\omega})$ induced by the inclusion $A^{\circ} \hookrightarrow A$ is an equivalence.*

Proposition 5.21 *In the situation of Proposition 5.11, suppose furthermore that A is lattice ordered. Then the morphism f is an isomorphism.*

We conclude this subsection by showing Proposition 5.20; we prove Proposition 5.21 in the next subsection.

Proof of Proposition 5.20 In this proof, we regard $\mathrm{Fun}(A^\circ, \mathcal{C})^\omega$ as a full subcategory of $\mathrm{Fun}(A, \mathcal{C})^\omega$ by left Kan extension.

Let J denote the quotient A/A°. For each $j \in J$ we choose an element $a_j \in j$. Then as a category, we can identify A with $\coprod_{j\in J}(A^\circ + a_j)$. Hence combining the translations $A+a_j \to A$ for all $j \in J$, we have a functor $A \to A^\circ$. This defines a functor $\mathrm{Fun}(A, \mathcal{C})^\omega \to \mathrm{Fun}(A^\circ, \mathcal{C})^\omega$ by left Kan extension; concretely, it is given by the formula $\bigoplus_{j\in J} F_j\{a_j\} \mapsto \bigoplus_{j\in J} F_j$ for any family $(F_j)_{j\in J}$ of objects of $\mathrm{Fun}(A^\circ, \mathcal{C})^\omega$ such that F_j is zero except for a finite number of indices $j \in J$.

Let s denote the composite $\mathrm{Fun}(A, \mathcal{C})^\omega \to \mathrm{Fun}(A^\circ, \mathcal{C})^\omega \to \mathrm{Zar}(\mathrm{Fun}(A^\circ, \mathcal{C})^\omega)$, where the second map is the initial support. We wish to show that s is a support for $(\mathrm{Fun}(A, \mathcal{C})^\omega)^\otimes$ by checking that the conditions given in Definition 3.6 are satisfied. The only nontrivial point is to prove that s satisfies condition (3). Since the case $n = 0$ follows from the fact that $\mathbf{1}\{-a_{[0]}\}$ is invertible, it is sufficient to consider the case $n = 2$. We take two objects $F, G \in \mathrm{Fun}(A, \mathcal{C})^\omega$ and decompose them as $F \simeq \bigoplus_{j\in J} F_j\{a_j\}$ and $G \simeq \bigoplus_{j\in J} G_j\{a_j\}$ using two families of objects $(F_j)_{j\in J}$, $(G_j)_{j\in J}$ of $\mathrm{Fun}(A^\circ, \mathcal{C})^\omega$ such that both F_j and G_j are zero except for a finite number of indices $j \in J$. To prove $s(F \otimes G) = s(F) \wedge s(G)$, unwinding the definitions, we need to show that the equality

$$\bigvee_{j,j'\in J} \sqrt{(F_j \otimes G_{j'})\{a_j + a_{j'} - a_{j+j'}\}} = \bigvee_{j\in J} \sqrt{F_j} \wedge \bigvee_{j\in J} \sqrt{G_j}$$

holds in $\mathrm{Zar}(\mathrm{Fun}(A^\circ, \mathcal{C})^\omega)$. This follows from the observation made in Example 5.13.

Hence we obtain a morphism $r\colon \mathrm{Zar}(\mathrm{Fun}(A, \mathcal{C})^\omega) \to \mathrm{Zar}(\mathrm{Fun}(A^\circ, \mathcal{C})^\omega)$ from s. The composite $r \circ i$ is the identity since we have $(r \circ i)\big(\sqrt{F}\big) = \sqrt{F\{-a_{[0]}\}} = \sqrt{F}$ for $F \in \mathrm{Fun}(A^\circ, \mathcal{C})^\omega$. To prove that $i \circ r$ is the identity, unwinding the definitions, it suffices to show that $\sqrt{\bigoplus_{j\in J} F_j\{a_j\}}$ equals to $\sqrt{\bigoplus_{j\in J} F_j}$ for any family $(F_j)_{j\in J}$ of objects of $\mathrm{Fun}(A^\circ, \mathcal{C})^\omega$ such that F_j is zero except for a finite number of indices $j \in J$. This also follows from the observation made in Example 5.13. $\square$

5.5 Postponed proofs

In this subsection, we give the proofs of Propositions 5.16 and 5.21. First we introduce some terminology.

Definition 5.22 (*Used only in this subsection*) Suppose that A is a lattice ordered abelian group.

(1) We call a subset $B \subset A$ *saturated* if it is finite and closed under binary joins. Note that for every finite set $B \subset A$ we can find the smallest saturated subset of A containing B.
(2) Let $\mathcal{C}$ be a compactly generated stable ∞-category. By applying Proposition 2.13 to the cosieve of A generated by a single element $a \in A$, we obtain a presentation of $\mathrm{Fun}(A, \mathcal{C})$ as a recollement. Hence for $F \in \mathrm{Fun}(A, \mathcal{C})$ we have a cofiber sequence $j_!j^*F \to F \to i_*i^*F$, where we use the notation of Lemma 2.14. We write $F_{\geq a}$ and $F_{\ngeq a}$ for $j_!j^*F$ and i_*i^*F, respectively.

The proof of Proposition 5.16 relies on the following lemma:

Lemma 5.23 *Suppose that A is a lattice ordered abelian group, $\mathcal{C}$ is a compactly generated stable ∞-category, and s is an A-semisupport for* $\operatorname{Fun}(A,\mathcal{C})^{\omega}$. *Then for $a \in A$ and $F \in$* $\operatorname{Fun}(A,\mathcal{C})^{\omega}$, *we have $s(F) = s(F_{\geq a}) \vee s(F_{\not\geq a})$.*

We first prove the following special case:

Lemma 5.24 *In the situation of Lemma 5.23, suppose that $a' \leq a$ are elements of A and $B \subset A$ is a saturated subset satisfying $b \wedge a \in \{a', a\}$ for $b \in B$. If an object $F \in$* $\operatorname{Fun}(A,\mathcal{C})^{\omega}$ *is obtained as the left Kan extension of $F|_B$, we have $s(F) = s(F_{\geq a}) \vee s(F_{\not\geq a})$.*

Proof We may assume that $a' = 0$ and B contains 0, which automatically becomes the least element of B.

We again consider the recollement description of $\operatorname{Fun}(A,\mathcal{C})$ given in (2) of 5.22 and continue to use the notation of Lemma 2.14.

We first show that $j^*((i_!i^*F)\{0/a\})$ is zero, where $i_!$ denotes the left adjoint of i^*. This is equivalent to the assertion that for any $c \in A_{\geq 0}$ the morphism $(i_!i^*F)(c) \to (i_!i^*F)(c+a)$ is an equivalence. We may assume that $b \wedge a = 0$ holds for any $b \in B$ to prove this. Unwinding the definitions, it is enough to show that $\{b \in B \mid b \leq c\}$ and $\{b \in B \mid b \leq c+a\}$ have the same greatest element. Let b be the greatest element of the latter set. Then we have $b = b \wedge (c+a) = c + ((b-c) \wedge a) \leq c + (b \wedge a) = c$, which means that b belongs to the former set.

Then we consider the following diagram:

$$\begin{tikzcd} i^*((i_!i^*F)\{0/a\}) \arrow[r, "\simeq"'] \arrow[d] & i^*(F\{0/a\}) \arrow[d] \\ i^*j_*j^*((i_!i^*F)\{0/a\}) \arrow[r] & i^*j_*j^*(F\{0/a\}). \end{tikzcd}$$

By what we have shown above, the bottom left object is zero, so that the right vertical morphism is zero. By applying Lemma 2.14, we see that $F_{\not\geq a} \simeq (F\{0/a\})_{\not\geq a}$ is a direct summand of $F\{0/a\}$. Hence we have $s(F) \geq s(F\{0/a\}) \geq s(F_{\not\geq a})$, which completes the proof. $\square$

Proof of Lemma 5.23 We take a saturated subset $B \subset A$ such that F is equivalent to the left Kan extension of $F|_B$. We may assume that B contains 0 as the least element. By replacing a with $a \vee 0$, we may assume that $a \geq 0$ and also $a \in B$. Now we take a maximal chain $0 = b_0 < \cdots < b_n = a$ in B. Then we can apply Lemma 5.24 iteratively to obtain an inequality

$$s(F) = s(F_{\geq 0}) = s(F_{\geq b_0}) \geq \cdots \geq s(F_{\geq b_n}) = s(F_{\geq a}),$$

which completes the proof. $\square$

Proof of Proposition 5.16 First we take a finite subset $B \subset A$ such that B is closed under binary joins and meets and F is equivalent to the left Kan extension of $F|_B$. For $b \in B$, we can take distinct elements $a_1, \ldots, a_n \in A_{\geq 0}$ (possibly $n = 0$) such that $\{b + a_1, \ldots, b + a_n\}$ is the set of minimal elements of $\{c \in B \mid b < c\}$. By the assumption on B, we see that $a_i \wedge a_j = 0$ for $i \neq j$. Let F_b denote the object $(\cdots(F_{\geq b})_{\not\geq b+a_1}\cdots)_{\not\geq b+a_n}$. Then applying Lemma 5.23 iteratively, we have $s(F) = \bigvee_{b \in B} s(F_b)$. Thus we wish to show that $F_b\{-b\}$ is equivalent to $F(b)\{0/a_1\}\cdots\{0/a_n\}$ for $b \in B$ to complete the proof.

We may assume that $b = 0$ and $F = F_{\geq 0}$. We define a subset of A by $B' = \left\{\sum_{i \in I} a_i \;\middle|\; I \subset \{1, \ldots, n\}\right\}$; note that here $\sum_{i \in I} a_i$ equals to the join $\bigvee_{i \in I} a_i$ taken in $A_{\geq 0}$. By induction we can see that $F(0)\{0/a_1\}\cdots\{0/a_n\}$ is equivalent to the left Kan extension of the object of $\mathrm{Fun}(B', \mathcal{C})$ which is the right Kan extension of $F(0) \in \mathrm{Fun}(\{0\}, \mathcal{C})$. Using the equivalence

$$F_0 \simeq (\cdots(F(0)\{0/a_1\}\cdots\{0/a_n\})_{\not\geq a_1}\cdots)_{\not\geq a_n},$$

we are reduced to showing that $(F(0)\{0/a_1\}\cdots\{0/a_n\})(c) \simeq 0$ if $c \in A$ satisfies $c \geq a_i$ for some i. This follows from the above description. □

Finally, we give the proof of Proposition 5.21.

Proof of Proposition 5.21 For a saturated subset $B \subset A$ and $a \in B$, we make the following definition:

$$\Theta(B, a) = \{J \in \mathrm{Arch}(A) \mid (a + J) \cap B = \{a\}\} \in \mathrm{P}(\mathrm{Arch}(A)).$$

We claim that this set is in the image of the monomorphism $\mathrm{Free}(\mathrm{Arch}(A)) \hookrightarrow \mathrm{P}(\mathrm{Arch}(A))$ described in Lemma 3.17. We note that $\Theta(\{a, b, a \vee b\}, a)$ is in the image for any $b \in A$ because an ideal J belongs to this set if and only if J does not contain the ideal generated by $(a \vee b) - a$. Hence $\Theta(B, a)$ is also in the image since it can be written as $\bigcap_{b \in B} \Theta(\{a, b, a \vee b\}, a)$.

For an object $F \in \mathrm{Fun}(A, \mathcal{C})^{\omega}$, we can take a saturated subset B such that F is equivalent to the left Kan extension of $F|_B$. Then we define $s(F)$, which is a priori dependent on B, as follows:

$$s(F) = \bigvee_{a \in B} \sqrt{F(a)} \wedge \Theta(B, a) \in \mathrm{Zar}(\mathcal{C}^{\omega}) \otimes \mathrm{Free}(\mathrm{Arch}(A)).$$

Here we abuse the notation by identifying $\mathrm{Free}(\mathrm{Arch}(A))$ with its image under the monomorphism $\mathrm{Free}(\mathrm{Arch}(A)) \hookrightarrow \mathrm{P}(\mathrm{Arch}(A))$.

First we wish to prove that $s(F)$ is independent of the choice of B. Let B' be another saturated subset such that F is equivalent to the left Kan extension of $F|_{B'}$. We need to prove the following equality:

$$\bigvee_{a \in B} \sqrt{F(a)} \wedge \Theta(B, a) = \bigvee_{a \in B'} \sqrt{F(a)} \wedge \Theta(B', a).$$

By considering a saturated set containing $B \cup B'$, we may assume that $B \subset B'$. For any minimal element b' of $B' \backslash B$, the subset $B' \backslash \{b'\}$ is also saturated. Hence by induction we may also assume that $B' \backslash B = \{b'\}$ for some $b' \in B'$. If b' is a minimal element of B', then we get the equality since in this case $F(b')$ is a zero object and $\Theta(B, a) = \Theta(B', a)$ for any $a \in B$. Let us consider the case when b' is not minimal. Since B is saturated, we can take the greatest element b of the set $\{a \in B \mid a \leq b'\}$. Consider an element $a \in B$. It is clear that $\Theta(B, a) \supset \Theta(B', a)$ holds and this inclusion becomes an equality if $a \not\leq b'$. In fact, it also becomes an equality if $a \leq b$ and $a \neq b$: Suppose that $J \in \Theta(B, a)$ fails to belong to $\Theta(B', a)$. Then we have $(a + J) \cap B' = \{a, b'\}$. But in this case, from the inequality $0 \leq b - a \leq b' - a$ we get $b \in (a + J) \cap B$, which contradicts our assumption. Therefore, it is enough to show that $\Theta(B, b) = \Theta(B', b) \cup \Theta(B', b')$ holds since we have $F(b) \simeq F(b')$. First we prove $\Theta(B, b) \subset \Theta(B', b) \cup \Theta(B', b')$. For $J \in \Theta(B, b) \backslash \Theta(B', b)$, we have $(b' + J) \cap B' \subset (b + J) \cap B' = \{b, b'\}$ and so $J \in \Theta(B', b')$. To prove the other inclusion, it remains to show that $\Theta(B, b) \supset \Theta(B', b')$ holds. For $J \in \Theta(B', b')$ and $a \in (b + J) \cap B$, by $0 \leq (a \vee b') - b' = a - (a \wedge b') \leq a - b \in J$, we have $a \vee b' \in (b' + J) \cap B' = \{b'\}$. This means $a = b$, which completes the proof of the well-definedness of $s(F)$.

Hence we have a function $s\colon \operatorname{Fun}(A,\mathcal{C})^\omega \to \operatorname{Zar}(\mathcal{C}^\omega)\otimes\operatorname{Free}(\operatorname{Arch}(A))$. We wish to prove that s is a support. Since it is an A-semisupport and $s(\mathbf{1})=1$ by definition, we are reduced to showing that $s(F\otimes G)=s(F)\wedge s(G)$ for $F,G\in\operatorname{Fun}(A,\mathcal{C})^\omega$. We note that the assignments $(F,G)\mapsto s(F\otimes G)$ and $(F,G)\mapsto s(F)\wedge s(G)$ are both A-semisupports in each variable. Hence by Proposition 5.16, it suffices to show that $s((C\otimes D)\{0/a_1\}\cdots\{0/a_{m+n}\})$ equals to $s(C\{0/a_1\}\cdots\{0/a_m\})\wedge s(D\{0/a_{m+1}\}\cdots\{0/a_{m+n}\})$ for $C,D\in\mathcal{C}^\omega$ and $a_1,\dots,a_{m+n}\in A_{\geq 0}$. This claim follows if we have $s(F\{0/b\})=s(F)\wedge s(\mathbf{1}\{0/b\})$ for $F\in\operatorname{Fun}(A,\mathcal{C})^\omega$ and $b\in A_{\geq 0}$. To prove this, by using Proposition 5.16 again, we may assume that F can be written as $C\{0/a_1\}\cdots\{0/a_n\}$ with $C\in\mathcal{C}^\omega$ and $a_1,\dots,a_n\in A_{\geq 0}$ satisfying $a_i\wedge a_j=0$ if $i\neq j$. We may furthermore suppose that $a_i>0$ for each i and $b>0$; otherwise the claim is trivial. For $I\subset\{1,\dots,n\}$ let a_I denote the sum $\sum_{i\in I}a_i$, which is equal to the join $\bigvee_{i\in I}a_i$ taken in $A_{\geq 0}$ by assumption. We now take the following two subsets of A, both of which are saturated by assumption:

$$\begin{aligned} B&=\{a_I \mid I\subset\{1,\dots,n\}\},\\ B'&=B\cup\{a_I+(a_{I'}\vee b)\mid I,I'\subset\{1,\dots,n\}\text{ satisfying } I\cap I'=\emptyset\}. \end{aligned}$$

Then F and $F\{0/b\}$ are left Kan extensions of $F|_B$ and $F\{0/b\}|_{B'}$, respectively. We first prove that $s(F\{0/b\})\leq s(F)\wedge s(\mathbf{1}\{0/b\})$. Since we have $s(F\{0/b\})\leq s(F)$ by the fact that s is an A-semisupport, we need to show $s(F\{0/b\})\leq s(\mathbf{1}\{0/b\})$. Unwinding the definitions, we are reduced to proving that $F\{0/b\}(c)\simeq 0$ or $\Theta(B',c)\subset\Theta(\{0,b\},0)$ holds for each $c\in B'$. If $F\{0/b\}(c)$ is not zero, $F(c)$ or $F\{b\}(c)\simeq F(c-b)$ is not zero. These two cases are treated separately as follows:

(1) If $F(c)$ is not zero, then we have either $c=0$ or $c=b$. In the former case, we have indeed $\Theta(B',0)\subset\Theta(\{0,b\},0)$ since $b\in B'$. In the latter case, the morphism $F(b)\to F(0)$ is equivalent to the identity of C. Hence $F\{0/b\}(b)$ is zero, which is a contradiction.
(2) If $F\{b\}(c)$ is not zero, then $c\in B$ or $c=a_I\vee b$ for some $I\subset\{1,\dots,n\}$. In the former case, we have $\Theta(B',c)\subset\Theta(\{0,b\},0)$ since $c+b\in B'$. In the latter case, we have $c=a_I\vee b<a_I+b\leq c+b$; if the first inequality is an equality we have $F\{b\}(c)=F\{b\}(a_I+b)\simeq F(a_I)\simeq 0$, which is a contradiction. Combining this with $a_I+b\in B'$, we have $\Theta(B',c)\subset\Theta(\{0,b\},0)$.

Next we prove that $s(F\{0/b\})\geq s(F)\wedge s(\mathbf{1}\{0/b\})$. Note that the right hand side equals to $\sqrt{C}\wedge\{J\in\Theta(B,0)\mid b\notin J\}$ by definition. Hence the claim follows from the observation that $b\notin J\in\Theta(B,0)$ implies $J\in\Theta(B',0)$.

Therefore we obtain a morphism $g\colon \operatorname{Zar}(\operatorname{Fun}(A,\mathcal{C})^\omega)\to\operatorname{Zar}(\mathcal{C}^\omega)\otimes\operatorname{Free}(\operatorname{Arch}(A))$ of distributive lattices. We observe that $g\circ f$ is the identity by checking it for elements of $\operatorname{Zar}(\mathcal{C}^\omega)$ and $\operatorname{Arch}(A_{\geq 0})$. Also, by applying Proposition 5.16, the computations $g\big(\sqrt{C}\big)=\sqrt{C}$ for $C\in\mathcal{C}^\omega$ and $g(\mathbf{1}\{0/a\})=\langle a\rangle$ for $a\in A_{\geq 0}$ show that $f\circ g$ is the identity. Hence we conclude that f is an isomorphism. □

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Data availability This is a purely theoretical paper, so I haven't used any data to produce this article.

Appendix A: Bases in higher topos theory

In this Appendix, we develop the theory of bases for sites in the ∞-categorical setting. See [14, Section B.6] for a discussion in the 1-categorical setting. The main result is Theorem A.6, which says that an ∞-site and its basis define the same ∞-topos after hypercompletion. Note that its 1-categorical version, which follows from Corollary A.7, originally appeared in [1, Exposé III, Théorème 4.1] under the name "lemme de comparaison".

The only result in this Appendix that we need in the main body of this paper is Example A.12, which is a corollary of Theorem A.6.

Remark A.1 In the previous version of the preprint [2], Asai and Shah claimed that the map i_* we describe in Example A.11 is an equivalence, which implies Example A.12. However, the claim is not true, as we see in Example A.13. As this paper is being written, its corrected version, which still covers the case we use in the main body of this paper, has appeared. They use a different argument to prove it. We also note that we prove a conjecture stated in their preprint; see Example A.11.

To begin, recall that giving a Grothendieck topology on an ∞-category is the same thing as giving that on its homotopy category (see [11, Remark 6.2.2.3]). Hence we can translate notions used for sites into the ∞-categorical setting without making any essential change. Note that we do not impose the existence of any pullbacks on ∞-categories.

Definition A.2 An ∞-*site* is a small ∞-category equipped with a Grothendieck topology.

A *basis* for an ∞-site $\mathcal{C}$ is a full subcategory $\mathcal{B} \subset \mathcal{C}$ such that for every object $C \in \mathcal{C}$ there exists a set of morphisms $\{B_i \to C \mid i \in I\}$ that satisfies $B_i \in \mathcal{B}$ for all $i \in I$ and generates a covering sieve on C. Note that in this case there exists a unique Grothendieck topology on $\mathcal{B}$ such that a sieve $\mathcal{B}^{(0)}_{/B}$ on B is covering if and only if its image under the inclusion $\mathcal{B}_{/B} \hookrightarrow \mathcal{C}_{/B}$ generates a covering sieve. We always regard a basis as an ∞-site by using this Grothendieck topology.

Example A.3 Considering the poset of open sets of a topological space equipped with the canonical topology, the notion of basis specializes to that of basis used in point-set topology.

Remark A.4 In ordinary topos theory, a basis is often referred to as a dense subsite. But in the ∞-categorical setting, the term "dense" can be misleading since a basis need not define the same ∞-topos; see [13, Example 20.4.0.1] or Example A.13.

Proposition A.5 *Let $\mathcal{C}$ be an ∞-site. Suppose that $\mathcal{B}$ is a basis for $\mathcal{C}$ and F is a presheaf on $\mathcal{B}$. Then F is a sheaf on $\mathcal{B}$ if and only if its right Kan extension is a sheaf on $\mathcal{C}$. Especially, by restricting the right Kan extension functor* $\mathrm{PShv}(\mathcal{B}) \hookrightarrow \mathrm{PShv}(\mathcal{C})$, *we obtain a geometric embedding* $\mathrm{Shv}(\mathcal{B}) \hookrightarrow \mathrm{Shv}(\mathcal{C})$.

We omit the proof of this fact because this is proven in the same way as in the 1-categorical setting; see the second and third paragraphs of the proof of [14, Proposition B.6.6], but beware that some arguments in the first paragraph cannot be translated to our setting.

The main result is the following:

Theorem A.6 *Let $\mathcal{B}$ be a basis for an ∞-site $\mathcal{C}$ and G a presheaf on $\mathcal{C}$. Then G is a hypercomplete object of the ∞-topos* $\mathrm{Shv}(\mathcal{C})$ *if and only if the following conditions are satisfied*:

(1) *The restriction $G|_{\mathcal{B}^{\mathrm{op}}}$ is a hypercomplete object of the ∞-topos* $\mathrm{Shv}(\mathcal{B})$.
(2) *The functor G is a right Kan extension of $G|_{\mathcal{B}^{\mathrm{op}}}$.*

We note that Jacob Lurie let the author know that this result could be proven using hypercoverings. Here we will give a different proof, which does not use (semi)simplicial machinery.

Before giving the proof of this theorem, let us collect its formal consequences.

Corollary A.7 *Let $\mathcal{B}$ be a basis for an ∞-site $\mathcal{C}$. Then the geometric embedding* $\mathrm{Shv}(\mathcal{B}) \hookrightarrow \mathrm{Shv}(\mathcal{C})$ *obtained in Proposition A.5 is cotopological. In particular, it induces equivalences between their hypercompletions, their Postnikov completions, their bounded reflections, and their n-localic reflections for any n.*

Corollary A.8 *Let $\mathcal{B}$ be a basis for an ∞-site $\mathcal{C}$. Suppose that both $\mathcal{B}$ and $\mathcal{C}$ are n-category for some n and have finite limits (but the inclusion need not preserve them). Then the geometric embedding obtained in Proposition A.5 is an equivalence.*

Remark A.9 In [6, Lemma C.3], Hoyois gave another sufficient condition under which the geometric embedding $\mathrm{Shv}(\mathcal{B}) \hookrightarrow \mathrm{Shv}(\mathcal{C})$ itself is an equivalence.

Our proof uses the following relative variant of [13, Lemma 20.4.5.4]:

Lemma A.10 *Let $f^*\colon \mathcal{Y} \to \mathcal{X}$ be the left adjoint of a geometric morphism between ∞-toposes and $\mathcal{D} \subset \mathcal{Y}$ an essentially small full subcategory. Suppose that for every $Y \in \mathcal{Y}$ there exists a family of objects $(V_i)_{i\in I}$ of $\mathcal{D}$ and a morphism $\coprod_{i\in I} V_i \to Y$ whose image under f^* is an effective epimorphism. Then the object $f^*\big(\varinjlim_{V\in\mathcal{D}} V\big) \in \mathcal{X}$ is ∞-connective.*

Proof This follows from a slight modification of the proof of [13, Lemma 20.4.5.4] by using the fact that f^* preserves finite limits and colimits and it determines a functor $\mathcal{Y}_{/Y} \to \mathcal{X}_{/f^*Y}$ for any object $Y \in \mathcal{Y}$, which is again the left adjoint of a geometric morphism. □

Proof of Theorem A.6 Let $i^*\colon \mathrm{PShv}(\mathcal{C}) \to \mathrm{PShv}(\mathcal{B})$ denote the restriction functor. We write i_* for its right adjoint. We let L be the sheafification functor associated to the ∞-site $\mathcal{C}$ and $j\colon \mathcal{C} \to \mathrm{PShv}(\mathcal{C})$ the Yoneda embedding.

Suppose that G is a hypercomplete object of $\mathrm{Shv}(\mathcal{C})$. We prove that the map $G \to i_*i^*G$ is an equivalence. For $C \in \mathcal{C}$, the map $G(C) \to (i_*i^*G)(C)$ can be identified with the image of the morphism $f\colon \varinjlim_{B\in\mathcal{B}_{/C}} j(B) \to j(C)$ under the functor $\mathrm{Map}_{\mathrm{PShv}(\mathcal{C})}(-, G)$. Since G is a hypercomplete object of $\mathrm{Shv}(\mathcal{C})$, it suffices to prove that the morphism Lf is ∞-connective. We can see this by applying Lemma A.10 to the geometric embedding $\mathrm{Shv}(\mathcal{C})_{/Lj(C)} \hookrightarrow \mathrm{PShv}(\mathcal{C})_{/j(C)} \simeq \mathrm{PShv}\big(\mathcal{C}_{/C}\big)$.

Let $\mathcal{X}$ denote the essential image of $\mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}}$ under i^*. It follows from what we have shown above that i^* restricts to determine an equivalence $\mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}} \to \mathcal{X}$. It also follows that the composition of left exact functors

$$\mathrm{PShv}(\mathcal{B}) \xrightarrow{i_*} \mathrm{PShv}(\mathcal{C}) \xrightarrow{X\mapsto (LX)^{\mathrm{hyp}}} \mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}} \xrightarrow{i^*} \mathcal{X}$$

is a left adjoint to the inclusion $\mathcal{X} \hookrightarrow \mathrm{PShv}(\mathcal{B})$: Indeed, for $F \in \mathrm{PShv}(\mathcal{B})$ and $G \in \mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}}$, we have

$$\begin{aligned}\mathrm{Map}(F, i^*G) &\simeq \mathrm{Map}(i_*F, i_*i^*G) \simeq \mathrm{Map}(i_*F, G) \simeq \mathrm{Map}((Li_*F)^{\mathrm{hyp}}, G)\\ &\simeq \mathrm{Map}((Li_*F)^{\mathrm{hyp}}, i_*i^*G) \simeq \mathrm{Map}(i^*((Li_*F)^{\mathrm{hyp}}), i^*G).\end{aligned}$$

Hence $\mathcal{X}$ is a subtopos of $\mathrm{PShv}(\mathcal{B})$. Using Proposition A.5, we obtain inclusions $\mathrm{Shv}(\mathcal{B})^{\mathrm{hyp}} \subset \mathcal{X} \subset \mathrm{Shv}(\mathcal{B})$ of subtoposes. Since $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}}$ is hypercomplete, the first inclusion is an equality. Therefore, the restriction of the adjoint pair (i^*, i_*) determines an equivalence $\mathrm{Shv}(\mathcal{B})^{\mathrm{hyp}} \simeq \mathrm{Shv}(\mathcal{C})^{\mathrm{hyp}}$, which is a restatement of what we wanted to show. □

We conclude this Appendix by specializing our results to the case directly related to the main body of this paper.

Example A.11 Let P be a poset. Consider the canonical topology on the poset of open sets of $\mathrm{Alex}(P)$ (see Definition 4.11 for the definition). Then the subposet of principal cosieves, which is equivalent to P^{op}, is a basis and the induced topology on it is trivial. Hence by Corollary A.7, we have a cotopological inclusion $i_*\colon \mathrm{PShv}(P^{\mathrm{op}}) \hookrightarrow \mathrm{Shv}(\mathrm{Alex}(P))$. Since $\mathrm{PShv}(P^{\mathrm{op}})$ is already hypercomplete, i_* can be identified with the inclusion $\mathrm{Shv}(\mathrm{Alex}(P))^{\mathrm{hyp}} \hookrightarrow \mathrm{Shv}(\mathrm{Alex}(P))$.

We note that this observation settles a conjecture posed by Asai and Shah in [2, Remark 2.6] affirmatively.

Example A.12 In the situation of Example A.11, the inclusion i_* is an equivalence when P is finite since the ∞-topos of sheaves on a Noetherian topological space of finite Krull dimension is hypercomplete; see [11, Section 7.2.4]. The same holds when P has finite joins since in this case $\mathrm{PShv}(P^{\mathrm{op}})$ is 0-localic. We note that the class of all posets whose associated geometric embedding is an equivalence is closed under coproducts.

Example A.13 In the situation of Example A.11, the inclusion i_* itself need not be an equivalence in general.

Consider the set $P = \mathbf{N} \times \{0, 1\}$ equipped with the ordering depicted as follows:

$$\begin{tikzcd}
(0,1) \arrow[r] \arrow[dr] & (1,1) \arrow[r] \arrow[dr] & (2,1) \arrow[r] \arrow[dr] & (3,1) \arrow[r] \arrow[dr] & (4,1) \arrow[r] \arrow[dr] & \cdots \\
(0,0) \arrow[r] \arrow[ur] & (1,0) \arrow[r] \arrow[ur] & (2,0) \arrow[r] \arrow[ur] & (3,0) \arrow[r] \arrow[ur] & (4,0) \arrow[r] \arrow[ur] & \cdots.
\end{tikzcd}$$

We can check that the locale of open sets of $\mathrm{Alex}(P)$ is coherent, so the final object of $\mathrm{Shv}(\mathrm{Alex}(P))$ is compact by [11, Corollary 7.3.5.4]. We now show that the final object of $\mathrm{PShv}(P^{\mathrm{op}})$, which is the constant functor taking the value $*$, is not compact. If so, by Corollary 2.11, we can take n such that the final object is the left Kan extension of that of $\mathrm{Fun}(\{0, \dots, n\} \times \{0, 1\}, \mathrm{S})$, but then the value at $(n + 1, 0)$ becomes S^n, which is a contradiction.

We note that this ∞-topos $\mathrm{Shv}(\mathrm{Alex}(P))$ essentially appears in [4, Example A9]. They show the failure of hypercompleteness using a different argument there.

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