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Estimating the set of bifurcation values of a smooth function

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Abstract
We introduce the framework for finding out conditions that imply that $y \in \mathbb{R}$ is a typical value of a smooth function f defined on an open subset of $\mathbb{R}^n$. We adopt this framework to obtain some currently known conditions for estimating the set of bifurcation values. Additionally, we give two new such conditions. Moreover, we show that the trivialization of f in a neighbourhood U of $f^{-1}(y)$ can always be obtained by integrating its gradient with respect to some metric on U.

Keywords Bifurcation values · Fibrations · Malgrange condition · Trivialization

Mathematics Subject Classification 14D06 · 34Cxx · 58K05

Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, i.e., a function of class C^∞. The smallest set $B \subset \mathbb{R}$, relative to the inclusion relation, such that the function

$$f|_{\mathbb{R}^n \setminus f^{-1}(B)} : \mathbb{R}^n \setminus f^{-1}(B) \to \mathbb{R} \setminus B$$

is a locally trivial smooth fibration is called *the bifurcation set of* f and is denoted by $B(f)$. In 1969 R. Thom (see [19]) proved that $B(f)$ is finite for polynomial functions f. In general, it is well known that $B(f) = K_0(f) \cup B_\infty(f)$, where $K_0(f)$ is *the set of critical values* of f and $B_\infty(f)$ is *the set of bifurcation values of* f *at infinity*, i.e. the set of points at which f is not locally trivial smooth fibration outside a compact set. In case $n = 2$, M. Coste and M.J. de la Puente in [2] gave an effective algorithm to determine the set $B(f)$ (for complex case see [7, 18]). In general, the computation of $B(f)$ is an open problem.

In order to estimate the set $B_\infty(f)$ some conditions on the function f in neighborhoods of fibers $f^{-1}(y)$ are introduced, which implies that the points y are *typical values* of f (i.e. $y \in \mathbb{R} \setminus B(f)$). One of the most frequently used is the Malgrange's condition. We say that f satisfies *Malgrange's condition* at a point $y \in \mathbb{R}$ if there exists a neighborhood $U \subset \mathbb{R}$ of the

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point y and constants $R, \delta > 0$ such that

$$|\nabla f(x)||x| \geq \delta \text{ for } x \in f^{-1}(U), \ |x| > R,$$

where $|x|$ is an Euclidean norm of $x \in \mathbb{R}^n$. By $K_\infty(f)$ we denote the set of *asymptotic critical values of* f, i.e. the set of points where f does not satisfy Malgrange's condition:

$$K_\infty(f) = \{y \in \mathbb{R} : \exists_{(x_k)_{k=1}^\infty \subset \mathbb{R}^n} \lim_{k\to\infty} |x_k| = +\infty, \lim_{k\to\infty} f(x_k) = y,$$
$$\lim_{k\to\infty} |x_k||\nabla f(x_k)| = 0\}.$$

It is well known (see i.e. [15, 17]) that $B_\infty(f) \subset K_\infty(f)$ and that the set $K_\infty(f)$ is finite, provided f is a polynomial (see i.e. [10–12]).

In this paper we prove a theorem which gives a sufficient and necessary conditions for a point y to be a typical value of a function f defined on an open set $D_f \subset \mathbb{R}^n$. To achive this we introduce sets $\overline{O}_y(f)$ and $O_y(f)$ (see the beginning of Sect. 2). They consist of parameters (v, h, f^*) that can be chosen accordingly to f and y to produce conditions for estimating the set $B(f)$. Roughly speaking:

- v is a vector field that is transversal to the fibers of f,
- h is a function such that $h(x)$ give us the information how far from infinity or the border of D_f the point x is,
- f^* is a function such that $f^*(x)$ measures how far from the fiber $f^{-1}(y)$ the point x is.

Given the above, the boundedness of $\partial_v h/\partial_v f^*$ on the solutions of system $x' = v(x)$ gives the sufficient condition to construct trivialization of f near the fiber $f^{-1}(y)$ by integrating v. More precisely, as show in Theorem 2.1, $\overline{O}_y(f) \neq \emptyset$ is sufficient to conclude that y is a typical value of f. The inverse to the above statement is also true (see Theorem 2.5). As a corollary we get that every trivialization of f near the typical value can be realized by integrating a vector field $\nabla_g f$ with respect to some metric tensor g (see Corollary 2.6).

We end the Sect. 2 showing how one can use the above theorems to get well known conditions for trivializing a function (see Corollary 2.7) and introduce two new conditions which can be regarded as an improvement to the corresponding classical conditions (see Proposition 2.8).

The last section is devoted to simple examples of calculations the set $B(f)$ when the Malgrange's condition does not give the optimal upper bound or when the domain of f is a proper subset of $\mathbb{R}$. These examples illustrate how to use the main theorems of the paper.

1 Preliminary

In this section we will present some definitions and notations that we will use later.

Let M, N be smooth manifolds and $k \in \mathbb{N} \cup \{0, \infty\}$. By $C^k(M, N)$ we denote the set of all mappings $f : M \to N$ that are C^k class. When $N = \mathbb{R}$ we omit the second parameter and write $C^k(M)$. If M is equipted with a metric tensor g we denote by $\nabla_g f$ the gradient of f with respect to g. In the case when M is an open subset of $\mathbb{R}^n$ and g is a standard inner product we write ∇f.

Let $D_f, D_v \subset \mathbb{R}^n$ be open sets. We say that a vector field $v \in C^\infty(D_v, \mathbb{R}^n)$ is *transversal to the level sets of* $f \in C^\infty(D_f)$ on $D \subset D_f \cap D_v$ if

$$\partial_{v(x)} f(x) \neq 0 \quad \text{for } x \in D. \tag{1}$$

We denote by $\pitchfork(f, D)$ the set of all vector fields transversal to the level sets of $f \in C^{\infty}(D_f, \mathbb{R})$ on $D \subset D_f$. Obviously, if $\nabla f(x) \neq 0$ for $x \in D$ then $\nabla f \in \pitchfork(f, D)$ and $\pitchfork(f, D) \neq \emptyset$. Conversely, if $\pitchfork(f, D) \neq \emptyset$ then $\nabla f(x) \neq 0$ for $x \in D$ and $\nabla f \in \pitchfork(f, D)$. Denote

$$\pitchfork_{+}(f, D) := \{v \in \pitchfork(f, D) : \partial_{v(x)} f(x) > 0 \text{ for } x \in D\}.$$

Obviously $\pitchfork_{+}(f, D) \subset \pitchfork(f, D)$. The set $\pitchfork_{+}(f, D)$ can be geometrically described as the set of all gradients of f with respect to some metric tensor g on D. More precisely let v be a continuous tangent vector field on a manifold M. We say that a continuously differentiable function $E : M \to \mathbb{R}$ is a strict Lyapunov function for $x' = -v(x)$ if

$$dE(x)(v(x)) > 0 \ \text{ for } x \in M, \ v(x) \neq 0. \tag{2}$$

In [1, Theorem 1] authors prove the following:

Theorem 1.1 *Let M be a manifold, v a continuous tangent vector field on M and let $E : M \to \mathbb{R}$ be a continuously differentiable, strict Lyapunov function for $x' = -v(x)$. Then there exists a Riemannian metric g on the open set $\tilde{M} := \{x \in M : v(x) \neq 0\}$ such that*

$$\nabla_g E(x) = v(x) \textit{ for } x \in \tilde{M}.$$

From the above we get

Corollary 1.2 *Let $v \in \pitchfork_{+}(f, D)$. Then there exists a Riemannian metric g on D such that*

$$\nabla_g f(x) = v(x) \textit{ for } x \in D.$$

Let $D \subset \mathbb{R}^n$ be an open set, $v \in C^{\infty}(D, \mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Define

$$\begin{aligned} &\pitchfork(v, D, A) := \{f \in C^0(D_f) : D \subset D_f,\ f|_{D\setminus A} \in C^{\infty}(D\setminus A), \\ &\partial_{v(x)} f(x) \neq 0 \text{ for } x \in D\setminus A\}, \\ &\pitchfork(v, D) := \pitchfork(v, D, \emptyset). \end{aligned} \tag{3}$$

Let $D \subset \mathbb{R}^n$ be an open set $f \in C^{\infty}(D)$, $v \in \pitchfork(f, D)$, $y \in \mathbb{R}$. For $x \in D$ denote $\varphi_x : I_x \to D$ the integral solution of $x' = v(x)$ satisfying $\varphi_x(0) = x$ and define

$$J_x := \{t \in I_x : \min\{f(x), y\} \leq f(\varphi_x(t)) \leq \max\{f(x), y\}\}.$$

By $[f, y]_v^D$ we denote the set of all functions $f^* \in \pitchfork(v, D, f^{-1}(y))$ such that for $x \in D$ the function

$$J_x \ni t \to f^*(\varphi_x(t)) \in \mathbb{R}$$

is bounded. Note that $f \in [f, y]_v^D$.

We say that $h \in C^1(D_h)$ is proper if for every compact set $K \subset \mathbb{R}$ the set $h^{-1}(K)$ is compact in D_h.

2 Main results

Let $f \in C^{\infty}(D_f)$ where D_f is an open subset of $\mathbb{R}^n$.

Denote by Γ the set of all triples (v, h, f^*) where $v \in C^{\infty}(D_v, \mathbb{R}^n)$, $h \in C^1(D_h)$, $f^* \in C^0(D_{f^*})$ and D_v, D_h, D_{f^*} are open in $\mathbb{R}^n$.

Define $\overline{O}_y(f)$ as the set of all triples $(v, h, f^*) \in \Gamma$ for which there exists a neighborhood U of y and a compact set $K \subset D_f$ such that the following conditions are satisfied :

($\overline{O}_y(f)$-i) $v \pitchfork (f, f^{-1}(U))$,
($\overline{O}_y(f)$-ii) $f^{-1}(U)\backslash K \subset D_h$ and for every compact sets $K_1 \subset U, K_2 \subset \mathbb{R}$ such that $y \in K_1$ the set $f^{-1}(K_1) \cap h^{-1}(K_2)$ is compact
($\overline{O}_y(f)$-iii) $f^* \in [f, y]_v^{f^{-1}(U)\backslash K}$ and for any solution $\varphi : I \to f^{-1}(U)\backslash K$ of the system

$$x' = v(x), \quad x \in f^{-1}(U)\backslash K \tag{4}$$

the function $H : f^{-1}(U)\backslash(K \cup f^{-1}(y)) \to \mathbb{R}$ defined by

$$H(x) := \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f^*(x)} \tag{5}$$

is bounded on $\varphi(I)\backslash f^{-1}(y)$.

Theorem 2.1 *If $\overline{O}_y(f) \neq \emptyset$ then y is a typical value of f. More precisely, then there is a neighborhood U of y, a vector field $v \pitchfork (f, f^{-1}(U))$ such that the trivialization of $f|_{f^{-1}(U)}$ can be realized by integrating the vector field v.*

Proof Choose $(v, h, f^*) \in \Gamma$ and $U \subset R$, $K \subset \mathbb{R}^n$ such that conditions ($\overline{O}_y(f)$-i),($\overline{O}_y(f)$-ii), ($\overline{O}_y(f)$-iii) are satisfied. Then $\partial_{v(x)} f(x) \neq 0$ for $x \in f^{-1}(U)$.

Consider the system of differential equations with a parameter $\mu \in U$

$$x' = \frac{(y - \mu)}{\partial_{v(x)} f(x)} v(x) \tag{6}$$

with right-hand side defined on $\mathbb{R} \times f^{-1}(U)$.

Denote by $\Phi_\mu : V_\mu \to f^{-1}(U)$ the general solution of (6), where

$$V_\mu := \{(\tau, \eta, t) \in \mathbb{R} \times f^{-1}(U) \times \mathbb{R} : t \in I_\mu(\tau, \eta)\},$$

and $I_\mu(\tau, \eta)$ is the domain of the integral solution $t \to \Phi_\mu(\tau, \eta, t)$ and the equation $\Phi_\mu(\tau, \eta, \tau) = \eta$ is satisfied.

We will show that $1 \in I_{f(x)}(0, x)$ for $x \in f^{-1}(U)$. Suppose the contrary that $1 \notin I_{f(x)}(0, x)$ for some $x \in f^{-1}(U)$. Then the right end point β of $I_{f(x)}(0, x)$ satisfies $0 < \beta \leq 1$. Let φ_x be the integral solution of (6) with $\mu = f(x)$ satisfying $\varphi_x(0) = x$. We easily check that

$$f \circ \varphi_x(t) = (y - f(x))t + f(x), \quad t \in I_{f(x)}(0, x) \tag{7}$$

and $f \circ \varphi_x(t) \in P$ for $t \in [0, \beta)$, where P is a closed interval with endpoints y and $f(x)$. Consider the set

$$K' := \{(t, x') \in \mathbb{R} \times f^{-1}(U) : t \in [0, 1],\ f(x') \in P,\ x' \in K\}.$$

Since $P \subset U$, we see that K' is compact. Therefore, there exists $\tau \in (0, \beta)$ such that $(t, \varphi_x(t)) \notin K'$ for $t \in [\tau, \beta)$. Given that $f \circ \varphi_x(t) \in P$ for $t \in [\tau, \beta)$ we have $\varphi_x(t) \notin K$ for $t \in [\tau, \beta)$.

Set $\overline{x} := \varphi_x(\tau)$. Given the above, $1 \notin I_{f(x)}(0, x)$ and (7), we have $\overline{x} \in f^{-1}(U)\backslash(K \cup f^{-1}(y))$. Denote by $\varphi_{\overline{x}} : I_{\overline{x}} \to f^{-1}(U)\backslash K$ the integral solution of the system (4)[1] satisfying $\varphi_{\overline{x}}(0) = \overline{x}$. Let $J_{\overline{x}} := \{t \in I_{\overline{x}} : \min\{f(\overline{x}), y\} \leq f(\varphi_{\overline{x}}(t)) \leq \max\{f(\overline{x}), y\}\}$. Obviously $J_{\overline{x}}$ is an interval and the set $f \circ \varphi_{\overline{x}}(J_{\overline{x}})$ is included in closed interval with endpoints $f(\overline{x})$ and y.

[1] note that $\varphi_{\overline{x}}$ and φ_x are the solutions of different systems.

From ($\overline{O}_y$-iii) there exist $L_{\overline{x}}, M_{\overline{x}} \in \mathbb{R}$ such that

$$|f^*(\varphi_{\overline{x}}(t))| \leq M_{\overline{x}} \text{ for } t \in J_{\overline{x}}, \tag{8}$$

$$|H(\varphi_{\overline{x}}(t))| \leq L_{\overline{x}} \text{ for } t \in J_{\overline{x}} \setminus \{t \in J_{\overline{x}} : f(\varphi_{\overline{x}}(t)) = y\}. \tag{9}$$

We will show that the function $J_{\overline{x}} \ni t \to |h(\varphi_{\overline{x}}(t))| \in \mathbb{R}$ is bounded. Indeed, from (9) for $t \in J_{\overline{x}} \setminus \{t \in J_{\overline{x}} : f(\varphi_{\overline{x}}(t)) = y\}$ we have

$$\begin{aligned}|h(\varphi_{\overline{x}}(t))| &\leq |h(\varphi_{\overline{x}}(t)) - h(\varphi_{\overline{x}}(0))| + |h(\varphi_{\overline{x}}(0))| \\ &= \left|\int_0^t \frac{d}{ds} h(\varphi_{\overline{x}}(s))ds\right| + |h(\varphi_{\overline{x}}(0))| \\ &= \left|\int_0^t (\partial_{v(\varphi_{\overline{x}}(s))} h)(\varphi_{\overline{x}}(s))ds\right| + |h(\varphi_{\overline{x}}(0))| \\ &\leq L_{\overline{x}} \int_0^t |(\partial_{v(\varphi_{\overline{x}}(s))} f^*)(\varphi_{\overline{x}}(s))|ds + |h(\varphi_{\overline{x}}(0))| \\ &= L_{\overline{x}} \left|\int_0^t (\partial_{v(\varphi_{\overline{x}}(s))} f^*)(\varphi_{\overline{x}}(s))ds\right| + |h(\varphi_{\overline{x}}(0))| \\ &= L_{\overline{x}} |f^*(\varphi_{\overline{x}}(t)) - f^*(\overline{x})| + |h(\varphi_{\overline{x}}(0))|.\end{aligned} \tag{10}$$

Therefore, from the continuity of the function $f^* \circ \varphi_{\overline{x}}$ and (8) we get that

$$|h(\varphi_{\overline{x}}(t))| \leq L_{\overline{x}}(M_{\overline{x}} + |f^*(\overline{x})|) + |h(\varphi_{\overline{x}}(0))| \text{ for } t \in J_{\overline{x}}.$$

From ($\overline{O}_y$-ii) we get that $\varphi_{\overline{x}}(J_{\overline{x}})$ is contained in a compact subset of $f^{-1}(U)$. Since $\varphi_x([0, \beta)) \subset \varphi_{\overline{x}}(J_{\overline{x}})$, $\varphi_x([0, \beta))$ is contained in a compact subset of $f^{-1}(U)$ which contradicts the assumption that $\varphi_x : I_{f(x)}(0, x) \to f^{-1}(U)$ is the integral solution of (6). In conclusion, we showed that $1 \in I_{f(x)}(0, x)$.

Consider the mapping

$$\Psi_1 : f^{-1}(U) \ni x \mapsto \Phi_{f(x)}(0, x, 1) \in f^{-1}(y).$$

The mapping Ψ_1 is defined correctly. Indeed, $1 \in I_{f(x)}(0, x)$ and from (7) we get $f(\Phi_{f(x)}(0, x, 1)) = f(\varphi_x(1)) = y$. Similarly as above we show that the mapping

$$\Theta : f^{-1}(y) \times U \ni (\xi, \mu) \mapsto \Phi_\mu(1, \xi, 0) \in f^{-1}(U)$$

is also well defined. It is easy to check that

$$\Psi : f^{-1}(U) \ni x \mapsto (\Psi_1(x), f(x)) \in f^{-1}(y) \times U$$

is a C^∞ diffeomorphism and $\Psi^{-1} = \Theta$. Therefore, $f|_{f^{-1}(U)}$ is a C^∞ trivial fibration. □

Define $O_y(f)$ as the set of all triples $(v, h, f^*) \in \Gamma$ for which there exists a neighborhood U of y and a compact set $K \subset D_f$ such that ($\overline{O}_y(f)$-i), ($\overline{O}_y(f)$-ii) and the following conditions are satisfied:

($O_y(f)$-iii) $f^* \in [f, y]_v^{f^{-1}(U)\setminus K}$ and the function $H : f^{-1}(U)\setminus(K \cup f^{-1}(y)) \to \mathbb{R}$ defined as in (5) is bounded.

Obviously $O_y(f) \subset \overline{O}_y(f)$, therefore directly from Theorem 2.1 we get the following

Theorem 2.2 *If $O_y(f) \neq \emptyset$ then y is a typical value of f. More precisely, then there is a neighborhood U of y, a vector field $v \Subset\pitchfork (f, f^{-1}(U))$ such that the trivialization of $f|_{f^{-1}(U)}$ can be realized by integrating the vector field v.*

The inverse to the above theorem is true. Indeed, we have the following

Theorem 2.3 *Let y be a typical value of $f \in C^\infty(D_f)$. Let $h_y \in C^\infty(f^{-1}(y))$ be a proper function. Then there is a neighborhood U of y, a vector field $v \Subset\pitchfork_+ (f, f^{-1}(U))$ and a function $h \in C^\infty(f^{-1}(U))$ satisfying the condition $(\overline{O}_y(f)$*-ii) such that

$$H(x) = \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f(x)} = 0 \tag{11}$$

for $x \in f^{-1}(U)$. In particular, $O_y(f) \neq \emptyset$.

Proof Let $U \subset \mathbb{R}$ be a neighborhood of y and $\Psi_1 : f^{-1}(U) \to f^{-1}(y)$ a mapping such that

$$\Psi = (\Psi_1, f) : f^{-1}(U) \ni x \mapsto (\Psi_1(x), f(x)) \in f^{-1}(y) \times U$$

is a C^∞ diffeomorphism. Shrinking U we can assume that $U = (-\varepsilon + y, \varepsilon + y)$. For $(\overline{x}, t) \in f^{-1}(y) \times U$ we have

$$(\Psi_1(\Psi^{-1}(\overline{x}, t)), f(\Psi^{-1}(\overline{x}, t))) = \Psi(\Psi^{-1}(\overline{x}, t)) = (\overline{x}, t).$$

Since for $(\overline{x}, t) \in f^{-1}(y) \times U$ we get

$$f(\Psi^{-1}(\overline{x}, t)) = t, \tag{12}$$

$$\frac{\partial}{\partial t} f(\Psi^{-1}(\overline{x}, t)) = 1. \tag{13}$$

Define a vector field $v \in C^\infty(f^{-1}(U), \mathbb{R}^n)$ as

$$v(x) := \frac{\partial}{\partial t} \Psi^{-1}(\overline{x}, t)|_{(\overline{x},t)=(\Psi_1(x), f(x))} \text{ for } x \in f^{-1}(U).$$

We will show that $v \Subset\pitchfork_+ (f, f^{-1}(U))$. Set $x \in f^{-1}(U)$ and put

$$\varphi(t) := \Psi^{-1}(\Psi_1(x), t + f(x)) \text{ for } t \in (-\varepsilon + y - f(x),\ \varepsilon + y - f(x)). \tag{14}$$

From $f(x) \in U = (-\varepsilon + y,\ \varepsilon + y)$ we get $0 \in (-\varepsilon + y - f(x),\ \varepsilon + y - f(x))$. Moreover

$$\varphi(0) = \Psi^{-1}(\Psi_1(x), f(x)) = \Psi^{-1}(\Psi(x)) = x, \tag{15}$$

$$\varphi'(0) = \frac{d}{dt}(\Psi^{-1}(\Psi_1(x), t + f(x))|_{t=0} = \frac{\partial}{\partial t} \Psi^{-1}(\overline{x}, t)|_{(\overline{x},t)=(\Psi_1(x), f(x))} = v(x).$$

Therefore from (13) we have

$$\partial_{v(x)} f(x) = \partial_{\varphi'(0)} f(\varphi(0)) = \frac{d}{dt} f(\varphi(t))|_{t=0} =$$
$$\frac{d}{dt} f(\Psi^{-1}(\Psi_1(x), t + f(x))|_{t=0} = 1.$$

In conclusion $v \Subset\pitchfork_+ (f, f^{-1}(U))$.

Define $h : f^{-1}(U) \to \mathbb{R}$ as

$$h(x) := h_y(\Psi_1(x)) \text{ for } x \in f^{-1}(U).$$

Note that for compact sets $K_1 \subset U, K_2 \subset \mathbb{R}$ such that $y \in K_1, h^{-1}(K_2) \neq \emptyset$ the set $f^{-1}(K_1) \cap h^{-1}(K_2)$ is diffeomorphic to $\Psi(f^{-1}(K_1) \cap h^{-1}(K_2)) = h_y^{-1}(K_2) \times K_1$ and therefore the condition ($\overline{O}_y(f)$-ii) is satisfied. Now we will show that

$$\partial_{v(x)} h(x) = 0 \text{ for } x \in f^{-1}(U).$$

Indeed, for every $(\overline{x}, t) \in f^{-1}(y) \times U$ we have $\Psi_1(\Psi^{-1}(\overline{x}, t)) = \overline{x}$. Hence

$$\partial_{v(x)} h(x) = \frac{\partial}{\partial t} h_y(\Psi_1(\Psi^{-1}(\Psi_1(x), t + f(x))))|_{t=0} = 0 \text{ for } x \in f^{-1}(U)$$

which proves (11) and ends the proof. □

Remark 2.4 Note that in the case when the boundary of the set D_f is described as the zero set of a function $g \in C^\infty(\mathbb{R}^n)$, for all $y \in \mathbb{R}$ one can choose $h_y \in C^\infty(f^{-1}(y))$ defined as

$$h_y(x) := |x|^2 + \frac{1}{(g(x))^2} \text{ for } x \in f^{-1}(y).$$

In particular, if $D_f = \mathbb{R}^n$ then for all $y \in \mathbb{R}$ we can put $h_y(x) := |x|^2$, for $x \in f^{-1}(y)$.

Using theorem 2.2 and theorem 2.3 we get

Theorem 2.5 *Let $f \in C^\infty(D_f)$, $y \in \mathbb{R}$. Then y is a typical value of f if and only if $O_y(f) \neq \emptyset$.*

Proof “ $\Rightarrow$” Assume that y is a typical value of f. If $D_f = \mathbb{R}^n$ put $h_y(x) := |x|^2$, for $x \in f^{-1}(y)$. If $D_f \subsetneq \mathbb{R}^n$ then from Whitney extension theorem there exists a function $g \in C^\infty(\mathbb{R}^n)$ such that $D_f = \{x \in \mathbb{R}^n : g(x) \neq 0\}$. In this case define $h_y \in C^\infty(f^{-1}(y))$ as in Remark 2.4. In both cases h_y is a proper function. Therefore, from Theorem 2.3 we get that $O_y(f) \neq \emptyset$.

“ $\Leftarrow$” Immediately follows from Theorem 2.2. □

Given the above, Corollary 1.2 and the proof of Theorem 2.1 we get

Corollary 2.6 *Let $f \in C^\infty(D_f)$, $y \in \mathbb{R}$. If y is a typical value of f then there exists a Riemannian metric g on the neighbourhood of $f^{-1}(y)$ such that the trivialization of f in the neighbourhood of $f^{-1}(y)$ can be realized by integrating a vector field $\nabla_g f$.*

Now we prove some known theorems with conditions implying the trivialization of the functions in a neighborhood of a fiber (see i.e. 3,12,15,17).

Corollary 2.7 *Let $f \in C^\infty(\mathbb{R}^n)$, $y \in U \subset \mathbb{R}$ and suppose that $\nabla f(x) \neq 0$ for $x \in f^{-1}(U)$. Then the following conditions are sufficient for y to be a typical value of f*

- *(E)* (Ehresmann's lemma) f *is a proper function,*
- *(F)* (Fedoryuk's condition) *there exist* $R, \delta > 0$ *such that*

$$|\nabla f(x)| \geq \delta \qquad \text{for } x \in f^{-1}(U), \ |x| > R,$$

- *(M)* (Malgrange's condition) *there exist* $R, \delta > 0$ *such that*

$$|\nabla f(x)||x| \geq \delta \qquad \text{for } x \in f^{-1}(U), \ |x| > R,$$

- *(K-Ł)* (Kurdyka-Łojasiewicz exponent) *there exist* $R, C > 0$ *and* $\theta < 1$ *such that*

$$|x| \cdot |\nabla f(x)| \geq C|f(x) - y|^\theta \text{ for } x \in f^{-1}(U), |x| > R,$$

(R) *there exists* $R > 0$ *and a continuous, non-negative function* $\lambda : (R, +\infty) \to \mathbb{R}$ *satisfying*

$$\int_R^{+\infty} \lambda(s)ds = +\infty,$$
$$|\nabla f(x)| \geq \lambda(|x|) \text{ for } x \in f^{-1}(U), |x| > R.$$

Proof From theorem 2.5 we need to show that each from the above conditions implies $O_y(f) \neq \emptyset$.

(E) From (E) we get $(\nabla f, f, f) \in O_y(f)$.

(F) From (F) we get $(\nabla f, h_1, f) \in O_y(f)$, where

$$h_1(x) := |x| \text{ for } x \neq 0.$$

(M) From (M) we get $(\nabla f, h_2, f) \in O_y(f)$, where

$$h_2(x) := \ln(|x|^2) \text{ for } x \neq 0.$$

(K-Ł) From (K-Ł) we get $(\nabla f, h_2, f_\theta) \in O_y(f)$, where h_2 as above and

$$f_\theta(x) := |f(x) - y|^{1-\theta} \text{ for } x \in \mathbb{R}^n.$$

(R) From (R) we get $(\nabla f, h_3, f) \in O_y(f)$, where

$$h_3(x) := \int_R^{|x|} \lambda(s)ds \text{ for } |x| > R.$$

□

Note that for proofs in (M) and (K-Ł) we actually need weaker assumptions. More precisely, let $a \in \mathbb{R}^n$ and put $h(x) = \ln(|x - a|^2)$ for $x \neq a$. Considering triples $(\nabla f, h, f)$ and $(\nabla f, h, f_\theta)$ as in Corollary 2.7 we get

Proposition 2.8 *Let* $f \in C^\infty(\mathbb{R}^n)$, $y \in U \subset \mathbb{R}$, $a \in \mathbb{R}^n$ *and suppose that* $\nabla f(x) \neq 0$ *for* $x \in f^{-1}(U)$. *Then the following conditions are sufficient for* y *to be a typical value of* f

(M_a) *there exist* $R, C > 0$ *such that*

$$|\langle \nabla f(x), x - a \rangle| \leq C|x - a|^2 |\nabla f(x)|^2 \quad \text{for } x \in f^{-1}(U),\ |x| > R,$$

(K-Ł$_a$) *there exist* $R, C > 0$ *and* $\theta < 1$ *such that*

$$|f(x) - y|^\theta |\langle \nabla f(x), x - a \rangle| \leq C|x - a|^2 |\nabla f(x)|^2 \text{ for } x \in f^{-1}(U),\ |x| > R.$$

If $a = 0 \in \mathbb{R}^n$ the conditions (M_a) and (K-Ł$_a$) can be seen as improvements of conditions (M) and (K-Ł) respectively in the sense that if the condition (M) is satisfied then (M_a) is satisfied (the sames goes for (K-Ł) and (K-Ł$_a$)). Therefore, they could give a better estimation of the set $B(f)$.

3 Examples

We give an example of a function with typical value at 0 that does not satisfied Malgrange's condition at 0 (and even weaker condition (M_a) with $a = (0, 0)$ from Proposition 2.8). We show that changing one element of the triple $(\nabla f, h, f)$ can result in a better estimation of the set $B(f)$. In the example we use the following:

Lemma 3.1 *Let* $v \in C^{\infty}(D, \mathbb{R}^n)$*,* $f \in\pitchfork (v, D)$*,* $y \in \mathbb{R}$*. If* $f^* \in\pitchfork (v, D, f^{-1}(y))$ *and there exists* $y^* \in \mathbb{R}$ *such that*

$$\forall_{x \in D \setminus f^{-1}(y)} \quad \partial_{v(x)} f(x) \partial_{v(x)} f^*(x)(f(x) - y)(f^*(x) - y^*) \geq 0, \tag{16}$$

then $f^* \in [f, y]_v^D$*.*

Proof Set $x \in D$ and denote by $\varphi_x : I_x \to D$ the solution of $x' = v(x)$ satisfying $\varphi_x(0) = x$. Define

$$J_x := \{t \in I_x : \min\{f(x), y\} \leq f(\varphi_x(t)) \leq \max\{f(x), y\}\}.$$

If $x \in f^{-1}(y)$ then J_x is a point and the function $f^* \circ \varphi$ is bounded on J_x. Assume that $x \in D \setminus f^{-1}(y)$. Consider cases with regard to signs of $\partial_{v(x)} f(x)$ and $f(x) - y$.

Case $\partial_{v(x)} f(x) > 0$, $f(x) - y < 0$.

Given the above we have $J_x = \{t \in I_x : f(x) \leq f(\varphi_x(t)) \leq y\}$. Therefore $\min(f \circ \varphi_x|_{J_x}) = f(x) = f(\varphi(0))$ and

$$f(\varphi_x(t))) \leq y \text{ for } t \in J_x. \tag{17}$$

Since $f \in\pitchfork (v, D)$, $\partial_{v(x)} f(x) > 0$ and J_x is connected, $f \circ \varphi_x|_{J_x}$ is increasing and $J_x \subset [0, +\infty)$. Consider subcases:

Subcase $\partial_{v(x)} f^*(x) > 0$.

Then $\partial_{v(\varphi_x(t))} f^*(\varphi_x(t)) > 0$ for $t \in \mathrm{Int} J_x$, where $\mathrm{Int} J_x$ is an interior of J_x. Therefore $f^* \circ \varphi_x|_{J_x}$ is increasing. Since $0 \in J_x \subset [0, +\infty)$ we have $\min(f^* \circ \varphi_x|_{J_x}) = f^*(\varphi_x(0)) = f^*(x)$. Moreover, from (16) i (17) we get $f^*(\varphi_x(t)) \leq y^*$ for $t \in J_x$. This proves $f^* \in [f, y]_v^D$.

Subcase $\partial_{v(x)} f^*(x) < 0$.

Then $\partial_{v(\varphi_x(t))} f^*(\varphi_x(t)) < 0$ for $t \in \mathrm{Int} J_x$, where $\mathrm{Int} J_x$ is an interior of J_x. Therefore $f^* \circ \varphi_x|_{J_x}$ is decreasing. Since $0 \in J_x \subset [0, +\infty)$ we have $\max(f^* \circ \varphi_x|_{J_x}) = f^*(\varphi_x(0)) = f^*(x)$. Moreover, from (16) i (17) we get $f^*(\varphi_x(t)) \geq y^*$ for $t \in J_x$. This proves $f^* \in [f, y]_v^D$.

The remaining cases can be proved analogously. □

Example 1 Let $f \in C^{\infty}(\mathbb{R}^2)$ be defined as

$$f(x, y) := \frac{y}{1 + x^2} \text{ for } (x, y) \in \mathbb{R}^2.$$

Using i.e. the Malgrange's condition it can be shown that $B(f) \subset \{0\}$. The function f does not satisfy the Malgrange's condition at 0. Moreover, we show that $(\nabla f, h, f) \notin O_0(f)$ where

$$h(x, y) := \ln(x^2 + y^2) \text{ for } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Indeed, consider the sequence $(x_n, y_n) = (n, \frac{1+n^2}{n})$, $n \in \mathbb{N}$. Obviously we have $\lim_{n \to \infty} |(x_n, y_n)| = +\infty$ and $\lim_{n \to \infty} f(x_n, y_n) = 0$. Furthermore

$$\lim_{n \to \infty} |H(x_n, y_n)| = \lim_{n \to \infty} \left| \frac{\partial_{\nabla f(x_n, y_n)} h(x_n, y_n)}{\partial_{\nabla f(x_n, y_n)} f(x_n, y_n)} \right|$$
$$= \lim_{n \to \infty} \frac{|1 - n^2|(1 + n^2)}{n^3 \left(1 + \left(\frac{1}{n^2} + 1\right)^2\right) 5} = +\infty,$$

which gives $(\nabla f, h, f) \notin O_0(f)$ (compare to condition (M_a) with $a = (0,0)$ from Proposition 2.8). In particular, f does not satisfy Malgrange's condition at 0.

Now we show how one can change the triple $(\nabla f, h, f)$ to prove that 0 is a typical value of f using Theorem 2.2.

Denote $U := (-1,1)$ and $K := \{0\} \times [-1,1]$.

a) Changing ∇f in $(\nabla f, h, f)$.

Define $v_1 \in C^\infty(f^{-1}(U)\backslash K)$ as

$$v_1(x,y) = (-2xy, 2x^2) \text{ for } (x,y) \in f^{-1}(U)\backslash K.$$

Then $v_1 \pitchfork_+ (f, f^{-1}(U)\backslash K)$ and obviously $\nabla f \pitchfork_+ (f, f^{-1}(U))$. Using partition of unity we construct a vector field $v \in C^\infty(f^{-1}(U))$ such that $v_1 \pitchfork_+ (f, f^{-1}(U))$ and

$$v(x,y) = v_1(x,y) \text{ for } (x,y) \in f^{-1}(U), |(x,y)| > R,$$

for some constant $R > 0$. Therefore

$$\partial_v h(x) = \frac{2}{|(x,y)|^2}\big| -2x^2y + 2x^2y\big| = 0 \text{ for } (x,y) \in f^{-1}(U), |(x,y)| > R.$$

That implies boundedness of $H = \frac{\partial_v h}{\partial_v f}$ on $f^{-1}(U)$ and proofs $(v, h, f) \in O_0(f)$. From Theorem 2.2 we get that 0 is a typical value of f.

b) Changing h in $(\nabla f, h, f)$.

Let $D_{h_2} := \mathbb{R}^n \backslash \{(x,y) \in \mathbb{R}^2 : x = 0\}$. Define $h_2 : D_{h_2} \to \mathbb{R}$ as

$$h_2(x,y) := \frac{1}{2}x^2 + y^2 + \frac{1}{2}\ln(x^2) \text{ for } (x,y) \in D_{h_2}.$$

The reader can check that the condition ($\overline{O}_y(f)$-ii) is satisfied. Moreover, for $(x,y) \in f^{-1}(U)\backslash K$ we have

$$\partial_{\nabla f(x,y)} h_2(x,y) = \left\langle \left(\frac{1+x^2}{x}, 2y\right), \left(\frac{-2xy}{(1+x^2)^2}, \frac{1}{1+x^2}\right)\right\rangle = 0.$$

Therefore $H = \frac{\partial_v h_2}{\partial_v f} = 0$ on $f^{-1}(U)\backslash K$ and $(v, h_2, f) \in O_0(f)$. From Theorem 2.2 we get that 0 is a typical value of f.

c) Changing f in $(\nabla f, h, f)$.

Let $K_2 := \{(x,y) \in \mathbb{R}^2 : x \in [-1,1], (x,y) \in \overline{f^{-1}(U)}\}$. Define $f^* \in C^\infty(\mathbb{R}^2)$ as

$$f^*(x,y) = y \text{ for } (x,y) \in \mathbb{R}^2.$$

Then for $(x,y) \in \mathbb{R}^2$ we have

$$\partial_{\nabla f(x,y)} f^*(x,y) = \frac{1}{1+x^2} > 0,$$

which gives $f^* \pitchfork (\nabla f, f^{-1}(U), f^{-1}(0))$. Obviously

$$f(x,y)f^*(x,y) = \frac{y^2}{1+x^2} \geq 0 \text{ for } (x,y) \in \mathbb{R}^2.$$

Therefore, the condition (16) is satisfied with $y^* = 0$ and from Lemma 3.1 we have $f^* \in [f, y]_{\nabla f}^{f^{-1}(U)}$. Moreover, for $(x, y) \in f^{-1}(U) \setminus K_2$, $y \neq 0$ we have

$$|H(x, y)| = \left| \frac{\partial_{\nabla f(x,y)} h(x, y)}{\partial_{\nabla f(x,y)} f^*(x, y)} \right|$$
$$= \frac{|2y|}{(x^2 + y^2)} \frac{|1 - x^2|}{|1 + x^2|} \leq \frac{|2y|}{|2xy|} \cdot 1 \leq 1.$$

For $(x, y) \in f^{-1}(U) \setminus K_2$, $y = 0$ we have $H(x, y) = 0$. Therefore, H is bounded on $f^{-1}(U) \setminus K_2$ and $(\nabla f, h, f^*) \in O_0(f)$. From Theorem 2.2 we get that 0 is a typical value of f.

Summing up, $B(f) = \emptyset$.

Before we give some examples how to use Theorem 2.2 when $D_f \neq \mathbb{R}^n$ we present some usefull lemmas.

Let $D \subset \mathbb{R}^n$, $(x_k)_{k=1}^{\infty} \subset D$. We say that the sequence $(x_k)_{k=1}^{\infty}$ *is escaping* D if the set $\{x_k : k \in \mathbb{N}\}$ has no accumulation points in D.

We leave the proofs of the next two lemmas as an exercise.

Lemma 3.2 *Let $D \subset \mathbb{R}^n$, $f \in C^{\infty}(D)$, $h \in C^1(D)$, $y \in U \subset \mathbb{R}$, $K = \emptyset$. The condition* ($\overline{O}_y(f)$-ii) *is satisfied if and only if*

$$\forall_{\substack{K_1-compact \\ K_1 \subset U, y \in K_1}} \forall_{(x_k)_{k=1}^{\infty} \subset f^{-1}(K_1)} \Big((x_k)_{k=1}^{\infty} \textit{is escaping } D \Rightarrow \lim_{k \to \infty} |h(x_k)| = +\infty \Big). \tag{18}$$

Lemma 3.3 *Let $D \subset \mathbb{R}^n$, $f \in C^{\infty}(D)$, $h \in C^1(D)$. If the condition*

$$\forall_{y \in \mathbb{R}} \forall_{(x_k)_{k=1}^{\infty} \subset D} \left(\lim_{k \to \infty} f(x_k) = y \wedge (x_k)_{k=1}^{\infty} \textit{ is escaping } D \Rightarrow \lim_{k \to \infty} |h(x_k)| = +\infty \right) \tag{19}$$

is satisfied then the condition ($\overline{O}_y(f)$-ii) *is satisfied for all $y \in \mathbb{R}$ with $U = \mathbb{R}$ and $K = \emptyset$. In particular, if the condition*

$$\forall_{(x_k)_{k=1}^{\infty} \subset D} \left((x_k)_{k=1}^{\infty} \textit{ is escaping } D \Rightarrow \lim_{k \to \infty} |h(x_k)| = +\infty \right) \tag{20}$$

is satisfied then the condition ($\overline{O}_y(f)$-ii) is satisfied for all $y \in \mathbb{R}$ with $U = \mathbb{R}$ and $K = \emptyset$.

Remark 3.4 Note that in general the condition

$$\forall_{(x_k)_{k=1}^{\infty} \subset D} \left(\lim_{k \to \infty} f(x_k) = y \wedge (x_k)_{k=1}^{\infty} \text{ is escaping } D \Rightarrow \lim_{k \to \infty} |h(x_k)| = +\infty \right) \tag{21}$$

does not imply the condition ($\overline{O}_y(f)$-ii) for some neighborhood U and compact set K. Indeed, one can check that if we denote $D := \mathbb{R}^2$ and

$$f(x, y) := y, \quad h(x, y) := \frac{(x)^2}{1 + (xy)^2} \quad \text{for } (x, y) \in \mathbb{R}^2,$$

then the condition (21) is satisfied at 0 but ($\overline{O}_0(f)$-ii) is not (for arbitrary U and K).

Below we give some easy examples how one can use theorem 2.2 when $D_f \neq \mathbb{R}^n$.

Example 2 Let $D := \mathbb{R}^n \setminus \{(0, .., 0)\}$ and put $f \in C^\infty(D)$ as

$$f(x) := x_n \quad \text{for } x = (x_1, x_2, \dots, x_n) \in D.$$

We will show that $B(f) = \{0\}$. Denote

$$h(x) := |x|^2 + \frac{1}{|x|^2} \text{ for } x \in D.$$

Obviously $h \in C^\infty(D)$ and the condition (20) is satisfied. Set $v = \nabla f$ and $f^* = f$. We have

$$H(x) = \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f^*(x)} = 2x_n \left(1 - \frac{1}{|x|^4}\right) = 2f(x)\left(1 - \frac{1}{|x|^4}\right) \text{ for } x \in D.$$

Therefore, if $y \neq 0$ the function H is bounded in $f^{-1}(U)$ for some neighborhood U of y. From Theorem 2.2 we get $B(f) \subset \{0\}$. One can check that this upper bound is optimal ($B(f) = \{0\}$).

Example 3 Let $D := \mathbb{R}^n \setminus \{x = (x_1, x_2, \dots, x_n) : x_1 = 0\}$ and put $f \in C^\infty(D)$ as

$$f(x) := x_n \text{ for } x = (x_1, x_2, \dots, x_n) \in D.$$

We will show that $B(f) = \emptyset$. Denote

$$h(x) := |x|^2 + \frac{1}{(x_1)^2} \text{ for } x \in D.$$

Obviously $h \in C^\infty(D)$ and the condition (20) is satisfied. Set $v = \nabla f$ and $f^* = f$. We have

$$H(x) = \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f^*(x)} = 2x_n = 2f(x) \text{ for } x \in D.$$

Therefore, for $y \in \mathbb{R}$ the function H is bounded in $f^{-1}(U)$ for some neighborhood U of y. From Theorem 2.2 we get $B(f) = \emptyset$.

Example 4 Let $D := \mathbb{R}^n \setminus \{x = (x_1, x_2, \dots, x_n) : x_n = 0\}$ and put $f \in C^\infty(D)$ as

$$f(x) := x_n \text{ for } x = (x_1, x_2, \dots, x_n) \in D.$$

We will show that $B(f) = \{0\}$. Denote

$$h(x) := |x|^2 + \frac{1}{(x_n)^2} \text{ for } x \in D.$$

Obviously $h \in C^\infty(D)$ and the condition (20) is satisfied. Set $v = \nabla f$ and $f^* = f$. We have

$$H(x) = \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f^*(x)} = 2\left(x_n - \frac{1}{(x_n)^3}\right) = 2\left(f(x) - \frac{1}{(f(x))^3}\right) \text{ for } x \in D.$$

Therefore, if $y \neq 0$ then the function H is bounded in $f^{-1}(U)$ for some neighborhood U of y. From Theorem 2.2 we get $B(f) \subset \{0\}$. Considering that for $y \neq 0$ we have $f^{-1}(y) \neq \emptyset$ and $f^{-1}(0) = \emptyset$ we conclude $B(f) = \{0\}$.

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