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A type of oscillatory integral operator and its applications

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Abstract

In this paper, we consider L^p- estimate for a class of oscillatory integral operators satisfying the Carleson–Sjölin conditions with further convex and straight assumptions. As applications, the multiplier problem related to a general class of hypersurfaces with nonvanishing Gaussian curvature, local smoothing estimates for the fractional Schrödinger equation and the sharp resolvent estimates outside of the uniform boundedness range are discussed.

Keywords Hörmander-type operator · Polynomial partitioning · k-broad "norm"

Mathematics Subject Classification Primary 42B10; Secondary 42B20

1 Introduction

Let $n \geq 2$, $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ be non-negative and supported in $B_1^n(0) \times B_1^{n-1}(0)$ and $\phi\colon B_1^n(0) \times B_1^{n-1}(0) \to \mathbb{R}$ be a smooth function which satisfies the following Carleson–Sjölin conditions:

(H1) rank $\partial^2_{\xi x}\phi(x,\xi) = n-1$ for all $(x,\xi) \in B_1^n(0) \times B_1^{n-1}(0)$;

(H2) Defining the map $G\colon B_1^n(0) \times B_1^{n-1}(0) \to S^{n-1}$ by $G(x,\xi) := \frac{G_0(x,\xi)}{|G_0(x,\xi)|}$ where

$$G_0(x,\xi) := \bigwedge_{j=1}^{n-1} \partial_{\xi_j}\partial_x \phi(x,\xi),$$

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the curvature condition

$$\det \partial^2_{\xi\xi} \langle \partial_x \phi(x, \xi), G(x, \xi_0)\rangle|_{\xi=\xi_0} \neq 0$$

holds for all $(x, \xi_0) \in \operatorname{supp} a$.

For any $\lambda \geq 1$, define the operator T^λ by

$$T^\lambda f(x) := \int_{B_1^{n-1}(0)} e^{2\pi i \phi^\lambda(x,\xi)} a^\lambda(x, \xi) f(\xi)\, d\xi \tag{1.1}$$

where $f\colon B_1^{n-1}(0) \to \mathbb{C}$, $a(x, \xi) \in C_c^\infty(B_1^n(0) \times B_1^{n-1}(0))$ and

$$a^\lambda(x, \xi) := a(x/\lambda, \xi),\ \phi^\lambda(x, \xi) := \lambda\phi(x/\lambda, \xi).$$

We say T^λ is a Hörmander type operator if ϕ satisfies the conditions (H1) and (H2). A typical example for the Hörmander-type operator is the following extension operator E defined by

$$Ef(x) := \int_{B_1^{n-1}(0)} e^{2\pi i(x'\cdot\xi + x_n\psi(\xi))} f(\xi) d\xi, \tag{1.2}$$

with

$$\operatorname{rank}\Big(\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j}\Big)_{(n-1)\times(n-1)} = n - 1.$$

Hörmander conjectured that if ϕ satisfies conditions H_1, H_2, then

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(B_1^{n-1}(0))}, \tag{1.3}$$

for $p > \frac{2n}{n-1}$. Hörmander [19] proved the above conjecture for $n = 2$. For the higher dimensional case, Stein [26] proved (1.3) for $p \geq 2\frac{n+1}{n-1}$ and $n \geq 3$. Later, Bourgain [2] disproved Hörmander's conjecture by constructing a kind of counterexample. Furthermore, he showed that Stein's result is sharp in the odd dimensions. For the even dimensions, up to the endpoint case, Bourgain, Guth [4] proved the sharp result. In summary, we may state the results as follows.

Theorem 1.1 [4, 26] *Let $n \geq 3$ and T^λ be a Hörmander type operator. For all $\varepsilon > 0$, $\lambda \geq 1$,*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\varepsilon,\phi,a} \lambda^\varepsilon \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.4}$$

holds whenever

$$p \geq \begin{cases} 2\frac{n+1}{n-1} & \text{for } n \text{ odd},\\ 2\frac{n+2}{n} & \text{for } n \text{ even}.\end{cases} \tag{1.5}$$

Lee [23] observed that if we further impose the following *convex condition*

(H3) The eigenvalues of the Hessian

$$\partial^2_{\xi\xi} \langle \partial_x \phi(x, \xi), G(x, \xi_0)\rangle|_{\xi=\xi_0}$$

are all positive for $(x, \xi_0) \in \operatorname{supp} a$;

on the phase, the range of p can be obtained beyond that in (1.5). Recently, Guth–Hickman–Iliopoulou [15] proved the sharp results for the operator T^λ with a convex phase. To be more precise, they showed

Theorem 1.2 [15] *Let $n \geq 3$ and T^λ be a Hörmander type operator satisfying the convex condition. For all $\varepsilon > 0$, $\lambda \geq 1$,*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\varepsilon,\phi,a} \lambda^\varepsilon \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.6}$$

holds whenever

$$p \geq \begin{cases} 2\frac{3n+1}{3n-3} & \text{for } n \text{ odd}, \\ 2\frac{3n+2}{3n-2} & \text{for } n \text{ even}. \end{cases} \tag{1.7}$$

The primary difference between the translation invariant case (1.2) and (1.1) is that the main contribution of $T^\lambda f$ may be concentrated in a small neighborhood of a lower dimensional submanifold which features slightly differently between the odd and even dimensions. However, such phenomena can not happen for the extension operator E if the Kakeya conjecture holds. The difference between Theorems 1.1 and 1.2 arises from the fact that in the convex setting, such concentration lies in an at least $\lambda^{1/2}$ neighborhood of a submanifold which can be manifested by the transverse equidistribution property, while for the general phase, it can be further squeezed into an 1-neighborhood of a submainifold.

As one can see, the Kakeya compression phenomena prohibit the sharp range of p in (1.5), (1.7) to be matched with the conjectured range $p > \frac{2n}{n-1}$. Therefore, it is natural to conjecture the potentially possible range of p in (1.3) will be $p > \frac{2n}{n-1}$ if the Kakeya compression phenomena does not happen. A probable way to preclude the Kakeya compression phenomena is to impose the following *straight condition* on the phase.

H4) For given ξ, $G(x, \xi)$ keeps invariant when x changes.

Formally, we may formulate the following conjecture.

Conjecture 1.3 *Let $n \geq 3$ and T^λ be a Hörmander type operator with the straight condition. For all $\varepsilon > 0$, the estimate*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\phi,a} \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.8}$$

holds uniformly for $\lambda \geq 1$ whenever $p > \frac{2n}{n-1}$.

Obviously, Conjecture 1.3 implies the restriction conjecture, and thus the Kakeya conjecture. Furthermore, we may see later Conjecture 1.3 also has many other applications. For example, Conjecture 1.3 implies the Bochner–Riesz conjecture related to a class of general hypersurfaces with nonvanishing Gaussian curvature and the local smoothing conjecture for the fractional Schrödinger equation and the sharp resolvent estimates outside of the uniform boundedness range.

In this paper, we prove certain L^p estimate for T^λ being a Hörmander type operator with the convex and straight conditions. Define p_n as follows[1]

$$p_n := \min_{2 \leq k \leq \left[\frac{2n+4}{3}\right]} \max\left\{2\frac{2n-k+2}{2n-k}, 2 + \frac{6}{2(n-1) + (k-1)\prod_{i=k}^{n-1}\frac{2i}{2i+1}}\right\}.$$

The exponent p_n originally comes from the work [18] of Hickman and Zahl. For some lower dimensional cases, the value of p_n may be found in Fig. 2 of [18]. We state our main results as follows.

1 $\left[\frac{2n+4}{3}\right]$ denotes the integer part of $\frac{2n+4}{3}$.

Theorem 1.4 *Let $n \geq 3$ and T^λ be a Hörmander type operator with the convex and straight conditions. For all $\varepsilon > 0$ the estimate*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\varepsilon,\phi,a} \lambda^\varepsilon \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.9}$$

holds uniformly for $\lambda \geq 1$ whenever $p \geq p_n$.

Under the straight conditions, from the Fig. 2 in [18], we may break the sharp range of p in [15] for oscillatory integrall operators satisfying the Carleson–Sjölin and convex conditions in some dimensions.

The proof of Theorem 1.4 relies on the polynomial partitioning method which was introduced by Guth [13, 14] to handle the restriction problem. Since then, it has been also used to study the pointwise convergence problem for the Schrödigner operator, Bochner–Riesz conjecture, Kakeya conjecture and local smoothing conjecture for the wave equation and the fractional Schrödinger equation, one may refer to [5, 9, 12, 17, 29] and references therein for more details. Technically speaking, the straight condition can not be kept under the change of variables in the spatial space which can be explicitly demonstrated in the Sect. 2.1. To overcome this obstacle, we need to work with a more general class of functions which satisfy the straight condition up to a diffeomorphism in the spatial variables. It should be noted that the proof of Theorem 1.4 is obtained by adapting the arguments in [12, 15]. Thus we only streamline the structure of the proof when there are too many overlaps.

The rest of this paper is organized as follows: In Sect. 2, we will show the applications of conjecture 1.3 to the multiplier problem, local smoothing estimates for the fractional Schrödinger equation and the sharp resolvent estimates outside of the uniform boundedness range. In Sect. 3, we perform some reductions. In particular, we introduce a special class of functions to make the induction arguments completed. In Sect. 4, we introduce the wave packet decomposition which is an important tool. In Section 5, we prepare some useful ingredients which play important roles in the proof of the broad "norm" estimate in Sect. 6. With the above preparations, finally, we prove Theorem 1.4 in Sect. 7.

Notations. For nonnegative quantities X and Y, we will write $X \lesssim Y$ to denote the inequality $X \leq CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$, we will write $X \sim Y$. Dependence of implicit constants on the spatial dimensions or integral exponents such as p will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, $X \lesssim_u Y$ indicates $X \leq CY$ for some $C = C(u)$. We write $A(R) \leq \mathrm{RapDec}(R)B$ to mean that for any power β, there is a constant C_β such that

$$|A(R)| \leq C_\beta R^{-\beta} B \quad \text{for all } R \geq 1.$$

We will also often abbreviate $\|f\|_{L^r_x(\mathbb{R}^n)}$ to $\|f\|_{L^r}$. For $1 \leq r \leq \infty$, we use r' to denote the dual exponent to r such that $\frac{1}{r} + \frac{1}{r'} = 1$. Throughout the paper, χ_E is the characteristic function of the set E. We usually denote by $B^n_r(a)$ a ball in $\mathbb{R}^n$ with center a and radius r. We will also denote by B^n_R a ball of radius R and arbitrary center in $\mathbb{R}^n$. Denote by $A(r) := B^n_{2r}(0)\backslash B^n_{r/2}(0)$. We denote $w_{B^n_R(x_0)}$ to be a nonnegative weight function adapted to the ball $B^n_R(x_0)$ such that

$$w_{B^n_R(x_0)}(x) \lesssim (1 + R^{-1}|x - x_0|)^{-M},$$

for some large constant $M \in \mathbb{N}$.

We define the Fourier transform on $\mathbb{R}^n$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x := \mathcal{F}f(\xi).$$

and the inverse Fourier transform by

$$\check{g}(x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} g(\xi)\mathrm{d}\xi := (\mathcal{F}^{-1}g)(x).$$

These help us to define the fractional differentiation operators $|\nabla|^s$ and $\langle\nabla\rangle^s$ for $s \in \mathbb{R}$ via

$$|\nabla|^s f(x) := \mathcal{F}^{-1}\Big\{|\xi|^s \hat{f}(\xi)\Big\}(x) \quad \text{and} \quad \langle\nabla\rangle^s f(x) := \mathcal{F}^{-1}\Big\{(1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\Big\}(x).$$

In this manner, we define the Sobolev norm of the space $L^p_\alpha(\mathbb{R}^n)$ by

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} := \Big\|\langle\nabla\rangle^\alpha f\Big\|_{L^p(\mathbb{R}^n)}.$$

2 Applications

In this section, we talk about the relations of Conjecture 1.3 to other associated problems. In particular, the multiplier problem with respect to a general class of hypersurfaces with non-vanishing Gaussian curvature, local smoothing conjecture for the fractional Schrödinger and the sharp resovent estimate outside of uniform boundedness range will be discussed.

Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function with

$$\operatorname{rank}\Big(\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j}\Big)_{(n-1)\times(n-1)} = n-1,$$

and

$$|\partial^\alpha\psi(\xi)| \le 1, \quad \alpha \in \mathbb{Z}^{n-1}, |\alpha| \le N,$$

where N is a large constant. Therefore, by inverse function theorem, there exists locally a function $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that

$$\partial_\xi\psi(g(x')) = -x', \quad x' \in \mathbb{R}^{n-1}. \tag{2.1}$$

2.1 Multiplier problem

Let $\delta \ge 0$, $\xi = (\xi', \xi_n)$ and $m^\delta(\xi) := (\xi_n - \psi(\xi'))^\delta_+ \chi(\xi')$, where χ is a smooth compactly supported function with supp$\chi \subset B_2^{n-1}(0)$ and

$$t^\delta_+ = \begin{cases} t^\delta, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

We consider the following multiplier problem: for which δ and p such that

$$\Big\|m^\delta(D)f\Big\|_{L^p(\mathbb{R}^n)} \lesssim_\delta \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.2}$$

It is conjectured that

Conjecture 2.1 *For $\delta \ge 0$ and $1 \le p \le \infty$, then*

$$\Big\|m^\delta(D)f\Big\|_{L^p(\mathbb{R}^n)} \lesssim_\delta \|f\|_{L^p(\mathbb{R}^n)}, \quad \delta > \delta(p) := \max\Big\{n\Big|\frac{1}{2}-\frac{1}{p}\Big| - \frac{1}{2}, 0\Big\}. \tag{2.3}$$

We will show how Conjecture 1.3 implies Conjecture 2.1. Let $p > \frac{2n}{n-1}$ and $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported function, with supp$\eta \subset (1/2, 1)$ satisfying

$$\sum_{j\in\mathbb{Z}} \eta(2^j t) \equiv 1, \quad t > 0.$$

We break m^δ into pieces

$$m^\delta(\xi) = \sum_{j\geq 1} \eta(2^j(\xi_n - \psi(\xi')))m^\delta(\xi) + r(\xi),$$

where $r(\xi)$ is a smooth function with supp$r \subset B_2^n(0)$.

Define an operator $m_j^\delta(D)$ as follows:

$$m_j^\delta(D)f(x) := \Big(\eta(2^j(\xi_n - \psi(\xi')))m^\delta(\xi)\hat{f}(\xi)\Big)^\vee(x).$$

Let $K_j^\delta(x)$ be the kernel of the multiplier $m_j^\delta(D)$, i.e.

$$K_j^\delta(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \eta(2^j(\xi_n - \psi(\xi')))m^\delta(\xi)d\xi.$$

Through changing of variables, we may reformulate $K_j^\delta(x)$ as follows:

$$K_j^\delta(x) = 2^{-j\delta} \int_{\mathbb{R}} e^{2\pi i x_n\xi_n}\widetilde{\eta}(2^j\xi_n) \int_{\mathbb{R}^{n-1}} e^{2\pi i(x'\cdot\xi' + x_n\psi(\xi'))}\chi(\xi')d\xi'd\xi_n,$$

where

$$\widetilde{\eta}(t) = \eta(t)t_+^\delta.$$

For convenience, define

$$K_j(x) := \int_{\mathbb{R}} e^{2\pi i x_n\xi_n}\widetilde{\eta}(2^j\xi_n) \int_{\mathbb{R}^{n-1}} e^{2\pi i(x'\cdot\xi' + x_n\psi(\xi'))}\chi(\xi')d\xi'd\xi_n.$$

To handle the inner part of the integral with respect to ξ', we use the stationary phase method. For this purpose, we borrow the following lemma from [22] with a slight modification. One may refer to [22] for the proof.

Lemma 2.2 *Define*

$$I_\psi(x) := \int_{\mathbb{R}^{n-1}} e^{2\pi i(x'\cdot\xi' + x_n\psi(\xi'))}\chi(\xi')d\xi',$$

then

- *If $|x_n| \geq 1/2$ and $2^5|x'| \leq |x_n|$, then for every $M \in \mathbb{N}$ satisfying $2M \leq N$ we have*

$$I_\psi(x) = \frac{c_n}{\sqrt{|K|}} e^{2\pi i\left(x'\cdot g\left(\frac{x'}{x_n}\right) + x_n\psi\left(g\left(\frac{x'}{x_n}\right)\right)\right)} \times \sum_{j=0}^{M-1} \mathcal{D}_j\chi(\xi')|_{\xi'=g\left(\frac{x'}{x_n}\right)}|x_n|^{-\frac{n-1}{2}-j} + \mathcal{E}_M(x), \tag{2.4}$$

where c_n is a constant depending on n, K denotes the Gaussian curvature of the hypersurface $(\xi', \psi(\xi'))$ at point $(g(\frac{x'}{x_n}), \psi(g(\frac{x'}{x_n})))$, $\mathcal{D}_0\chi = \chi$ and $\mathcal{D}_j$ is a differential operator in ξ' of order $2j$. For $\mathcal{E}$, we have the estimate

$$|\mathcal{E}_M(x)| \lesssim_{M,\psi} |x_n|^{-M}.$$

- *If* $2^6|x'| \geq |x_n|$ *or* $|x_n| \leq 2$, *then for every* $0 \leq M \leq N$ *there exists a constant* C_M, *such that*
$$|I_\psi(x)| \leq C_M(1+|x|)^{-M}.$$

Let $\widetilde{\chi} \in C_c^\infty(\mathbb{R})$ with supp$\widetilde{\chi} \subset (-2^{-5}, 2^{-5})$ equaling to 1 in $(-2^{-6}, 2^{-6})$, $\beta \in C_0^\infty(\mathbb{R})$ with supp$\beta \subset [-9/8, -3/8] \cup [3/8, 9/8]$ and

$$\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell}t) = 1, \ t \neq 0.$$

We split K_j as follows:

$$K_j(x) = K_{j,0}(x) + \sum_{\ell \geq 1} K_{j,\ell}(x),$$

where

$$K_{j,\ell}(x) = \widetilde{\chi}\Big(\frac{|x'|}{x_n}\Big)\beta(2^{-\ell}x_n)K_j(x).$$

Using Lemma 2.2, we have

$$\|K_{j,0} * f\|_{L^p} \lesssim \|f\|_{L^p}.$$

For $\ell \geq 1$, the main contribution to $K_{j,\ell}$ comes from $\widetilde{K}_{j,\ell}$ defined by

$$\widetilde{K}_{j,\ell}(x) = \widetilde{\chi}\left(\frac{|x'|}{x_n}\right)\beta(2^{-\ell}x_n)|x_n|^{-\frac{n-1}{2}} e^{2\pi i\left(x'\cdot g\left(\frac{x'}{x_n}\right)+x_n\psi\left(g\left(\frac{x'}{x_n}\right)\right)\right)} \times \int_{\mathbb{R}} e^{2\pi i x_n\xi_n}\widetilde{\eta}(2^j\xi_n)d\xi_n.$$

Thus it suffices to show

$$\|\widetilde{T}_{j,\ell}f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{\left(\frac{n+1}{2}-\frac{n}{p}\right)\ell}2^{-j}(1+2^{\ell-j})^{-M}\|f\|_{L^p},$$

where $\widetilde{T}_{j,\ell}$ is defined by

$$\widetilde{T}_{j,\ell}f(x) = \int_{\mathbb{R}^n} \widetilde{K}_{j,\ell}(x-y)f(y)dy.$$

Then by a standard optimization argument, we have

$$\sum_{\ell=1}^{\infty} \|\tilde{T}_{j,\ell}f\|_{L^p} \lesssim 2^{\left(\frac{n-1}{2}-\frac{n}{p}\right)j}\|f\|_{L^p}. \tag{2.5}$$

Therefore, by a localization argument, it suffices to show

$$\|2^{\ell n}\widetilde{K}_{j,\ell}(2^\ell\cdot) * f\|_{L^p(B_1^n(0))} \lesssim 2^{\left(\frac{n+1}{2}-\frac{n}{p}\right)\ell}2^{-j}(1+2^{\ell-j})^{-M}\|f\|_{L^p(B_1^n(0))}.$$

Note that

$$\begin{aligned} 2^{\ell n}\widetilde{K}_{j,\ell}(2^\ell\cdot) * f &= 2^{\frac{n+1}{2}\ell}\int_{\mathbb{R}^n} e^{2\pi i 2^\ell\left((x'-y')\cdot g\left(\frac{x'-y'}{x_n-y_n}\right)+(x_n-y_n)\psi\left(g\left(\frac{x'-y'}{x_n-y_n}\right)\right)\right)}a_{\ell,j}(x,y)f(y)dy \\ &= 2^{\frac{n+1}{2}\ell}\int_{\mathbb{R}} T_{y_n}^{2^\ell}f_{y_n}dy_n, \end{aligned} \tag{2.6}$$

where

$$a_{\ell,j}(x,y) := \widetilde{\beta}(x_n-y_n)\widetilde{\chi}\Big(\frac{x'-y'}{x_n-y_n}\Big) \times \int_{\mathbb{R}} e^{2\pi i 2^\ell(x_n-y_n)\xi_n}\widetilde{\eta}(2^j\xi_n)d\xi_n, \ \widetilde{\beta}(t) := \beta(t)|t|^{-\frac{n-1}{2}}$$

and

$$T_{y_n}^{2^\ell} f_{y_n}(x) := \int_{\mathbb{R}^{n-1}} e^{2\pi i 2^\ell \left((x'-y')\cdot g\left(\frac{x'-y'}{x_n-y_n}\right)+(x_n-y_n)\psi\left(g\left(\frac{x'-y'}{x_n-y_n}\right)\right)\right)}$$
$$a_{\ell,j}(x,y) f(y',y_n)dy', \;\; f_{y_n}(\cdot) := f(\cdot, y_n).$$

It's easy to show that

$$|\partial_x^\alpha a_{\ell,j}(x,y)| \lesssim_{\alpha,M} 2^{-j}(1+2^{j-\ell})^M. \tag{2.7}$$

Indeed, since $\partial_x^\alpha \left(\widetilde{\beta}(x_n-y_n)\widetilde{\chi}\left(\frac{x'-y'}{x_n-y_n}\right)\right)$ is bounded for any $|\alpha| \geq 0$, it suffices to show

$$\left|\partial_{x_n}^\alpha \int_{\mathbb{R}} e^{2\pi i 2^\ell (x_n-y_n)\xi_n} \widetilde{\eta}(2^j\xi_n)d\xi_n\right| \lesssim_{\alpha,M} 2^{-j}(1+2^{j-\ell})^M. \tag{2.8}$$

By integration by parts, (2.8) follows easily.

For fixed y_n, by changing of variables

$$\frac{x'}{x_n-y_n} \to x', \quad \frac{1}{x_n-y_n} \to x_n,$$

under the new coordinates, the phase $(x'-y')\cdot g\left(\frac{x'-y'}{x_n-y_n}\right)+(x_n-y_n)\psi\left(g\left(\frac{x'-y'}{x_n-y_n}\right)\right)$ becomes

$$\Psi(x,y') := \left(\frac{x'}{x_n}-y'\right)\cdot g(x'-x_ny') + \frac{1}{x_n}\psi(g(x'-x_ny')). \tag{2.9}$$

A direct computation shows that the associated Gauss map $G(x,y')$ related to the hypersurface $\{\partial_x \Psi(x,y')\}$ at y' is given by

$$G(x,y') = \frac{(y',1)}{\sqrt{1+|y'|^2}}, \tag{2.10}$$

which is obviously independent of the spatial variables x. Indeed, suppose that $g(\xi) = (g_1(\xi), g_2(\xi), \cdots, g_{n-1}(\xi))$, $\xi \in \mathbb{R}^{n-1}$. For $1 \leq i \leq n-1$,

$$\begin{aligned}\frac{\partial\Psi}{\partial x_i}(x',y') &= \frac{g_i(x'-x_ny')}{x_n} + \sum_{1\leq j\leq n-1}\left(\frac{x_j}{x_n}-y_j\right)\frac{\partial g_j}{\partial \xi_i}(x'-x_ny') \\ &\quad + \frac{\sum_{j=1}^{n-1}\left((\partial_j\psi)(g(x'-x_ny'))\right)\frac{\partial g_j}{\partial\xi_i}(x'-x_ny')}{x_n} \\ &= \frac{g_i(x'-x_ny')}{x_n},\end{aligned}$$

here we have used the fact that $(\partial_j\psi)(g(x'-x_ny')) = x_ny_j - x_j$ which follows from (2.1). We also have

$$\frac{\partial\Psi}{\partial x_n}(x',y') = -\frac{x'\cdot g(x'-x_ny') + \psi(g(x'-x_ny'))}{x_n^2}.$$

Therefore, for $1 \leq i,k \leq n-1$,

$$\frac{\partial^2\Psi}{\partial y_k \partial x_i}(x',y') = -\frac{\partial g_i}{\partial \xi_k}(x'-x_ny'),$$

and

$$\frac{\partial^2 \Psi}{\partial y_k \partial x_n}(x', y') = -\frac{\sum_{j=1}^{n-1}(-x_j x_n)\frac{\partial g_j}{\partial \xi_k}(x' - x_n y') + \sum_{j=1}^{n-1}(-x_n^2 y_j + x_j x_n)\frac{\partial g_j}{\partial \xi_k}(x' - x_n y')}{x_n^2}$$
$$= \sum_{j=1}^{n-1} y_j \frac{\partial g_j}{\partial \xi_k}(x' - x_n y').$$

Then it is obvious that for each x, $(y', 1)$ is orthogonal to $\partial_{y_k}\partial_x \Psi(x, y')$, $1 \le k \le n-1$. Hence we can get (2.10). Furthermore, $\Psi(x, y')$ satisfies the Carleson–Sjölin conditions by our assumption that the Hessian of ψ is nondegerate.

Recall (2.7), we may use Conjecture 1.3 to obtain that

$$\|T_{y_n}^{2^\ell} f_{y_n}\|_{L^p(B_1^n(0))} \lesssim 2^{-\frac{n\ell}{p}} 2^{-j}(1 + 2^{\ell-j})^{-M}\|f_{y_n}\|_{L^p(B_1^n(0))},$$

uniformly for y_n. Finally, integrating with respect to y_n, we will obtain the desired results.

If we impose an additional condition that all eigenvalues of the Hessian of ψ are positive, then $\Psi(x, y')$ also satisfies the convex condition. From the above discussion, as a direct consequence of Theorem 1.4, we also have

Corollary 2.3 *Let* $1 \le p \le \infty$, $\psi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ *be smooth and* $\left(\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j}\right)_{(n-1)\times(n-1)}$ *has* $(n-1)$ *positive eigenvalues, then*

$$\|m^\delta(D) f\|_{L^p(\mathbb{R}^n)} \lesssim_{\delta,\psi,p} \|f\|_{L^p(\mathbb{R}^n)} \tag{2.11}$$

for all p *such that* $\max\{p, p'\} > p_n$ *and* $\delta > \delta(p)$.

2.2 Local smoothing estimates for the fractional Schrödinger equation

Let $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ be the solution to the following equation

$$\begin{cases} i\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = f(x), \end{cases} \tag{2.12}$$

where $\alpha \in (0, 1) \cup (1, \infty)$ and f is a Schwartz function. The solution u can be expressed by

$$u(x, t) = e^{it(-\Delta)^{\frac{\alpha}{2}}} f(x) := \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + t|\xi|^\alpha)} \hat{f}(\xi) d\xi. \tag{2.13}$$

We are concerned with L^p-regularity of the solution u. For a fixed time t, Fefferman and Stein [7], Miyachi [24] showed the following optimal L^p estimate:

$$\|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L^p(\mathbb{R}^n)} \le C_{t,p}\|f\|_{L^p_{s_{\alpha,p}}(\mathbb{R}^n)}, \quad s_{\alpha,p} := \alpha n\left|\frac{1}{2} - \frac{1}{p}\right|, \quad 1 < p < \infty. \tag{2.14}$$

It is conjectured that:

Conjecture 2.4 (Local smoothing for the fractional Schrödinger operator) *Let* $\alpha \in (0, 1) \cup (1, \infty)$, $p > 2 + \frac{2}{n}$ *and* $s \ge \alpha n(\frac{1}{2} - \frac{1}{p}) - \frac{\alpha}{p}$. *Then*

$$\|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L^p(\mathbb{R}^n \times [1,2])} \le C_{p,s}\|f\|_{L^p_s(\mathbb{R}^n)}. \tag{2.15}$$

We will show Conjecture 1.3 implies Conjecture 2.4. Indeed, following the reduction in [10], up to the endpoint regularity, to show (2.15), it suffices to prove

$$\|e^{it\psi(D)}f\|_{L^p_{x,t}(B^n_{R^2}\times[R^2/2,R^2])} \lesssim_\varepsilon R^{2n(\frac{1}{2}-\frac{1}{p})+\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}, \quad \operatorname{supp}\hat{f}\subset B^n_1(0), \tag{2.16}$$

where ψ also satisfies

- $\psi(0)=0$, $\nabla\psi(0)=0$;
- For $\xi_0\in B^n_1(0)$, the absolute value of all eigenvalues of the Hessian $\left(\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j}\right)|_{\xi=\xi_0}$ falls into $[1/2,1)$.

By a localization argument, we may also assume supp$f\subset B^n_{R^2}$. Note that

$$e^{it\psi(D)}f(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{2\pi i\left((x-y)\cdot\xi+t\psi(\xi)\right)}a(\xi)f(y)d\xi dy,$$

where $a\in C^\infty_c(B^n_2(0))$. We denote $K(x,t,y)$ the kernel of the operator $e^{it\psi(D)}$, then

$$K(x,t,y)=\int_{\mathbb{R}^n}e^{2\pi i((x-y)\cdot\xi+t\psi(\xi))}a(\xi)d\xi.$$

Through a standard stationary phase argument, we have

$$K(x,t,y)\sim|t|^{-\frac{n}{2}}e^{2\pi i\left((x-y)\cdot g\left(\frac{x-y}{t}\right)+t\psi\left(g\left(\frac{x-y}{t}\right)\right)\right)}a(\tfrac{x-y}{t}),$$

where g is defined as in (2.1) with $n-1$ being replaced by n. Note $t\sim R^2$, therefore, it suffices to consider the following oscillatory integral operators

$$R^{-n}\int_{\mathbb{R}^n}e^{2\pi i\left((x-y)\cdot g\left(\frac{x-y}{t}\right)+t\psi\left(g\left(\frac{x-y}{t}\right)\right)\right)}a(\tfrac{x-y}{t})f(y)dy.$$

By changing of variables,

$$x\to R^2x,\, t\to R^2t,\, y\to R^2y,$$

we have

$$\begin{aligned}&\left\|\int_{\mathbb{R}^n}e^{2\pi i\left((x-y)\cdot g(\frac{x-y}{t})+t\psi\left(g\left(\frac{x-y}{t}\right)\right)\right)}a\left(\tfrac{x-y}{t}\right)f(y)dy\right\|_{L^p(B^n_{R^2}\times[R^2/2,R^2])}\\&\lesssim R^{2n+\frac{2(n+1)}{p}}\left\|\int_{\mathbb{R}^n}e^{2\pi iR^2\left((x-y)\cdot g\left(\frac{x-y}{t}\right)+t\psi\left(g\left(\frac{x-y}{t}\right)\right)\right)}a\left(\tfrac{x-y}{t}\right)f(R^2y)dy\right\|_{L^p(B^n_1\times[1/2,1])}.\end{aligned} \tag{2.17}$$

Performing change of variables as follows

$$\frac{x}{t}\to x,\quad \frac{1}{t}\to t,$$

the corresponding phase under the new coordinates becomes

$$\Psi(x,t,y):=\left(\frac{x}{t}-y\right)g(x-ty)+\frac{1}{t}\psi(g(x-ty)).$$

By our assumption, $\Psi(x,t,y)$ satisfies the straight condition, as well as the Carleson–Sjölin conditions. Therefore, by Conjecture 1.3, we have

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^n} e^{2\pi i R^2\left((x-y)\cdot g\left(\frac{x-y}{t}\right)+t\psi\left(g\left(\frac{x-y}{t}\right)\right)\right)} a\left(\tfrac{x-y}{t}\right) f(R^2 y)dy \Big\|_{L^p(B_1^n\times[1/2,1])} \\ &\lesssim R^{-\frac{2(n+1)}{p}} \|f(R^2\cdot)\|_{L^p(B_1^n)} \\ &\lesssim R^{-\frac{2(n+1)}{p}-\frac{2n}{p}} \|f\|_{L^p(B^n_{R^2})}. \end{aligned} \tag{2.18}$$

Thus, we complete the proof.

It should be noted that Gan–Oh–Wu [8] considered the local smoothing problem for the fractional Schrödinger equation via a different approach and mentioned essentially the same method as above discussed. Furthermore, it is possible to further improve Gan-Oh-Wu's result by considering the Hörmanger type operator with the convex and straight conditions using Wang's method [28] at least in dimension $n = 2$.

2.3 Sharp resolvent estimates outside of the uniform boundedness range

The resolvent estimate for the Laplatian is of the form

$$\|(-\Delta - z)^{-1} f\|_{L^q(\mathbb{R}^n)} \le C(z,p,q)\|f\|_{L^p(\mathbb{R}^n)}, \quad \forall z \in \mathbb{Z}\backslash[0,\infty). \tag{2.19}$$

This inequality and its variants have been applied to study the problems of uniform Sobolev estimates, unique continuation properties and limiting absorption principles, etc, one may refer to [6, 11, 21] for more details.

Let's briefly review the results related to (2.19). In [21], Kenig, Ruiz and Sogge showed, for $z \in \mathbb{C}\backslash[0,\infty)$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n}, \quad \frac{2n}{n+3} < p < \frac{2n}{n+1},$$

with $n \ge 3$, the constant $C(p,q,n) > 0$ can be obtained independent of z. By homogeneity, a simple calculation shows that

$$\|(-\Delta - z)^{-1}\|_{p\to q} = |z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \Big\| \left(-\Delta - \frac{z}{|z|}\right)^{-1} \Big\|_{p\to q}, \quad \forall z \in \mathbb{C}\backslash[0,\infty).$$

For $z \in \mathbb{S}^1\backslash\{1\}$, Gutiérrez [16] obtained the optimal range of p, q with $n \ge 3$ in the sense that the constant $C(p,q,n)$ is independent of z. To be more precise, if $z \in \mathbb{S}^1\backslash\{1\}$ and $(\frac{1}{p},\frac{1}{q})$ lies in the set

$$\left\{(x,y) : \frac{2}{n+1} \le x - y \le \frac{2}{n}, x > \frac{n+1}{2n}, y < \frac{n-1}{2n}\right\}, \quad n \ge 3,$$

the sharp constant $C(p,q,z)$ in (2.19) can be obtained uniformly independent of z.

To formally state results regarding $C(z,p,q)$ with $z \in \mathbb{S}^1\backslash\{1\}$ and $(\frac{1}{p},\frac{1}{q})$ lying outside of the uniform boundedness range, let's firstly introduce some notations. Let I^2 be a closed square defined by

$$I^2 := \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}.$$

For each $(x,y) \in I^2$, define

$$(x,y)' := (1-x, 1-y).$$

Similarly, for any subset $\mathcal{R} \subset I^2$, define $\mathcal{R}'$ to be

$$\mathcal{R}' := \{(x, y) \in I^2 : (x, y)' \in \mathcal{R}\}.$$

Definition 2.5 For $X_1, \ldots, X_\ell \in I^2$, we denote by $[X_1, \ldots, X_\ell]$ the convex hull of the points $X_1, \ldots, X_\ell$. In particular, $[X, Y]$ will denote the closed line segment jointing X and Y. We also denote by (X, Y) and $[X, Y)$ for the open interval $[X, Y]\backslash\{X, Y\}$ and the half-open interval $[X, Y]\backslash\{Y\}$ respectively.

Set $C = (\frac{1}{2}, \frac{1}{2})$ and

$$\begin{aligned} B &:= \left(\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)}\right), & B' &:= \left(\frac{n^2+4n-1}{2n(n+1)}, \frac{n-1}{2n}\right), \\ D &:= \left(\frac{n-1}{2n}, \frac{n-1}{2n}\right), & D' &:= \left(\frac{n+1}{2n}, \frac{n+1}{2n}\right), \\ E &:= \left(\frac{n+1}{2n}, 0\right), & E' &:= \left(\frac{n-1}{2n}, 1\right), \end{aligned}$$

and

$$\mathcal{R}_0 = \mathcal{R}_0(n) := \begin{cases} \Big\{(x, y) : 0 \le x, y \le 1,\ 0 \le x - y < 1\Big\} & \text{if } n = 2, \\ \Big\{(x, y) : 0 \le x, y \le 1,\ 0 \le x - y \le \frac{2}{n}\Big\}\backslash\Big\{\left(1, \frac{n-2}{n}\right), \left(\frac{2}{n}, 0\right)\Big\} & \text{if } n \ge 3. \end{cases}$$

It is conjectured that:

Conjecture 2.6 *Let $n \ge 2$. If $(\frac{1}{p}, \frac{1}{q})$ lies in $\mathcal{R}_0\backslash\Big([B, E] \cup [B', E'] \cup [D, C) \cup [D', C)\Big)$, then*

$$\|(-\Delta - z)^{-1}\|_{p\to q} \simeq_{p,q,n} |z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \operatorname{dist}(z, [0, \infty))^{-\gamma_{p,q}} \tag{2.20}$$

holds for $z \in \mathbb{C}\backslash[0, \infty)$, where $\gamma_{p,q}$ is defined as follows:

$$\gamma_{p,q} := \max\left\{0,\ 1 - \frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right),\ \frac{n+1}{2} - \frac{n}{p},\ \frac{n}{q} - \frac{n-1}{2}\right\}. \tag{2.21}$$

Remark 1 If $\left(\frac{1}{p}, \frac{1}{q}\right) \in \{B, B'\}$, the restricted weak type estimate

$$\|(-\Delta - z)^{-1} f\|_{q,\infty} \le C|z|^{-1+\frac{n}{n+1}} \|f\|_{p,1} \tag{2.22}$$

holds. One may refer to [22] for more details.

From [22], we know that the lower bounded of (2.20) is true for all $n \ge 2$. For $n = 2$, this conjecture has been completely established, one may refer to [22]. However, for $n \ge 3$, only partial positive results of Conjecture 2.6 have been proved, see Fig. 1. More presicely, for $n \ge 3$, setting

$$\mathcal{R}_1 := [P_*, P_0, C]\backslash\{C\}$$

Kown–Lee [22] showed Conjecture 2.20 holds except for $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{R}_1 \cup \mathcal{R}_1'$. Among other things, following the proof of Proposition 4.1 in [22], we have

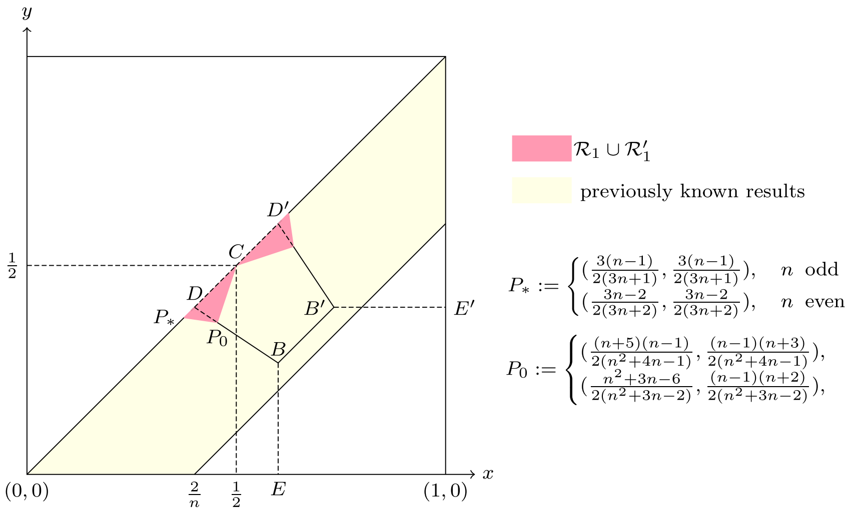

Fig. 1 The conjectured range for the resolvent estimates outside of the uniform boundedness and current progress

Theorem 2.7 *Let T^λ be a Hörmander type operator with the convex and straight conditions. If $p > \frac{2n}{n-1}$ and*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(B_1^{n-1}(0))}, \tag{2.23}$$

then

$$\|(-\Delta - z)^{-1}\|_{p\to p} \lesssim_{p,n} |z|^{-1+\gamma_{p,p}} \mathrm{dist}(z, [0, \infty))^{-\gamma_{p,p}}. \tag{2.24}$$

Remark 2 Indeed, the proof of (2.24) can be reduced to showing a multiplier estimate

$$\left\|\mathcal{F}^{-1}\left(\frac{\tilde{\chi}(\xi)\hat{f}(\xi)}{|\xi|^2 - 1 - i\delta}\right)\right\|_p \lesssim |\delta|^{-\gamma_{p,p}} \|f\|_p$$

where $\tilde{\chi} \in C_0^\infty(1 - 2\delta_0, 1 + 2\delta_0)$ for a small $\delta_0 > 0$ and $0 < |\delta| \ll 1$. Using the Carleson–Sjölin reduction as displayed in Sect. 2.1, an important ingredient in the approach is the following oscillatory integral operator estimate

$$\left\|\int_{\mathbb{R}^{n-1}} e^{2\pi i \Psi(x,y')} a(x, y') f(y') dy'\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p},$$

where Ψ is defined as in (2.9) and $a \in C_c^\infty(B_1^n(0) \times B_1^{n-1}(0))$. Since $\Psi(x, y')$, up to a diffemorphism in x, satisfies the conditions H_1, H_2, H_3, H_4, we may apply Conjecture 1.3 to get the desired results.

As a direct consequence of Theorem 2.7 and (2.22), by interpolation and the epsilon removal arguments, up to a pair of intervals $(B, D) \cup (B', D')$, we obtain Conjecture 1.3 implies Conjecture 2.6. Furthermore, we may use the new oscillatory integral estimates in Theorem 1.4 to further improve the range of p in (2.24).

3 Reductions

Typically speaking, the phase $\phi(x,\xi)$ which satisfies the conditions $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3,\mathrm{H}_4$ can be viewed as a small perturbation of the translation invariant case. More precisely, through changing of variables, it can be rewritten as

$$\phi(x,\xi)=\langle x',\xi\rangle+x_n h(\xi)+\mathcal{E}(x,\xi), \tag{3.1}$$

where h and $\mathcal{E}$ are smooth functions, h is quadratic in ξ and $\mathcal{E}$ is quadratic in x,ξ. However, under the new coordinate, the formula of ϕ in (3.1) may not satisfy the straight condition, even though $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3$ can be ensured. In other words, the straight condition may not be kept under a general diffeomorphism in the spatial variables. Therefore, we should be careful when performing the change of variables in x and, meanwhile, keeping track of the straight condition.

Basic reductions As mentioned above, the straight condition may be destroyed while performing a diffeomorphism with respect to the spatial variables, which inspires us to consider a wider class of functions which, upon a diffeomorphism in the spatial variables, satisfy the straight condition. To formalize that, we introduce a notion of $\mathbf{\Phi}_{\mathrm{cs}}$.

Definition 3.1 We say a function $\phi(x,\xi)$ lies in the class $\mathbf{\Phi}_{\mathrm{cs}}$, if, modulo a diffeomorphism in the spatial variables x, $\phi(x,\xi)$ satisfies the conditions $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3,\mathrm{H}_4$.

Remark 3 In terms of $\mathbf{\Phi}_{\mathrm{cs}}$, it is an interesting problem to investigate the influence of the higher order terms of $\phi(x,\xi)$ in x.

Example 3.2 In [2], Bourgain disproved Hormander's conjecture by constructing a counterexample where the Kakeya compression phenomena happen which roughly say that the main contribution to the oscillatory integral may be concentrated in a lower dimensional submanifold. Next we will analyse Bourgain's counterexample to vividly show that there does not exist a diffeomorphism in the spatial variables such that the Gauss map $G(x,\xi)$ is invariant when x changes.

Let

$$P(x,y)=x_1y_1+x_2y_2+2x_3y_1y_2+x_3^2y_1^2.$$

Assume that there exists a diffeomorphism

$$x\to\kappa(\tilde{x}),$$

such that for given ξ, the associated $G(x,\xi)$ keeps invariant when x changes, where $x=(x_1,x_2,x_3)=(\kappa_1(\tilde{x}),\kappa_2(\tilde{x}),\kappa_3(\tilde{x}))$. Then the tangent space of the hypersurface $\{\partial_{\tilde{x}}P(\tilde{x},y): y\in B_1^{n-1}(0)\}$ at point $(\tilde{x},y)$ can be spanned by the following two linear independent vectors

$$\partial_{xy}P(x,y)|_{x=\kappa(\tilde{x})}\left(\frac{\partial\kappa}{\partial\tilde{x}}\right),$$

where

$$\partial_{xy}P(x,y)=\begin{pmatrix}1&0&2y_2+4x_3y_1\\0&1&2y_1\end{pmatrix}. \tag{3.2}$$

We claim that there does not exist a diffeomorphism κ such that

$$\begin{pmatrix}1&0&2y_2+4\kappa_3(\tilde{x})y_1\\0&1&2y_1\end{pmatrix}\left(\frac{\partial\kappa}{\partial\tilde{x}}\right)=\begin{pmatrix}1&0&C_1(y_1,y_2)\\0&1&C_2(y_1,y_2)\end{pmatrix}, \tag{3.3}$$

where $C_1(y_1, y_2)$, $C_2(y_1, y_2)$ only depend on y_1, y_2. Indeed, if (3.3) holds, we have

$$\frac{\partial \kappa_1(\tilde{x})}{\partial \tilde{x}_1} + (2y_2 + 4\kappa_3(\tilde{x})y_1)\frac{\partial \kappa_3(\tilde{x})}{\partial \tilde{x}_1} = 1,$$
$$\frac{\partial \kappa_1(\tilde{x})}{\partial \tilde{x}_2} + (2y_2 + 4\kappa_3(\tilde{x})y_1)\frac{\partial \kappa_3(\tilde{x})}{\partial \tilde{x}_2} = 0,$$
$$\frac{\partial \kappa_1(\tilde{x})}{\partial \tilde{x}_3} + (2y_2 + 4\kappa_3(\tilde{x})y_1)\frac{\partial \kappa_3(\tilde{x})}{\partial \tilde{x}_3} = C_1(y_1, y_2).$$

By solving the equations, one has $\kappa_1(\tilde{x}) = -2y_2\kappa_3(\tilde{x}) - 2y_1\kappa_3(\tilde{x})^2 + \tilde{x}_1 + C_1(y_1, y_2)\tilde{x}_3 + C_3(y_1, y_2)$. Since $C_1(y_1, y_2)$, $C_3(y_1, y_2)$ are constants when y_1, y_2 are fixed, we get

$$\begin{aligned}\kappa_1(\tilde{x}) &= -2\kappa_3(\tilde{x}) - 2\kappa_3(\tilde{x})^2 + \tilde{x}_1 + C_1(1,1)\tilde{x}_3 + C_3(1,1)\\ &= -\kappa_3(\tilde{x}) - \kappa_3(\tilde{x})^2 + \tilde{x}_1 + C_1(1/2,1/2)\tilde{x}_3 + C_3(1/2,1/2),\end{aligned}$$

then $\kappa_3(\tilde{x}) + \kappa_3(\tilde{x})^2 = c_1\tilde{x}_3 + c_3$, where $c_1 = C_1(1,1) - C_1(1/2,1/2)$ and $c_3 = C_3(1,1) - C_3(1/2,1/2)$. By the same argument, $\kappa_2(\tilde{x}) = -2y_1\kappa_3(\tilde{x}) + \tilde{x}_2 + C_2(y_1, y_2)\tilde{x}_3 + C_4(y_1, y_2)$ and $\kappa_3(\tilde{x}) = c_2\tilde{x}_3 + c_4$. Thus $c_2\tilde{x}_3 + c_4 + (c_2\tilde{x}_3 + c_4)^2 = c_1\tilde{x}_3 + c_3$ holds for all $\tilde{x}_3 \in \mathbb{R}$, which implies $c_1 = c_2 = 0$ and $\kappa_3(\tilde{x}) = c_4$, this is a contradiction since we assume κ is a diffemorphism.

In addition, we also assume some additional quantitative conditions on ϕ. Firstly, let's introduce a notion of *reduced form*.

Definition 3.3 We say a function $\phi(x, \xi)$ is of reduced form if $\phi \in \mathbf{\Phi}_{\mathrm{cs}}$ with the following conditions hold: let $\varepsilon > 0$ be a fixed constant and $a(x, \xi)$ be supported on $X \times \Omega$, where $X := X' \times X_n$ and $X' \subset B_1^{n-1}(0)$, $X_n \subset (-1, 1)$ and $\Omega \subset B_1^{n-1}(0)$, upon which the phase ϕ has the form

$$\phi(x, \xi) = \langle x', \xi\rangle + x_n h(\xi) + \mathcal{E}(x, \xi),$$

with

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \le C_{\alpha,\beta}, \quad |\alpha|, |\beta| \le N_{\mathrm{par}}, \tag{3.4}$$

here h and $\mathcal{E}$ are smooth functions and h is quadratic in ξ, $\mathcal{E}$ is quadratic in x, ξ and N_{par} is a given large constant.

Furthermore, ϕ also satisfies the following conditions:

$\mathbf{C_1}$: The eigenvalues of the Hessian $\left(\frac{\partial^2 h}{\partial \xi_i \partial \xi_j}\right)_{(n-1)\times(n-1)}$ all fall into $[1/2, 2]$.

$\mathbf{C_2}$: Let $c_{\mathrm{par}} > 0$ be a small constant, $N_{\mathrm{par}} > 0$ be a given large constant as above,

$$|\partial_x^\alpha \partial_\xi^\beta \mathcal{E}(x, \xi)| \le c_{\mathrm{par}}, \quad |\alpha|, |\beta| \le N_{\mathrm{par}}.$$

Let $1 \le R \le \lambda$, T^λ be defined with the reduced form and $Q_p(\lambda, R)$ be the optimal constant such that

$$\|T^\lambda f\|_{L^p(B_R^n(0))} \le Q_p(\lambda, R)\|f\|_{L^2}^{\frac{2}{p}}\|f\|_{L^\infty}^{1-\frac{2}{p}}. \tag{3.5}$$

We claim that the proof of Theorem 1.4 can be reduced to showing that for $p \ge p_n$ and for each $\varepsilon > 0$,

$$Q_p(\lambda, R) \lesssim_{\varepsilon,p} R^\varepsilon. \tag{3.6}$$

Indeed, we firstly claim that:

Claim: If $\mathcal{T}^\lambda$ is an operator satisfying the conditions H_1, H_2, H_3, H_4, then

$$\|\mathcal{T}^\lambda f\|_{L^p(B^n_R(0))} \lesssim_\phi \|T^{\lambda\tilde{r}^2}\tilde{f}\|_{L^p(B^n_{CR}(0))}, \tag{3.7}$$

where T^λ is defined with the reduced form, $\tilde{r} > 0$ is an appropriate constant depending on ϕ and

$$\|\tilde{f}\|_{L^p} \lesssim_\phi \|f\|_{L^p}. \tag{3.8}$$

We take the above claim for granted and prove (3.6) implies Theorem 1.4. To be more precise, we need to show (3.6) implies

$$\|\mathcal{T}^\lambda f\|_{L^p(B^n_R(0))} \lesssim_{\varepsilon,p,\phi} R^\varepsilon \|f\|_{L^p}.$$

Indeed, by (3.6),(3.7), (3.8), we have

$$\|\mathcal{T}^\lambda f\|_{L^p(B^n_R(0))} \lesssim_{\varepsilon,p,\phi} R^\varepsilon \|f\|_{L^2}^{\frac{2}{p}} \|f\|_{L^\infty}^{1-\frac{2}{p}}. \tag{3.9}$$

By taking $f = \chi_E$, we get

$$\|\mathcal{T}^\lambda f\|_{L^p(B^n_R(0))} \lesssim_{\varepsilon,p,\phi} R^\varepsilon \|f\|_{L^p}.$$

Then the desired results follows by interpolation argument. Therefore, it suffices to verify the claim. For convenience, we just need to track the phase when changing of variables.

The proof can be obtained by modifying the associated part in [15]. Without loss of generality, we may assume

$$\partial_x^\alpha \phi(x,0) = 0, \quad \partial_\xi^\alpha \phi(0,\xi) = 0, \quad \alpha \in \mathbb{Z}^n.$$

Otherwise, we take ϕ to be

$$\phi(x,\xi) + \phi(0,0) - \phi(0,\xi) - \phi(x,0).$$

By Taylor's formula, we have

$$\phi(x,\xi) = \partial_\xi \phi(x,0)\cdot\xi + \rho(x,\xi),$$

where $\rho(x,\xi)$ is quadratic in ξ. By the condition H_1, we may assume $\mathrm{rank}\,\partial_{x'\xi}\phi = n-1$ and $G(0,0) = (0,\dots,1)$, thus we may find a smooth function $\Phi(x', x_n, 0)$ such that

$$\partial_\xi \phi(\Phi(x', x_n, 0), x_n, 0) = x'.$$

By our assumption, one may also get

$$\Phi(0,0) = 0, \quad \partial_{x_n}\Phi(0,0) = 0, \quad \partial_{x'}\Phi(0,0) = \partial^2_{x'\xi}\phi(0,0)^{-1}. \tag{3.10}$$

By changing of variables

$$x' \longrightarrow \Phi(x', x_n, 0), \quad x_n \longrightarrow x_n,$$

thus it suffices to consider

$$\langle x', \xi\rangle + \rho(\Phi(x', x_n, 0), x_n, \xi).$$

Then taking another expansion in x and using (3.10) yield

$$\begin{aligned}\rho(\Phi(x', x_n, 0), x_n, \xi)) = \rho(\Phi(0,0), 0, \xi) + \partial_{x'}\rho(0,\xi)\partial_{x'}\Phi(0)x' \\ + \left(\partial_{x_n}\rho\right)(0,\xi)x_n + O(|x|^2|\xi|^2).\end{aligned} \tag{3.11}$$

Finally, from (3.10), one deduces that

$$\phi(x,\xi)=\langle x',\xi+\partial_{x'\xi}\phi(0,0)^{-T}\partial_{x'}\rho(0,\xi)\rangle+x_n\partial_{x_n}\rho(0,\xi)+O(|x|^2|\xi|^2).$$

Then by changing of variables

$$\xi+\partial_{x'\xi}\phi(0,0)^{-T}\partial_{x'}\rho(0,\xi)\longrightarrow\xi,$$

and taking $h(\xi)=\partial_{x_n}\rho(0,\xi)$, we have

$$\phi(x,\xi)=\langle x',\xi\rangle+x_nh(\xi)+O(|x|^2|\xi|^2).$$

Since $\Omega\subset B_1^{n-1}(0)$, we partition Ω into a family of balls $\{B_\alpha\}$ of radius r and center ξ_α, such that

$$\Omega\subset\bigcup_\alpha B_\alpha.$$

By triangle inequality, it suffices to consider a single ball B_α. By changing of variables

$$\xi\longrightarrow\tilde r\xi+\xi_\alpha,$$

where $\tilde r\geq r$, under the new coordinates, we just need to consider $\xi\in B_{r/\tilde r}^{n-1}(0)$ and

$$e^{2\pi i\phi^\lambda(x,\xi_\alpha)}\int e^{2\pi i(\phi^\lambda(x,\xi)-\phi^\lambda(x,\xi_\alpha))}a^\lambda(x,\xi)f(\xi)d\xi.$$

Since $\phi(x,\xi)=\langle x',\xi\rangle+x_nh(\xi)+\mathcal{E}(x,\xi)$, we have

$$\phi^\lambda(x,\xi)-\phi^\lambda(x,\xi_\alpha)=\tilde r\partial_\xi\phi^\lambda(x,\xi_\alpha)\cdot\xi+\tilde r^2x_n\tilde h(\xi)+\tilde r^2\widetilde{\mathcal{E}}^\lambda(x,\xi),$$

where

$$\begin{aligned}\tilde h(\xi)&:=\tilde r^{-2}(h(\tilde r\xi+\xi_\alpha)-h(\xi_\alpha)-\tilde r\partial_\xi h(\xi_\alpha)\cdot\xi)\\ \widetilde{\mathcal{E}}^\lambda(x,\xi)&:=\tilde r^{-2}(\mathcal{E}^\lambda(x,\tilde r\xi+\xi_\alpha)-\mathcal{E}^\lambda(x,\xi_\alpha)-\tilde r\partial_\xi\mathcal{E}^\lambda(x,\xi_\alpha)\cdot\xi).\end{aligned}\tag{3.12}$$

By another change of variables in x as follows

$$x'\longrightarrow\lambda\Phi\Big(\frac{x'}{\lambda\tilde r},\frac{x_n}{\lambda\tilde r^2},\xi_\alpha\Big),\quad x_n\longrightarrow\tilde r^{-2}x_n,$$

finally, it suffices to consider

$$\tilde\phi:=\langle x',\xi\rangle+x_n\tilde h(\xi)+\bar{\mathcal{E}}^{\lambda\tilde r^2}(x,\xi),\tag{3.13}$$

where $\bar{\mathcal{E}}(x,\xi):=\widetilde{\mathcal{E}}(\Phi(\tilde rx',x_n,\xi_\alpha),x_n,\xi)$.

From (3.12), $\tilde h$ is quadratic in ξ and $\bar{\mathcal{E}}(x,\xi)$ is quadratic in x,ξ. Furthermore, through an affine change of variables in ξ and by choosing appropriate small $\tilde r$ such that $r/\tilde r$ is also sufficiently small, we may ensure the condition C_1 and C_2. Define T^λ with the phase in (3.13) and note that all the implicit constants arising when performing the change of variables depend on ϕ, we will obtain (3.7) and (3.8).

To prove (3.6), we use the induction approach. For $\lambda\leq 1000$, (3.6) holds trivially by choosing the implicit constants sufficiently large.

3.1 Further remarks on the phase

Let ϕ be of the reduced form, by our assumption, we may choose a smooth function $p(x)$ such that $\phi(p(x), \xi)$ satisfies the straight condition and

$$|\partial_x^\alpha \partial_\xi^\beta \phi(p(x), \xi)| \le \bar{C}_{\alpha,\beta}, \quad |\alpha|, \beta| \le N_{\text{par}}, \tag{3.14}$$

uniformly. Indeed, we may always choose a function $p : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ with $|\partial_x p(x)| \lesssim 1$ such that $\phi(Ap(\frac{x}{A}), \xi)$ satisfies the straight condition where A is large enough to ensure (3.14).

4 wave packet decomposition

Let $r \ge 1$ and Θ_r be a collection of cubes $\{\theta\}$ of sidelength $\frac{9}{11} r^{-1/2}$ and center ξ_θ which cover the ball $B_2^{n-1}(0)$. Correspondingly, we take a smooth partition of unity $\{\psi_\theta\}_{\theta\in\Theta_r}$ with respect to the cover Θ_r. Let $\tilde{\psi}_\theta$ be a non-negative smooth cut-off function supported on $\frac{11}{9}\theta$ and equal to 1 on $\frac{11}{10}\theta$. Given a function g, by taking Fourier series expansion, we have

$$g(\xi)\psi_\theta(\xi) \cdot \tilde{\psi}_\theta(\xi) = \left(\frac{r^{1/2}}{2\pi}\right)^{n-1} \sum_{v\in r^{1/2}\mathbb{Z}^{n-1}} \widehat{(g\psi_\theta)}(v) e^{2\pi i v\cdot\xi} \tilde{\psi}_\theta(\xi).$$

Define

$$g_{\theta,v}(\xi) := \left(\frac{r^{1/2}}{2\pi}\right)^{n-1} \widehat{(g\psi_\theta)}(v) e^{2\pi i v\cdot\xi} \tilde{\psi}_\theta(\xi).$$

Correspondingly, we may make the following decomposition

$$g = \sum_{(\theta,v)\in\Theta_r\times r^{1/2}\mathbb{Z}^{n-1}} g_{\theta,v}.$$

Let $1 \le r \le R$ and $B_r^n(x_0) \subset B_R^n(0)$, define

$$\phi_{x_0}^\lambda(x,\xi) := \phi^\lambda(x,\xi) - \phi^\lambda(x_0,\xi).$$

By the assumption of the phase, there exists $\gamma_{\theta,v,x_0}^\lambda(x_n)$ such that

$$\partial_\xi \phi_{x_0}^\lambda(\gamma_{\theta,v,x_0}^\lambda(x_n), x_n, \xi_\theta) + v = 0.$$

Given θ, v, define a tube $T_{\theta,v} = T_{\theta,v}(x_0)$ to be

$$T_{\theta,v}(x_0) := \{(x', x_n) : |x' - \gamma_{\theta,v,x_0}^\lambda(x_n)| \lesssim r^{\frac{1+\delta}{2}}, |x_n - x_0^n| \le Cr\}, \tag{4.1}$$

and

$$g_{T_{\theta,v}} := e^{-2\pi i\phi^\lambda(x_0,\xi)} (g(\cdot) e^{2\pi i \phi^\lambda(x_0,\cdot)})_{\theta,v}.$$

Thus, we have

$$T^\lambda g(x) = \sum_{\theta,v} T^\lambda g_{T_{\theta,v}}(x).$$

We define a collection of tubes associated to the ball $B_r^n(x_0)$ by

$$\mathbb{T}[B_r^n(x_0)] := \{T_{\theta,v}(x_0) : (\theta, v) \in \Theta_r \times r^{1/2}\mathbb{Z}^{n-1}\}.$$

The main contribution of $T^\lambda g_{T_{\theta,v}}$ is concentrated on $T_{\theta,v}$ and rapidly decays outside of the tube which can be manifested in the following lemma.

Lemma 4.1 *If* $x \in B^n_r(x_0)\backslash T_{\theta,v}$, *then*

$$|T^\lambda g_{T_{\theta,v}}(x)| \lesssim_N (1 + r^{-1/2}|\nabla_\xi \phi^\lambda_{x_0}(x, \xi_\theta) + v|)^{-N}\mathrm{RapDec}(r)\|g\|_{L^2}.$$

Proof For convenience, we use T to denote $T_{\theta,v}$ and use g_{x_0} to denote $ge^{2\pi i\phi^\lambda(x_0,\xi)}$. Recall the definition of g_T, we have

$$T^\lambda g_T(x) = \left(\frac{r^{\frac{1}{2}}}{2\pi}\right)^{n-1} (g_{x_0}\psi_\theta)\widehat{\ }(v) \int e^{2\pi i\phi^\lambda(x,\xi)-2\pi i\phi^\lambda(x_0,\xi)} e^{2\pi i v\cdot\xi} a^\lambda(x, \xi)\widetilde{\psi}_\theta(\xi)d\xi.$$

By changing of variables: $\xi \longrightarrow r^{-1/2}\xi + \xi_\theta$, it suffices to consider the integral

$$\int e^{2\pi i\phi^\lambda(x,r^{-1/2}\xi+\xi_\theta)-2\pi i\phi^\lambda(x_0,r^{-1/2}\xi+\xi_\theta)} e^{2\pi i r^{-1/2}v\cdot\xi} a^\lambda(x, r^{-1/2}\xi + \xi_\theta)\widetilde{\psi}(\xi)d\xi.$$

Taking the derivative in ξ, we get

$$\begin{aligned}
&\partial_\xi(\phi^\lambda(x, r^{-1/2}\xi + \xi_\theta) - \phi^\lambda(x_0, r^{-1/2}\xi + \xi_\theta) + r^{-1/2}v \cdot \xi)\\
&\quad= r^{-1/2}(\partial_\xi\phi^\lambda(x, r^{-1/2}\xi + \xi_\theta)) - \partial_\xi\phi^\lambda(x, \xi_\theta) - (\partial_\xi\phi^\lambda(x_0, r^{-1/2}\xi + \xi_\theta) - \partial_\xi\phi^\lambda(x_0, \xi_\theta))\\
&\qquad + r^{-1/2}(v + \partial_\xi\phi^\lambda_{x_0}(x, \xi_\theta))\\
&\quad= r^{-1/2}(v + \partial_\xi\phi^\lambda_{x_0}(x, \xi_\theta)) + O(1).
\end{aligned}$$

Integration by parts, we will obtain the desired results. □

We also have the following L^2-orthogonality properties.

Lemma 4.2 (L^2-orthogonality) *For any* $\mathbb{T} \subset \mathbb{T}[B^n_r(x_0)]$, *it holds that*

$$\left\|\sum_{T\in\mathbb{T}} g_T\right\|_2^2 \lesssim \sum_{T\in\mathbb{T}} \|g_T\|_2^2 \lesssim \|g\|_2^2. \tag{4.2}$$

Moreover, if $\mathbb{T}$ *is any collection of tubes with the same* θ, *then*

$$\left\|\sum_{T\in\mathbb{T}} g_T\right\|_2^2 \sim \sum_{T\in\mathbb{T}} \|g_T\|_2^2.$$

Comparing wave-packet at different scales. Let $r^{1/2} < \rho < r$. Consider another smaller ball $B^n_\rho(\tilde{x}_0) \subset B^n_r(x_0)$. Similarly, we may define the wave-packet decomposition with respect to the ball $B^n_\rho(\tilde{x}_0)$. To distinguish the wave packet of different scales, we use $\widetilde{\mathbb{T}}[B^n_\rho(\tilde{x}_0)]$ to denote the smaller scale wave-packet.

Definition 4.3 We say a function h is concentrated on wave packets from a tube set $\mathbb{T}_\alpha$, if

$$h = \sum_{T\in\mathbb{T}_\alpha} h_T + \mathrm{RapDec}(r)\|h\|_2. \tag{4.3}$$

Definition 4.4 Let $(\theta, v) \in \Theta_r \times r^{1/2}\mathbb{Z}^{n-1}$ and let $(\tilde{\theta}, \tilde{v}) \in \Theta_\rho \times \rho^{1/2}\mathbb{Z}^{n-1}$. We define a set $\widetilde{\mathbb{T}}_{\theta,v}[B^n_\rho(\tilde{x}_0)]$ collection of smaller tubes as follows

$$\widetilde{\mathbb{T}}_{\theta,v}[B^n_\rho(\tilde{x}_0)] := \{\tilde{T}_{\tilde{\theta},\tilde{v}} \in \widetilde{\mathbb{T}}[B^n_\rho(\tilde{x}_0)] : \mathrm{dist}(\theta, \tilde{\theta}) \lesssim \rho^{-1/2}, |\tilde{v} - (\partial_\xi\phi^\lambda_{x_0}(\tilde{x}_0, \xi_\theta) + v)| \lesssim r^{(1+\delta)/2}\}.$$

One may carry over the approach verbatim in [12] to obtain the following two lemmas.

Lemma 4.5 [12] *Let $T_{\theta,v} \in \mathbb{T}[B^n_r(x_0)]$. Then it holds that*

$$g_{T_{\theta,v}} = (g_{T_{\theta,v}})|_{\widetilde{\mathbb{T}}_{\theta,v}[B^n_\rho(\tilde{x}_0)]} + \mathrm{RapDec}(r)\|g\|_2.$$

Lemma 4.6 [12] *Assume $T_{\theta,v} \subset \mathbb{T}[B^n_r(x_0)]$. If $\tilde{T}_{\tilde{\theta},\tilde{v}} \in \widetilde{\mathbb{T}}_{\theta,v}[B^n_\rho(\tilde{x}_0)]$, then it holds that*

$$\mathrm{HausDist}(\tilde{T}_{\tilde{\theta},\tilde{v}}, T_{\theta,v} \cap B^n_\rho(\tilde{x}_0)) \lesssim r^{(1+\delta)/2},$$

and

$$\measuredangle(G(\xi_\theta), G(\xi_{\tilde{\theta}})) \lesssim \rho^{-1/2}.$$

5 Transverse equidistribution property

Transverse equidistribution property is based on a simple observation which can be roughly stated as follows: if $\mathrm{supp}\hat{g} \subset B^n_r(0)$, $r > 0$, then g can not be concentrated in a ball of radius less than r^{-1}. Starting from this fact and other geometric assumptions, Guth [13] established the transverse equidistribution lemma for the extension operator. Then Guth–Hickman–Iliopoulou [15] extended it to the Hörmander type operator with the convex condition. It should be noted that the proof of the transverse equidistribution lemma in [15] relies on the phase that belongs to a category which may not satisfy the straight condition. To overcome this obstacle, we follow the approach in [12] which deals with the input function directly without recourse to a further operation under T^λ. Since T^λ satisfies the straight condition (i.e., for given ξ, $G^\lambda(x,\xi)$ keeps invariant when x changes), we may use $G(\xi)$ to denote $G^\lambda(x,\xi)$. It is worth noting that there are still some differences between [12] and our case at this point, for example, in [12],

$$G(\xi) = \frac{(-\xi, 1)}{\sqrt{1+|\xi|^2}},$$

therefore, if $V \subset \mathbb{R}^n$ is a subspace, then the set

$$\mathcal{S} := \{\xi \in \mathbb{R}^n : G(\xi) \in V\}$$

falls into an affine subspace in $\mathbb{R}^n$. However, in general cases, for example,

$$G(\xi) = \frac{(-\partial h(\xi), 1)}{\sqrt{1+|\partial h(\xi)|^2}},$$

where h is a smooth function with non-degenerate Hessian, the associated set $\mathcal{S}$ may be a curved submanifold which requires more technical handling.

Since $\phi(x,\xi)$ satisfies the straight condition, in terms of the Gauss map, it suffices to consider the following class of varying hypersurfaces $\{\partial_x \phi(x,\xi) : \xi \in \Omega\}$ at $x = 0$. By the condition H_1, we may find locally a function $q : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ such that

$$\partial_{x'}\phi(0, q(\xi)) = \xi.$$

Define

$$h(\xi) := \partial_{x_n}\phi(0, q(\xi)). \tag{5.1}$$

We may assume the hypersurface $\{\partial_x \phi(0,\xi) : \xi \in \Omega\}$ can be reparameterized by $\{(\xi, h(\xi) : \xi \in \Omega)\}$ with

$$\|\partial^2_{\xi\xi} h(\xi) - I_{n-1}\|_{\mathrm{op}} \ll 1, \quad \xi \in \Omega. \tag{5.2}$$

Otherwise, we can choose a non-degenerate matrix A such that

$$\partial^2_{\xi\xi} h(A\xi)|_{\xi=0} = I_{n-1}.$$

By choosing the support of ξ sufficiently small, it holds that

$$\left\| \partial^2_{\xi\xi} h(A\xi) - I_{n-1} \right\|_{\mathrm{op}} \ll 1.$$

Correspondingly, we make another affine transformation in x and replace $\phi(x, \xi)$ by $\tilde{\phi}(x, \xi)$ which is defined by

$$\tilde{\phi}(x, \xi) := \phi(A^{-1}x', x_n, q(A\xi)).$$

Obviously, $\tilde{\phi}(x, \xi)$ satisfies the straight conditions and

$$\partial_x \tilde{\phi}(x, \xi)|_{x=0} = (\xi, \partial_{x_n} \phi(0, q(A\xi))).$$

Definition 5.1 Let $P_1, \ldots, P_{n-m} : \mathbb{R}^n \to \mathbb{R}$ be polynomials. We consider the common zero set

$$Z(P_1, \ldots, P_{n-m}) := \{x \in \mathbb{R}^n : P_1(x) = \cdots = P_{n-m}(x) = 0\}. \tag{5.3}$$

Suppose that for all $z \in Z(P_1, \ldots, P_{n-m})$, one has

$$\bigwedge_{j=1}^{n-m} \nabla P_j(x) \neq 0.$$

Then a connected branch of this set, or a union of connected branches of this set, is called an m-dimensional transverse complete intersection. Given a set Z of the form (5.3), the degree of Z is defined by

$$\min \Big(\prod_{j=1}^{n-m} \deg(P_i) \Big),$$

where the minimum is taken over all possible representations of $Z = Z(P_1, \ldots, P_{n-m})$.

Definition 5.2 Let $r \geq 1$ and Z be an m-dimensional transverse complete intersection. A tube $T_{\theta,v}(x_0) \in \mathbb{T}[B^n_r(x_0)]$ is said to be $r^{-1/2+\delta_m}$-tangent to Z in $B^n_r(x_0)$ if it satisfies

- $T_{\theta,v}(x_0) \subset N_{r^{1/2+\delta_m}}(Z) \cap B^n_r(x_0)$;
- For every $z \in Z \cap B^n_r(x_0)$, if there is $y \in T_{\theta,v}(x_0)$ with $|z - y| \lesssim r^{1/2+\delta_m}$, then one has

$$\measuredangle(G(\theta), T_z Z) \lesssim r^{-1/2+\delta_m}.$$

Here, $T_z Z$ is the tangent space of Z at z and

$$G(\theta) := \{G(\xi) : \xi \in \theta\}.$$

Definition 5.3 Let $1 \leq \rho \leq r$ and Z be an m-dimensional transverse complete intersection and let $B^n_\rho(\tilde{x}_0) \subset B^n_r(x_0)$. Define a collection of tangent tubes inside a ball as

$$\mathbb{T}_Z[B^n_r(x_0)] := \{T \in \mathbb{T}[B^n_r(x_0)] : T \text{ is } r^{-1/2+\delta_m}\text{-tangent to } Z \text{ in } B^n_r(x_0)\}.$$

Given an arbitrary translation $b \in \mathbb{R}^n$, define

$$\widetilde{\mathbb{T}}_b[B^n_\rho(\tilde{x}_0)] := \{\tilde{T} \in \widetilde{\mathbb{T}}[B^n_\rho(\tilde{x}_0)] : \tilde{T} \text{ is } \rho^{-1/2+\delta_m} \text{ -tangent to } Z + b \text{ in } B^n_\rho(\tilde{x}_0)\}.$$

For simplicity, we also abbreviate T_Z and $\widetilde{\mathbb{T}}_b$ for $\mathbb{T}_Z[B^n_r(x_0)]$ and $\widetilde{\mathbb{T}}_b[B^n_\rho(\tilde{x}_0)]$ respectively.

We may state our main results in this section as follows.

Lemma 5.4 *Let $|b| \lesssim r^{1/2+\delta_m}$. Suppose that h is concentrated on large wave packets from $\mathbb{T}_Z \cap \mathbb{T}_{\tilde{\theta},w}$ for some $(\tilde{\theta}, w) \in \Theta_\rho \times r^{1/2}\mathbb{Z}^{n-1}$. Then for every $\widetilde{W} \subset \widetilde{\mathbb{T}}_b$, we have*

$$\left\| h|_{\widetilde{W}} \right\|_2^2 \lesssim r^{O(\delta_m)}(r/\rho)^{-\frac{n-m}{2}} \|h\|_2^2. \tag{5.4}$$

As a direct consequence of Lemma 5.4, we may obtain the following results.

Corollary 5.5 *Let $|b| \lesssim r^{1/2+\delta_m}$. Suppose that h is concentrated on large wave packet from $\mathbb{T}_Z$. Then for every $\widetilde{W} \subset \widetilde{\mathbb{T}}_b$, we have*

$$\left\| h|_{\widetilde{W}} \right\|_2^2 \lesssim r^{O(\delta_m)}(r/\rho)^{-\frac{n-m}{2}} \|h\|_2^2.$$

Proof We may rewrite

$$h|_{\widetilde{W}} = \sum_{(\tilde{\theta},w)} h_{\tilde{\theta},w},$$

such that $h_{\tilde{\theta},w}$ is concentrated on wave packets from $\mathbb{T}_Z \cap \mathbb{T}_{\tilde{\theta},w}$. Then, we may apply (5.4) to each $h_{\tilde{\theta},w}$ and recall the L^2-orthogonality in Lemma 4.2. □

Let $\tilde{x}_0 := (\tilde{x}_0', \tilde{x}_0^n)$, and define $x_\gamma := \gamma^\lambda_{\tilde{\theta},\tilde{v},\tilde{x}_0}(\tilde{x}_0^n)$, $B := B^{n-1}_{Cr^{1/2+\delta}}(x_\gamma)$. Let Z_0 be the intersection of $Z + b$ of the hyperplane $\{x : x_n = \tilde{x}_0\}$. Up to a harmless small perturbation[2] in the x_n direction, we may assume Z_0 is a transverse complete intersection in $\mathbb{R}^{n-1}$. Define a smooth map $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as follows

$$\Phi(x') := -\partial_\xi \phi(x', \tilde{x}_0^n, \xi_{\tilde{\theta}}).$$

Proposition 5.6 *Let h be concentrated on bigger wave packets from $\mathbb{T}_{\tilde{\theta},w} \cap \mathbb{T}_Z$. Then*

$$\|\widehat{(h|_{\widetilde{W}})}\|_2 \lesssim \|\hat{h}\chi_{N_{C\rho^{1/2+\delta_m}}(\Phi(Z_0)\cap\Phi(CB))}\|_2 + \mathrm{RapDec}(\rho)\|h\|_2. \tag{5.5}$$

Proof First we claim that
Claim: Let $\tilde{T}_{\tilde{\theta},\tilde{v}} \in \widetilde{W}$, then

$$\Big| \sum_{\tilde{T}_{\tilde{\theta},\tilde{v}} \in \widetilde{W}} \widehat{\left(\rho^{\frac{n-1}{2}} e^{2\pi i \phi^\lambda(\tilde{x}_0,\xi)} \psi_{\tilde{\theta}}\right)}(\tilde{v} - y) \Big| \le C\chi_{N_{C\rho^{1/2+\delta_m}}(\Phi(Z_0)\cap\Phi(CB))}(y) + \mathrm{RapDec}(\rho)\|h\|_2. \tag{5.6}$$

We firstly take the above claim for granted and continue the proof of (5.5). By the definition,

$$h|_{\widetilde{W}} = \sum_{\tilde{T}\in\widetilde{W}} h_T.$$

Therefore, by the orthogonality property, we have

$$\|(h|_{\widetilde{W}})\|_2^2 \lesssim \sum_{\tilde{T}\in\widetilde{W}} \|h_{\tilde{T}}\|_2^2.$$

[2] see the appendix in [12]

By the Plancherel's theorem, we get

$$\sum_{\tilde{T}\in\widetilde{W}} \|h_{\tilde{T}}\|_2^2 \lesssim \rho^{n-1} \sum_{\tilde{T}\in\widetilde{W}} |(h_{\tilde{x}_0}\psi_{\tilde{\theta}})\widehat{\ }(v)|^2 \|\psi_{\tilde{\theta}}\|_2^2 = \rho^{\frac{n-1}{2}} \sum_{\tilde{T}\in\widetilde{W}} |(h_{\tilde{x}_0}\psi_{\tilde{\theta}})\widehat{\ }(v)|^2.$$

Note that

$$(h_{\tilde{x}_0}\psi_{\tilde{\theta}})\widehat{\ }(\tilde{v}) = \hat{h} * (e^{2\pi i\phi^\lambda(\tilde{x}_0,\cdot)}\psi_{\tilde{\theta}})\widehat{\ }(\tilde{v}).$$

Using Hölder's inequality, we obtain

$$\rho^{\frac{n-1}{2}} \sum_{\tilde{T}\in\widetilde{W}} |(h_{\tilde{x}_0}\psi_{\tilde{\theta}})\widehat{\ }(v)|^2 \lesssim \int |\hat{h}(y)|^2 \Big(\sum_{\tilde{T}\in\widetilde{W}} |(\rho^{\frac{n-1}{2}} e^{2\pi i\phi^\lambda(\tilde{x}_0,\cdot)}\psi_{\tilde{\theta}})\widehat{\ }(\tilde{v}-y)| \Big) dy. \tag{5.7}$$

Then (5.5) follows from (5.6). Therefore, it remains to show the claim. By changing of variables: $\xi \to \rho^{-1/2}\xi + \xi_{\tilde{\theta}}$, we have

$$\Big(e^{2\pi i\phi^\lambda(\tilde{x}_0,\xi)}\psi_{\tilde{\theta}}\Big)\widehat{\ }(\tilde{v}-y) = \rho^{-\frac{n-1}{2}} \int e^{2\pi i\phi^\lambda(\tilde{x}_0,\rho^{-1/2}\xi+\xi_{\tilde{\theta}}) - 2\pi i(\tilde{v}-y)(\rho^{-1/2}\xi+\xi_{\tilde{\theta}})}\psi(\xi)d\xi.$$

Recall that

$$\partial_\xi \phi^\lambda_{\tilde{x}_0}(\gamma^\lambda_{\tilde{\theta},\tilde{v},\tilde{x}_0}(t), t, \xi_{\tilde{\theta}}) = \tilde{v}.$$

A stationary phase argument shows that the above integral is essentially nontrivial when $y \in B^n_{\rho^{1/2+\delta_m}}(\tilde{v} - \partial_\xi\phi^\lambda(\tilde{x}_0, \xi_{\tilde{\theta}}))$. By our assumption and definition, we have

$$\gamma^\lambda_{\tilde{\theta},\tilde{v},\tilde{x}_0}(\tilde{x}_0^n) \subset N_{C\rho^{1/2+\delta_m}}(Z_0) \cap CB,$$

Thus

$$\tilde{v} \subset N_{C\rho^{1/2+\delta_m}}(\Psi(Z_0)) \cap \Psi(CB),$$

where $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is defined by

$$\Psi(x') := \partial_\xi \phi^\lambda_{x_0}(x', \tilde{x}_0^n, \xi_{\tilde{\theta}}).$$

Recall that

$$\phi^\lambda_{\tilde{x}_0}(x,\xi) = \phi^\lambda(x,\xi) - \phi^\lambda(\tilde{x}_0,\xi),$$

thus we obtain the desired results. □

Proposition 5.7 *Assume* $Z_0 = (Z + b) \cap \{x_n = \tilde{x}_0^n\}$ *and* $B = B^n_{r^{1/2+\delta_m}}(\tilde{x}_0)$. *Suppose* h *is concentrated on scale* r *wave packets in* $\mathbb{T}_{\tilde{\theta},w} \cap \mathbb{T}_Z$. *Then*

$$\int |\hat{h}|^2 \cdot \chi_{N_{C\rho^{1/2+\delta_m}}(\Phi(Z_0)\cap\Phi(CB))} \lesssim r^{O(\delta_m)} \Big(\frac{\rho}{r}\Big)^{(n-m)/2} \|h\|_2^2. \tag{5.8}$$

The proof of Proposition 5.7 is left to the end of this section.

Define $T_{Z,B,\tilde{\theta}}$ as follows

$$\mathbb{T}_{V,B,\tilde{\theta}} := \{(\theta, v) : T_{\theta,v} \cap B \neq \emptyset, \measuredangle(G(\xi_\theta), V) \lesssim r^{-1/2+\delta_m},\ \mathrm{dist}(\theta,\tilde{\theta}) \lesssim \rho^{-1/2}\}.$$

Let V be an m-dimensional subspace defined by

$$V := \{x \in \mathbb{R}^n : \sum_{j=1}^n a_{i,j}x_j = 0, i = 1, \dots, n-m\}.$$

If $\phi(x,\xi) = x'\cdot\xi + \frac{1}{2}x_n|\xi|^2$, then

$$G(\xi) = \frac{(-\xi, 1)}{\sqrt{1+|\xi|^2}}.$$

It is easy to see that $\{\xi \in \mathbb{R}^n : G(\xi) \in V\}$ defines an affine subspace. Therefore, the set

$$\{\xi \in \mathbb{R}^n : \measuredangle(G(\xi), V) \lesssim r^{-1/2+\delta_m}\}$$

is contained in a $Cr^{-1/2+\delta_m}$-neighborhood of an affine subspace. However, if we only know $\phi(x,\xi)$ satisfies the Carleson–Sjölin conditions with the convex and straight assumptions, things may be a little trickier. Since in the general setting, $\{\xi \in \mathbb{R}^n : G(\xi) \in V\}$ may be a curved submanifold. Specially, for our case,

$$G(\xi) = \frac{(-\partial_\xi h(\xi), 1)}{\sqrt{1+|\partial_\xi h(\xi)|^2}},$$

where $h(\xi)$ is defined in (5.1) and satisfies (5.2). Let L denote the submanifold

$$L := \{\xi \in \mathbb{R}^n : G(\xi) \in V\},$$

by the implicit function theorem, we know the dimension of L is $m-1$. Define V' to be the tangent space $T_{\tilde\xi}L$ of L at a given point $\tilde\xi \in L$ with $\mathrm{dist}(\tilde\xi, \tilde\theta) \lesssim \rho^{-1/2}$.

Lemma 5.8 *The set*

$$\{\xi \in \mathbb{R}^n : \measuredangle(G(\xi), V) \lesssim r^{-1/2+\delta_m},\ \mathrm{dist}(\xi, \tilde\theta) \lesssim \rho^{-1/2}\}$$

is contained in a $Cr^{-1/2+\delta_m}$-neighborhood of an affine subspace V'.

Proof Obviously, the set

$$\{\xi \in \mathbb{R}^n : \measuredangle(G(\xi), V) \lesssim r^{-1/2+\delta_m}\}$$

is contained in a $Cr^{-1/2+\delta_m}$-neighborhood of L. Recall that

$$V' = T_{\tilde\xi}L, \text{ and } \mathrm{dist}(\tilde\xi, \tilde\theta) \lesssim r^{-1/2+\delta_m},$$

thus

$$\mathrm{dist}(\tilde\xi, \xi) \lesssim \rho^{-1/2}.$$

Therefore, it suffices to show

$$N_{Cr^{-1/2+\delta_m}}(L) \cap \{\xi : \mathrm{dist}(\xi, \tilde\xi) \lesssim \rho^{-1/2}\} \subset N_{Cr^{-1/2+\delta_m}}(V').$$

Without loss of generality, we may assume $\tilde\xi$ and L can be parametrized by $(0, u(0))$ and

$$L := \{(\xi', u(\xi')) : \xi' \in \mathbb{R}^{m-1}\}, \quad \text{with} \quad u'(0) = 0,$$

respectively. Therefore, it remains to show : if $|\xi'| \lesssim \rho^{-1/2}$

$$|u(\xi')| \lesssim r^{-1/2+\delta_m}. \tag{5.9}$$

Indeed, (5.9) can be easily obtained from Taylor's formula and the fact the second order derivatives of u can be uniformly bounded. □

Define $\widetilde{V}$ to be the orthogonal complement in $\mathbb{R}^{n-1}$, that is $\widetilde{V} := (V')^\perp$, and $\bar V \subset \mathbb{R}^{n-1}$ to be identified with $V \cap \{x_n = 0\}$.

Lemma 5.9 *Let* V *and* $\widetilde{V}$ *be defined as above, then* $\widetilde{V}$ *and* V *are transverse in the sense that*

$$\operatorname*{Angle}_{v\in\bar{V}\backslash\{0\},\tilde{v}\in\widetilde{V}\backslash\{0\}}(v,\tilde{v})\gtrsim 1. \tag{5.10}$$

Proof Since $G(\tilde{\xi})\in V$, we may write it explicitly as follows

$$\sum_{j=1}^{n-1} a_{i,j}\partial_{\xi_j}h(\tilde{\xi})-a_{i,n}=0,\ i=1,\ldots,n-m. \tag{5.11}$$

Define $\alpha_i:=(a_{i,1},\ldots,a_{i,n})$ and $\alpha_i':=(a_{i,1},\ldots,a_{i,n-1})$. From (5.11), we have

$$\operatorname{rank}(\alpha_1',\ldots\alpha_{n-m}')=n-m.$$

Since $V'=T_{\tilde{\xi}}L$, and $\widetilde{V}=(V')^{\perp}$, by (5.11), we have

$$\widetilde{V}=\operatorname{span}\langle\partial^2_{\xi\xi}h(\tilde{\xi})\alpha_1',\ldots,\partial^2_{\xi\xi}h(\tilde{\xi})\alpha_{n-m}'\rangle.$$

To prove (5.10), it suffices to show: for each $\bar{v}\in\bar{V}\backslash\{0\}$, then

$$\langle\partial^2_{\xi\xi}h(\tilde{\xi})\alpha_i',\bar{v}\rangle\ll 1. \tag{5.12}$$

Since

$$\langle\alpha_i',\bar{v}\rangle=0$$

and

$$\|\partial^2_{\xi\xi}h(\tilde{\xi})-I_{n-1}\|_{\mathrm{op}}\le\varepsilon_0,$$

thus (5.12) follows immediately by choosing ε_0 sufficiently small. □

Lemma 5.10 [14] *Suppose that* $G:\mathbb{R}^n\to\mathbb{C}$ *is a function, and* $\hat{G}$ *is supported in a ball* $B^n_{r_1}(\xi_0)$ *of radius* r_1*. Then, for any ball* $B^n_{r_2}(x_0)$ *of radius* $r_2\le {r_1}^{-1}$*,*

$$\int_{B^n_{r_2}(x_0)}|G|^2\lesssim\frac{|B^n_{r_2}|}{|B^n_{{r_1}^{-1}}|}\int|G|^2. \tag{5.13}$$

Finally, as a consequence of the above preparations, we have

Proposition 5.11 *Let* $V,\bar{V},\widetilde{V},V'$ *be defined as above. If* g *is concentrated on wave packets from* $\mathbb{T}_{V,B,\tilde{\theta}}$*, if* $\Pi\subset\{x_n=\tilde{x}_n^0\}$ *is any affine subspace parallel to* $\widetilde{V}$ *and* $y\in\Pi\cap\Phi(CB)$*, then*

$$\int_{\Pi\cap B^n_{\rho^{1/2+\delta_m}}(y)}|\hat{g}|^2\lesssim r^{O(\delta_m)}\left(\frac{\rho^{1/2}}{r^{1/2}}\right)^{\dim(\widetilde{V})}\int_{\Pi}|\hat{g}|^2.$$

Proof Note that g is concentrated from $\mathbb{T}_{V,B,\tilde{\theta}}$, from the above discussion, we have g is supported in the $r^{-1/2+\delta_m}$ neighborhood of V'. Thus, $\left(\hat{g}|_{\Pi}\right)^{\vee}$ is supported in an $n-m$ dimensional $r^{-1/2+\delta_m}$ ball centered at $\operatorname{proj}_{\widetilde{V}}(\xi_V)$, by Lemma 5.10, we obtain the desired results. □

Before the proof Proposition 5.7, we still needs some additional inputs. One may follow the approach in Section 6 of [12] to obtain

Proposition 5.12 1. $\Phi(Z_0)$ *is quantitatively transverse to* $\widetilde{V}$ *at every point* $z\in\Phi(Z_0)\cap\Phi(CB)$.

2. $\Phi^{-1}(\Pi)$ *is an* $n-1-\dim(V')$ *dimensional transverse complete intersection in* $\mathbb{R}^{n-1}$.
3. $\Pi \cap N_{C\rho^{1/2+\delta_m}}(\Phi(Z_0) \cap \Phi(CB))$ *can be covered by* $\left(\frac{r^{1/2}}{\rho^{1/2}}\right)^{\dim Z_0 - \dim V_0}$ *many balls in* Π *of radius* $\rho^{1/2+\delta_m}$.

Proof of Proposition 5.7 Since the wave packets in $\mathbb{T}_{Z,B,\tilde{\theta}}$ are tangent to Z in B, thus

$$\measuredangle(G(\theta), T_z Z) \lesssim r^{-1/2+\delta_m}$$

for every $z \in Z \cap 2B$ and $T_{\theta,v} \in \mathbb{T}_{Z,B,\tilde{\theta}}$. There is a subspace V of minimal dimension and $\dim V \leq \dim Z$ such that for all θ making contribution to $\mathbb{T}_{Z,B,\tilde{\theta}}$, we have

$$\measuredangle(G(\theta), V) \lesssim r^{-1/2+\delta_m},$$

which indicates that h is concentrated on wave packets from $T_{V,B,\tilde{\theta}}$. Therefore, by Lemma 5.11, we have

$$\int_{\Pi \cap B^n_{\rho^{1/2+\delta_m}}(y)} |\hat{h}|^2 \lesssim r^{O(\delta_m)} \left(\frac{\rho^{1/2}}{r^{1/2}}\right)^{\dim(\widetilde{V})} \int_{\Pi} |\hat{h}|^2.$$

Finally, by Proposition 5.12, we get

$$\int |\hat{h}|^2 \cdot \chi_{N_{C\rho^{1/2+\delta_m}}(\Phi(Z_0)\cap\Phi(CB))} \lesssim r^{O(\delta_m)} \left(\frac{\rho}{r}\right)^{(n-m)/2} \int_{\Pi} |\hat{h}|^2. \tag{5.14}$$

Integrate over the affine subspace Π which is parallel to $\widetilde{V}$, we will obtain the desired results. □

6 Broad-norm estimate

In this section, we assume the operator T^λ satisfies the straight condition. In this setting, the tubes introduced in (4.1) is straight. To prove Theorem 1.4, we will use the broad-narrow analysis developed by Bourgain–Guth [4], which deduces the linear estimates from the multilinear ones. In [13], Guth observed that full power of the $k-$linear inequality could be replaced by a certain weakened version of the multilinear estimate for the Fourier extension operators known as $k-$broad "norm" estimates. Following the approach developed by Guth in [13], we shall divide $T^\lambda f$ into narrow and broad parts in the frequency space, and one part is around a neighborhood of $(k-1)$-dimensional subspace, another comes from its outside. We estimate the contribution of the first part through the decoupling theorem and an induction on scales argument, and then use the k-broad "norm" estimates to handle the broad part.

First, we shall introduce a notion of broad "norm". Let $V \subset \mathbb{R}^n$ be a $(k-1)$-dimensional subspace. Assume $\{\tau\}$ are a collection of balls in $\mathbb{R}^{n-1}$ of radius K^{-1} which form a partition of $B_1^{n-1}(0)$. We denote by $\measuredangle(G(\tau), V)$ the smallest angle between the non-zero vectors $v \in V$ and $v' \in G(\tau)$, where

$$G(\tau) := \{G(\xi) : \xi \in \tau\}.$$

Define

$$f_\tau := f\chi_\tau.$$

For each ball $B^n_{K^2} \subset B^n_R$, define

$$\mu_{T^\lambda}(B^n_{K^2}) := \min_{V_1,\dots,V_L} \max_{\substack{\tau\notin V_\ell \\ 1\le\ell\le L}} \Big(\int_{B^n_{K^2}} |T^\lambda f_\tau|^p dx \Big),$$

where for each $1 \le \ell \le L$, V_ℓ is a $(k-1)-$dimensional subspace and $\tau \notin V_\ell$ means $\mathrm{Ang}(G(\tau), V_\ell) > K^{-1}$.

Let $\{B^n_{K^2}\}$ be a collection of finitely overlapping balls which form a cover of B^n_R. We define the k-broad "norm" by

$$\left\| T^\lambda f \right\|^p_{\mathrm{BL}^p_{k,L}(B^n_R)} := \sum_{B^n_{K^2}\subset B^n_R} \mu_{T^\lambda}(B^n_{K^2}).$$

We will establish the following broad norm estimate.

Theorem 6.1 *Let $\mathscr{T}^\lambda$ be defined with ϕ satisfying the conditions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4$ *and*

- *The eigenvalues of the Hessian*

$$\partial_{\xi\xi}\langle \partial_x \phi(x,\xi), G(x,\xi_0)\rangle|_{\xi=\xi_0}$$

  all fall into $[1/2, 2]$ for $x \in X, \xi_0 \in \Omega$.
- *Let $N_{\mathrm{par}} > 0$ be a given large constant as above,*

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| \le C_{\alpha,\beta}, \quad |\alpha|, |\beta| \le N_{\mathrm{par}}.$$

If $2 \le k \le n-1$ and

$$p \ge p_n(k) := 2 + \frac{6}{2(n-1) + (k-1)\prod_{i=k}^{n-1}\frac{2i}{2i+1}}, \tag{6.1}$$

then for every $\varepsilon > 0$, there exits L such that

$$\left\| \mathscr{T}^\lambda f \right\|_{\mathrm{BL}^p_{k,L}(B^n_R(0))} \lesssim_{\varepsilon,L,K} R^\varepsilon \|f\|^{2/p}_{L^2} \|f\|^{1-2/p}_{L^\infty}, \tag{6.2}$$

for every $K \ge 1$, $1 \le R \le \lambda$. Furthermore, the implicit constant depends polynomially on K.

By combining the material in Sect. 4, 5 and using the polynomial partitioning method, we may obtain the proof of Theorem (6.1). At this point, there is no difference between our case and that in [12]. Therefore, one may refer to [12] for details.

7 Going from k-broad to linear estimates

Proposition 7.1 *Let T^λ be defined with the reduced form and $\mathscr{T}^\lambda$ be defined as above. Suppose that for all $K \ge 1$, $\varepsilon > 0$, the operator $\mathscr{T}^\lambda$ obeys the k-broad inequality*

$$\|\mathscr{T}^\lambda f\|_{\mathrm{BL}^p_{k,L}(B^n_R(0))} \lesssim_{K,\varepsilon,L} R^\varepsilon \|f\|^{\frac{2}{p}}_{L^2} \|f\|^{1-\frac{2}{p}}_{L^\infty}, \tag{7.1}$$

for some fixed k, p, L and all $R \ge 1$. If

$$2\frac{2n-k+2}{2n-k} \le p \le 2\frac{k-1}{k-2},$$

then

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^2}^{\frac{2}{p}} \|f\|_{L^\infty}^{1-\frac{2}{p}}.$$

Therefore, as a consequence of Theorem 6.1 and Proposition 7.1, Theorem 1.4 holds for all $p \geq p_n$. To prove Proposition 7.1, we also need the decoupling inequality.

Lemma 7.2 (Decoupling inequality) *Let T^λ be a Hörmander-type operator with the convex condition and $V \subset \mathbb{R}^n$ be an m-dimensional linear subspace, then for $2 \leq p \leq 2m/(m-1)$ and $\delta > 0$, we have*

$$\Big\|\sum_{\tau \in V} T^\lambda g_\tau\Big\|_{L^p(B^n_{K^2})} \lesssim_\delta K^{(m-1)(1/2-1/p)+\delta} \left(\sum_{\tau\in V} \|T^\lambda g_\tau\|^p_{L^p(w_{B^n_{K^2}})}\right)^{1/p}.$$

Here, the sum over all caps τ for which $\measuredangle(G(\tau), V) \leq K^{-1}$.

Heuristically, if $K^2 \leq \lambda^{1/2-\delta}$ with $0 < \delta < 1/2$, T^λ is essentially equivalent to the translation invariant case on $B^n_{K^2}$, the fact can be seen by expanding the phase using Taylor's formula. Then Lemma 7.2 can be obtained directly by using the sharp ℓ^2-decoupling theorem of Bourgain-Demeter [3] and Hölder's inequality. For more details, One may refer to [1, 20].

Lemma 7.3 *Let $\mathcal{D}$ is a maximal R^{-1}-separated discrete subset of Ω, then*

$$\Big\|\sum_{\xi_\theta\in\mathcal{D}} e^{2\pi i\phi^\lambda(\cdot,\xi_\theta)} F(\xi_\theta)\Big\|_{L^p(B^n_R(0))} \lesssim Q_p(\lambda, R) R^{(n-1)/p'} \|F\|_{l^2(\mathcal{D})}^{\frac{2}{p}} \|F\|_{l^\infty(\mathcal{D})}^{1-\frac{2}{p}} \tag{7.2}$$

for all $F : \mathcal{D} \to \mathbb{C}$, where

$$\|F\|_{\ell^p(\mathcal{D})} := \Big(\sum_{\xi_\theta\in\mathcal{D}} |F(\xi_\theta)|^p\Big)^{\frac{1}{p}},$$

for $1 \leq p < \infty$ and $p = \infty$ with a usual modification.

Proof Here our proof is essentially the same as that of Lemma 11.8 in [15]. Let η be a bump smooth function on $\mathbb{R}^{n-1}$, which is supported on $B_2^{n-1}(0)$ and equals to 1 on $B_1^{n-1}(0)$. For each $\xi_\theta \in \mathcal{D}$, we set $\eta_\theta(\xi) := \eta(10R(\xi - \xi_\theta))$.

Then as in Lemma 11.8 of [15], we have

$$\Big|\sum_{\xi_\theta\in\mathcal{D}} e^{2\pi i\phi^\lambda(\cdot,\xi_\theta)} F(\xi_\theta)\Big| \lesssim R^{n-1} \sum_{k\in\mathbb{Z}^n} (1 + |k|)^{-(n+1)} |T^\lambda f_k(x)|, \tag{7.3}$$

where T^λ is defined with the reduced form and

$$f_k(\xi) := \sum_{\xi_\theta\in\mathcal{D}} F(\xi_\theta) c_{k,\theta}(\xi)\eta_\theta(\xi)$$

with $\|c_{k,\theta}(\xi)\|_\infty \le 1$. By the definition of $Q_p(\lambda, R)$ and (7.3),

$$\Big\|\sum_{\xi_\theta\in\mathcal{D}} e^{2\pi i\phi^\lambda(\cdot,\xi_\theta)}F(\xi_\theta)\Big\|_{L^p(B^n_R(0))}$$
$$\lesssim Q_p(\lambda, R)R^{n-1}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)}\|f_k\|^{\frac{2}{p}}_{L^2(B^{n-1}_2)}\|f_k\|^{1-\frac{2}{p}}_{L^\infty(B^{n-1}_2)}.$$

The support of η_θ are pairwise disjoint, for any $q>0$, we have

$$\|f_k\|_{L^q(B^{n-1}_2)} \lesssim R^{-(n-1)/q}\|F\|_{l^q(\mathcal{D})}.$$

Thus we get

$$\Big\|\sum_{\xi_\theta\in\mathcal{D}} e^{2\pi i\phi^\lambda(\cdot,\xi_\theta)}F(\xi_\theta)\Big\|_{L^p(B^n_R(0))}$$
$$\lesssim Q_p(\lambda, R)R^{n-1}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)}R^{-(n-1)/p}\|F\|^{\frac{2}{p}}_{l^2(\mathcal{D})}\|F\|^{1-\frac{2}{p}}_{l^\infty(\mathcal{D})}$$
$$\lesssim Q_p(\lambda, R)R^{(n-1)/p'}\|F\|^{\frac{2}{p}}_{l^2(\mathcal{D})}\|F\|^{1-\frac{2}{p}}_{l^\infty(\mathcal{D})}.$$

□

Lemma 7.4 (Parabolic rescaling) *Let $1\le R\le\lambda$, and f supported in a ball of radius K^{-1}, where $1\le K\le R$. Then for all $p\ge 2$ and $\delta>0$, we have*

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \lesssim_\delta Q_p\Big(\frac{\lambda}{K^2}, \frac{R}{K^2}\Big)R^\delta K^{2n/p-(n-1)}\|f\|^{\frac{2}{p}}_{L^2(B^{n-1}_1)}\|f\|^{1-\frac{2}{p}}_{L^\infty(B^{n-1}_1)}.$$

Proof Without loss of generality, we may assume the ball to be $B^{n-1}_{K^{-1}}(\bar{\xi})$. Doing the same argument as in Sect. 3, we obtain

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \lesssim_\delta K^{(n+1)/p}\|\widetilde{T}^{\lambda/K^2}\tilde{f}\|_{L^p(\tilde{D}_R)}$$

where $\widetilde{T}^{\lambda/K^2}$ is defined with phase $\tilde{\phi}$ as in (3.13) and $\tilde{D}_R$ is an ellipse with principle axes parallel to the coordinate axes and dimensions $O(R/K)\times\cdots\times O(R/K)\times O(R/K^2)$ and $\tilde{f}(\xi) := K^{-(n-1)}f(\bar{\xi}+K^{-1}\xi)$, note that for each $q>0$,

$$\|\tilde{f}\|_{L^q} \lesssim K^{-(n-1)+(n-1)/q}\|f\|_{L^q}.$$

Then it suffice to show that

$$\|\widetilde{T}^{\lambda/K^2}\tilde{f}\|_{L^p(\tilde{D}_R)} \lesssim_\delta Q_p\Big(\frac{\lambda}{K^2}, \frac{R}{K^2}\Big)R^\delta\|\tilde{f}\|^{\frac{2}{p}}_{L^2(B^{n-1}_1)}\|\tilde{f}\|^{1-\frac{2}{p}}_{L^\infty(B^{n-1}_1)}.$$

Since the phase $\tilde{\phi}$ is also of reduced form, to ease notations, we just need to show

$$\|T^\lambda f\|_{L^p(D_R)} \lesssim_\delta Q_p(\lambda, R)R^\delta\|f\|^{\frac{2}{p}}_{L^2(B^{n-1}_1)}\|f\|^{1-\frac{2}{p}}_{L^\infty(B^{n-1}_1)}.$$

for all $1\ll R\le R'\le\lambda$ and $\delta>0$, where

$$D_R := \Big\{x\in\mathbb{R}^n : \Big(\frac{|x'|}{R'}\Big)^2+\Big(\frac{|x_n|}{R}\Big)^2\le 1\Big\}$$

is an ellipse and T^λ is an operator with the reduced form. Choose a collection of essentially disjoint R^{-1}-caps θ covers B^{n-1}, denote the center of θ by ξ_θ and decompose f as $f = \sum_\theta f_\theta$. Set

$$T_\theta^\lambda f(x) := e^{-2\pi i\phi^\lambda(x,\xi_\theta)} T^\lambda(x),$$

hence we have

$$T^\lambda f(x) = \sum_\theta e^{-2\pi i\phi^\lambda(x,\xi_\theta)} T_\theta^\lambda f_\theta(x).$$

Fix $\delta > 0$ to be sufficiently small for the purpose of the forthcoming argument. We may also write

$$T_\theta^\lambda f_\theta(x) = T_\theta^\lambda f_\theta * \eta_{R^{1-\delta}}(x) + \mathrm{RapDec}(R)\|f\|_{L^2(B^{n-1})}$$

for some choice of smooth, rapidly decreasing function η such that $|\eta|$ admits a smooth rapidly decreasing majorant $\zeta : \mathbb{R}^n \to [0, +\infty)$ which is locally constant at scale 1. In particular, one has

$$\zeta_{R^{1-\delta}}(x) \lesssim R^\delta \zeta_{R^{1-\delta}}(y) \quad \text{if} \quad |x - y| \lesssim R. \tag{7.4}$$

Cover D_R by finitely-overlapping R-balls, and let B_R^n be some member of this cover with the center denoted by $\bar{x}$, by the above observation, for $z \in B_R^n(0)$, we have

$$|T^\lambda f(\bar{x} + z)| \lesssim R^\delta \int_{\mathbb{R}^n} \left| \sum_\theta e^{2\pi i\phi^\lambda(\bar{x}+z,\xi_\theta)} T_\theta^\lambda f_\theta(y) \right| \zeta_{R^{1-\delta}}(\bar{x} - y)dy.$$

By taking the L^p-norm in z and modifying the proof of Lemma 7.3 for the phase $\phi^\lambda(\bar{x}+\cdot, \xi_\theta)$, we have

$$\begin{aligned}\|T^\lambda f\|_{L^p(B_R^n(0))} &\lesssim R^\delta \int_{\mathbb{R}^n} \left\| \sum_\theta e^{2\pi i\phi^\lambda(\bar{x}+z,\xi_\theta)} T_\theta^\lambda f_\theta(y) \right\|_{L^p(B_R^n(0))} \zeta_{R^{1-\delta}}(\bar{x} - y)dy \\ &\lesssim Q_p(\lambda, R) R^{(n-1)/p'} R^\delta \int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(y)\|_{l^2(\theta)}^{2/p} \|T_\theta^\lambda f_\theta(y)\|_{l^\infty(\theta)}^{1-2/p} \zeta_{R^{1-\delta}}(\bar{x} - y)dy,\end{aligned}$$

where we use $\|a_\theta\|_{\ell^p(\theta)}$ to denote $\left(\sum_\theta |a_\theta|^p\right)^{1/p}$.

By property (7.4), for $z \in B_R^n(0)$

$$\begin{aligned}&\int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(y)\|_{l^2(\theta)}^{2/p} \|T_\theta^\lambda f_\theta(y)\|_{l^\infty(\theta)}^{1-2/p} \zeta_{R^{1-\delta}}(\bar{x} - y)dy \\ &= \int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^2(\theta)}^{2/p} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^\infty(\theta)}^{1-2/p} \zeta_{R^{1-\delta}}(y - z)dy \\ &\lesssim R^{O(\delta)} \int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^2(\theta)}^{2/p} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^\infty(\theta)}^{1-2/p} \zeta_{R^{1-\delta}}(y)dy \\ &\lesssim R^{O(\delta)} \left(\int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^2(\theta)}^2 \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^\infty(\theta)}^{p-2} \zeta_{R^{1-\delta}}(y)dy \right)^{1/p}\end{aligned}$$

Then we deduces that for all $z \in B_R^n(0)$

$$\begin{aligned}\|T^\lambda f\|_{L^p(B_R^n(0))} &\lesssim Q_p(\lambda, R) R^{(n-1)/p'} R^{O(\delta)} \\ &\quad \times \left(\int_{\mathbb{R}^n} \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^2(\theta)}^2 \|T_\theta^\lambda f_\theta(\bar{x} + z - y)\|_{l^\infty(\theta)}^{p-2} \zeta_{R^{1-\delta}}(y)dy \right)^{1/p}\end{aligned}$$

By raising both sides of this estimate to the pth power, averaging in z and summing over all balls $B^n_R(0)$ in the covering, it follows that $\|T^\lambda f\|_{L^p(D_R)}$ is dominated by

$$Q_p(\lambda, R)R^{(n-1)/p'-n/p}R^{O(\delta)}\left(\int_{\mathbb{R}^n}\sum_\theta \|T^\lambda_\theta f_\theta\|^2_{L^2(D_R-y)}\sup_\theta \|T^\lambda_\theta f_\theta\|^{p-2}_{L^\infty(D_R-y)}\zeta_{R^{1-\delta}}(y)dy\right)^{1/p}$$

We have the trivial estimate

$$\|T^\lambda_\theta f_\theta\|_{L^\infty(D_R-y)} \lesssim \|f_\theta\|_{L^1} \lesssim R^{-(n-1)}\|f_\theta\|_{L^\infty}$$

and

$$\|T^\lambda_\theta f_\theta\|_{L^2(D_R-y)} \lesssim R^{1/2}\|f_\theta\|_{L^2}.$$

Hence $\|T^\lambda f\|_{L^p(D_R)}$ is dominated by $Q_p(\lambda, R)R^{O(\delta)}\|f\|^{\frac{2}{p}}_{L^2}\|f\|^{1-\frac{2}{p}}_{L^\infty}$. □

Proof of Proposition 7.1 Let T^λ be defined with the reduced form. Then there exits a smooth function $p(x)$ such that $\phi(p(x), \xi)$ satisfies the straight condition with (3.14) holding. For convenience, we denote $\bar{\phi}(x, \xi) := \phi(p(x), \xi)$ and $\mathscr{T}^\lambda$ be defined with $\bar{\phi}$. Hence, we have

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \lesssim \|\mathscr{T}^\lambda f\|_{L^p(B^n_{CR}(0))}.$$

For a given ball $B^n_{K^2}$, we chose a collection of $(k-1)$- subspaces $V_1, \ldots, V_L$ which achieve the minimum under the definition of k- board "norm". Then

$$\int_{B^n_{K^2}}|\mathscr{T}^\lambda f|^p \lesssim K^{O(1)}\max_{\tau\notin V_\ell, 1\le\ell\le L}\int_{B^n_{K^2}}|\mathscr{T}^\lambda f_\tau|^p + \sum_{\ell=1}^L\int_{B^n_{K^2}}|\sum_{\tau\in V_\ell}\mathscr{T}^\lambda f_\tau|^p.$$

We can use the k-broad hypothesis to dominate the first term, indeed, let $\mathcal{B}_{K^2}$ be a collection of finitely overlapping balls of radius K^2 which cover $B^n_{CR}(0)$, then one has

$$\begin{aligned}\int_{B^n_{CR}(0)}|\mathscr{T}^\lambda f|^p &\lesssim K^{O(1)}\|\mathscr{T}^\lambda f\|^p_{\mathrm{BL}^p_{k,L}(B^n_{CR}(0))} + \sum_{B^n_{K^2}\in\mathcal{B}_{K^2}}\sum_{\ell=1}^L\int_{B^n_{K^2}}\Big|\sum_{\tau\in V_\ell}\mathscr{T}^\lambda f_\tau|^p\\ &\lesssim K^{O(1)}C(K, \varepsilon_1, L)R^{p\varepsilon_1}\|f\|^p_{L^p} + \sum_{B^n_{K^2}\in\mathcal{B}^n_{K^2}}\sum_{\ell=1}^L\int_{B^n_{K^2}}\Big|\sum_{\tau\in V_\ell}\mathscr{T}^\lambda f_\tau|^p,\end{aligned}$$

where $\varepsilon_1 > 0$ is a small constant which we will chose later.

By Lemma 7.2, for any δ', we have

$$\int_{B^n_{K^2}}|\sum_{\tau\in V_\ell}\mathscr{T}^\lambda f_\tau|^p \lesssim_{\delta'} K^{(k-2)(p/2-1)+\delta'}\sum_{\tau\in V_\ell}\int_{\mathbb{R}^n}|\mathscr{T}^\lambda f_\tau|^p w_{B^n_{K^2}}$$

for each $1 \le \ell \le L$. Since $w_{B^n_R(0)} = \sum_{B^n_{K^2}\in\mathcal{B}_{K^2}} w_{B^n_{K^2}}$, one has

$$\sum_{B^n_{K^2}\in\mathcal{B}_{K^2}}\sum_{\ell=1}^L\int_{B^n_{K^2}}|\sum_{\tau\in V_\ell}\mathscr{T}^\lambda f_\tau|^p \lesssim_{\delta'} K^{(k-2)(p/2-1)+\delta'}\sum_\tau\int_{\mathbb{R}^n}|\mathscr{T}^\lambda f_\tau|^p w_{B^n_R(0)}.$$

For each τ, we take the same approach as in Sect. 3 which obtains the reduced form from a general phase. To ease the notations, under the new coordinates, we use $T^\lambda f_\tau$ to denote the new operator which belongs to the reduced form and the new function. Therefore,

$$\int_{\mathbb{R}^n}|\mathscr{T}^\lambda f_\tau|^p w_{B^n_R(0)} \lesssim \int_{\mathbb{R}^n}|T^\lambda f_\tau|^p w_{B^n_{CR}(0)}.$$

Note that $w_{B^n_{CR}(0)}$ rapidly decay outside $B^n_{2CR}(0)$, we get

$$\sum_{B^n_{K^2}\in\mathcal{B}_{K^2}}\sum_{\ell=1}^{L}\int_{B^n_{K^2}}|\sum_{\tau\in V_\ell}T^\lambda f_\tau|^p\lesssim_{\delta'}K^{(k-2)(p/2-1)+\delta'}\sum_\tau\int_{B^n_{2CR}(0)}|T^\lambda f_\tau|^p.$$

Let $\delta>0$ be a small number to be determined later. By a finitely-overlapping decomposition and translation, from Lemma 7.4, we obtain

$$\int_{B^n_{2CR}(0)}|T^\lambda f_\tau|^p\lesssim Q_p\Big(\frac{\lambda}{K^2},\frac{R}{K^2}\Big)^pR^\delta K^{2n-(n-1)p}\|f_\tau\|^2_{L^2}\|f_\tau\|^{p-2}_{L^\infty}.$$

Let

$$e(k,p):=(k-2)(1-\frac{1}{2}p)-2n+(n-1)p.$$

Recall

$$\sum_\tau\|f_\tau\|^2_{L^2}\lesssim\|f\|^2_{L^2},$$

therefore, we have

$$\sum_{B^n_{K^2}\in\mathcal{B}^n_{K^2}}\sum_{\ell=1}^{L}\int_{B_{K^2}}\left|\sum_{\tau\in V_\ell}T^\lambda f_\tau\right|^p\lesssim_{\delta,\delta'}Q_p\Big(\frac{\lambda}{K^2},\frac{R}{K^2}\Big)^pR^\delta K^{-e(k,p)+\delta'}\|f\|^2_{L^2}\|f\|^{p-2}_{L^\infty}.$$

Combining above estimates, we get

$$\int_{B^n_R(0)}|T^\lambda f|^p\le(K^{O(1)}C(K,\varepsilon_1,L)R^{p\varepsilon_1}$$
$$+C_{\delta,\delta'}Q_p\Big(\frac{\lambda}{K^2},\frac{R}{K^2}\Big)^pR^\delta K^{-e(k,p)+\delta'})\|f\|^2_{L^2}\|f\|^{p-2}_{L^\infty}.$$

Then our induction assumption, it holds

$$Q_p(\lambda,R)^p\le K^{O(1)}C(K,\varepsilon_1,L)R^{p\varepsilon_1}+C_{\delta,\delta'}Q_p\Big(\frac{\lambda}{K^2},\frac{R}{K^2}\Big)^pR^\delta K^{-e(k,p)+\delta'}.$$

When $p\ge2\frac{2n-k+2}{2n-k}$, $e(k,p)\ge0$, thus

$$Q_p(\lambda,R)^p\le K^{O(1)}C(K,\varepsilon_1,L)R^{p\varepsilon_1}+C^p_\varepsilon R^{\varepsilon p}C_{\delta,\delta'}K^{-2\varepsilon p}R^\delta K^{\delta'}.$$

If we choose $K=K_0R^{\varepsilon^2}$ where $K_0>0$ is a sufficiently large constant depending on ε,δ,p, then

$$Q_p(\lambda,R)^p\le K^{O(1)}C(K,\varepsilon_1,L)R^{p\varepsilon_1}+C^p_\varepsilon R^{\varepsilon p}C_{\delta,\delta'}K_0^{-2\varepsilon p+\delta'}R^{-2\varepsilon^3p+\varepsilon^2\delta'+\delta}.$$

We choose $\delta'=\varepsilon p$, $\delta=\frac{1}{2}\varepsilon^3p$ and K_0 sufficiently large such that

$$K_0^{-2\varepsilon p+\delta'}C_{\delta,\delta'}\le\frac{1}{2},\quad -2\varepsilon^3p+\varepsilon^2\delta'+\delta<0.$$

Recall that $C(K,\varepsilon_1,L)$ depends polynomially on K, then we will complete the proof by choosing suitable $0<\varepsilon_1\ll\varepsilon$. □

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