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H^2-regularity for a two-dimensional transmission problem with geometric constraint

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Abstract
The H^2-regularity of variational solutions to a two-dimensional transmission problem with geometric constraint is investigated, in particular when part of the interface becomes part of the outer boundary of the domain due to the saturation of the geometric constraint. In such a situation, the domain includes some non-Lipschitz subdomains with cusp points, but it is shown that this feature does not lead to a regularity breakdown. Moreover, continuous dependence of the solutions with respect to the domain is established.

Keywords Transmission problem · Regularity · Non-Lipschitz domain

Mathematics Subject Classification 35B65 · 35J25 · 35J20

1 Introduction

The H^2-regularity of variational solutions to a two-dimensional transmission problem with geometric constraint is investigated, in particular when part of the interface becomes part of the outer boundary of the domain due to the geometric constraint, a situation in which the domain includes some non-Lipschitz subdomains with cusp points. Such a regularity is required in particular to guarantee that the variational solutions satisfy the strong formulation of the transmission problem. H^2-regularity is, however, not true in general and known to depend heavily on the geometry and smoothness of the domain and the interfaces. In

An extended version of this manuscript with the same title is available at https://arxiv.org/pdf/2103.07301.pdf. Some proofs being similar to [7] are only sketched herein but detailed proofs are supplied in the extended version for the sake of completeness.

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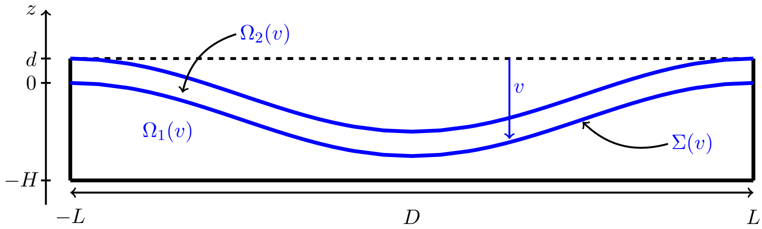

Fig. 1 Geometry of $\Omega(v)$ for a state $v \in \mathcal{S}$ with empty coincidence set

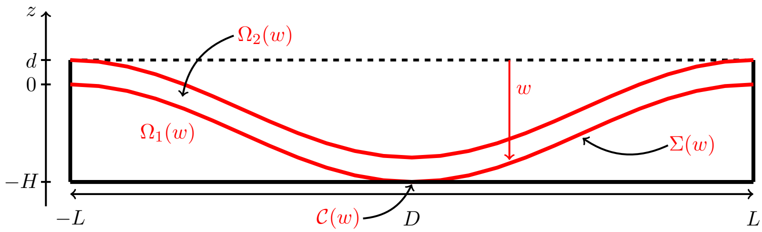

Fig. 2 Geometry of $\Omega(w)$ for a state $w \in \bar{\mathcal{S}}$ with non-empty coincidence set

fact, when interfaces intersect the outer boundary of the domain, regularity of variational solutions to transmission problems in non-smooth domains is a challenging issue, even for transversal intersections, see [1–3, 5, 10, 11, 13] and the references therein. Motivated by the mathematical study of microelectromechanical systems (MEMS), we identify herein a class of two-dimensional domains possibly featuring cusps for which H^2-regularity is true. We actually derive H^2-estimates which hold uniformly with respect to suitable perturbations of the underlying domain. We point out that such quantitative estimates are not contained in the above mentioned literature, but they turn out to be instrumental for a thorough study of MEMS models [9].

To set up the geometric framework, let $D := (-L, L)$ be a finite interval of $\mathbb{R}$, $L > 0$, and let $H > 0$ and $d > 0$ be two positive parameters. Given a function $u \in C(\bar{D}, [-H, \infty))$ with $u(\pm L) = 0$, we define the subdomain $\Omega(u)$ of $D \times (-H, \infty)$ by

$$\Omega(u) := \{(x, z) \in D \times \mathbb{R} \,:\, -H < z < u(x) + d\} = \Omega_1(u) \cup \Omega_2(u) \cup \Sigma(u)\,,$$

where

$$\Omega_1(u) := \{(x, z) \in D \times \mathbb{R} \,:\, -H < z < u(x)\}$$

and

$$\Omega_2(u) := \{(x, z) \in D \times \mathbb{R} \,:\, u(x) < z < u(x) + d\}$$

are separated by the interface

$$\Sigma(u) := \{(x, z) \in D \times \mathbb{R} \,:\, z = u(x) > -H\}\ .$$

Owing to the (geometric) constraint $u \geq -H$, the lower boundary of $\Omega_2(u)$, given by the graph of the function u, cannot go beyond the lower boundary $D \times \{-H\}$ of $\Omega_1(u)$ but may coincide partly with it, along the so-called coincidence set

$$\mathcal{C}(u) := \{x \in D \,:\, u(x) = -H\}\,, \tag{1.1}$$

see Figs. 1 and 2. Clearly, the geometry of $\Omega(u)$, as well as the regularity of its boundary, heavily depends on whether $\min_D\{u\} > -H$ or $\min_D\{u\} = -H$. Indeed, if $\min_D\{u\} > -H$ (i.e. the graph of u is strictly separated from $D \times \{-H\}$ as in Fig. 1), then the coincidence set $\mathcal{C}(u)$ is empty and $\Omega_1(u)$ is connected. In contrast, if $\min_D\{u\} = -H$ so that the graph of u intersects $D \times \{-H\}$, then $\mathcal{C}(u) \neq \emptyset$ and $\Omega_1(u)$ is disconnected with at least two (and possibly infinitely many) connected components, see Figs. 2 and 3.

For such a geometry, we study the regularity of variational solutions to the transmission problem

$$\operatorname{div}(\sigma\nabla\psi_u) = 0 \quad \text{in } \Omega(u)\,, \tag{1.2a}$$

$$[\![\psi_u]\!] = [\![\sigma\nabla\psi_u]\!] \cdot \mathbf{n}_{\Sigma(u)} = 0 \quad \text{on } \Sigma(u)\,, \tag{1.2b}$$

$$\psi_u = h_u \quad \text{on } \partial\Omega(u)\,, \tag{1.2c}$$

where

$$\sigma := \sigma_1 \mathbf{1}_{\Omega_1(u)} + \sigma_2 \mathbf{1}_{\Omega_2(u)}$$

for some positive constants $\sigma_1 \neq \sigma_2$, and $\mathbf{n}_{\Sigma(u)}$ denotes the unit normal vector field to $\Sigma(u)$ (pointing into $\Omega_2(u)$) given by

$$\mathbf{n}_{\Sigma(u)} := \frac{(-\partial_x u, 1)}{\sqrt{1 + (\partial_x u)^2}}\,.$$

In (1.2c), h_u is a suitable function reflecting the boundary behavior of ψ_u, see Section 2 for details. In addition, $[\![\cdot]\!]$ denotes the (possible) jump across the interface $\Sigma(u)$; that is,

$$[\![f]\!](x, u(x)) := f|_{\Omega_1(u)}(x, u(x)) - f|_{\Omega_2(u)}(x, u(x))\,, \qquad x \in D\,,$$

whenever meaningful for a function $f : \Omega(u) \to \mathbb{R}$.

Let us already mention that there are several features of the specific geometry of $\Omega(u)$ which may hinder the H^2-regularity of the solution ψ_u to (1.2). Indeed, on the one hand, the interface $\Sigma(u)$ always intersects with the boundary $\partial\Omega(u)$ of $\Omega(u)$ and it follows from [10] that this sole property prevents the H^2-regularity of ψ_u, unless σ and the angles between $\Sigma(u)$ and $\partial\Omega(u)$ at the intersection points satisfy some additional conditions. On the other hand, $\Omega(u)$ and $\Omega_2(u)$ are at best Lipschitz domains, while $\Omega_1(u)$ may consist of non-Lipschitz domains with cusp points.

The particular geometry $\Omega(u) = \Omega_1(u) \cup \Omega_2(u) \cup \Sigma(u)$, in which the boundary value problem (1.2) is set, is encountered in the investigation of an idealized electrostatically actuated MEMS as already pointed out and described in detail in [6, 14]. Such a device consists of an elastic plate of thickness d which is fixed at its boundary $\{\pm L\} \times (0, d)$ and suspended above a rigid conducting ground plate located at $z = -H$. The elastic plate is made up of a dielectric material and deformed by a Coulomb force induced by holding the ground plate and the top of the elastic plate at different electrostatic potentials. In this context, u represents the vertical deflection of the bottom of the elastic plate, so that the elastic plate is given by $\Omega_2(u)$, while $\Omega_1(u)$ denotes the free space between the elastic plate and the ground plate. An important feature of the model is that the elastic plate cannot penetrate the ground plate, resulting in the geometric constraint $u \geq -H$. Still, a contact between the elastic plate and the ground plate – corresponding to a non-empty coincidence set $\mathcal{C}(u)$ – is explicitly allowed. The dielectric properties of $\Omega_1(u)$ and $\Omega_2(u)$ are characterized by positive constants σ_1 and σ_2, respectively. The electrostatic potential ψ_u is then supposed to satisfy (1.2) and is completely determined by the deflection u. The state of the MEMS device is thus described by the deflection u, and equilibrium configurations of the device are obtained as critical points of the total energy which is the sum of the mechanical and electrostatic energies, the former being a functional of u while the latter is the Dirichlet integral of ψ_u. Owing to the nonlocal dependence of ψ_u on u, minimizing the total energy and deriving the associated Euler-Lagrange equation demand quite precise information on the regularity of the electrostatic potential ψ_u for an arbitrary, but *fixed* function u and its continuous dependence thereon. This first step of provisioning the required information is the main purpose of the present research. In the companion paper [9], we use the results obtained herein to analyze the minimizing problem leading to the determination of u and compute the associated Euler-Lagrange equation.

Since the regularity of the variational solution ψ_u to (1.2) is intimately connected with the regularity of the boundaries of $\Omega(u)$, $\Omega_1(u)$, and $\Omega_2(u)$, let us first mention that $\Omega(u)$ and $\Omega_2(u)$ are always Lipschitz domains and that the measures of the angles at their vertices do not exceed π, a feature which complies with the H^2-regularity of ψ_u away from the interface $\Sigma(u)$ [4]. This property is shared by $\Omega_1(u)$ when the coincidence set $\mathcal{C}(u)$ is empty, see Fig. 1, so that it is expected that $\psi|_{\Omega_i(u)}$ belongs to $H^2(\Omega_i(u))$, $i = 1, 2$, in that case. However, when $\mathcal{C}(u)$ is non-empty, the open set $\Omega_1(u)$ is no longer connected and the boundary of its connected components is no longer Lipschitz, but features cusp points. Moreover, there is an interplay between the transmission conditions (1.2b) and the boundary condition (1.2c) when $\mathcal{C}(u) \neq \emptyset$. Whether $\psi|_{\Omega_i(u)}$ still belongs to $H^2(\Omega_i(u))$, $i = 1, 2$, in this situation is thus an interesting question, that we answer positively in our first result. For the precise statement, we introduce the functional setting we shall work with in the sequel. Specifically, we set

$$\bar{\mathcal{S}} := \{v \in H^2(D) \cap H_0^1(D) \,:\, v \geq -H \text{ in } D \;\text{ and }\; \pm [\![\sigma]\!] \partial_x v(\pm L) \leq 0\}$$

and

$$\mathcal{S} := \{v \in H^2(D) \cap H_0^1(D) \,:\, v > -H \text{ in } D \;\text{ and }\; \pm [\![\sigma]\!] \partial_x v(\pm L) \leq 0\}\,.$$

Clearly, the coincidence set $\mathcal{C}(u)$ is empty if and only if $u \in \mathcal{S}$. In addition, the situation already alluded to, where $\mathcal{C}(u)$ is non-empty and $\Omega_1(u)$ is a disconnected open set in $\mathbb{R}^2$ with a non-Lipschitz boundary, corresponds to functions $u \in \bar{\mathcal{S}} \setminus \mathcal{S}$. Also, we include the constraint $\pm[\![\sigma]\!]\partial_x u(\pm L) \leq 0$ in the definition of $\mathcal{S}$ and $\bar{\mathcal{S}}$ to guarantee that the way $\Sigma(u)$ and $\partial\Omega(u)$ intersect does not prevent the H^2-regularity of ψ_u in smooth situations (i.e. $u \in \mathcal{S} \cap W_\infty^2(D)$), see [10].

Theorem 1.1 *Suppose* (2.1) *below.*

(a) *For each $u \in \bar{\mathcal{S}}$, there is a unique variational solution $\psi_u \in h_u + H_0^1(\Omega(u))$ to (1.2). Moreover, $\psi_{u,1} := \psi_u|_{\Omega_1(u)} \in H^2(\Omega_1(u))$ and $\psi_{u,2} := \psi_u|_{\Omega_2(u)} \in H^2(\Omega_2(u))$, and ψ_u is a strong solution to the transmission problem* (1.2).
(b) *Given $\kappa > 0$, there is $c(\kappa) > 0$ such that, for every $u \in \bar{\mathcal{S}}$ satisfying $\|u\|_{H^2(D)} \leq \kappa$,*

$$\|\psi_u\|_{H^1(\Omega(u))} + \|\psi_{u,1}\|_{H^2(\Omega_1(u))} + \|\psi_{u,2}\|_{H^2(\Omega_2(u))} \leq c(\kappa)\,.$$

It is worth emphasizing that, for $i \in \{1, 2\}$, the restriction of ψ_u to $\Omega_i(u)$ belongs to $H^2(\Omega_i(u))$ for all $u \in \bar{\mathcal{S}}$. In particular, there is no regularity breakdown when the coincidence set $\mathcal{C}(u)$ is non-empty. Moreover, the H^2-regularity of ψ_u is uniformly valid when u ranges in a bounded subset of $\bar{\mathcal{S}}$. A similar observation is made in [7] for a different geometric setting when one of the two subsets does not depend on the function u. Identifying other non-smooth geometries for which H^2-regularity of the variational solution to a transmission problem depends in a somewhat uniform way on some specific features of the domain is an interesting issue, which is worth a forthcoming investigation.

Remark 1.2 When the upper part $\Omega_2(v)$ is clamped at its lateral boundaries in the sense that

$$u \in H_0^2(D) := \{v \in H^2(D) \cap H_0^1(D) \,:\, \partial_x v(\pm L) = 0\}\,,$$

Theorem 1.1 applies whatever the values of σ_1 and σ_2.

Theorem 1.1 is an immediate consequence of Proposition 4.9 below. Its proof begins with quantitative H^2-estimates on ψ_u depending only on $\|u\|_{H^2(D)}$ for sufficiently smooth functions in $\mathcal{S}$, the H^2-regularity of ψ_u being guaranteed by [10] in that case. Since the class of functions for which these estimates are valid is dense in $\bar{\mathcal{S}}$, we complete the proof with a compactness argument, the main difficulty to be faced being the dependence of $\Omega(u)$ on u. More precisely, we begin with a variational approach to (1.2) and first show in Section 3 by classical arguments that, given $u \in \bar{\mathcal{S}}$, the variational solution ψ_u to (1.2) corresponds to the minimizer on $h_u + H_0^1(\Omega(u))$ of the associated Dirichlet energy

$$\mathcal{J}(u)[\theta] := \frac{1}{2}\int_{\Omega(u)} \sigma |\nabla\theta|^2 \,\mathrm{d}(x,z)\,, \qquad \theta \in h_u + H_0^1(\Omega(u))\,.$$

Thanks to this characterization, we use Γ-convergence tools to show the H^1-stability of ψ_u with respect to u in Sect. 3.2. Section 4 is devoted to the study of the H^2-regularity of ψ_u which we first establish in Sect. 4.1 for smooth functions $u \in \mathcal{S} \cap W_\infty^2(D)$ (thus having an empty coincidence set), relying on the analysis performed in [10]. It is worth mentioning that the constraint involving $[\![\sigma]\!]$ in the definition of $\mathcal{S}$ comes into play here. For $u \in \mathcal{S} \cap W_\infty^2(D)$, we next derive quantitative H^2-estimates on ψ_u which only depend on $\|u\|_{H^2(D)}$ as stated in Theorem 1.1 (b), see Sect. 4.2. The building block is an identity in the spirit of [4, Lemma 4.3.1.2] allowing us to interchange derivatives with respect to x and z in some integrals involving second-order derivatives, its proof being provided in Appendix 1. We then combine these estimates with the already proved H^1-stability of variational solutions to (1.2) and use a compactness argument to extend the H^2-regularity of ψ_u to arbitrary functions $u \in \bar{\mathcal{S}}$ in Sect. 4.3. In this step, special care is required to cope with the variation of the functional spaces with u. In fact, as a side product of the proof of Theorem 1.1, we obtain qualitative information on the continuous dependence of ψ_u with respect to u, which we collect in the next result.

Theorem 1.3 *Suppose (2.1) below. Let $\kappa > 0$, $u \in \bar{\mathcal{S}}$, and consider a sequence $(u_n)_{n\ge 1}$ in $\bar{\mathcal{S}}$ such that*

$$\|u_n\|_{H^2(D)} \le \kappa\,, \quad n \ge 1\,, \qquad \lim_{n\to\infty} \|u_n - u\|_{H^1(D)} = 0\,. \tag{1.3}$$

Setting $M := d + \max\left\{\|u\|_{L_\infty(D)}\,,\ \sup_{n\ge 1}\{\|u_n\|_{L_\infty(D)}\}\right\}$,

$$\lim_{n\to\infty} \left\|(\psi_{u_n} - h_{u_n}) - (\psi_u - h_u)\right\|_{H^1(\Omega_M)} = 0\,. \tag{1.4a}$$

In addition, if $i \in \{1,2\}$ and U_i is an open subset of $\Omega_i(u)$ such that $\bar{U}_i$ is a compact subset of $\Omega_i(u)$, then

$$\psi_{u_n,i} \rightharpoonup \psi_{u,i} \quad in \quad H^2(U_i)\,. \tag{1.4b}$$

Also, for any $p \in [1, \infty)$,

$$\begin{aligned} \lim_{n\to\infty} \left\| \nabla\psi_{u_n,2}(\cdot, u_n) - \nabla\psi_{u,2}(\cdot, u) \right\|_{L_p(D,\mathbb{R}^2)} = 0\,, \\ \lim_{n\to\infty} \left\| \nabla\psi_{u_n,2}(\cdot, u_n + d) - \nabla\psi_{u,2}(\cdot, u + d) \right\|_{L_p(D,\mathbb{R}^2)} = 0\,. \end{aligned} \tag{1.4c}$$

Clearly, the quantity M introduced in Theorem 1.3 is finite due to (1.3) and the continuous embedding of $H^1(D)$ in $C(\bar{D})$.

Remark 1.4 An interesting issue is the extension of the above results to a three-dimensional setting, where D is a bounded domain of $\mathbb{R}^2$ instead of an interval. There are, however, at least two difficulties to overcome, which are both of geometric nature. On the one hand, the coincidence set $\mathcal{C}(u)$ defined in (1.1) is no longer a countable union of open intervals when D is a two-dimensional domain and it might have a much more complicated structure. The former property plays an essential role in the proof of Proposition 4.9 (a) below. On the other hand, the Γ-convergence argument involved in the proof of Proposition 3.3 strongly makes use of the two-dimensional geometry of $\Omega(u)$. In fact, the literature on regularity of solutions to transmission problems in non-smooth three-dimensional domains when the interfaces intersect the outer boundary seems to be rather sparse and restricted to specific geometries. We refer to [1, 3, 5, 11, 13] for results in that direction.

Notation Given $v \in \bar{\mathcal{S}}$, $f \in L_2(\Omega(v))$, and $i \in \{1, 2\}$, we denote the restriction of f to $\Omega_i(v)$ by f_i; that is, $f_i := f|_{\Omega_i(v)}$.

Throughout the paper, c and $(c_k)_{k\geq 1}$ denote positive constants depending only on L, H, d, σ_1, and σ_2. The dependence upon additional parameters will be indicated explicitly.

2 The boundary values

We state the precise assumptions on the function h_v occurring in (1.2c). Roughly speaking, we assume that it is the trace on $\partial\Omega(v)$ of a function $h_v \in H^1(\Omega(v))$ which is such that $h|_{\Omega_i(v)}$ belongs to $H^2(\Omega_i(v))$ for $i = 1, 2$ and satisfies the transmission conditions (1.2b), as well as suitable boundedness and continuity properties with respect to v.

Specifically, for every $v \in \bar{\mathcal{S}}$, let

$$h_v : D \times (-H, \infty) \to \mathbb{R}$$

be such that

$$h_v \in H^1(\Omega(v))\,, \qquad h_{v,i} := h_v|_{\Omega_i(v)} \in H^2\big(\Omega_i(v)\big)\,, \quad i = 1, 2\,, \tag{2.1a}$$

and suppose that h_v satisfies the transmission conditions

$$[\![h_v]\!] = [\![\sigma\nabla h_v]\!] \cdot \mathbf{n}_{\Sigma(v)} = 0 \ \text{ on } \ \Sigma(v)\,. \tag{2.1b}$$

For $\kappa > 0$ given, there is $c(\kappa) > 0$ such that, for all $v \in \bar{\mathcal{S}}$ satisfying $\|v\|_{H^2(D)} \leq \kappa$,

$$\|h_{v,i}\|_{H^2(\Omega_i(v))} \leq c(\kappa)\,, \quad i = 1, 2\,. \tag{2.1c}$$

Moreover, given $v \in \bar{\mathcal{S}}$ and a sequence $(v_n)_{n\geq 1}$ in $\bar{\mathcal{S}}$ satisfying

$$\lim_{n\to\infty} \|v_n - v\|_{H^1(D)} = 0\,,$$

we assume that

$$\lim_{n\to\infty} \|h_{v_n} - h_v\|_{H^1(D\times(-H,M))} = 0 \tag{2.1d}$$

and

$$\lim_{n\to\infty} \|h_{v_n}(\cdot, v_n + d) - h_v(\cdot, v + d)\|_{C(\bar{D})} = 0\,, \tag{2.1e}$$

where

$$M := d + \max\left\{ \|v\|_{L_\infty(D)}\,,\ \sup_{n\ge 1}\{\|v_n\|_{L_\infty(D)}\} \right\} < \infty\,.$$

Observe that the convergence of $(v_n)_{n\ge1}$, the continuous embedding of $H^1(D)$ in $C(\bar{D})$, and (2.1d) imply that

$$\lim_{n\to\infty} \int_{\Omega(v_n)} \sigma|\nabla h_{v_n}|^2\,\mathrm{d}(x,z) = \int_{\Omega(v)} \sigma|\nabla h_v|^2\,\mathrm{d}(x,z)\,. \tag{2.2}$$

From now on, we impose the conditions (2.1) throughout.

We finish this short section by providing an example of h_v satisfying the imposed conditions (2.1).

Example 2.1 Let $\zeta \in C^2(\mathbb{R})$ be such that $\zeta|_{(-\infty,1]} \equiv 0$ and $\zeta|_{[1+d,\infty)} \equiv V$ for some $V > 0$. Given $v \in \bar{\mathcal{S}}$, put

$$h_v(x,z) := \zeta(z - v(x) + 1)\,, \qquad -H \le z\,, \quad x \in \bar{D}\,. \tag{2.3}$$

Then (2.1a)–(2.1e) are satisfied. In addition,

$$h_v(x,-H) = 0\,, \quad h_v(x, v(x)+d) = V\,, \qquad x \in D\,.$$

In the context of a MEMS device alluded to in the introduction, these additional properties mean that the ground plate and the top of the elastic plate are kept at constant potential. For instance, $\zeta(r) := V\min\{1, (r-1)^2/d^2\}$ for $r > 1$ and $\zeta \equiv 0$ on $(-\infty, 1]$ will do.

3 Variational solution to (1.2)

In this section we investigate the properties of the variational solution ψ_v to (1.2) for $v \in \bar{\mathcal{S}}$ and, in particular, its H^1-stability.

3.1 A variational approach to (1.2)

Given $v \in \bar{\mathcal{S}}$ we introduce the set of admissible potentials

$$\mathcal{A}(v) := h_v + H^1_0(\Omega(v))\,,$$

on which we define the functional

$$\mathcal{J}(v)[\theta] := \frac{1}{2}\int_{\Omega(v)} \sigma|\nabla\theta|^2\,\mathrm{d}(x,z)\,, \qquad \theta \in \mathcal{A}(v)\,. \tag{3.1}$$

The variational solution ψ_v to the transmission problem (1.2) is then the minimizer of the functional $\mathcal{J}(v)$ on the set $\mathcal{A}(v)$:

Lemma 3.1 *For each $v \in \bar{\mathcal{S}}$ there is a unique minimizer $\psi_v \in \mathcal{A}(v)$ of $\mathcal{J}(v)$ on $\mathcal{A}(v)$; that is,*

$$\mathcal{J}(v)[\psi_v] = \min_{\theta \in \mathcal{A}(v)} \mathcal{J}(v)[\theta] . \tag{3.2}$$

In addition,

$$\int_{\Omega(v)} \sigma |\nabla \psi_v|^2 \, \mathrm{d}(x,z) \le \int_{\Omega(v)} \sigma |\nabla h_v|^2 \, \mathrm{d}(x,z) . \tag{3.3}$$

Proof Let $v \in \bar{\mathcal{S}}$ and recall that $h_v \in H^1(\Omega(v))$ according to (2.1a). Thus, the existence of a minimizer ψ_v of $\mathcal{J}(v)$ on $\mathcal{A}(v)$ readily follows from the direct method of calculus of variations due to the lower semicontinuity and coercivity of $\mathcal{J}(v)$ on $\mathcal{A}(v)$, the latter being ensured by the assumption $\sigma \ge \min\{\sigma_1, \sigma_2\} > 0$ and Poincaré's inequality. The uniqueness of ψ_v is guaranteed by the strict convexity of $\mathcal{J}(v)$. Next, since obviously $h_v \in \mathcal{A}(v)$, the inequality (3.3) is an immediate consequence of the minimizing property (3.2) of ψ_v. □

For further use, we report the following version of Poincaré's inequality for functions in $H_0^1(\Omega(v))$ with a constant depending mildly on $v \in \bar{\mathcal{S}}$.

Lemma 3.2 *Let $v \in \bar{\mathcal{S}}$ and $\theta \in H_0^1(\Omega(v))$. Then*

$$\|\theta\|_{L_2(\Omega(v))} \le 2\|H + d + v\|_{L_\infty(D)} \|\partial_z \theta\|_{L_2(\Omega(v))} .$$

Proof For $x \in D$ and $z \in (-H, v(x) + d)$,

$$\theta(x,z)^2 = 2 \int_{-H}^{z} \theta(x,y) \partial_z \theta(x,y) \, \mathrm{d}y .$$

Hence, after integration with respect to (x,z) over $\Omega(v)$,

$$\begin{aligned} \|\theta\|_{L_2(\Omega(v))}^2 &= \int_{\Omega(v)} \theta(x,z)^2 \, \mathrm{d}(x,z) \\ &\le 2\|H + d + v\|_{L_\infty(D)} \int_{\Omega(v)} |\theta(x,y)| |\partial_z \theta(x,y)| \, \mathrm{d}(x,z) \\ &\le 2\|H + d + v\|_{L_\infty(D)} \|\theta\|_{L_2(\Omega(v))} \|\partial_z \theta\|_{L_2(\Omega(v))} , \end{aligned}$$

from which we deduce the stated inequality. □

3.2 H^1-stability of ψ_v

The purpose of this section is to study the continuity properties of the solution ψ_v to (3.2) with respect to v. More precisely, we aim at establishing the following result.

Proposition 3.3 *Consider $v \in \bar{\mathcal{S}}$ and a sequence $(v_n)_{n\ge1}$ in $\bar{\mathcal{S}}$ such that*

$$v_n \to v \ \ in \ \ H_0^1(D) , \tag{3.4}$$

and set

$$M := d + \max\left\{ \|v\|_{L_\infty(D)} \, , \, \sup_{n \ge 1}\{\|v_n\|_{L_\infty(D)}\} \right\} , \tag{3.5}$$

which is finite by (3.4) *and the continuous embedding of* $H^1(D)$ *in* $C(\bar{D})$. *Then*

$$\lim_{n\to\infty} \left\|(\psi_{v_n} - h_{v_n}) - (\psi_v - h_v)\right\|_{H_0^1(D\times(-H,M))} = 0$$

and

$$\lim_{n\to\infty} \mathcal{J}(v_n)[\psi_{v_n}] = \mathcal{J}(v)[\psi_v]\,.$$

To prove Proposition 3.3, we make use of a Γ-convergence approach and argue as in [7, Section 3.2] with minor changes. We thus omit the proof here and refer to the extended version of this paper [8] for details.

4 H^2-regularity

In the previous section we introduced the variational solution $\psi_v \in H^1(\Omega(v))$ to (1.2) for arbitrary $v \in \bar{\mathcal{S}}$ and noticed its continuous dependence in $H^1(\Omega(v))$ with respect to v. We now aim at improving the H^1-regularity of $\psi_v|_{\Omega_i(v)}$ to $H^2(\Omega_i(v))$ for $i = 1, 2$. To this end we first consider the case of smooth functions $v \in \mathcal{S} \cap W^2_\infty(D)$ with empty coincidence sets and provide in Sects. 4.1 and 4.2 the corresponding H^2-estimates that depend only on the norm of v in $H^2(D)$ (but *not* on its $W^2_\infty(D)$-norm). In Sect. 4.3 we extend these estimates to the general case $v \in \bar{\mathcal{S}}$ by means of a compactness argument.

4.1 H^2-regularity for $v \in \mathcal{S} \cap W^2_\infty(D)$

Assuming that v is smoother with an empty coincidence set, see Fig. 1, the existence of a strong solution ψ_v to (1.2) is a consequence of the analysis performed in [10].

Proposition 4.1 *If* $v \in \mathcal{S} \cap W^2_\infty(D)$, *then the variational solution* ψ_v *to* (3.2) *satisfies*

$$\psi_{v,i} := \psi_v|_{\Omega_i(v)} \in H^2(\Omega_i(v))\,, \quad i = 1, 2\,,$$

and the transmission problem

$$\operatorname{div}(\sigma\nabla\psi_v) = 0 \quad in\ \ \Omega(v)\,, \tag{4.1a}$$

$$[\![\psi_v]\!] = [\![\sigma\nabla\psi_v]\!]\cdot\mathbf{n}_{\Sigma(v)} = 0 \quad on\ \ \Sigma(v)\,, \tag{4.1b}$$

$$\psi_v = h_v \quad on\ \ \partial\Omega(v)\,. \tag{4.1c}$$

Moreover, $\partial_x\psi_v + \partial_x v\partial_z\psi_v$ *and* $-\sigma\partial_x v\partial_x\psi_v + \sigma\partial_z\psi_v$ *both belong to* $H^1(\Omega(v))$.

Besides [10], the proof of Proposition 4.1 requires the following auxiliary result.

Lemma 4.2 *Let* $v \in \bar{\mathcal{S}}$ *and consider* $\phi \in L_2(\Omega(v))$ *such that*

$$\phi_i := \phi|_{\Omega_i(v)} \in H^1(\Omega_i(v))\,, \quad i = 1, 2\,,$$

and $[\![\phi]\!] = 0$ *on* $\Sigma(v)$. *Then* $\phi \in H^1(\Omega(v))$ *and*

$$\|\phi\|_{H^1(\Omega(v))} \le \|\phi_1\|_{H^1(\Omega_1(v))} + \|\phi_2\|_{H^1(\Omega_2(v))}\,. \tag{4.2}$$

Proof We set $e_x = (1,0)$ and $e_z = (0,1)$. Given $\theta \in C_c^\infty\big(\Omega(v)\big)$ and $j \in \{x,z\}$ we note that

$$\begin{aligned}\int_{\Omega(v)} \phi \partial_j \theta \,\mathrm{d}(x,z) &= \int_{\Omega(v)} \mathrm{div}(\phi\theta e_j)\,\mathrm{d}(x,z) - \sum_{i=1}^{2}\int_{\Omega_i(v)} \theta \partial_j \phi_i \,\mathrm{d}(x,z)\\ &= \int_{\Sigma(v)} [\![\phi]\!]\,\theta e_j \cdot \mathbf{n}_{\Sigma(v)}\,\mathrm{d}\sigma_{\Sigma(v)} - \sum_{i=1}^{2}\int_{\Omega_i(v)} \theta \partial_j \phi_i \,\mathrm{d}(x,z)\,,\end{aligned}$$

due to Gauß' theorem. Thus, since $[\![\phi]\!] = 0$ on $\Sigma(v)$,

$$\left|\int_{\Omega(v)} \phi \partial_j \theta \,\mathrm{d}(x,z)\right| \le \big(\|\phi_1\|_{H^1(\Omega_1(v))} + \|\phi_2\|_{H^1(\Omega_2(v))}\big)\,\|\theta\|_{L_2(\Omega(v))}\,,$$

for $j = x, z$ and $\theta \in C_c^\infty\big(\Omega(v)\big)$. Consequently, $\phi \in H^1(\Omega(v))$. □

Proof of Proposition 4.1 We check that the transmission problem (4.1) fits into the framework of [10]. Since $v \in \mathcal{S} \cap W_\infty^2(D)$ and $v(\pm L) = 0$, the boundaries of $\Omega_1(v)$ and $\Omega_2(v)$ are W_∞^2-smooth curvilinear polygons and the interface $\Sigma(v)$ meets the boundary $\partial\Omega(v)$ of $\Omega(v)$ at the vertices $A_\pm := (\pm L, 0)$. Moreover, at the vertex $A_\pm$, the measures $\omega_{\pm,1}$ and $\omega_{\pm,2}$ of the angles between $-e_z$ and $(1, \mp\partial_x v(\pm L))$ and between $(1, \mp\partial_x v(\pm L))$ and e_z, respectively, satisfy $\omega_{\pm,1} + \omega_{\pm,2} = \pi$, as well as

$$\begin{aligned}\omega_{\pm,2} &\ge \frac{\pi}{2} \quad \text{if } [\![\sigma]\!] < 0\,,\\ \omega_{\pm,2} &\le \frac{\pi}{2} \quad \text{if } [\![\sigma]\!] > 0\,,\end{aligned}$$

by definition of $\mathcal{S}$. According to the analysis performed in [10], these conditions guarantee that the variational solution ψ_v to (3.2) provided by Lemma 3.1 satisfies $\psi_{v,i} = \psi_v|_{\Omega_i(v)} \in H^2(\Omega_i(v))$ for $i = 1, 2$ and solves the transmission problem (1.2) in a strong sense.

Next, owing to the just established H^2-regularity of $\psi_{v,1}$ and $\psi_{v,2}$, we may differentiate with respect to x the transmission condition $[\![\psi_v]\!](x, v(x)) = 0$, $x \in D$, and find that

$$[\![\partial_x \psi_v + \partial_x v \partial_z \psi_v]\!] = 0 \quad \text{on } \Sigma(v)\,.$$

The stated H^1-regularity of $\partial_x \psi_v + \partial_x v \partial_z \psi_v$ then follows from Lemma 4.2 and the boundedness of $\partial_x v$ and $\partial_x^2 v$. In the same vein, due to (1.2b), the regularity of v, and the identity

$$\frac{[\![-\sigma\partial_x v \partial_x \psi_v + \sigma \partial_z \psi_v]\!]}{\sqrt{1 + (\partial_x v)^2}} = [\![\sigma \nabla \psi_v]\!] \cdot \mathbf{n}_{\Sigma(v)} = 0\,,$$

the claimed H^1-regularity of $-\sigma\partial_x v \partial_x \psi_v + \sigma \partial_z \psi_v$ is again a consequence of Lemma 4.2 and the boundedness of $\partial_x v$ and $\partial_x^2 v$. □

4.2 H^2-Estimates on ψ_v for $v \in \mathcal{S} \cap W_\infty^2(D)$

The H^2-regularity of ψ_v being guaranteed by Proposition 4.1 for $v \in \mathcal{S} \cap W_\infty^2(D)$, the next step is to show that this property extends to any $v \in \bar{\mathcal{S}}$. To this end, we shall now derive quantitative H^2-estimates on ψ_v, paying special attention to their dependence upon the regularity of v. As in [7], it turns out to be more convenient to study a non-homogeneous transmission problem with homogeneous Dirichlet boundary conditions instead of (4.1). Specifically, for $v \in \mathcal{S} \cap W^2_\infty(D)$, we define

$$\chi = \chi_v := \psi_v - h_v \in H^1_0(\Omega(v)) , \tag{4.3}$$

where $\psi_v \in H^1(\Omega(v))$ is the unique solution to (4.1) provided by Proposition 4.1. Since $\psi_{v,i} = \psi_v|_{\Omega_i(v)}$ belongs to $H^2(\Omega_i(v))$ for $i = 1, 2$, we readily infer from (2.1a) and (4.3) that

$$\chi_i := \chi_v|_{\Omega_i(v)} \in H^2(\Omega_i(v)) , \quad i = 1, 2 . \tag{4.4}$$

We omit in the following the dependence of χ on v for ease of notation.

According to (2.1a), (2.1b), and Proposition 4.1, χ solves the transmission problem

$$\operatorname{div}(\sigma\nabla\chi) = -\operatorname{div}(\sigma\nabla h_v) \quad \text{in } \Omega(v) , \tag{4.5a}$$

$$[\![\chi]\!] = [\![\sigma\nabla\chi]\!] \cdot \mathbf{n}_{\Sigma(v)} = 0 \quad \text{on } \Sigma(v) , \tag{4.5b}$$

$$\chi = 0 \quad \text{on } \partial\Omega(v) , \tag{4.5c}$$

and it follows from (2.1a) that it is equivalent to derive H^2-estimates on $(\psi_{v,1}, \psi_{v,2})$ or (χ_1, χ_2).

For that purpose, we transform (4.5) to a transmission problem on the rectangle $\mathcal{R} := D \times (0, 1 + d)$. More precisely, we introduce the transformation

$$T_1(x, z) := \left(x, \frac{z + H}{v(x) + H} \right) , \qquad (x, z) \in \Omega_1(v) , \tag{4.6}$$

mapping $\Omega_1(v)$ onto the rectangle $\mathcal{R}_1 := D \times (0, 1)$, and the transformation

$$T_2(x, z) := (x, z - v(x) + 1) , \qquad (x, z) \in \Omega_2(v) , \tag{4.7}$$

mapping $\Omega_2(v)$ onto the rectangle $\mathcal{R}_2 := D \times (1, 1 + d)$. The interface separating $\mathcal{R}_1$ and $\mathcal{R}_2$ is

$$\Sigma_0 := D \times \{1\} ,$$

so that

$$\mathcal{R} = D \times (0, 1 + d) = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \Sigma_0 .$$

It is worth pointing out here that T_1 is well-defined due to $v \in \mathcal{S}$. Let (x, η) denote the new variables in $\mathcal{R}$; that is, $(x, \eta) = T_1(x, z)$ for $(x, z) \in \mathcal{R}_1$ and $(x, \eta) = T_2(x, z)$ for $(x, z) \in \mathcal{R}_2$. Then, (4.4) implies

$$\Phi := \Phi_1 \mathbf{1}_{\mathcal{R}_1} + \Phi_2 \mathbf{1}_{\mathcal{R}_2} \in H^1_0(\mathcal{R}) , \qquad \Phi_i := \chi_i \circ T_i^{-1} \in H^2(\mathcal{R}_i) , \quad i = 1, 2 . \tag{4.8}$$

For further use, we also introduce

$$\hat{\sigma}(x, \eta) := \begin{cases} \dfrac{\sigma_1}{v(x) + H} , & (x, \eta) \in \mathcal{R}_1 , \\ \sigma_2 , & (x, \eta) \in \mathcal{R}_2 , \end{cases}$$

and derive the following fundamental identity for Φ, which provides a connection between some integrals involving products of second-order derivatives of Φ and is in the spirit of [4, Lemma 4.3.1.2], [7, Lemma 3.4], and [10, Lemme II.2.2].

Lemma 4.3 *Given $v \in \mathcal{S} \cap W^2_\infty(D)$, the function Φ defined in (4.8) satisfies*

$$
\begin{aligned}
\sum_{i=1}^{2} \int_{\mathcal{R}_i} \hat{\sigma}\, \partial_x^2 \Phi_i\, \partial_\eta^2 \Phi_i \, \mathrm{d}(x,\eta) = \sum_{i=1}^{2} \int_{\mathcal{R}_i} \hat{\sigma} |\partial_x \partial_\eta \Phi_i|^2 \, \mathrm{d}(x,\eta) \\
- \sigma_1 \int_{\mathcal{R}_1} \frac{\partial_x v}{(v+H)^2} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 \, \mathrm{d}(x,\eta) \\
+ \frac{1}{2} \int_D \frac{\partial_x^2 v \big((\partial_x v)^2 - 1\big)}{(1+(\partial_x v)^2)^2} \left[\!\left[\sigma (\partial_x \Phi)^2 \right]\!\right] (x,1)\, \mathrm{d}x\,.
\end{aligned}
$$

Proof We adapt the proof of [7, Lemma 3.4] and [10, Lemme II.2.2]. Note that (4.5b), (4.6), (4.7), and (4.8) imply $[\![\Phi]\!] = 0$ on Σ_0, so that

$$
[\![\partial_x \Phi]\!] = 0 \quad \text{on} \quad \Sigma_0\,. \tag{4.9}
$$

Consequently, since $(\partial_x \Phi_1, \partial_x \Phi_2)$ lies in $H^1(\mathcal{R}_1) \times H^1(\mathcal{R}_2)$ by (4.8), we may argue as in the proof of Lemma 4.2 and deduce from (4.9) that

$$
F := \partial_x \Phi \in H^1(\mathcal{R})\,.
$$

Moreover, by (4.8),

$$
F(x,0) = F(x,1+d) = 0\,, \qquad x \in D\,. \tag{4.10}
$$

Similarly, setting

$$
G := -\sigma \frac{\partial_x v}{1+(\partial_x v)^2} \partial_x \Phi + \hat{\sigma} \partial_\eta \Phi,
$$

we derive from (4.8) that $G_i := G|_{\mathcal{R}_i} \in H^1(\mathcal{R}_i)$ for $i = 1, 2$, while (4.5b), (4.6), (4.7), and (4.8) imply that, for $x \in D$,

$$
\begin{aligned}
G_1(x,1) &= \frac{\sigma_1}{\sqrt{1+(\partial_x v(x))^2}} \big[-\partial_x v(x) \partial_x \chi_1(x, v(x)) + \partial_z \chi_1(x, v(x)) \big] \\
&= \frac{\sigma_2}{\sqrt{1+(\partial_x v(x))^2}} \big[-\partial_x v(x) \partial_x \chi_2(x, v(x)) + \partial_z \chi_2(x, v(x)) \big] = G_2(x,1)\,;
\end{aligned}
$$

that is, $[\![G]\!] = 0$ on Σ_0, and we argue as in the proof of Lemma 4.2 to conclude that

$$
G \in H^1(\mathcal{R})\,.
$$

In addition, by (4.8),

$$
\begin{aligned}
G(\pm L, \eta) &= -\sigma(\pm L, \eta) \left(\frac{\partial_x v}{1+(\partial_x v)^2} \right) (\pm L) \partial_x \Phi(\pm L, \eta) + \hat{\sigma}(\pm L, \eta) \partial_\eta \Phi(\pm L, \eta) \\
&= -\sigma(\pm L, \eta) \left(\frac{\partial_x v}{1+(\partial_x v)^2} \right) (\pm L) \partial_x \Phi(\pm L, \eta)
\end{aligned}
$$

for $\eta \in (0, 1+d)$. Hence,

$$
G(\pm L, \eta) + \sigma(\pm L, \eta) \left(\frac{\partial_x v}{1+(\partial_x v)^2} \right) (\pm L) F(\pm L, \eta) = 0\,, \qquad \eta \in (0, 1+d)\,. \tag{4.11}
$$

Owing to (4.10), (4.11), and the H^1-regularity of F and G, we are in a position to apply Lemma A.1 (see Appendix 1) with

$$
(V, W) = (F, G) \quad \text{and} \quad \tau^{\pm} = \sigma \left(\frac{\partial_x v}{1+(\partial_x v)^2} \right) (\pm L)\,,
$$

to obtain the identity

$$\int_{\mathcal{R}} \partial_x F \partial_\eta G \, \mathrm{d}(x,\eta) = \int_{\mathcal{R}} \partial_\eta F \partial_x G \, \mathrm{d}(x,\eta) . \tag{4.12}$$

Using the definitions of F and G, the identity (4.12) reads

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{\mathcal{R}_i} \partial_x^2 \Phi_i \left(-\sigma \frac{\partial_x v}{1+(\partial_x v)^2} \partial_x \partial_\eta \Phi_i + \hat{\sigma} \partial_\eta^2 \Phi_i \right) \mathrm{d}(x,\eta) \\
&= \sum_{i=1}^{2} \int_{\mathcal{R}_i} \partial_x \partial_\eta \Phi_i \left(-\sigma \frac{\partial_x v}{1+(\partial_x v)^2} \partial_x^2 \Phi_i - \sigma \frac{\partial_x^2 v [1-(\partial_x v)^2]}{[1+(\partial_x v)^2]^2} \partial_x \Phi_i \right) \mathrm{d}(x,\eta) \\
&\quad + \sum_{i=1}^{2} \int_{\mathcal{R}_i} \partial_x \partial_\eta \Phi_i \Big(\partial_x \hat{\sigma} \partial_\eta \Phi_i + \hat{\sigma} \partial_x \partial_\eta \Phi_i \Big) \mathrm{d}(x,\eta) .
\end{aligned}$$

Noticing that the first terms on both sides of the above identity are the same and that

$$\partial_x \Phi_i \partial_x \partial_\eta \Phi_i = \frac{1}{2} \partial_\eta \big((\partial_x \Phi_i)^2 \big)$$

implies that

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{\mathcal{R}_i} \sigma \frac{\partial_x^2 v \big[(\partial_x v)^2) - 1 \big]}{[1+(\partial_x v)^2]^2} \partial_x \Phi_i \partial_x \partial_\eta \Phi_i \, \mathrm{d}(x,\eta) \\
&\qquad = \frac{1}{2} \int_D \frac{\partial_x^2 v \big[(\partial_x v)^2 - 1 \big]}{[1+(\partial_x v)^2]^2} \left[\!\left[\sigma (\partial_x \Phi)^2 \right]\!\right] (x,1) \, \mathrm{d}x ,
\end{aligned}$$

the assertion follows, recalling that $\partial_x \hat{\sigma} = 0$ in $\mathcal{R}_2$. □

Remark 4.4 If $\partial_x v(\pm L) = 0$, then (4.11) reduces to $G(\pm L, \eta) = 0$ for $\eta \in (0, 1+d)$ and the crucial identity (4.12) used in the proof of Lemma 4.3 directly follows from [4, Lemma 4.3.1.2]. For the general case $v \in \mathcal{S}$, we require the extension given in Lemma A.1.

We now translate the outcome of Lemma 4.3 in terms of the solution χ to (4.5).

Lemma 4.5 *Let $v \in \mathcal{S} \cap W_\infty^2(D)$. The solution $\chi = \psi_v - h_v$ to (4.5) satisfies*

$$\begin{aligned}
\sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma \, \partial_x^2 \chi_i \, \partial_z^2 \chi_i \, \mathrm{d}(x,z) &= \sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma |\partial_x \partial_z \chi_i|^2 \, \mathrm{d}(x,z) \\
&\quad - \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \big(\partial_z \chi_2(x, v(x)+d) \big)^2 \mathrm{d}x \\
&\quad - \frac{1}{2} \int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2} \left[\!\left[\sigma |\nabla \chi|^2 \right]\!\right] \big(x, v(x) \big) \, \mathrm{d}x .
\end{aligned}$$

Proof Let us first recall the regularity of Φ stated in (4.8) which validates the subsequent computations. Using the transformations T_1 and T_2 introduced in (4.6) and (4.7), respectively, we obtain

$$\sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma \, \partial_x^2 \chi_i \, \partial_z^2 \chi_i \, \mathrm{d}(x,z)$$

$$
\begin{aligned}
&= \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[\partial_x^2 \Phi_1 + \eta \Big(2 \Big(\frac{\partial_x v}{v+H} \Big)^2 - \frac{\partial_x^2 v}{v+H} \Big) \partial_\eta \Phi_1 - 2\eta \frac{\partial_x v}{v+H} \partial_x \partial_\eta \Phi_1 \\
&\qquad + \eta^2 \Big(\frac{\partial_x v}{v+H} \Big)^2 \partial_\eta^2 \Phi_1 \Big] \partial_\eta^2 \Phi_1 \, \mathrm{d}(x,\eta) \\
&\quad + \int_{\mathcal{R}_2} \sigma_2 \Big[\partial_x^2 \Phi_2 - 2\partial_x v \partial_x \partial_\eta \Phi_2 - \partial_x^2 v \partial_\eta \Phi_2 + (\partial_x v)^2 \partial_\eta^2 \Phi_2 \Big] \partial_\eta^2 \Phi_2 \, \mathrm{d}(x,\eta) \\
&= \sum_{i=1}^2 \int_{\mathcal{R}_i} \hat{\sigma}\, \partial_x^2 \Phi_i \, \partial_\eta^2 \Phi_i \, \mathrm{d}(x,\eta) \\
&\quad + \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[\eta \Big(2 \Big(\frac{\partial_x v}{v+H} \Big)^2 - \frac{\partial_x^2 v}{v+H} \Big) \partial_\eta \Phi_1 - 2\eta \frac{\partial_x v}{v+H} \partial_x \partial_\eta \Phi_1 \\
&\qquad + \eta^2 \Big(\frac{\partial_x v}{v+H} \Big)^2 \partial_\eta^2 \Phi_1 \Big] \partial_\eta^2 \Phi_1 \, \mathrm{d}(x,\eta) \\
&\quad + \int_{\mathcal{R}_2} \sigma_2 \Big[- 2\partial_x v \partial_x \partial_\eta \Phi_2 - \partial_x^2 v \partial_\eta \Phi_2 + (\partial_x v)^2 \partial_\eta^2 \Phi_2 \Big] \partial_\eta^2 \Phi_2 \, \mathrm{d}(x,\eta) \, .
\end{aligned}
$$

We use Lemma 4.3 to express the first integral on the right-hand side and get

$$
\begin{aligned}
&\sum_{i=1}^2 \int_{\Omega_i(v)} \sigma \, \partial_x^2 \chi_i \, \partial_z^2 \chi_i \, \mathrm{d}(x,z) \\
&\quad = \int_{\mathcal{R}_1} \hat{\sigma} |\partial_x \partial_\eta \Phi_1|^2 \, \mathrm{d}(x,\eta) + \int_{\mathcal{R}_2} \hat{\sigma} |\partial_x \partial_\eta \Phi_2|^2 \, \mathrm{d}(x,\eta) \\
&\qquad + \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[- \frac{\partial_x v}{v+H} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 - 2\eta \frac{\partial_x v}{v+H} \partial_x \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \\
&\qquad + \eta^2 \Big(\frac{\partial_x v}{v+H} \Big)^2 \big| \partial_\eta^2 \Phi_1 \big|^2 + 2\eta \Big(\frac{\partial_x v}{v+H} \Big)^2 \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \\
&\qquad - \eta \frac{\partial_x^2 v}{v+H} \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \Big] \, \mathrm{d}(x,\eta) \\
&\qquad + \int_{\mathcal{R}_2} \sigma_2 \Big[- 2\partial_x v \partial_x \partial_\eta \Phi_2 \partial_\eta^2 \Phi_2 - \partial_x^2 v \partial_\eta \Phi_2 \partial_\eta^2 \Phi_2 + (\partial_x v)^2 \big| \partial_\eta^2 \Phi_2 \big|^2 \Big] \, \mathrm{d}(x,\eta) \\
&\qquad + \frac{1}{2} \int_D \frac{\partial_x^2 v \big((\partial_x v)^2 - 1 \big)}{(1 + (\partial_x v)^2)^2} \, \big[\!\big[\sigma (\partial_x \Phi)^2 \big]\!\big] (x,1) \, \mathrm{d}x \, . \qquad (4.13)
\end{aligned}
$$

We then compute separately the integrals over $\mathcal{R}_i$, $i = 1, 2$, and begin with the contribution of $\mathcal{R}_1$. We complete the square to get

$$
\begin{aligned}
I_1 &:= \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[|\partial_x \partial_\eta \Phi_1|^2 - \frac{\partial_x v}{v+H} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 - 2\eta \frac{\partial_x v}{v+H} \partial_x \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \\
&\quad + \eta^2 \Big(\frac{\partial_x v}{v+H} \Big)^2 \big| \partial_\eta^2 \Phi_1 \big|^2 + 2\eta \Big(\frac{\partial_x v}{v+H} \Big)^2 \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \\
&\quad - \eta \frac{\partial_x^2 v}{v+H} \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \Big] \, \mathrm{d}(x,\eta) \\
&= \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[\big| \partial_x \partial_\eta \Phi_1 \big|^2 + \Big(\frac{\partial_x v}{v+H} \Big)^2 \big| \partial_\eta \Phi_1 \big|^2 + \eta^2 \Big(\frac{\partial_x v}{v+H} \Big)^2 \big| \partial_\eta^2 \Phi_1 \big|^2
\end{aligned}
$$

$$
\begin{aligned}
&- 2\eta \frac{\partial_x v}{v+H} \partial_x \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 + 2\eta \Big(\frac{\partial_x v}{v+H}\Big)^2 \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \\
&- 2 \frac{\partial_x v}{v+H} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 \Big] \,\mathrm{d}(x,\eta) \\
&+ \int_{\mathcal{R}_1} \frac{\sigma_1}{v+H} \Big[- \Big(\frac{\partial_x v}{v+H}\Big)^2 |\partial_\eta \Phi_1|^2 + \frac{\partial_x v}{v+H} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 \\
&- \eta \frac{\partial_x^2 v}{v+H} \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \Big] \,\mathrm{d}(x,\eta) \\
= &\int_{\mathcal{R}_1} \sigma_1 (v+H) \Big[\frac{\partial_x \partial_\eta \Phi_1}{v+H} - \frac{\partial_x v}{(v+H)^2} \partial_\eta \Phi_1 - \eta \frac{\partial_x v}{(v+H)^2} \partial_\eta^2 \Phi_1 \Big]^2 \,\mathrm{d}(x,\eta) \\
&+ \int_{\mathcal{R}_1} \sigma_1 \partial_x v \Big[\frac{1}{(v+H)^2} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 - \frac{\partial_x v}{(v+H)^3} \big(\partial_\eta \Phi_1\big)^2 \Big] \,\mathrm{d}(x,\eta) \\
&- \int_{\mathcal{R}_1} \sigma_1 \frac{\partial_x^2 v}{(v+H)^2}\, \eta\, \partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 \,\mathrm{d}(x,\eta)\,.
\end{aligned}
$$

Thanks to the identities

$$
\begin{aligned}
\frac{1}{(v+H)^2} \partial_\eta \Phi_1 \partial_x \partial_\eta \Phi_1 - \frac{\partial_x v}{(v+H)^3} \big(\partial_\eta \Phi_1\big)^2 &= \frac{1}{2} \partial_x \left(\Big(\frac{\partial_\eta \Phi_1}{v+H} \Big)^2 \right), \\
\partial_\eta \Phi_1 \partial_\eta^2 \Phi_1 &= \frac{1}{2} \partial_\eta \big(\partial_\eta \Phi_1\big)^2,
\end{aligned}
$$

and the property $\partial_\eta \Phi_1(\pm L, \eta) = 0$ for $\eta \in (0,1)$ stemming from (4.8), we may perform integration by parts in the last two integrals on the right-hand side of the previous identity and obtain

$$
\begin{aligned}
I_1 = &\int_{\mathcal{R}_1} \sigma_1 (v+H) \Big[\frac{\partial_x \partial_\eta \Phi_1}{v+H} - \frac{\partial_x v}{(v+H)^2} \partial_\eta \Phi_1 - \eta \frac{\partial_x v}{(v+H)^2} \partial_\eta^2 \Phi_1 \Big]^2 \,\mathrm{d}(x,\eta) \\
&- \frac{\sigma_1}{2} \int_D \frac{\partial_x^2 v}{(v+H)^2} \big(\partial_\eta \Phi_1(x,1)\big)^2 \,\mathrm{d}x\,.
\end{aligned}
$$

Transforming the above identity back to $\Omega_1(v)$ yields

$$
I_1 = \int_{\Omega_1(v)} \sigma_1 \big|\partial_x \partial_z \chi_1\big|^2 \,\mathrm{d}(x,z) - \frac{\sigma_1}{2} \int_D \partial_x^2 v(x) \big(\partial_z \chi_1(x, v(x))\big)^2 \,\mathrm{d}x\,. \tag{4.14}
$$

Next, arguing in a similar way,

$$
\begin{aligned}
I_2 := \sigma_2 &\int_{\mathcal{R}_2} \Big[\big|\partial_x \partial_\eta \Phi_2\big|^2 - 2 \partial_x v \partial_x \partial_\eta \Phi_2 \partial_\eta^2 \Phi_2 - \partial_x^2 v \partial_\eta \Phi_2 \partial_\eta^2 \Phi_2 + (\partial_x v)^2 \big|\partial_\eta^2 \Phi_2\big|^2 \Big] \,\mathrm{d}(x,\eta) \\
= \sigma_2 &\int_{\mathcal{R}_2} \Big[\big|\partial_x \partial_\eta \Phi_2\big|^2 - 2 \partial_x v \partial_x \partial_\eta \Phi_2 \partial_\eta^2 \Phi_2 + (\partial_x v)^2 \big|\partial_\eta^2 \Phi_2\big|^2 \Big] \,\mathrm{d}(x,\eta) \\
&- \frac{\sigma_2}{2} \int_{\mathcal{R}_2} \partial_x^2 v \partial_\eta \big(\partial_\eta \Phi_2\big)^2 \,\mathrm{d}(x,\eta) \\
= \sigma_2 &\int_{\mathcal{R}_2} \Big[\partial_x \partial_\eta \Phi_2 - \partial_x v \partial_\eta^2 \Phi_2 \Big]^2 \,\mathrm{d}(x,\eta) - \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \big(\partial_\eta \Phi_2(x, 1+d)\big)^2 \,\mathrm{d}x \\
&+ \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \big(\partial_\eta \Phi_2(x,1)\big)^2 \,\mathrm{d}x\,.
\end{aligned}
$$

Transforming this formula back to $\Omega_2(v)$ yields

$$I_2 = \sigma_2 \int_{\Omega_2(v)} \left|\partial_x \partial_z \chi_2\right|^2 \mathrm{d}(x,z) - \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \left(\partial_z \chi_2(x, v(x)+d)\right)^2 \mathrm{d}x \\ + \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \left(\partial_z \chi_2(x, v(x))\right)^2 \mathrm{d}x \,. \tag{4.15}$$

Finally,

$$\int_D \frac{\partial_x^2 v\big[(\partial_x v)^2 - 1\big]}{[1+(\partial_x v)^2]^2} \left[\!\left[\sigma (\partial_x \Phi)^2\right]\!\right] (x,1)\,\mathrm{d}x \\ = \int_D \frac{\partial_x^2 v\big[(\partial_x v)^2 - 1\big]}{[1+(\partial_x v)^2]^2} \left[\!\left[\sigma (\partial_x \chi + \partial_x v \partial_z \chi)^2\right]\!\right] (x,1)\,\mathrm{d}x \,,$$

and we deduce from (4.13), (4.14), (4.15), and the above identity that

$$\sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma\, \partial_x^2 \chi_i\, \partial_z^2 \chi_i\, \mathrm{d}(x,z) \\ = \sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma \left|\partial_x \partial_z \chi_i\right|^2 \mathrm{d}(x,z) - \frac{\sigma_2}{2} \int_D \partial_x^2 v(x) \left(\partial_z \chi_2(x, v(x)+d)\right)^2 \mathrm{d}x \\ -\frac{1}{2} \int_D \partial_x^2 v(x) \left[\!\left[\sigma \left(\partial_z \chi_2\right)^2\right]\!\right] (x, v(x))\,\mathrm{d}x \\ +\frac{1}{2} \int_D \frac{\partial_x^2 v\big[(\partial_x v)^2 - 1\big]}{[1+(\partial_x v)^2]^2} \left[\!\left[\sigma \left(\partial_x \chi + \partial_x v \partial_z \chi\right)^2\right]\!\right] (x, v(x))\,\mathrm{d}x \,. \tag{4.16}$$

It remains to simplify the last two integrals on the right-hand side of (4.16). To this end, we first recall that the regularity of χ allows us to differentiate with respect to x the transmission condition $[\![\chi]\!] = 0$ on $\Sigma(v)$ to deduce that

$$[\![\partial_x \chi + \partial_x v \partial_z \chi]\!] = 0 \quad \text{on} \quad \Sigma(v)\,, \tag{4.17}$$

while the second transmission condition in (4.5b) reads

$$[\![\sigma\left(\partial_x v \partial_x \chi - \partial_z \chi\right)]\!] = 0 \quad \text{on} \quad \Sigma(v)\,. \tag{4.18}$$

In particular, (4.17) and (4.18) imply that, on $\Sigma(v)$,

$$[\![\sigma\left(\partial_x v \partial_x \chi - \partial_z \chi\right)\left(\partial_x \chi + \partial_x v \partial_z \chi\right)]\!] = \left(\partial_x \chi_1 + \partial_x v \partial_z \chi_1\right)[\![\sigma\left(\partial_x v \partial_x \chi - \partial_z \chi\right)]\!] \\ + \sigma_2\left(\partial_x v \partial_x \chi_2 - \partial_z \chi_2\right)[\![\left(\partial_x \chi + \partial_x v \partial_z \chi\right)]\!] \\ = 0\,.$$

Therefore,

$$J := \left[(\partial_x v)^2 - 1\right]\left[\!\left[\sigma\left(\partial_x \chi + \partial_x v \partial_z \chi\right)^2\right]\!\right] - [1+(\partial_x v)^2]^2 \left[\!\left[\sigma\left(\partial_z \chi\right)^2\right]\!\right] \\ = \left[(\partial_x v)^2 - 1\right]\left[\!\left[\sigma\left(\partial_x \chi + \partial_x v \partial_z \chi\right)^2\right]\!\right] - [1+(\partial_x v)^2]^2 \left[\!\left[\sigma\left(\partial_z \chi\right)^2\right]\!\right] \\ - 2\partial_x v \left[\!\left[\sigma\left(\partial_x v \partial_x \chi - \partial_z \chi\right)\left(\partial_x \chi + \partial_x v \partial_z \chi\right)\right]\!\right] \\ = \left[\!\left[\sigma\left[(\partial_x v)^2 - 1 - 2(\partial_x v)^2\right]\left(\partial_x \chi\right)^2\right]\!\right] \\ + \left[\!\left[\sigma\left[2\partial_x v\left((\partial_x v)^2 - 1\right) - 2(\partial_x v)^3 + 2\partial_x v\right]\partial_x \chi \partial_z \chi\right]\!\right]$$

$$
\begin{aligned}
&+ \left[\!\left[\sigma\left[(\partial_x v)^2\left((\partial_x v)^2-1\right)+2(\partial_x v)^2-[1+(\partial_x v)^2]^2\right]\left(\partial_z\chi\right)^2\right]\!\right] \\
&= -\left[1+(\partial_x v)^2\right]\left[\!\left[\sigma\left(\partial_x\chi\right)^2+\sigma\left(\partial_z\chi\right)^2\right]\!\right] \\
&= -\left[1+(\partial_x v)^2\right]\left[\!\left[\sigma|\nabla\chi|^2\right]\!\right].
\end{aligned}
$$

Hence,

$$
\frac{(\partial_x v)^2-1}{[1+(\partial_x v)^2]^2}\left[\!\left[\sigma\left(\partial_x\chi+\partial_x v\partial_z\chi\right)^2\right]\!\right]-\left[\!\left[\sigma\left(\partial_z\chi\right)^2\right]\!\right]=-\frac{1}{1+(\partial_x v)^2}\left[\!\left[\sigma|\nabla\chi|^2\right]\!\right]. \tag{4.19}
$$

Consequently, (4.16) and (4.19) entail

$$
\begin{aligned}
\sum_{i=1}^{2}\int_{\Omega_i(v)}\sigma\,\partial_x^2\chi_i\,\partial_z^2\chi_i\,\mathrm{d}(x,z) &= \sum_{i=1}^{2}\int_{\Omega_i(v)}\sigma|\partial_x\partial_z\chi_i|^2\,\mathrm{d}(x,z) \\
&\quad -\frac{1}{2}\int_D \sigma_2\partial_x^2 v(x)\left(\partial_z\chi_2(x,v(x)+d)\right)^2\mathrm{d}x \\
&\quad -\frac{1}{2}\int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2}\left[\!\left[\sigma|\nabla\chi|^2\right]\!\right]\left(x,v(x)\right)\mathrm{d}x\,,
\end{aligned}
$$

as claimed. □

In order to estimate the boundary and the transmission terms in Lemma 4.5, we first report the following trace estimates.

Lemma 4.6 *Given $\kappa>0$ and $\alpha\in(0,1/2]$, there is $c(\alpha,\kappa)>0$ such that, for any $v\in\bar{\mathcal{S}}$ satisfying $\|v\|_{H^2(D)}\le\kappa$ and $\theta\in H^1(\Omega_2(v))$,*

$$
\|\theta(\cdot,v)\|_{H^\alpha(D)}+\|\theta(\cdot,v+d)\|_{H^\alpha(D)}\le c(\alpha,\kappa)\,\|\theta\|_{L_2(\Omega_2(v))}^{(1-2\alpha)/2}\,\|\theta\|_{H^1(\Omega_2(v))}^{(2\alpha+1)/2}.
$$

Proof Let $\theta\in H^1(\Omega_2(v))$. Using the transformation T_2 defined in (4.7) which maps $\Omega_2(v)$ onto the rectangle $\mathcal{R}_2=D\times(1,1+d)$, we note that $\phi:=\theta\circ T_2^{-1}$ belongs to $H^1(\mathcal{R}_2)$ with

$$
\|\phi\|_{L_2(\mathcal{R}_2)}=\|\theta\|_{L_2(\Omega_2(v))} \tag{4.20}
$$

and

$$
\|\nabla\phi\|_{L_2(\mathcal{R}_2)}^2=\|\partial_x\theta+\partial_x v\partial_z\theta\|_{L_2(\Omega_2(v))}^2+\|\partial_z\theta\|_{L_2(\Omega_2(v))}^2\,,
$$

so that the continuous embedding of $H^2(D)$ in $W_\infty^1(D)$ and the assumed bound on v readily imply that

$$
\|\phi\|_{H^1(\mathcal{R}_2)}\le c(\kappa)\|\theta\|_{H^1(\Omega_2(v))}\,. \tag{4.21}
$$

By complex interpolation,

$$
[L_2(\mathcal{R}_2),H^1(\mathcal{R}_2)]_{\alpha+1/2}\doteq H^{\alpha+1/2}(\mathcal{R}_2)\,,
$$

from which we deduce that

$$
\|\phi\|_{H^{\alpha+1/2}(\mathcal{R}_2)}\le c(\alpha)\|\phi\|_{L_2(\mathcal{R}_2)}^{(1-2\alpha)/2}\|\phi\|_{H^1(\mathcal{R}_2)}^{(2\alpha+1)/2}.
$$

Since $\alpha > 0$, the trace maps $H^{\alpha+1/2}(\mathcal{R}_2)$ continuously on $H^\alpha(D \times \{1\})$, and we thus infer from (4.20) and (4.21) that

$$\begin{aligned}\|\theta(\cdot, v)\|_{H^\alpha(D)} = \|\phi(\cdot, 1)\|_{H^\alpha(D)} &\le c(\alpha)\|\phi\|_{H^{\alpha+1/2}(\mathcal{R}_2)} \\ &\le c(\alpha)\|\phi\|_{L_2(\mathcal{R}_2)}^{(1-2\alpha)/2}\|\phi\|_{H^1(\mathcal{R}_2)}^{(2\alpha+1)/2} \\ &\le c(\alpha,\kappa)\|\theta\|_{L_2(\Omega_2(v))}^{(1-2\alpha)/2}\|\theta\|_{H^1(\Omega_2(v))}^{(2\alpha+1)/2}.\end{aligned}$$

The estimate for $\|\theta(\cdot, v+d)\|_{H^\alpha(D)}$ is proved in a similar way. □

Based on Lemma 4.6 we are in a position to estimate the boundary and transmission terms in the identity provided by Lemma 4.5.

Lemma 4.7 *Let $\zeta \in (3/4, 1)$ and $\kappa > 0$. There is $c(\zeta, \kappa) > 0$ such that, if $v \in \mathcal{S} \cap W_\infty^2(D)$ satisfies $\|v\|_{H^2(D)} \le \kappa$, then the solution $\chi = \chi_v$ to (4.5) satisfies*

$$\begin{aligned}&\left|\frac{\sigma_2}{2}\int_D \partial_x^2 v(x)\big(\partial_z\chi_2(x, v(x)+d)\big)^2\,\mathrm{d}x\right| \\ &\qquad\le c(\zeta,\kappa)\,\|\partial_z\chi_2\|_{L_2(\Omega_2(v))}^{2(1-\zeta)}\,\|\partial_z\chi_2\|_{H^1(\Omega_2(v))}^{2\zeta}\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\left|\frac{1}{2}\int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2}\,\big[\!\big[\sigma|\nabla\chi|^2\big]\!\big]\,\big(x, v(x)\big)\,\mathrm{d}x\right| \\ &\qquad\le c(\zeta,\kappa)\,\|\nabla\chi_2\|_{L_2(\Omega_2(v))}^{2(1-\zeta)}\,\|\nabla\chi_2\|_{H^1(\Omega_2(v))}^{2\zeta}.\end{aligned} \tag{4.23}$$

Proof To prove (4.22), let us first note that $H^{\zeta-1/2}(D)$ embeds continuously into $L_4(D)$. We use the Cauchy-Schwarz inequality and Lemma 4.6 with $\alpha = \zeta - 1/2$ and deduce

$$\begin{aligned}\left|\frac{\sigma_2}{2}\int_D \partial_x^2 v(x)\big(\partial_z\chi_2(x, v(x)+d)\big)^2\,\mathrm{d}x\right| &\le \frac{\sigma_2}{2}\|\partial_x^2 v\|_{L_2(D)}\,\|\partial_z\chi_2(\cdot, v+d)\|_{L_4(D)}^2 \\ &\le c(\kappa)\,\|\partial_z\chi_2(\cdot, v+d)\|_{H^{\zeta-1/2}(D)}^2 \\ &\le c(\zeta,\kappa)\,\|\partial_z\chi_2\|_{L_2(\Omega_2(v))}^{2(1-\zeta)}\,\|\partial_z\chi_2\|_{H^1(\Omega_2(v))}^{2\zeta}.\end{aligned}$$

As for (4.23) we obtain analogously

$$\begin{aligned}&\left|\frac{\sigma_2}{2}\int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2}\left[\big(\partial_x\chi_2(x, v(x))\big)^2 + \big(\partial_z\chi_2(x, v(x))\big)^2\right]\mathrm{d}x\right| \\ &\quad\le \frac{\sigma_2}{2}\|\partial_x^2 v\|_{L_2(D)}\|\nabla\chi_2(\cdot, v)\|_{L_4(D)}^2 \\ &\quad\le c(\zeta,\kappa)\,\|\nabla\chi_2\|_{L_2(\Omega_2(v))}^{2(1-\zeta)}\,\|\nabla\chi_2\|_{H^1(\Omega_2(v))}^{2\zeta}\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}&\left|\frac{\sigma_1}{2}\int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2}\left[\big(\partial_x\chi_1(x, v(x))\big)^2 + \big(\partial_z\chi_1(x, v(x))\big)^2\right]\mathrm{d}x\right| \\ &\quad\le \frac{\sigma_1}{2}\|\partial_x^2 v\|_{L_2(D)}\|\nabla\chi_1(\cdot, v)\|_{L_4(D)}^2.\end{aligned} \tag{4.25}$$

At this point, we use (4.17) and (4.18) to show that

$$\partial_x\chi_1 = \frac{\sigma_1+\sigma_2(\partial_x v)^2}{\sigma_1\big(1+(\partial_x v)^2\big)}\partial_x\chi_2 + \frac{[\![\sigma]\!]\partial_x v}{\sigma_1\big(1+(\partial_x v)^2\big)}\partial_z\chi_2 \quad \text{on} \;\; \Sigma(v),$$

$$\partial_z \chi_1 = \frac{[\![\sigma]\!] \partial_x v}{\sigma_1\big(1+(\partial_x v)^2\big)} \partial_x \chi_2 + \frac{\sigma_1 + \sigma_2 (\partial_x v)^2}{\sigma_1\big(1+(\partial_x v)^2\big)} \partial_z \chi_2 \quad \text{on } \Sigma(v)\,.$$

Consequently,

$$|\partial_x \chi_1| \le \frac{\max\{\sigma_1, \sigma_2\}}{\sigma_1} \left(|\partial_x \chi_2| + |\partial_z \chi_2|\right) \quad \text{on } \Sigma(v)\,,$$
$$|\partial_z \chi_1| \le \frac{\max\{\sigma_1, \sigma_2\}}{\sigma_1} \left(|\partial_x \chi_2| + |\partial_z \chi_2|\right) \quad \text{on } \Sigma(v)\,,$$

so that

$$\|\nabla \chi_1(\cdot, v)\|_{L_4(D)} \le c \|\nabla \chi_2(\cdot, v)\|_{L_4(D)}\,.$$

Owing to (4.25) and the above inequality, we may then argue as in the proof of (4.24) to conclude that

$$\left| \frac{\sigma_1}{2} \int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2} \left[\big(\partial_x \chi_1(x, v(x))\big)^2 + \big(\partial_z \chi_1(x, v(x))\big)^2 \right] \mathrm{d}x \right|$$
$$\le c(\zeta, \kappa)\, \|\nabla \chi_2\|_{L_2(\Omega_2(v))}^{2(1-\zeta)}\, \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^{2\zeta}\,,$$

as claimed in (4.23). □

We now gather the previous findings to deduce the following crucial H^2-estimate on the solution ψ_v to (4.1) for $v \in \mathcal{S} \cap W_\infty^2(D)$, which only depends on the $H^2(D)$-norm of v (but not on its $W_\infty^2(D)$-norm).

Proposition 4.8 *Let $\kappa > 0$ and $v \in \mathcal{S} \cap W_\infty^2(D)$ be such that $\|v\|_{H^2(D)} \le \kappa$. There is a constant $c_0(\kappa) > 0$ such that the solution ψ_v to (4.1) satisfies*

$$\|\chi\|_{H^1(\Omega(v))} + \|\chi_1\|_{H^2(\Omega_1(v))} + \|\chi_2\|_{H^2(\Omega_2(v))} \le c_0(\kappa) \tag{4.26a}$$

and

$$\|\psi_v\|_{H^1(\Omega(v))} + \|\psi_{v,1}\|_{H^2(\Omega_1(v))} + \|\psi_{v,2}\|_{H^2(\Omega_2(v))} \le c_0(\kappa)\,, \tag{4.26b}$$

recalling that $\chi = \psi_v - h_v$ and $\chi_i = \chi|_{\Omega_i(v)}$, $i = 1, 2$.

Proof Let $v \in \mathcal{S} \cap W_\infty^2(D)$ with $\|v\|_{H^2(D)} \le \kappa$. Since σ is constant on $\Omega_1(v)$ and on $\Omega_2(v)$, it readily follows from (4.5a) that

$$\sum_{i=1}^2 \int_{\Omega_i(v)} \sigma |\Delta \chi_i|^2 \,\mathrm{d}(x,z) = \sum_{i=1}^2 \int_{\Omega_i(v)} \sigma |\Delta h_{v,i}|^2 \,\mathrm{d}(x,z)\,.$$

Since

$$|\Delta \chi_i|^2 = |\partial_x^2 \chi_i|^2 + |\partial_z^2 \chi_i|^2 + 2\partial_x^2 \chi_i \partial_z^2 \chi_i\,, \qquad i = 1, 2\,,$$

we infer from Lemma 4.5 and the above two formulas that

$$\sum_{i=1}^2 \int_{\Omega_i(v)} \sigma \left\{ |\partial_x^2 \chi_i|^2 + 2|\partial_x \partial_z \chi_i|^2 + |\partial_z^2 \chi_i|^2 \right\} \mathrm{d}(x,z)$$
$$= \sum_{i=1}^2 \int_{\Omega_i(v)} \sigma |\Delta h_{v,i}|^2 \,\mathrm{d}(x,z) + 2 \sum_{i=1}^2 \int_{\Omega_i(v)} \sigma \left(|\partial_x \partial_z \chi_i|^2 - \partial_x^2 \chi_i \partial_z^2 \chi_i \right) \mathrm{d}(x,z)$$

$$\leq \sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma |\Delta h_{v,i}|^2 \, \mathrm{d}(x,z) + \sigma_2 \int_D \partial_x^2 v(x) \left(\partial_z \chi_2(x, v(x)+d)\right)^2 \mathrm{d}x$$
$$+ \int_D \frac{\partial_x^2 v(x)}{1+(\partial_x v(x))^2} \left[\!\left[\sigma |\nabla \chi|^2\right]\!\right] \left(x, v(x)\right) \mathrm{d}x \, .$$

Using Lemma 4.7 with $\zeta = 7/8$, along with the identity

$$\sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma \left\{ |\partial_x^2 \chi_i|^2 + 2|\partial_x \partial_z \chi_i|^2 + |\partial_z^2 \chi_i|^2 \right\} \mathrm{d}(x,z)$$
$$= \sigma_1 \|\nabla \chi_1\|_{H^1(\Omega_1(v))}^2 + \sigma_2 \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^2 \, ,$$

we further obtain

$$\sigma_1 \|\nabla \chi_1\|_{H^1(\Omega_1(v))}^2 + \sigma_2 \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^2$$
$$\leq \sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma |\Delta h_{v,i}|^2 \, \mathrm{d}(x,z) + c(\kappa) \, \|\nabla \chi_2\|_{L_2(\Omega_2(v))}^{1/4} \, \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^{7/4} \, .$$

Hence, thanks to Young's inequality,

$$\sigma_1 \|\nabla \chi_1\|_{H^1(\Omega_1(v))}^2 + \sigma_2 \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^2$$
$$\leq \sum_{i=1}^{2} \int_{\Omega_i(v)} \sigma |\Delta h_{v,i}|^2 \, \mathrm{d}(x,z) + \frac{\sigma_2}{2} \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^2 + c(\kappa) \|\nabla \chi_2\|_{L_2(\Omega_2(v))}^2 \, .$$

Recalling that

$$\|\nabla \chi_2\|_{L_2(\Omega_2(v))}^2 \leq \frac{1}{\sigma_2} \int_{\Omega(v)} \sigma |\nabla \chi|^2 \, \mathrm{d}(x,z) \leq \frac{1}{\sigma_2} \int_{\Omega(v)} \sigma |\nabla h_v|^2 \, \mathrm{d}(x,z)$$
$$\leq \frac{\max\{\sigma_1, \sigma_2\}}{\sigma_2} \|\nabla h_v\|_{L_2(\Omega(v))}^2$$

by (4.5) and that $\min\{\sigma_1, \sigma_2\} > 0$, we conclude that

$$\|\nabla \chi_1\|_{H^1(\Omega_1(v))}^2 + \|\nabla \chi_2\|_{H^1(\Omega_2(v))}^2$$
$$\leq c(\kappa) \left(\|\Delta h_{v,1}\|_{L_2(\Omega_1(v))}^2 + \|\Delta h_{v,2}\|_{L_2(\Omega_2(v))}^2 + \|\nabla h_v\|_{L_2(\Omega(v))}^2 \right) . \tag{4.27}$$

Owing to the continuous embedding of $H^2(D)$ in $C(\bar{D})$, combining (4.27) and Lemma 3.2 leads us to the estimate

$$\|\chi\|_{H^1(\Omega(v))} + \|\chi_1\|_{H^2(\Omega_1(v))} + \|\chi_2\|_{H^2(\Omega_2(v))}$$
$$\leq c(\kappa) \left(\|\nabla h_v\|_{L_2(\Omega(v))} + \|\Delta h_{v,1}\|_{L_2(\Omega_1(v))}^2 + \|\Delta h_{v,2}\|_{L_2(\Omega_2(v))}^2 \right) .$$

The bound (4.26a) then readily follows from the assumptions (2.1a) and (2.1c). Finally, (4.26a), together with (2.1a) and (2.1c), yields (4.26b). □

4.3 H^2-regularity and H^2-estimates on ψ_v for $v \in \bar{\mathcal{S}}$

Finally, we extend Propositions 4.1 and 4.8 by showing the H^2-regularity of ψ_v and the corresponding H^2-estimates for an arbitrary $v \in \bar{\mathcal{S}}$; that is, we drop the additional W_∞^2-

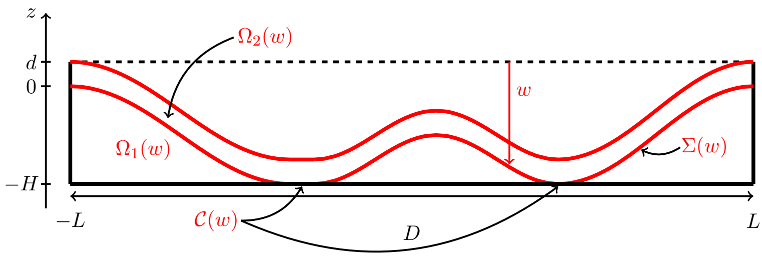

Fig. 3 Geometry of $\Omega(w)$ for a state $w \in \bar{\mathcal{S}}$ with non-empty and disconnected coincidence set

regularity of v assumed in the previous sections and also allow for a non-empty coincidence set.

Proposition 4.9 *Let $\kappa > 0$ and $v \in \bar{\mathcal{S}}$ be such that $\|v\|_{H^2(D)} \le \kappa$.*

(a) *The unique minimizer $\psi_v \in \mathcal{A}(v)$ of $\mathcal{J}(v)$ on $\mathcal{A}(v)$ provided by Lemma 3.1 satisfies*

$$\psi_{v,i} = \psi_v|_{\Omega_i(v)} \in H^2(\Omega_i(v)), \qquad i = 1, 2,$$

and is a strong solution to the transmission problem (4.1). Moreover, there is $c_1(\kappa) > 0$ such that

$$\|\psi_v\|_{H^1(\Omega(v))} + \|\psi_{v,1}\|_{H^2(\Omega_1(v))} + \|\psi_{v,2}\|_{H^2(\Omega_2(v))} \le c_1(\kappa). \tag{4.28}$$

(b) *Consider a sequence $(v_n)_{n\ge1}$ in $\bar{\mathcal{S}}$ satisfying*

$$\|v_n\|_{H^2(D)} \le \kappa, \quad n \ge 1, \quad \text{and} \quad \lim_{n\to\infty} \|v_n - v\|_{H^1(D)} = 0. \tag{4.29}$$

If $i \in \{1, 2\}$ and U_i is an open subset of $\Omega_i(v)$ such that $\bar{U}_i$ is a compact subset of $\Omega_i(v)$, then

$$\psi_{v_n,i} \rightharpoonup \psi_{v,i} \quad \text{in} \quad H^2(U_i),$$

recalling that $\psi_{v_n,i} = \psi_{v_n}|_{\Omega_i(v_n)}$.

The proof involves three steps: we first establish Proposition 4.9 (b) under the additional assumption

$$\sup_{n\ge1} \left\{ \|\psi_{v_n,1}\|_{H^2(\Omega_1(v_n))} + \|\psi_{v_n,2}\|_{H^2(\Omega_2(v_n))} \right\} < \infty.$$

Building upon this result, we take advantage of the density of $\mathcal{S} \cap W^2_\infty(D)$ in $\bar{\mathcal{S}}$ and of the estimates derived in Proposition 4.8 to verify Proposition 4.9 (a) by a compactness argument. Combining the previous steps leads us finally to a complete proof of Proposition 4.9 (b). We thus start with the proof of Proposition 4.9 (b) when the solutions $(\psi_{v_n})_{n\ge1}$ to (4.1) associated with the sequence $(v_n)_{n\ge1}$ satisfies the above additional bound. We state this result as a separate lemma for definiteness.

Lemma 4.10 *Let $\kappa > 0$ and $v \in \bar{\mathcal{S}}$ be such that $\|v\|_{H^2(D)} \le \kappa$ and consider a sequence $(v_n)_{n\ge1}$ in $\bar{\mathcal{S}}$ satisfying (4.29). Assume further that, for each $n \ge 1$, $(\psi_{v_n,1}, \psi_{v_n,2})$ belongs to $H^2(\Omega_1(v_n)) \times H^2(\Omega_2(v_n))$ and that there is $\mu > 0$ such that*

$$\|\psi_{v_n,1}\|_{H^2(\Omega_1(v_n))} + \|\psi_{v_n,2}\|_{H^2(\Omega_2(v_n))} \le \mu, \qquad n \ge 1. \tag{4.30}$$

Then $\psi_{v,i} \in H^2(\Omega_i(v))$, $i = 1, 2$. *In addition, if* $i \in \{1, 2\}$ *and* U_i *is an open subset of* $\Omega_i(v)$ *such that* $\bar{U}_i$ *is a compact subset of* $\Omega_i(v)$, *then*

$$\psi_{v_n,i} \rightharpoonup \psi_{v,i} \quad \text{in} \quad H^2(U_i)$$

and

$$\|\psi_{v,1}\|_{H^2(\Omega_1(v))} + \|\psi_{v,2}\|_{H^2(\Omega_2(v))} \le \mu\,. \tag{4.31}$$

The proof is very close to that of [7, Proposition 3.13 & Corollary 3.14], so that we omit the details here and refer to the extended version of this paper [8] instead.

Proof of Proposition 4.9 (a) Let $v \in \bar{\mathcal{S}}$ be such that $\|v\|_{H^2(D)} \le \kappa$. We may choose a sequence $(v_n)_{n\ge1}$ in $\mathcal{S} \cap W^2_\infty(D)$ satisfying

$$v_n \to v \ \text{ in } \ H^2(D)\,, \qquad \sup_{n\ge1} \|v_n\|_{H^2(D)} \le 2\kappa\,. \tag{4.32}$$

Owing to (4.32) and the regularity property $v_n \in \mathcal{S} \cap W^2_\infty(D)$, $n \ge 1$, Proposition 3.3 guarantees that $(\psi_{v_n,1}, \psi_{v_n,2})$ belongs to $H^2(\Omega_1(v_n)) \times H^2(\Omega_2(v_n))$ and $(\psi_{v_n})_{n\ge1}$ satisfies (4.30) with $\mu = c_0(2\kappa)$. We then infer from Lemma 4.10 that $(\psi_{v,1}, \psi_{v,2})$ belongs to $H^2(\Omega_1(v)) \times H^2(\Omega_2(v))$ and satisfies

$$\|\psi_{v,1}\|_{H^2(\Omega_1(v))} + \|\psi_{v,2}\|_{H^2(\Omega_2(v))} \le c_0(2\kappa)\,.$$

Combining the above bound with (2.1d) and Lemma 4.10 gives (4.28). Checking that ψ_v is a strong solution to (4.1) is then done as in [7, Corollary 3.14], see also the extended version of this paper [8] for a complete proof. □

Proof of Proposition 4.9 (b) Proposition 4.9 (b) is now a straightforward consequence of Proposition 4.9 (a) and Lemma 4.10. □

Proof of Theorem 1.1 The proof of Theorem 1.1 readily follows from Proposition 4.9 (a). □

We supplement the H^2-weak continuity of ψ_v with respect to v reported in Proposition 4.9 with the continuity of the traces of $\nabla\psi_{v,2}$ on the upper and lower boundaries of $\Omega_2(v)$.

Proposition 4.11 *Let* $\kappa > 0$ *and* $v \in \bar{\mathcal{S}}$ *be such that* $\|v\|_{H^2(D)} \le \kappa$ *and consider a sequence* $(v_n)_{n\ge1}$ *in* $\bar{\mathcal{S}}$ *satisfying* (4.29). *Then, for* $p \in [1, \infty)$,

$$\nabla\psi_{v_n,2}(\cdot, v_n) \to \nabla\psi_{v,2}(\cdot, v) \quad \text{in} \quad L_p(D, \mathbb{R}^2)\,, \tag{4.33}$$

$$\nabla\psi_{v_n,2}(\cdot, v_n + d) \to \nabla\psi_{v,2}(\cdot, v + d) \quad \text{in} \quad L_p(D, \mathbb{R}^2)\,, \tag{4.34}$$

and

$$\|\nabla\psi_{v,2}(\cdot, v)\|_{L_p(D,\mathbb{R}^2)} + \|\nabla\psi_{v,2}(\cdot, v + d)\|_{L_p(D,\mathbb{R}^2)} \le c(p, \kappa)\,. \tag{4.35}$$

Proof Recall first from (4.28) that

$$\|\psi_{v_n,2}\|_{H^2(\Omega_2(v_n))} \le c_1(\kappa)\,, \qquad n \ge 1\,. \tag{4.36}$$

As in the proof of Lemma 4.6 we map $\Omega_2(v)$ onto the rectangle $\mathcal{R}_2 = D \times (1, 1 + d)$ and define, for $(x, \eta) \in \mathcal{R}_2$ and $n \ge 1$,

$$\phi_n(x, \eta) := \psi_{v_n,2}(x, \eta + v_n(x) - 1)\,, \qquad \phi(x, \eta) := \psi_{v,2}(x, \eta + v(x) - 1)\,.$$

Let $q \in (1,2)$. Since

$$\nabla \phi_n(x,\eta) = \Big(\partial_x \psi_{v_n} + \partial_x v_n \partial_z \psi_{v_n}\,,\ \partial_z \psi_{v_n}\Big)(x,\eta+v_n(x)-1)\,,$$
$$\partial_x^2 \phi_n(x,\eta) = \Big(\partial_x^2 \psi_{v_n} + 2\partial_x v_n \partial_x \partial_z \psi_{v_n} + (\partial_x v_n)^2 \partial_z^2 \psi_{v_n} + \partial_x^2 v_n \partial_z \psi_{v_n}\Big)(x,\eta+v_n(x)-1)\,,$$
$$\partial_x \partial_\eta \phi_n(x,\eta) = \Big(\partial_x \partial_z \psi_{v_n} + \partial_x v_n \partial_z^2 \psi_{v_n}\Big)(x,\eta+v_n(x)-1)\,,$$
$$\partial_\eta^2 \phi_n(x,\eta) = \partial_z^2 \psi_{v_n}(x,\eta+v_n(x)-1)\,,$$

it follows from (4.29), (4.36), the continuous embedding of $H^2(D)$ in $C^1(\bar{D})$, and that of $H^1(\mathcal{R}_2)$ in $L^{2q/(2-q)}(\mathcal{R}_2)$ that

$$\phi_n \in W_q^2(\mathcal{R}_2) \quad \text{with} \quad \|\phi_n\|_{W_q^2(\mathcal{R}_2)} \le c(q,\kappa)\,, \qquad n \ge 1\,. \tag{4.37}$$

Now, given $p \in [1,\infty)$, we choose $q \in (1, \min\{2,p\})$ satisfying $1 < 2/q < 1 + 1/p$ and $s \in (2/q - 1/p, 1)$. Since

$$\phi_n \rightharpoonup \phi \quad \text{in} \quad W_q^2(\mathcal{R}_2)$$

by (2.1d), (4.37), and Proposition 4.9, the continuity of the trace as a mapping from $W_q^1(\mathcal{R}_2)$ to $W_q^{1-1/q}(D \times \{1\})$ and the compactness of the embedding of $W_q^{1-1/q}(D)$ in $L_p(D)$ imply that

$$\nabla \phi_n(\cdot,1) \to \nabla \phi(\cdot,1) \quad \text{in} \quad W_q^{s-1/q}(D) \tag{4.38}$$

and

$$\|\nabla \phi(\cdot,1)\|_{L_p(D)} \le c(p,\kappa)\,. \tag{4.39}$$

That is,

$$\partial_z \psi_{v_n,2}(\cdot,v_n) = \partial_\eta \phi_n(\cdot,1) \to \partial_\eta \phi(\cdot,1) = \partial_z \psi_{v,2}(\cdot,v) \quad \text{in} \quad L_p(D)$$

and, recalling (4.29) and the continuous embedding of $H^2(D)$ in $C^1(\bar{D})$,

$$\begin{aligned}\partial_x \psi_{v_n,2}(\cdot,v_n) &= \partial_x \phi_n(\cdot,1) - \partial_x v_n \partial_\eta \phi_n(\cdot,1) \\ &\to \partial_x \phi(\cdot,1) - \partial_x v \partial_\eta \phi(\cdot,1) = \partial_x \psi_{v,2}(\cdot,v) \quad \text{in} \quad L_p(D)\,.\end{aligned}$$

Furthermore, (4.38) and (4.39), along with the bound $\|v\|_{H^2(D)} \le \kappa$ and the continuous embedding of $H^2(D)$ in $C^1(\bar{D})$, entail that

$$\|\nabla \psi_{v,2}(\cdot,v)\|_{L_p(D)} \le c(p,\kappa)\,,$$

which proves (4.33) and the first bound in (4.35). Clearly, (4.34) and the second bound in (4.35) are shown in the same way. □

Proof of Theorem 1.3 The proof of Theorem 1.3 is now a consequence of Proposition 3.3 for (1.4a), Proposition 4.9 (b) for (1.4b), and Proposition 4.11 for (1.4c). □

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Appendix A. The Identity (4.12)

This appendix is devoted to the proof of the identity (4.12), which can be seen as a variant of [4, Lemma 4.3.1.2] with piecewise constant linear constraints on the boundaries instead of constant ones.

Lemma A.1 *Let $\mathcal{R} = D \times (0, 1+d)$ and consider $(V, W) \in H^1(\mathcal{R}, \mathbb{R}^2)$ satisfying*

$$V(x,0) = V(x,1+d) = 0\,, \qquad x \in D = (-L, L)\,, \tag{A.1a}$$

$$W(\pm L, \eta) + \tau^{\pm}(\eta) V(\pm L, \eta) = 0\,, \qquad \eta \in (0, 1+d)\,, \tag{A.1b}$$

where $\tau^{\pm}$ are piecewise constant functions of the form

$$\tau^{\pm} = \tau_1^{\pm}\mathbf{1}_{(0,1)} + \tau_2^{\pm}\mathbf{1}_{(1,1+d)} \tag{A.2}$$

with $(\tau_1^+, \tau_1^-, \tau_2^+, \tau_2^-) \in \mathbb{R}^4$, featuring possibly a jump discontinuity at $\eta = 1$. Then

$$\int_{\mathcal{R}} \partial_x V \partial_\eta W \,\mathrm{d}(x,\eta) = \int_{\mathcal{R}} \partial_\eta V \partial_x W \,\mathrm{d}(x,\eta)\,.$$

When $\tau_1^{\pm} = \tau_2^{\pm}$, Lemma A.1 is a straightforward consequence of [4, Lemma 4.3.1.2]. The novelty here is the possibility of handling the jump discontinuity in (A.1b) when $\tau_1^{\pm} \neq \tau_2^{\pm}$ in (A.2).

The proof follows the lines of that of [4, Lemma 4.3.1.2]. For $s \geq 1$, we introduce the space

$$\mathcal{G}^s(\mathcal{R}) := \{(V, W) \in H^s(\mathcal{R}, \mathbb{R}^2) \,:\, (V, W) \text{ satisfies } (A.1)\}$$

and first report the density of $\mathcal{G}^2(\mathcal{R})$ in $\mathcal{G}^1(\mathcal{R})$.

Lemma A.2 *$\mathcal{G}^2(\mathcal{R})$ is dense in $\mathcal{G}^1(\mathcal{R})$.*

As in the proof of [4, Lemma 4.3.1.3], the core of the proof of Lemma A.2 is to establish the density of the space $\mathcal{Z}^2(\partial\mathcal{R})$ of traces of functions in $\mathcal{G}^2(\mathcal{R})$ in the space $\mathcal{Z}^1(\partial\mathcal{R})$ of traces of functions in $\mathcal{G}^1(\mathcal{R})$, after identifying these two trace spaces. Since the proof is almost identical to that of [4, Lemma 4.3.1.3], we omit it here, but refer to the extended version of this paper [8].

Proof of Lemma A.1 Due to Lemma A.2, it suffices to prove the identity in Lemma A.1 when (V, W) belongs to $\mathcal{G}^2(\mathcal{R})$. This additional regularity allows us to use integration by parts to interchange the derivatives and guarantees the continuity of both V and W on $\bar{\mathcal{R}}$. Indeed, $H^2(\mathcal{R})$ embeds continuously in $C^\alpha(\bar{\mathcal{R}})$ for all $\alpha \in (0,1)$ by [12, Chapter 2, Theorem 3.8] and we deduce that

$$(V, W) \in C(\bar{\mathcal{R}}, \mathbb{R}^2)\,. \tag{A.3}$$

Next, after integrating by parts,

$$\begin{aligned} J(V,W) &:= \int_{\mathcal{R}} \left(\partial_x V \partial_\eta W - \partial_\eta V \partial_x W\right) \mathrm{d}(x,\eta) \\ &= \int_0^{1+d} \Big[(V\partial_\eta W)(x,\eta)\Big]_{x=-L}^{x=L} \mathrm{d}\eta - \int_{\mathcal{R}} V\partial_x\partial_\eta W \,\mathrm{d}(x,\eta) \\ &\quad - \int_D \Big[(V\partial_x W)(x,\eta)\Big]_{\eta=0}^{\eta=1+d} + \int_{\mathcal{R}} V\partial_x\partial_\eta W \,\mathrm{d}(x,\eta)\,. \end{aligned}$$

Since $V(x,0) = V(x,1+d) = 0$ for $x \in D$ by (A.1a) and the second and fourth terms cancel each other out, we obtain

$$J(V,W) = \int_0^{1+d} V(L,\eta)\partial_\eta W(L,\eta)\,\mathrm{d}\eta - \int_0^{1+d} V(-L,\eta)\partial_\eta W(-L,\eta)\,\mathrm{d}\eta\,.$$

Now, according to (A.1b) and the regularity of V and W,

$$\begin{aligned} \partial_\eta W(\pm L,\eta) &= -\tau_1^\pm \partial_\eta V(\pm L,\eta)\,, \qquad \eta \in (0,1)\,, \\ \partial_\eta W(\pm L,\eta) &= -\tau_2^\pm \partial_\eta V(\pm L,\eta)\,, \qquad \eta \in (1,1+d)\,, \end{aligned}$$

so that, since $[\eta \mapsto V(\pm L,\eta)] \in C([0,1+d])$ by (A.3),

$$\begin{aligned} J(V,W) &= -\tau_1^+ \int_0^1 (V\partial_\eta V)(L,\eta)\,\mathrm{d}\eta - \tau_2^+ \int_1^{1+d} (V\partial_\eta V)(L,\eta)\,\mathrm{d}\eta \\ &\quad + \tau_1^- \int_0^1 (V\partial_\eta V)(-L,\eta)\,\mathrm{d}\eta + \tau_2^- \int_1^{1+d} (V\partial_\eta V)(-L,\eta)\,\mathrm{d}\eta \\ &= -\tau_1^+ \frac{V(L,1)^2 - V(L,0)^2}{2} - \tau_2^+ \frac{V(L,1+d)^2 - V(L,1)^2}{2} \\ &\quad + \tau_1^- \frac{V(-L,1)^2 - V(-L,0)^2}{2} + \tau_2^- \frac{V(-L,1+d)^2 - V(-L,1)^2}{2} \\ &= \frac{\tau_1^+}{2} V(L,0)^2 - \frac{\tau_1^-}{2} V(-L,0)^2 - \frac{\tau_2^+}{2} V(L,1+d)^2 + \frac{\tau_2^-}{2} V(-L,1+d)^2 \\ &\quad - \frac{\tau_1^+ - \tau_2^+}{2} V(L,1)^2 + \frac{\tau_1^- - \tau_2^-}{2} V(-L,1)^2\,. \end{aligned} \tag{A.4}$$

On the one hand, it follows from (A.1) and the continuity (A.3) of V that

$$\begin{aligned} V(\pm L,0) &= \lim_{x\to\pm L} V(x,0) = 0\,, \\ V(\pm L,1+d) &= \lim_{x\to\pm L} V(x,1+d) = 0\,. \end{aligned} \tag{A.5}$$

On the other hand, using (A.1b) along with the continuity (A.3) gives

$$\begin{aligned} \tau_1^\pm V(\pm L,1) &= \lim_{\eta\nearrow 1} \tau^\pm(\eta)V(\pm L,\eta) = -\lim_{\eta\nearrow 1} W(\pm L,\eta) \\ &= -W(\pm L,1) = -\lim_{\eta\searrow 1} W(\pm L,\eta) = \lim_{\eta\searrow 1}\tau^\pm(\eta)V(\pm L,\eta) \\ &= \tau_2^\pm V(\pm L,1)\,. \end{aligned}$$

Consequently,

$$\left(\tau_1^\pm - \tau_2^\pm\right) V(\pm L,1) = 0\,. \tag{A.6}$$

Combining (A.4), (A.5), and (A.6) leads us to $J(V, W) = 0$ and we have proved that

$$J(V, W) = 0, \qquad (V, W) \in \mathcal{G}^2(\mathcal{R}). \tag{A.7}$$

In other words, the identity stated in Lemma A.1 is valid for $(V, W) \in \mathcal{G}^2(\mathcal{R})$. □

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