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Two-dimensional cycle classes on $\overline{\mathcal{M}}_{0,n}$

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Abstract

For each $n \geq 5$, we give an S_n-equivariant basis for $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, as well as for $H_{2(n-5)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Such a basis exists for $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ and for $H_{2(n-4)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, but it is not known whether one exists for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ when $3 \leq k \leq n-6$.

1 Introduction

The moduli space $\overline{\mathcal{M}}_{0,n}$ is an $(n-3)$-dimensional smooth projective variety that parametrizes stable n-marked genus-zero curves. $\overline{\mathcal{M}}_{0,n}$ admits a stratification whose intricate combinatorics are reflected in the structure of its homology groups. These groups have been described, in terms of generators and relations, first by Keel [10] and then by Kontsevich and Manin [12]. The groups were described further by Kapranov [9], Fulton-MacPherson [7], Manin [13], Getzler [8], Yuzvinsky [18] and others.

$\overline{\mathcal{M}}_{0,n}$ carries a natural S_n-action; this induces S_n-actions on the homology groups of $\overline{\mathcal{M}}_{0,n}$. Indeed, for $n \geq 5$, S_n is the full automorphism group of $\overline{\mathcal{M}}_{0,n}$ [3, 5]. Getzler [8] gave an algorithm for computing the character of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ as an S_n-representation, and Bergström and Minabe [2] later gave a recursive formula for the character using spaces of weighted stable curves.

Recall that a finite-dimensional representation V of a finite group G is called a *permutation representation* if V has a *permutation basis*; that is, a basis B of V such that the action of G on V restricts to an action on B. We ask:

Question 1 *Is $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ a permutation representation of S_n for all $n \geq 3$ and $k \geq 0$?*

Farkas and Gibney [6] gave an affirmative answer if $k = n - 4$ by producing a permutation basis of divisors. Since Poincaré duality induces S_n-equivariant isomorphisms $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \rightarrow H_{2(n-3-k)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, this also implies an affirmative answer if $k = 1$. The first author found [15] a permutation basis for $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ that is not Poincaré dual to the basis of Farkas and Gibney—interestingly, the two bases are not isomorphic as S_n-sets for even n.

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We give an affirmative answer in the case $k = 2$ (and therefore also $k = n - 5$):

Theorem 2.8. *Let $\mathbf{P}_{2,n}^1$ denote the S_n-set of all subsets $A \subseteq \{1, \ldots, n\}$ such that $5 \leq |A| \leq n$ and $|A|$ is odd. Let $\mathbf{P}_{2,n}^2$ denote the S_n-set of unordered pairs $\{P_1, P_2\}$, where P_1 and P_2 are disjoint subsets of $\{1, \ldots, n\}$ of cardinality at least 3. Then $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is a permutation representation of S_n, with a permutation basis in equivariant bijection with $\mathbf{P}_{2,n}^1 \sqcup \mathbf{P}_{2,n}^2$, for all $n \geq 3$.*

For general k, Question 1 appears to be open. The known bases of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ are constructed via recursion on n, which involves treating the n-th marked point as special. For example, Kapranov [9] and Yuzvinsky [18] described bases (recursively and in closed form, respectively) for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ that are permutation bases with respect to the S_{n-1} action induced by permuting all but the nth point. In contrast, a permutation basis for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, if it exists, would treat the n marked points symmetrically.

Very recent work of Castravet and Tevelev [4] on the derived category of $\overline{\mathcal{M}}_{0,n}$ implies that $H_*(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = \bigoplus_{k=0}^{n-3} H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is a permutation representation; however, this does not answer Question 1, as the permutation basis given for $H_*(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ consists of classes not of pure degree. (They are Chern characters of certain vector bundles.)

1.1 An S_n-equivariant filtration of $H_{2k}(\overline{\mathcal{M}}_{0,n})$ and the proof of Theorem 2.8

Theorem 2.8 follows from the stronger Theorem 2.7, which we now explain. In [15], the first author defined (see Definition 4) an S_n-equivariant decomposition

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \bigoplus_{r=1}^{\min\{k, n-2-k\}} Q_{k,n}^r. \tag{1}$$

The first author also showed (Theorem 2.3) that for all $k \geq 0$ and $n \geq 3$, $Q_{k,n}^1$ has a permutation basis, in equivariant bijection with the S_n-set $\mathbf{P}_{k,n}^1$ defined in Theorem 2.3. We prove:

Theorem 2.7. *For all $k \geq 0$ and $n \geq 3$, $Q_{k,n}^2$ has a permutation basis, in equivariant bijection with the S_n-set $\mathbf{P}_{k,n}^2$ defined in Definition 2.5.*

Corollary 2.9. *The S_n-permutation representation $\mathbb{Q}(\mathbf{P}_{k,n}^1 \sqcup \mathbf{P}_{k,n}^2)$ is a subrepresentation of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ for all k and n.*

By (1), $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = Q_{1,n}^1$ and $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong Q_{2,n}^1 \oplus Q_{2,n}^2$. Thus Theorems 2.3 and 2.7 imply Theorem 2.8. Note also that by Poincaré duality, for $n \leq 8$, these express $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ as a permutation representation for all k.

1.2 Experimental evidence towards Question 1

Let V be a finite-dimensional permutation representation of G with permutation basis B. Then B is a disjoint union of G-orbits, i.e. transitive G-sets. It is important to note that if B_1 and B_2 are two permutation bases for V, they need not be isomorphic as G-sets. However, the number of orbits of B_1 is equal to the dimension of the G-fixed subspace of V, thus equal to the number of orbits of B_2. Recall that every transitive G-set is equivariantly isomorphic to the G-set of left cosets of some subgroup of G. This gives a computational strategy for checking whether a given G-representation W is a permutation representation — namely, one enumerates the subgroups of G and the associated characters, then does an exhaustive

search for the character of W as a non-negative integer combination of those characters. This is doable in the GAP computer algebra system for $n \leq 10$.

J. Bergström and S. Minabe shared with us the characters of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ for $n \leq 20$. We now give some observations inferred from their data, from Theorem 2.7, and from a GAP analysis as above. The smallest example where Theorems 2.3 and 2.7 do not answer Question 1 is $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$.

- $Q_{3,9}^3$ is not a permutation representation. We computed the character of $Q_{3,9}^3$ by subtracting those of $Q_{3,9}^1$ and $Q_{3,9}^2$ (from Theorems 2.3 and 2.7) from that of $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ (from Bergström and Minabe's data). We then used the strategy outlined in the above paragraph to show that $Q_{3,9}^3$ is not a permutation representation.

- Despite the previous point, $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ is a permutation representation. We showed this by again applying the strategy of the previous paragraph; in fact, GAP produced two inequivalent decompositions of $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ into transitive permutation representations. It follows that in these two decompositions, the two permutation subrepresentations $\mathbb{Q}\mathbf{P}_{3,9}^1$ and $\mathbb{Q}\mathbf{P}_{3,9}^2$ *cannot* both appear.

- $H_{2k}(\overline{\mathcal{M}}_{0,10}, \mathbb{Q})$ is a permutation representation for all k, by the same exhaustive computer search.

1.3 Outline of the paper

In Sect. 2, we recall fundamental results about the homology groups of $\overline{\mathcal{M}}_{0,n}$, and define $Q_{k,n}^r$, $\mathbf{P}_{k,n}^1$, and $\mathbf{P}_{k,n}^2$. In Sect. 3.1 we outline the proof of Theorem 2.7, and the complete proof is in Sect. 3.2. In Sect. 4, we state a conjectural formula for the dimension of $Q_{k,n}^r$ in general.

2 Background and notation

Definition 2.1 Let $\mathbf{S}_{k,n}$ denote the set of stable trees with $n-2-k$ vertices, $n-3-k$ edges, and n marked half-edges (labeled by the set $\{1, \ldots, n\}$). (A tree is *stable* if each vertex has valence at least 3.) $\mathbf{S}_{k,n}$ is in canonical bijection with the set of (closed) k-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$.

We will abuse notation, and refer to an element of $\mathbf{S}_{k,n}$ interchangeably as a marked tree, as a closed k-dimensional subvariety of $\overline{\mathcal{M}}_{0,n}$, and as a cycle class in $H_{2k}(\overline{\mathcal{M}}_{0,n})$ or various quotients thereof. For example, it is a well-known result of Keel [10] that $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is spanned by $\mathbf{S}_{k,n}$. Kontsevich and Manin [12] gave a spanning set for the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Let $\sigma \in \mathbf{S}_{k+1,n}$, let v be a vertex of σ with valence at least 4, and let A, B, C, D be four edges or half-edges incident to v. Let $\mathbf{T}$ denote the set of edges and

half-edges incident to v, other than A, B, C, D. We define a relation

$$\mathcal{R} = \mathcal{R}(\sigma, v, A, B, C, D) = \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AB\mathbf{U}_1 | CD\mathbf{U}_2) - \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AC\mathbf{U}_1 | BD\mathbf{U}_2) \in \mathbb{Q}\mathbf{S}_{k,n}.$$

(2)

We now explain this notation, which we will continue to use. Here $(AB\mathbf{U}_1 | CD\mathbf{U}_2) \in \mathbf{S}_{k,n}$ is the tree obtained from σ by replacing v with two vertices v', v'' connected by an edge e', with the edges or half-edges $\{A, B\} \cup \mathbf{U}_1$ incident to v' and the edges or half-edges $\{C, D\} \cup \mathbf{U}_2$ incident to v''. Note that contracting e' yields σ again. Kontsevich and Manin showed that the relations obtained this way (varying over all choices of σ, v, A, B, C, D) span the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

Note that for any $\sigma \in \mathbf{S}_{k,n}$, we have

$$\sum_v (\mathrm{val}(v) - 3) = k,$$

(3)

where the sum is over vertices of σ. Thus to any $\sigma \in \mathbf{S}_{k,n}$ we may associate a partition λ_σ of k, namely the set of nonzero summands on the right-hand side of (3). Since $|V(\sigma)| = n - 2 - k$, λ_σ has at most $n - 2 - k$ parts. (As a partition of k, λ_σ also has at most k parts.)

Definition 2.2 [16] For $r \in \{1, \dots, \min\{k, n-2-k\}\}$, let $\mathbf{S}_{k,n}^{\geq r} \subseteq \mathbf{S}_{k,n}$ be the subset consisting of those $\sigma \in \mathbf{S}_{k,n}$ such that λ_σ has at least r parts. This defines an S_n-invariant filtration

$$\mathbf{S}_{k,n} = \mathbf{S}_{k,n}^{\geq 1} \supseteq \mathbf{S}_{k,n}^{\geq 2} \supseteq \cdots \supseteq \mathbf{S}_{k,n}^{\geq k}.$$

Since $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is spanned by $\mathbf{S}_{k,n}$, this also defines an S_n-invariant filtration

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = Q_{k,n}^{\geq 1} \supseteq Q_{k,n}^{\geq 2} \supseteq \cdots \supseteq Q_{k,n}^{\geq k},$$

(4)

where

$$Q_{k,n}^{\geq r} = \mathrm{Span}(\mathbf{S}_{k,n}^{\geq r}) \subseteq H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}).$$

Finally, let $\mathbf{S}_{k,n}^r = \mathbf{S}_{k,n}^{\geq r} \setminus \mathbf{S}_{k,n}^{\geq r+1}$ and $Q_{k,n}^r = Q_{k,n}^{\geq r} / Q_{k,n}^{\geq r+1}$. Note that $\mathbf{S}_{k,n}^r$ spans $Q_{k,n}^r$. Note also that the S_n-invariant filtration (4), together with Maschke's Theorem, induces an isomorphism of S_n-representations

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \bigoplus_{r=1}^{\min\{k, n-2-k\}} Q_{k,n}^r.$$

In [16], it is shown that the filtration (4) is preserved by pushforwards along forgetful morphisms of moduli spaces as well as along "Hurwitz correspondences", which are algebraic correspondences that generalize forgetful maps. In [17], $Q_{k,n}^1$ is described explicitly as a permutation representation. We briefly recall the relevant results here. Recall that Mumford [14] and Arbarello-Cornalba [1] defined, for every k, a codimension-k class $\kappa_k \in H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Given a subset $T \subset \{1, \dots, n\}$ with $|T| \geq 3$, let $\pi_T : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,|T|}$ be the forgetful map remembering only the marked points in T, and set $\kappa_k^T := \pi_T^*(\kappa_k) \in H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Let

$$\mathcal{K}^k = \mathrm{Span}\{\kappa_k^T \mid T \subset \{1, \dots, n\}\} \subset H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}).$$

Theorem 2.3 ([17])

1. *The subspace $\mathcal{K}^k \subset H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is the annihilator, under the intersection pairing, of the subspace $Q^{\geq 2}_{k,n} \subset H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, and there is therefore a natural S_n-equivariant isomorphism $\mathcal{K}^k \cong (Q^1_{k,n})^\vee$.*
2. *The set $\{\kappa_k^T \mid T \subset \{1, \ldots, n\}, |T| \geq 3, |T| \equiv (k+3) \mod 2\}$ is a permutation basis for $\mathcal{K}^k$.*

Let $\mathbf{P}^1_{k,n} := \{T \subseteq \{1, \ldots, n\} : k + 3 \leq |T| \leq n, |T| \equiv k + 3 \mod 2\}$. Since every finite-dimensional S_n representation is isomorphic to its dual, we obtain that $Q^1_{k,n} \cong \mathbb{Q}\mathbf{P}^1_{k,n}$ as S_n-representations. Since $Q^{\geq 2}_{1,n} \subseteq H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is zero by definition, this implies:

Corollary 2.4 ([17]) *For $n \geq 3$, there are isomorphisms of S_n-representations*

$$H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong Q^1_{1,n} \cong \mathbb{Q}\mathbf{P}^1_{1,n} \cong \mathbb{Q}\{T \subseteq \{1, \ldots, n\} : 4 \leq |T| \leq n, |T| \text{ even}\}.$$

Definition 2.5 Let $\mathbf{P}^2_{k,n}$ denote the set of unordered pairs $\{(P_1, \alpha_1), (P_2, \alpha_2)\}$, where:

- P_1 and P_2 are disjoint subsets of $\{1, \ldots, n\}$, and
- α_1 and α_2 are positive integers whose sum is k, and
- $P_i \geq \alpha_i + 2$ for $i \in \{1, 2\}$.

Observe that Definition 2.5 is consistent with the notation $\mathbf{P}^2_{2,n}$ from Theorem 2.8.

Definition 2.6 We define an S_n-equivariant map $W_n : \mathbf{S}^2_{k,n} \to \mathbf{P}^2_{k,n}$ as follows. For $\sigma \in \mathbf{S}^2_{k,n}$, let v_1 and v_2 be the two vertices of σ with valence at least 4. There is a unique edge e_1 incident to v_1 that is contained in any path from v_1 to v_2, and similarly an edge e_2 incident to v_2. Cutting the two edges e_1 and e_2 (into unmarked half-edges) yields three connected components σ_1, σ', and σ_2. (Note: the graphs σ_1 and σ_2 have a single "central" vertex, with trivalent subtrees incident to it. The graph σ' is itself a trivalent subtree.)

Let $\alpha_1 = \text{val}(v_1) - 3$, and let $P_1 \subseteq \{1, \ldots, n\}$ be the set of marked half-edges in σ_1. Similarly let $\alpha_2 = \text{val}(v_2) - 3$, and let $P_2 \subseteq \{1, \ldots, n\}$ be the set of marked half-edges in σ_2.

Then we define $W_n(\sigma) := \{(P_1, \alpha_1), (P_2, \alpha_2)\}$. Observe that $W_n(\sigma) \in \mathbf{P}^2_{k,n}$, since:

- P_1 and P_2 are disjoint (by definition),
- $\alpha_1 + \alpha_2 = k$ (this follows from the fact that σ is k-dimensional), and
- $|P_i| \geq \alpha_i + 2$ for $i \in \{1, 2\}$. (This is equivalent to $|P_i| \geq \text{val}(v_i) - 1$, which follows from the fact that by stability, every edge incident to v_i, except e_1, is on a nonrepeating path from v_i to at least one marked half-edge in σ_1.)

Theorem 2.7 *For any $n > 3$ and $2 \leq k \leq n - 4$, $Q^2_{k,n}$ is isomorphic (as an S_n-module) to $\mathbb{Q}\mathbf{P}^2_{k,n}$.*

Together with Theorem 2.3 and the fact that $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ consists of the two graded pieces $Q^1_{2,n}$ and $Q^2_{2,n}$, Theorem 2.7 implies:

Theorem 2.8 *For $n \geq 3$, $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \mathbb{Q}\mathbf{P}^1_{2,n} \oplus \mathbb{Q}\mathbf{P}^2_{2,n}$ as S_n-representations.*

Corollary 2.9 *For $n \geq 3$ and $k \geq 0$, $\mathbb{Q}\mathbf{P}^1_{k,n} \oplus \mathbb{Q}\mathbf{P}^2_{k,n}$ is a subrepresentation of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.*

3 Proof of Theorem 2.7

3.1 Outline of Proof

We now introduce further filtrations

$$\mathbf{S}_{k,n}^2 = (\mathbf{S}_{k,n}^2)^{\geq 0} \supseteq (\mathbf{S}_{k,n}^2)^{\geq 1} \supseteq \cdots \supseteq (\mathbf{S}_{k,n}^2)^{\geq n-k-4}$$
$$\mathbf{P}_{k,n}^2 = (\mathbf{P}_{k,n}^2)^{\geq 0} \supseteq (\mathbf{P}_{k,n}^2)^{\geq 1} \supseteq \cdots \supseteq (\mathbf{P}_{k,n}^2)^{\geq n-k-4}$$
$$Q_{k,n}^2 = (Q_{k,n}^2)^{\geq 0} \supseteq (Q_{k,n}^2)^{\geq 1} \supseteq \cdots \supseteq (Q_{k,n}^2)^{\geq n-k-4},$$

defined as follows.

Definition 3.1 For $\sigma \in \mathbf{S}_{k,n}^2$, let σ_1, σ_2, and σ' be as in Definition 2.6. Then for $0 \leq b \leq n-k-4$, we let $(\mathbf{S}_{k,n}^2)^{\geq b}$ denote the subset consisting of those trees σ such that σ' has at least b marked half-edges. Let

$$(Q_{k,n}^2)^{\geq b} = \mathrm{Span}((\mathbf{S}_{k,n}^2)^{\geq b}) \subseteq Q_{k,n}^2 \qquad \text{and} \qquad (\mathbf{P}_{k,n}^2)^{\geq b} = W_n((\mathbf{S}_{k,n}^2)^{\geq b}) \subseteq \mathbf{P}_{k,n}^2.$$

We also define the graded pieces

$$(\mathbf{S}_{k,n}^2)^b = (\mathbf{S}_{k,n}^2)^{\geq b} \setminus (\mathbf{S}_{k,n}^2)^{\geq b+1}$$
$$(Q_{k,n}^2)^b = (Q_{k,n}^2)^{\geq b} / (Q_{k,n}^2)^{\geq b+1}$$
$$(\mathbf{P}_{k,n}^2)^b = (\mathbf{P}_{k,n}^2)^{\geq b} \setminus (\mathbf{P}_{k,n}^2)^{\geq b+1}.$$

We write $\rho_{n,b}$ for the natural surjective linear map $\mathbb{Q}(\mathbf{S}_{k,n}^2)^b \to (Q_{k,n}^2)^b$. Observe that W_n restricts to a surjection $W_{n,b} : (\mathbf{S}_{k,n}^2)^b \to (\mathbf{P}_{k,n}^2)^b$. In other words, we have

$$\mathbb{Q}(\mathbf{S}_{k,n}^2)^b$$

$$\overset{\rho_{n,b}}{\swarrow} \qquad \overset{W_{n,b}}{\searrow}$$
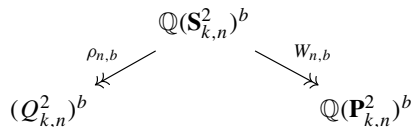
$$(Q_{k,n}^2)^b \qquad\qquad \mathbb{Q}(\mathbf{P}_{k,n}^2)^b$$

where we abuse notation and use $W_{n,b}$ to refer to the map induced on free $\mathbb{Q}$-vector spaces by $W_{n,b}$. We will show (Lemma 3.2) that $W_{n,b}$ factors equivariantly through $\rho_{n,b}$, and (Lemma 3.7) that $\rho_{n,b}$ factors equivariantly through $W_{n,b}$; the two resulting maps $(Q_{k,n}^2)^b \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^b$ and $\mathbb{Q}(\mathbf{P}_{k,n}^2)^b \to (Q_{k,n}^2)^b$ are therefore inverses, by a straightforward diagram chase.

3.2 Proofs of Lemmas 3.2 and 3.7

We now prove:

Lemma 3.2 *Fix nonnegative integers $n \geq 4$, $k \leq n - 3$, and $b \leq n - k - 4$. The map $W_{n,b} : \mathbb{Q}(\mathbf{S}_{k,n}^2)^b \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^b$ descends equivariantly to an S_n-equivariant map $\overline{W_{n,b}} : (Q_{k,n}^2)^b \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^b$.*

The proof is by induction on b. The base case $b = 0$ (for all n) is Proposition 3.3. The inductive step is Proposition 3.5.

Proposition 3.3 $W_{n,0} : \mathbb{Q}(\mathbf{S}_{k,n}^2)^0 \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^0$ *descends equivariantly to an S_n-equivariant map $\overline{W_{n,0}} : (Q_{k,n}^2)^0 \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^0$.*

The proof is complicated, so we first give a discussion. Note that

$$(Q_{k,n}^2)^0 = \frac{\mathbb{Q}\mathbf{S}_{k,n}^{\geq 2}}{R^{\geq 2} + \mathbb{Q}\mathbf{S}_{k,n}^{\geq 3} + \mathbb{Q}(\mathbf{S}_{k,n}^2)^{\geq 1}}.$$

Here $R^{\geq 2} := R \cap \mathbb{Q}\mathbf{S}_{k,n}^{\geq 2} \subseteq \mathbb{Q}\mathbf{S}_{k,n}^{\geq 2}$, where $R \subseteq \mathbb{Q}\mathbf{S}_{k,n}$ is the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. It is difficult to work directly with $R^{\geq 2}$, due to the fact that the filtration $(\mathbf{S}_{k,n}^{\geq 2})^\bullet$ does not interact in a predictable way with the Kontsevich-Manin relations (2). Our strategy is as follows. We will find a map $\tilde{W}_n : \mathbb{Q}\mathbf{S}_{k,n} \to \mathbb{Q}\mathbf{P}_{k,n}^2$ such that:

(i)   The restriction of $\tilde{W}_n$ to $\mathbb{Q}(\mathbf{S}_{k,n}^2)^0$ is $W_{n,0}$,
(ii) The image of the restriction of $\tilde{W}_n$ to $\mathbb{Q}(\mathbf{S}_{k,n}^2)^{\geq 1}$ lies in $\mathbb{Q}(\mathbf{P}_{k,n}^2)^{\geq 1}$,
(iii) The restriction of $\tilde{W}_n$ to $\mathbb{Q}\mathbf{S}_{k,n}^{\geq 3}$ is zero, and
(iv) For any Kontsevich-Manin relation $\mathcal{R} \in \mathbb{Q}\mathbf{S}_{k,n}$, $\tilde{W}_n(\mathcal{R}) \in \mathbb{Q}(\mathbf{P}_{k,n}^2)^{\geq 1}$.

Conditions 3.2, 3.2, and 3.2 imply that $\tilde{W}_n$ descends to a map

$$\frac{\mathbb{Q}\mathbf{S}_{k,n}}{R + \mathbb{Q}\mathbf{S}_{k,n}^{\geq 3} + \mathbb{Q}(\mathbf{S}_{k,n}^2)^{\geq 1}} \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^0.$$

Restricting to $\mathbb{Q}\mathbf{S}_{k,n}^{\geq 2} \subseteq \mathbb{Q}\mathbf{S}_{k,n}$, Condition 3.2 gives a map $\overline{W_{n,0}} : (Q_{k,n}^2)^0 \to \mathbb{Q}(\mathbf{P}_{k,n}^2)^0$ descended from $W_{n,0}$.

Remark 3.4 The existence of a map satisfying conditions 3.2–3.2 is quite mysterious to us; we found $\tilde{W}_n$ by computer experimentation. The fact that the definition of $\tilde{W}_n$ (see (5) below) is quite complicated makes us wonder if there is some simpler, more conceptual description that might simplfy the following proof.

Proof of Proposition 3.3 Let $\sigma \in \mathbf{S}_{k,n}^1$, and let v be the unique vertex of σ with valence ≥ 4. Then $\mathrm{val}(v) = k + 3$. We define a set partition $\Pi \vdash \{1, \ldots, n\}$ induced by σ, where i_1 and i_2 are in the same part if and only if the corresponding marked half-edges are in the same connected component of the complement of v in σ. Note that Π has $k + 3$ parts.

Suppose $\Gamma = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}_{k,n}^2)^0$ is such that P_1 is a union of s_1 parts of Π (so necessarily P_2 is a union of the remaining $s_2 := k + 3 - s_1$ parts of Π). Let $e_\Pi(\Gamma) = \min\{s_1 - \alpha_1, s_2 - \alpha_2\}$. We define:

$$\tilde{W}_n(\sigma) := \sum_{\substack{\Gamma \in (\mathbf{P}_{k,n}^2)^0 \\ \Gamma = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \\ P_1 \text{ is a union of parts of } \Pi.}} \frac{(-1)^{e_\Pi(\Gamma)}}{2} \Gamma. \tag{5}$$

This defines $\tilde{W}_n$ on $\mathbf{S}_{k,n}^1$. We also define

$$\tilde{W}_n(\sigma) = \begin{cases} W_n(\sigma) & \sigma \in \mathbf{S}_{k,n}^2 \\ 0 & \sigma \in \mathbf{S}_{k,n}^{\geq 3}. \end{cases} \tag{6}$$

Extending by linearity, these collectively define a map $\tilde{W}_n : \mathbb{Q}\mathbf{S}_{k,n} \to \mathbb{Q}\mathbf{P}_{k,n}^2$, which clearly satisfies conditions 3.2, 3.2, and 3.2 above.

Let $\mathcal{R} = \mathcal{R}(\sigma, v, A, B, C, D) \in \mathbb{Q}\mathbf{S}_{k,n}$ be a Kontsevich-Manin relation as in (2). We must show $\tilde{W}_n(\mathcal{R}) \in \mathbb{Q}(\mathbf{P}_{k,n}^2)^{\geq 1}$. There are several cases, which we treat separately – Case VI is the hardest by far:

I σ has ≥ 4 vertices with valence ≥ 4.
II σ has 3 vertices v, v', v'' with valence ≥ 4, and val$(v) \geq 5$.
III σ has 3 vertices v, v', v'' with valence ≥ 4, and val$(v) = 4$.
IV σ has exactly 2 vertices v, v' with valence ≥ 4, and val$(v) > 4$.
V σ has exactly 2 vertices v, v' with valence ≥ 4, and val$(v) = 4$.
VI v is the unique vertex of σ with val$(v) \geq 4$.

Case I. In this case, every term of $\mathcal{R}$ is in $\mathbf{S}_{k,n}^{\geq 3}$, so $\tilde{W}_n(\mathcal{R}) = 0$.
Case II. Again, every term of $\mathcal{R}$ is in $\mathbf{S}_{k,n}^{\geq 3}$, so $\tilde{W}_n(\mathcal{R}) = 0$.
Case III.

In this case, $\mathcal{R}$ has exactly two terms, with opposite signs. These two trees (which both have exactly two vertices with valence ≥ 4) differ only in the rearrangement of trivalent subtrees. Thus $\tilde{W}_n$ does not distinguish between them, i.e. $\tilde{W}_n(\mathcal{R}) = 0$.
Case IV. All but four terms of $\mathcal{R}$ are in $\mathbf{S}_{k,n}^{\geq 3}$, so we may ignore them. If v and v' are not adjacent, then these four terms are all in $(\mathbf{S}_{k,n}^2)^{\geq 1}$, hence $\tilde{W}_n(\mathcal{R}) \in (\mathbf{P}_{k,n}^2)^{\geq 1}$. If v and v' are adjacent, connected by an edge e, there are two subcases:

(1) One of A, B, C, D is e (without loss of generality, $D = e$), or
(2) None of A, B, C, D is e.

In Sect. 3.2, let $\mathbf{T}$ be the set of edges or half-edges incident to v other than A, B, C, D. In the notation of Sect. 2, the four remaining terms of $\mathcal{R}$ are:

$$(AB\mathbf{T}|Ce) + (AB|Ce\mathbf{T}) - (AC\mathbf{T}|Be) - (AC|Be\mathbf{T}).$$

Note that by definition $\tilde{W}_n(AB|Ce\mathbf{T}) = \tilde{W}_n(AC|Be\mathbf{T})$. The first and third terms are sent to $(\mathbf{P}_{k,n}^2)^{\geq 1}$. Thus $\tilde{W}_n(\mathcal{R}) \in (\mathbf{P}_{k,n}^2)^{\geq 1}$.
In Sect. 3.2, let $\mathbf{T}$ be the set of edges or half-edges incident to v other than A, B, C, D, e. The four remaining terms of $\mathcal{R}$ are:

$$(ABe\mathbf{T}|CD) + (AB|CDe\mathbf{T}) - (ACe\mathbf{T}|BD) - (AC|BDe\mathbf{T}).$$

Observe that

$$\tilde{W}_n(ABe\mathbf{T}|CD) = \tilde{W}_n(AB|CDe\mathbf{T}) = \tilde{W}_n(ACe\mathbf{T}|BD) = \tilde{W}_n(AC|BDe\mathbf{T}).$$

Thus $\tilde{W}_n(\mathcal{R}) = 0$.
Case V. $\mathcal{R}$ has two terms with opposite sign, each of which is a tree with exactly one vertex with valence ≥ 4. They induce the same partition $\Pi \vdash \{1, \ldots, n\}$ (as in (5)), hence they cancel after applying $\tilde{W}_n$.
Case VI. We have val$(v) = k + 4$. We write $\mathbf{T}$ for the set of edges and half-edges incident to v other than A, B, C, D. Note that $|\mathbf{T}| = k$.

Recall that

$$\mathcal{R} = \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AB\mathbf{U}_1|CD\mathbf{U}_2) - \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AC\mathbf{U}_1|BD\mathbf{U}_2) \in \mathbb{Q}\mathbf{S}_{k,n}.$$

Note that terms where $\mathbf{U}_1 = \emptyset$ or $\mathbf{U} = \emptyset$ are in $\mathbf{S}_{k,n}^1$, and terms where $\mathbf{U}_1, \mathbf{U}_2 \neq \emptyset$ are in $\mathbf{S}_{k,n}^2$. We will apply (5) or (6) accordingly.

We also introduce an abuse of notation that will now be convenient. By definition, A, B, C, D, and the elements of $\mathbf{T}$ denote edges or half-edges of σ incident to v. We use the same symbols to refer to subsets of $\{1, \ldots, n\}$, where e.g. A is the set of all $i \in \{1, \ldots, n\}$

such that the path from v to the ith marked half-edge contains the edge A. (If A is itself the ith marked half-edge, then we write $A = \{i\}$.)

Let $\Gamma = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}_{k,n}^2)^0$. Note that by definition of $(\mathbf{P}_{k,n}^2)^0$, we have $P_1 \cup P_2 = \{1, \ldots, n\}$. We need to show that the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. We do this argument in cases again; Γ must be of one of the following types:

Type 0. At least one of the sets $A, B, C, D \subseteq \{1, \ldots, n\}$, or a part of $\mathbf{T}$, has nonempty intersection with both P_1 and P_2.

Type 1. Two of A, B, C, D are subsets of P_1, and the other two are subsets of P_2, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 1.1 $A, B \subseteq P_1$ and $C, D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 1.2. $A, C \subseteq P_1$ and $B, D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 1.3. $A, D \subseteq P_1$ and $B, C \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 2. Three of A, B, C, D are subsets of P_1, and the remaining one is a subset of P_2, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 2.1 $A, B, C \subseteq P_1$ and $D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 2.2. $A, C, D \subseteq P_1$ and $B \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 2.3. $A, B, D \subseteq P_1$ and $C \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 2.4. $B, C, D \subseteq P_1$ and $A \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Type 3. $A, B, C, D \subseteq P_1$, and no part of $\mathbf{T}$ has nonempty intersection with both P_1 and P_2.

Caution. It is tempting to use the S_4-action that permutes A, B, C, D, but one must be very careful in doing so. The sets A, B, C, D have been fixed, and may have different cardinalities; we may only invoke symmetry if our arguments do not refer to any specific properties of A, B, C, D.

If Γ is of type 0, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. By (5) and (6), no terms of $\mathcal{R}$ contribute a term of type 0 to $\tilde{W}_n(\mathcal{R})$.

If Γ is of type 1.1, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. Let Γ be of type 1.1, so we may write

$$\Gamma = \left\{ \left(A \cup B \cup \bigcup_{S \in \mathbf{T}_1} S, \alpha_1 \right), \left(C \cup D \cup \bigcup_{S \in \mathbf{T}_2} S, \alpha_2 \right) \right\}$$

for some partition $\mathbf{T}_1 \sqcup \mathbf{T}_2 = \mathbf{T}$, and for some α_1, α_2. Note that $\alpha_1 + \alpha_2 = k = |\mathbf{T}| = |\mathbf{T}_1| + |\mathbf{T}_2|$. Let

$$\ell = \alpha_1 - |\mathbf{T}_1| = |\mathbf{T}_2| - \alpha_2.$$

Claim: If $\ell = 0$, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero.

Proof of claim First, note that in this case $|\mathbf{T}_1| = \alpha_1 \geq 1$, i.e. $\mathbf{T}_1 \neq \emptyset$. Similarly $\mathbf{T}_2 \neq \emptyset$. By definition, every term of $\mathcal{R}$ is of the form $(AB\mathbf{U}_1 | CD\mathbf{U}_2)$ or $-(AC\mathbf{U}_1 | BD\mathbf{U}_2)$, for some partition $\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}$. In terms of the latter form, Γ appears with coefficient zero by (5) and (6).

In a term $(AB\mathbf{U_1}|CD\mathbf{U_2})$ with $\mathbf{U_1}, \mathbf{U_2} \neq \emptyset$ and $\mathbf{U_1} \neq \mathbf{T_1}$, Γ also appears with coefficient zero by (6). Thus the only terms of $\mathcal{R}$ that contribute to the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB\mathbf{T_1}|CD\mathbf{T_2}), \qquad (AB|CD\mathbf{T}), \qquad \text{and} \qquad (AB\mathbf{T}|CD). \qquad (7)$$

By definition,

$$\tilde{W}_n((AB\mathbf{T_1}|CD\mathbf{T_2})) = 1 \cdot \Gamma.$$

The boundary stratum $(AB|CD\mathbf{T})$ has a single vertex with valence ≥ 4, with corresponding set partition $\Pi = \{A \cup B, C, D\} \cup \mathbf{T} \vdash \{1, \ldots, n\}$. Using the notation in the paragraph preceding (5) (with respect to our fixed $\Gamma \in (\mathbf{P}_{k,n}^2)^0$), we have $s_1 = 1 + |\mathbf{T_1}| = 1 + \alpha_1$, so $s_1 - \alpha_1 = 1$. Thus $s_2 - \alpha_2 = 2$, so $e_\Pi(\Gamma) = s_1 - \alpha_1 = 1$. The coefficient of Γ in $\tilde{W}_n((AB|CD\mathbf{T}))$ is thus $-1/2$.

The set partition associated to $(AB\mathbf{T}|CD)$ is $\Pi = \{A, B, C \cup D\} \cup \mathbf{T}$. In this case we have $s_1 = 2 + |\mathbf{T_1}| = 2 + \alpha_1$, so $s_1 - \alpha_1 = 2$. Thus $s_1 - \alpha_1 = 1$, so $e_\Pi(\Gamma) = s_2 - \alpha_2 = 1$. The coefficient of Γ in $\tilde{W}_n((AB\mathbf{T}|CD))$ is thus $-1/2$.

In total, the coefficient is $1 - 1/2 - 1/2 = 0$, proving the claim.

Claim: If $\ell \neq 0$, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero.

Proof of claim Note that $\alpha_1, \alpha_2 \geq 1$ imply $-|\mathbf{T_1}| + 1 \leq \ell \leq k - |\mathbf{T_2}| - 1$.

By the same argument as the one preceding (7), the only terms of $\mathcal{R}$ that contribute to the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ are

$$(AB\mathbf{T}|CD) \qquad\qquad \text{and} \qquad\qquad (AB|CD\mathbf{T}).$$

(Note that $(AB\mathbf{T_1}|CD\mathbf{T_2})$ does not contribute because the values of α_1, α_2 in $\tilde{W}_n((AB\mathbf{T_1}|CD\mathbf{T_2}))$ do not match those in Γ.)

As above, $(AB|CD\mathbf{T})$ induces set partition $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$. Using the notation in the paragraph preceding (5) (with respect to our fixed $\Gamma \in (\mathbf{P}_{k,n}^2)^0$), we have $s_1 = 1 + |\mathbf{T_1}| = 1 + \alpha_1 - \ell$, so $s_1 - \alpha_1 = 1 - \ell$ (and $s_2 - \alpha_2 = 2 + \ell$). Thus $e_{\Pi_1}(\Gamma) = \min\{1 - \ell, 2 + \ell\}$.

On the other hand, $(AB\mathbf{T}|CD)$ induces set partition $\Pi_2 = \{A, B, C \cup D\} \cup \mathbf{T}$. In this case $s_1 = 2 + |\mathbf{T_1}| = 2 + \alpha_1 - \ell$, so $s_1 - \alpha_1 = 2 - \ell \leq 1$ (and $s_2 - \alpha_2 = 1 + \ell \geq 2$). Thus $e_{\Pi_2}(\Gamma) = \min\{1 + \ell, 2 - \ell\}$.

If $\ell > 0$, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is $(-1)^{1-\ell}/2 + (-1)^{2-\ell}/2 = 0$. If $\ell < 0$, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is $(-1)^{2+\ell}/2 + (-1)^{1+\ell}/2 = 0$. This proves the claim.

We conclude that the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero.
If Γ is of type 1.2, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. The argument for the type 1.1 case did not refer to any properties of the sets A, B, C, D, so this case follows by symmetry.
If Γ is of type 1.3, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. No terms of $\mathcal{R}$ contribute a term of type 1.3 to $\tilde{W}_n(\mathcal{R})$.
If Γ is of type 2.1, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. The only terms of $\mathcal{R}$ that contribute to the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB|CD\mathbf{T}) \qquad\qquad \text{and} \qquad\qquad -(AC|BD\mathbf{T}).$$

The corresponding set partitions are $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$ and $\Pi_1 = \{A \cup C, B, D\} \cup \mathbf{T}$. By definition, $e_{\Pi_1}(\Gamma) = e_{\Pi_2}(\Gamma)$, so the coefficients of Γ in $\tilde{W}_n((AB|CD\mathbf{T}))$ and $\tilde{W}_n(-(AC|BD\mathbf{T}))$ cancel.
If Γ is of type 2.2, 2.3, or 2.4, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. The argument for type 2.1 did not refer to any properties of the sets A, B, C, D, so these cases follow by symmetry.

If Γ is of type 3, the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ is zero. The only terms of $\mathcal{R}$ that contribute to the coefficient of Γ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB|CD\mathbf{T}), \qquad (AB\mathbf{T}|CD), \qquad -(AC|BD\mathbf{T}), \qquad \text{and} \qquad -(AC\mathbf{T}|BD).$$

The corresponding set partitions are $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$, $\Pi_2 = \{A, B, C \cup D\} \cup \mathbf{T}$, $\Pi_3 = \{A \cup C, B, D\} \cup \mathbf{T}$, and $\Pi_1 = \{A, C, B \cup D\} \cup \mathbf{T}$. By definition, $e_{\Pi_1}(\Gamma) = e_{\Pi_2}(\Gamma) = e_{\Pi_3}(\Gamma) = e_{\Pi_4}(\Gamma)$, so the coefficients of Γ in $\tilde{W}_n((AB|CD\mathbf{T}))$, $\tilde{W}_n((AB\mathbf{T}|CD))$, $\tilde{W}_n(-(AC|BD\mathbf{T}))$, and $\tilde{W}_n(-(AC\mathbf{T}|BD))$ cancel.

Altogether, we conclude that $\tilde{W}_n(\mathcal{R}) = 0$. This completes Case VI and the proof of the base case.

Next we prove the inductive step.

Proposition 3.5 *Suppose* $W_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ *descends* S_n-*equivariantly to* $(Q^2_{k,n})^b$ *for a fixed b and n. Then* $W_{n,b+1} : \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1}$ *descends* S_{n+1}-*equivariantly to* $(Q^2_{k,n+1})^{b+1}$. *In particular, if Lemma 3.2 holds for (b, n) for all n (and fixed b), then it holds for $(b + 1, n)$ for all n.*

Proof Suppose $W_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ descends equivariantly to $(Q^2_{k,n})^b$. Let π_{n+1} be the forgetful map that forgets the last point. Note that $(\pi_{n+1})_*((Q^2_{k,n+1})^{\geq b+1}) \subseteq (Q^2_{k,n})^{\geq b}$. (This holds because for any tree in $(\mathbf{S}^2_{k,n+1})^{\geq b+1}$, the corresponding boundary stratum is either contracted in dimension by π_{n+1}, or maps isomorphically to a boundary stratum in $(\mathbf{S}^2_{k,n})^{\geq b}$.) Thus there is an induced map on quotients

$$(\pi_{n+1})_* : (Q^2_{k,n+1})^{b+1} \to (Q^2_{k,n})^b.$$

By the inductive hypothesis, there is a map

$$\overline{W_{n,b}} : (Q^2_{k,n})^b \to (\mathbf{P}^2_{k,n})^b,$$

such that the lower-right triangle commutes in the following diagram:
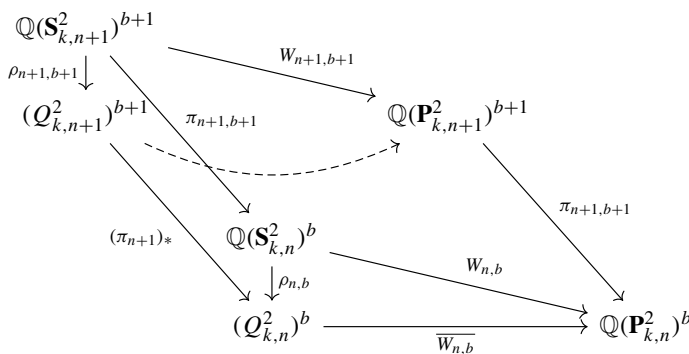
Here the map $\pi_{n+1,b+1} : \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{S}^2_{k,n})^b$ sends a tree σ to the tree obtained by contracting the $(n + 1)$-st marked half-edge and stabilizing (see e.g. [11]) if the $(n + 1)$-st marked half-edge is on σ' (using notation from Definition 2.6), and to zero otherwise. The map $\pi_{n+1,b+1} : \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ sends a pair $\{(P_1, \alpha_1), (P_2, \alpha_2)\}$ to itself if $n + 1 \in (P_1 \cup P_2)^C$, and to zero otherwise. The entire diagram is easily checked to commute, essentially using the fact that if $\sigma \in (\mathbf{S}^2_{k,n+1})^{b+1}$ is such that $n + 1 \in \sigma_1$, then

$(\pi_{n+1})_*(\rho_{n+1,b+1}(\sigma)) \in (Q^2_{k,n})^{\geq b+1}$. We are trying to prove the existence of the dashed arrow.

Let $r \in \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1}$ be such that $\rho_{n+1,b+1}(r) = 0$. Then certainly

$$\overline{W_{n,b}}((\pi_{n+1})_*(\rho_{n+1,b+1}(r))) = 0,$$

so by commutativity, $W_{n+1,b+1}(r)$ lies in

$$\ker(\pi_{n+1,b+1}) = \mathrm{Span}\{\{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n+1})^{b+1} : n+1 \in P_1 \cup P_2\}.$$

Then by S_{n+1}-symmetry, $W_{n+1,b+1}(r)$ actually lies in the smaller subspace

$$\mathrm{Span}\{\{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n+1})^{b+1} : P_1 \cup P_2 = \{1, \ldots, n+1\}\}.$$

Since $b + 1 > 0$, this subspace is trivial, so $W_{n+1,b+1}(r) = 0$. Thus $W_{n+1,b+1}$ descends to a (clearly S_{n+1}-equivariant) map $\overline{W_{n+1,b+1}} : (Q^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1}$. This completes the proof, and we conclude Lemma 3.2. □

We have just shown that $W_{n,b}$ descends to $(Q^2_{k,n})^b$. Next we must prove that $\rho_{n,b}$ descends to $(\mathbf{P}^2_{k,n})^b$. We will use the following well-known fact.

Proposition 3.6 *Let* $\sigma \in \mathbf{S}_{k,n}$, *and suppose* σ' *can be obtained from* σ *by rearranging a trivalent subtree. (That is, suppose there exist trivalent subtrees of* σ *and* σ' *whose complements are isomorphic as marked trees, and such that the two subtrees have the same marking set.) Then* σ *and* σ' *have the same class in* $H_{2k}(\overline{\mathcal{M}}_{0,n})$.

Sketch of proof Contracting the relevant trivalent subtrees, σ and σ' can both be written as the pushforward from the same product $\overline{\mathcal{M}}_{0,n_0} \times \overline{\mathcal{M}}_{0,n_1} \times \cdots \times \overline{\mathcal{M}}_{0,n_m}$, of the class $[\mathrm{pt}] \times 1 \times \cdots \times 1$, for different choices of pt $\in \overline{\mathcal{M}}_{0,n_0}$, the factor corresponding to the contracted subtrees. □

Lemma 3.7 *Fix nonnegative integers* $n \geq 4$, $k \leq n - 3$, *and* $b \leq n - k - 4$. *The map* $\rho_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to (Q^2_{k,n})^b$ *descends equivariantly to a map* $\overline{\rho_{n,b}} : \mathbb{Q}(\mathbf{P}^2_{k,n})^b \to (Q^2_{k,n})^b$.

Proof We must check that $\rho_{n,b}(\ker(W_{n,b})) = 0$. Note that as an induced map on free vector spaces, $W_{n,b}$ has kernel spanned by

$$\{\sigma - \sigma' : \sigma, \sigma' \in (S^2_{k,n})^b, W_{n,b}(\sigma) = W_{n,b}(\sigma')\}.$$

Thus it suffices to show that if $\sigma, \sigma' \in (S^2_{k,n})^b$ and $W_{n,b}(\sigma) = W_{n,b}(\sigma')$, then $\rho_{n,b}(\sigma) = \rho_{n,b}(\sigma')$. We consider two cases: $b = n - k - 4$ and $b < n - k - 4$.

First, suppose $b = n - k - 4$. Then for any $\sigma \in (S^2_{k,n})^{\geq b}$, the two vertices v_1 and v_2 are leaves, and $W_{n,b}(\sigma)$ determines σ up to rearrangement of the trivalent subtree obtained by deleting v_1 and v_2. Thus $\rho_{n,b}$ descends to $(\mathbf{P}^2_{k,n})^b$ by Proposition 3.6. Next, suppose $b < n - k - 4$. Let $\tau \in (S^2_{k,n})^b$. We will next define an algorithm ((8) below) for producing trees $\tau' \in (S^2_{k,n})^b$ such that $\rho_{n,b}(\tau') = \rho_{n,b}(\tau)$. Note that Proposition 3.6 is another such algorithm. We will then prove that if $\sigma, \sigma' \in (S^2_{k,n})^b$ and $W_{n,b}(\sigma) = W_{n,b}(\sigma')$, then σ' can be obtained from σ via these two algorithms.

Let $\tau \in (S^2_{k,n})^{\geq b}$. Since $b < n - k - 4$, at least one of v_1 and v_2 is not a leaf. Suppose without loss of generality that v_1 is not a leaf. Let e_1 be the edge incident to v_1 that is on the path from v_1 to v_2. Fix an edge $e \neq e_1$, and write $e = \{v_1, v'\}$. Let $\overline{\tau}$ be the tree obtained by contracting e to a vertex $v_1 = v'$. We consider Kontsevich-Manin relations

$\mathcal{R} = \mathcal{R}(\overline{\tau}, v_1 = v', A, B, C, e_1)$, where $A, B \neq e$ are incident to v' in τ, and $C \neq e$ is incident to v_1 in τ. All but four terms of $\mathcal{R}$ are in $\mathbf{S}_{k,n}^{\geq 3}$. The remaining terms are:

$$(AB\mathbf{T}|Ce_1) + (AB|Ce_1\mathbf{T}) - (AC\mathbf{T}|Be_1) + (AC|Be_1\mathbf{T})$$

where $\mathbf{T}$ is the set of edges and half-edges incident to $v_1 = v'$ other than A, B, C, e_1. Note that the first and third terms are in $(\mathbf{S}_{k,n}^2)^{\geq b+1}$, hence are zero in $(Q_{k,n}^2)^b$. We are left with the relation

$$(AB|Ce_1\mathbf{T}) = (AC|Be_1\mathbf{T}). \tag{8}$$

The left side is τ, and the right side is another element of $(\mathbf{S}_{k,n}^2)^b$.

Fix $P = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}_{k,n}^2)^b$. We must now show that if $\sigma, \sigma' \in (W_{n,b})^{-1}(P)$, then σ' can be obtained from σ via the two algorithms (8) and Proposition 3.6. We will instead show that any $\sigma \in (W_{n,b})^{-1}(P)$ can be manipulated via (8) into a standard form. That is, we will find $\sigma_0 \in (W_{n,b})^{-1}(P)$ such that for any $\sigma \in (W_{n,b})^{-1}(P)$, σ_0 can be obtained from σ via the two algorithms. Let σ_0 be as follows: On v_1, the first $\alpha_1 + 1$ elements of P_1 (in increasing order) are half-edges incident to v_1, and the remaining elements of P_1 form a trivalent subtree incident to v_1. Similarly for v_2, and the remaining points $(P_1 \cup P_2)^C$ of course form a trivalent subtree connecting v_1 and v_2. This defines σ_0 up to rearrangement of trivalent subtrees; this is sufficient in light of Proposition 3.6.

Fix $\sigma \in (W_{n,b})^{-1}(P)$. If the first $\alpha_1 + 1$ marked half-edges of P_1 are incident to v, then v_1 is already in the desired form. If not, let $e, \neq e_1$ be incident to v_1 such that subtree corresponding to $e = \{v_1, v'\}$ contains (at least) one of the first $\alpha_1 + 1$ marked half-edges. Using Proposition 3.6, rearrange the subtree so that this half-edge is incident to v', then apply (8), where B is the chosen half-edge, and C is any edge or half-edge incident to v_1 that is not equal to e, nor to any of the first $\alpha_1 + 1$ marked half-edges. (Such a flag must exist, since there are $\alpha_1 + 1$ edges and half-edges other than e and e_1, and we have assumed that one of the first $\alpha_1 + 1$ marked half-edges is on the subtree corresponding to e.) In the resulting tree, the number of the first $\alpha_1 + 1$ marked half-edges incident to v_1 has increased by 1. We repeat until this number is $\alpha_1 + 1$, then apply the same argument to v_2. The result is σ_0; this completes the proof.

4 A conjectural dimension formula

We end with a conjectural formula for the dimension of $Q_{k,n}^r$.

Conjecture 4.1 *The dimension of $Q_{k,n}^r$ is*

$$\sum_{\substack{\alpha_1,\ldots,\alpha_r \geq 1 \\ \alpha_1+\cdots+\alpha_r=k}} \sum_{\substack{p_1,\ldots,p_r,b \geq 0 \\ p_1+\cdots+p_r+b=n+r-2 \\ p_i \geq \alpha_i+2}} \frac{1}{r!}\binom{n+r-2}{p_1,\ldots,p_r,b}.$$

The case $r = 1$ appears in [15]. The case $r = 2$ follows from Theorem 2.7, since the formula gives the cardinality of $\mathbf{P}_{k,n}^2$. Summing over $r \in \{1, \ldots, \min\{k, n-2-k\}\}$ gives a conjectural closed formula for the dimension of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. This formula agrees with the actual dimension for $n \leq 20$ and all k, but we have not been able to find it in the literature. (It is not obvious to us whether or not the formula is recoverable from e.g. [18].)

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