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A p-specific spectral multiplier theorem with sharp regularity bound for Grushin operators

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Abstract

In a recent work, Chen and Ouhabaz proved a p-specific L^p-spectral multiplier theorem for the Grushin operator acting on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ which is given by

$$L = -\sum_{j=1}^{d_1} \partial_{x_j}^2 - \left(\sum_{j=1}^{d_1} |x_j|^2\right) \sum_{k=1}^{d_2} \partial_{y_k}^2.$$

Their approach yields an L^p-spectral multiplier theorem within the range $1 < p \le \min\{2d_1/(d_1+2), 2(d_2+1)/(d_2+3)\}$ under a regularity condition on the multiplier which is sharp only when $d_1 \ge d_2$. In this paper, we improve on this result by proving L^p-boundedness under the expected sharp regularity condition $s > (d_1+d_2)(1/p-1/2)$. Our approach avoids the usage of weighted restriction type estimates which played a key role in the work of Chen and Ouhabaz, and is rather based on a careful analysis of the underlying sub-Riemannian geometry and restriction type estimates where the multiplier is truncated along the spectrum of the Laplacian on $\mathbb{R}^{d_2}$.

Keywords Grushin operator · Spectral multiplier · Mikhlin–Hörmander multiplier · Bochner–Riesz mean · Restriction type estimate

Mathematics Subject Classification Primary 43A85 · 42B15; Secondary 47F05

1 Introduction

Let L be a positive self-adjoint linear differential operator on $L^2(M)$, where M is a smooth d-dimensional manifold endowed with a smooth positive measure μ. If E denotes the spectral measure of L, we can define for every Borel measurable function $F : \mathbb{R} \to \mathbb{C}$ the (possibly unbounded) operator

$$F(L) = \int_0^\infty F(\lambda)\, dE(\lambda).$$

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By the spectral theorem, $F(L)$ is a bounded operator on $L^2(M)$ if and only if the spectral multiplier F is E-essentially bounded. The L^p-*spectral multiplier problem* asks for identifying multipliers F for which $F(L)$ extends from $L^2(M) \cap L^p(M)$ to a bounded operator $F(L) : L^p(M) \to L^p(M)$.

For instance, in the case of the Laplacian $L = -\Delta$ on $\mathbb{R}^d$, the celebrated Mikhlin–Hörmander multiplier theorem [12] provides the following sufficient condition for the question of L^p-boundedness: The operator $F(-\Delta)$ is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ whenever $F : \mathbb{R} \to \mathbb{C}$ satisfies the regularity condition

$$\|F\|_{\mathrm{sloc},s} := \sup_{t>0} \|\eta F(t\,\cdot\,)\|_{L^2_s(\mathbb{R})} < \infty \quad \text{for some } s > d/2.$$

Here $\eta : \mathbb{R} \to \mathbb{C}$ shall denote some generic nonzero bump function supported in $(0, \infty)$, while $L^2_s(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is the Sobolev space of (fractional) order $s \in \mathbb{R}$. In the case $p = 1$, the operator $F(-\Delta)$ is of weak type $(1, 1)$, i.e., bounded as an operator between $L^1(\mathbb{R}^d)$ and the Lorentz space $L^{1,\infty}(\mathbb{R}^d)$. The threshold $d/2$ of the order s is optimal and cannot be decreased.

A lot of attention has been paid to the question whether an analogous result of the Mikhlin–Hörmander multiplier theorem holds true for more general classes of (sub)-elliptic differential operators, most notably *sub-Laplacians*. For left-invariant sub-Laplacians on Carnot groups, Christ [7], and Mauceri and Meda [23] showed that $F(L)$ extends to a bounded operator on all L^p-spaces for $1 < p < \infty$ and is of weak type $(1, 1)$ whenever

$$\|F\|_{\mathrm{sloc},s} < \infty \quad \text{for some } s > Q/2,$$

where Q is the so-called *homogeneous dimension* of the underlying Carnot group. It came therefore as a surprise when Müller and Stein [28], and independently Hebisch [11], discovered in the early nineties that in the case of Heisenberg (-type) groups the threshold $s > Q/2$ can be even pushed down to $s > d/2$, with d being the *topological dimension* of the underlying group. The question whether this holds true for any sub-Laplacian L is still open, although there has been extensive research on this problem and many partial results are available, including, e.g., sub-Laplacians on all 2-step stratified Lie groups of dimension ≤ 7 [20], certain classes of 2-step stratified Lie groups of higher dimension [18], Grushin operators [19], as well as various classes of compact sub-Riemannian manifolds [1, 4, 8, 9]. So far a counterexample requiring the threshold to be larger than $d/2$ is not known.

A refinement of asking for boundedness on all L^p-spaces for $1 < p < \infty$ simultaneously is the question which order of differentiability s is needed if p is given (*p-specific L^p-spectral multiplier estimates*). Again in the case of the Laplacian $L = -\Delta$, it is by now well-known (see [17, Theorem 1.4] for instance) that if $1 < p \leq 2(d+1)/(d+3)$ and if $F : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function satisfying

$$\|F\|_{\mathrm{sloc},s} < \infty \quad \text{for some } s > \max\{d\,|1/p - 1/2|\,, 1/2\},$$

then the operator $F(-\Delta)$ is bounded on $L^p(\mathbb{R}^d)$. The condition on the range of p derives from the celebrated Stein–Tomas Fourier restriction theorem [34] which is used for the proof of this result. It is an open problem (the famous *Bochner–Riesz conjecture*, cf. [2, 3, 10, 30, 32]) in the case of Bochner–Riesz means (where $F = (1 - |\cdot|)^\delta_+$, $\delta > 0$) whether the operators $(1 + \Delta)^\delta_+$ are bounded on $L^p(\mathbb{R}^d)$ whenever $\delta > \max\{d\,|1/p - 1/2| - 1/2, 0\}$.

Regarding *p-specific L^p*-spectral multiplier theorems for sub-Laplacians in more general settings, much fewer results featuring the topological dimension d are available so far. However, in [21], Martini et al. showed for a large class of smooth second-order real differential operators associated to a sub-Riemannian structure on smooth d-dimensional manifolds that regularity of order $s \geq d\,|1/p - 1/2|$ is necessary for having L^p-spectral multiplier estimates. In particular, this result applies to all sub-Laplacians on Carnot groups, and Grushin operators, which are the subject of the present paper.

Quite recently, Chen and Ouhabaz [5] proved a partial result for a p-specific L^p-spectral multiplier estimate in the case of the Grushin operator L acting on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1, d_2 \geq 1$, given by

$$L = -\sum_{j=1}^{d_1} \partial_{x_j}^2 - \Bigg(\sum_{j=1}^{d_1} |x_j|^2\Bigg) \sum_{k=1}^{d_2} \partial_{y_k}^2 = -\Delta_x - |x|^2 \Delta_y.$$

Here $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$ shall denote the two layers of a given point in $\mathbb{R}^d$, while Δ_x, Δ_y are the corresponding partial Laplacians, and $|x|$ is the Euclidean norm of x. The Grushin operator is positive, self-adjoint, and hypoelliptic according to a celebrated theorem by Hörmander [13], but not elliptic on the plane $x = 0$. In [5], it is proved that $F(L)$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ whenever

$$\|F\|_{\mathrm{sloc},s} < \infty \quad \text{for some } s > D(1/p - 1/2),$$

where $D := \max\{d_1 + d_2, 2d_2\}$ and $1 < p \leq p_{d_1,d_2}$, with

$$p_{d_1,d_2} := \min\left\{\frac{2d_1}{d_1+2}, \frac{2(d_2+1)}{d_2+3}\right\}. \tag{1.1}$$

As suspected by Chen and Ouhabaz in [5], one might expect that this result holds true with D being replaced by the topological dimension $d = d_1 + d_2$. However, their result yields the optimal threshold at least if $d_1 \geq d_2$.

A similar phenomenon as in [5] had already occurred earlier in [22], where Martini and Sikora proved a Mikhlin–Hörmander type result for the Grushin operator L with threshold $s > D/2$, which was later improved in [19] by Martini and Müller to hold for the topological dimension d in place of D. The approaches of [22] and [19] rely both on weighted Plancherel estimates for the integral kernels of $F(L)$, which are derived by pointwise estimates for Hermite functions. In [22], the employed weights are given by $w_\gamma(x, y) = |x|^\gamma$, $\gamma > 0$. In principle, the arguments work out for $\gamma < d_2/2$, but unfortunately, it is necessary to take an integral over the weight $|x|^\gamma$ at some point, which forces $\gamma < d_1/2$, which in turn yields $s > D/2$ in place of $s > d/2$ as a threshold. In [19], Martini and Müller employ the weights $w_\gamma(x, y) = |y|^\gamma$ in the second layer y, together with a rescaling factor in the first layer. Using the weights $|y|^\gamma$ does only force $\gamma < d_2/2$ when taking the integral over the weight, whence this approach provides the optimal threshold $s > d/2$. However, the weights $|y|^\gamma$ are harder to handle since a sub-elliptic estimate, which goes back to Hebisch [11], is not applicable for these weights.

The proof of Chen and Ouhabaz relies on weighted restriction type estimates using $|x|^\gamma$ as a weight. Similar to [22], they employ Hebisch's sub-elliptic estimate and have to take an integral over the weight $|x|^\gamma$ which forces $\gamma < d_1(1/p - 1/2)$, and in turn yields $s > D(1/p - 1/2)$ in place of $s > d(1/p - 1/2)$ as a threshold.

In this paper, we improve the result of [5] and prove a p-specific spectral multiplier estimate with optimal threshold for s. Similar as in [5], we also prove a corresponding result for Bochner–Riesz multipliers. Note that Theorem 1.1 only provides results if $d_1 \geq 3$ and $d_2 \geq 2$, and Theorem 1.2 if $d_1 \geq 2$ and $d_2 \geq 1$.

Theorem 1.1 *Let $1 < p \le p_{d_1,d_2}$. Suppose that $F : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function such that*

$$\|F\|_{\mathrm{sloc},s} < \infty \quad \textit{for some } s > (d_1 + d_2)(1/p - 1/2).$$

Then the operator $F(L)$ is bounded on $L^p(\mathbb{R}^d)$, and

$$\|F(L)\|_{L^p \to L^p} \le C_{p,s} \|F\|_{\mathrm{sloc},s}.$$

Theorem 1.2 *Let $1 \le p \le p_{d_1,d_2}$. Suppose that $\delta > (d_1 + d_2)(1/p - 1/2) - 1/2$. Then the Bochner–Riesz means $(1 - tL)_+^\delta$ are bounded on $L^p(\mathbb{R}^d)$ uniformly in $t \in [0, \infty)$.*

Our strategy when reaching for the optimal threshold $s > d(1/p - 1/2)$ is to follow the approach by Chen and Ouhabaz, but instead of showing *weighted* restriction type estimates, we prove restriction type estimates where the operator $F(L)$ is additionally truncated along the spectrum of the Laplacian on $\mathbb{R}^{d_2}$. On a heuristic level, this key idea may be illustrated as follows: Via Fourier transform in the second component, the study of the operator L translates into studying the family of operators $-\Delta_x + |x|^2|\eta|^2$, $\eta \in \mathbb{R}^{d_2}$, on $L^2(\mathbb{R}^{d_1})$. For fixed $\eta \in \mathbb{R}^{d_2}$, this operator is a rescaled version of the Hermite operator, and has discrete spectrum consisting of the eigenvalues $[k]|\eta|$, where $[k] = 2k + d_1$ and $k \in \mathbb{N}$. Moreover, the operator $T := (-\Delta_y)^{1/2}$ translates into the multiplication operator $|\eta|$ via Fourier transform in the second component. The operators L and T admit a joint functional calculus, and since $[k]|\eta|/|\eta| = [k]$, multiplication with the operator $\chi_k(L/T)$ (where $\chi_k : \mathbb{R} \to \mathbb{C}$ shall denote the indicator function of $\{2k + d_1\}$) corresponds to picking the k-th eigenvalue on $L^2(\mathbb{R}^{d_1})$ for every $\eta \in \mathbb{R}^{d_2}$ simultaneously. This is an observation that has been already been exploited earlier, for instance in [19, Lemma 11], and in [26, 27]. Since $r \sim [k]^{-1}$ on the support of a joint multiplier $F(\lambda)\chi_k(\lambda/r)$ whenever F is compactly supported away from the origin, the multiplication of an operator $F(L)$ by $\chi_k(L/T)$ is referred to as a *truncation along the spectrum of* T in the following. The benefit of this truncation is as follows: Since L and T admit a joint functional calculus, we have

$$F(L)\chi_k(L/T) = F([k]T)\chi_k(L/T).$$

Thus for every $k \in \mathbb{N}$, we may replace the operator L by the Laplacian in the second layer $y \in \mathbb{R}^{d_2}$, whence one might hope that on each "eigenspace" associated to k the underlying sub-Riemannian geometry behaves Euclidean up to a scaling by k in the second layer. In the proofs of Theorem 1.1 and Theorem 1.2, we will take advantage of this perspective in the case where $k \in \mathbb{N}$ is small.

This article is organized as follows: in Sect. 2, we recall the main facts concerning the sub-Riemannian geometry that is naturally associated to the Grushin operator L. In Sect. 3, we recall the essentials of the joint functional calculus of L and T and prove the truncated restriction type estimates mentioned above. Section 4 is devoted to the proofs of Theorem 1.1 and Theorem 1.2, where also a closer analysis of the underlying sub-Riemannian geometry takes place.

Finally, we briefly fix our notation. For us, zero shall be contained in the set of all natural numbers $\mathbb{N}$. The space of (equivalence classes of) integrable simple functions on $\mathbb{R}^n$ will be denoted by $D(\mathbb{R}^n)$, while $\mathcal{S}(\mathbb{R}^n)$ shall denote the space of Schwartz functions on $\mathbb{R}^n$. The indicator function of a subset $A \subseteq \mathbb{R}^n$ will be denoted by χ_A. For a function $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx, \quad \xi \in \mathbb{R}^n,$$

while the inverse Fourier transform $\check{f}$ is given by

$$\check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi) e^{ix\xi}\, d\xi, \quad x \in \mathbb{R}^n.$$

Constants may vary from line to line, but they will be occasionally denoted by the same letter. We write $A \lesssim B$ if $A \leq CB$ for a constant C. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. Moreover, we fix the following dyadic decomposition throughout this article. Let $\chi : \mathbb{R} \to [0, 1]$ be an even bump function supported in $[-2, -1/2] \cup [1/2, 2]$ such that

$$\sum_{j \in \mathbb{Z}} \chi_j(\lambda) = 1 \quad \text{for } \lambda \neq 0,$$

where χ_j is given by

$$\chi_j(\lambda) := \chi(\lambda/2^j) \quad \text{for } j \in \mathbb{Z}. \tag{1.2}$$

With this setup, we have in particular $|\lambda| \sim 2^j$ for all $\lambda \in \operatorname{supp} \chi_j$.

2 The sub-Riemannian geometry of the Grushin operator

Let ϱ denote the Carnot–Carathéodory distance associated to the Grushin operator L, i.e., for $z, w \in \mathbb{R}^d$, the distance $\varrho(z, w)$ is given by the infimum over all lengths of horizontal curves $\gamma : [0, 1] \to \mathbb{R}^d$ joining z with w (cf. Section III.4 of [35]). Due to the Chow–Rashevkii theorem (cf. Proposition III.4.1 in [35]), ϱ is indeed a metric on $\mathbb{R}^d$, which induces the Euclidean topology on $\mathbb{R}^d$. In our setting, the Carnot–Carathéodory distance possesses the following characterization (cf. Proposition 3.1 in [14]): If $z, w \in \mathbb{R}^d$, then

$$\varrho(z, w) = \sup_{\psi \in \Lambda} (\psi(z) - \psi(w)), \tag{2.1}$$

where Λ denotes the set of all locally Lipschitz continuous functions $\psi : \mathbb{R}^d \to \mathbb{R}$ such that

$$|\nabla_x \psi(x, y)|^2 + |x|^2 |\nabla_y \psi(x, y)|^2 \leq 1 \quad \text{for a.e. } (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \tag{2.2}$$

In the following, let $B_R^{\varrho}(a, b)$ denote the ball of radius $R \geq 0$ centered at $(a, b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with respect to the distance ϱ. The following statement summarizes the main properties of the sub-Riemannian geometry associated to L that we need later.

Proposition 2.1 *The following statements hold:*

(1) *For all* $(x, y), (a, b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$$\varrho((x, y), (a, b)) \sim |x - a| + \begin{cases} \frac{|y-b|}{|x|+|a|} & \text{if } |y - b|^{1/2} < |x| + |a|, \\ |y - b|^{1/2} & \text{if } |y - b|^{1/2} \geq |x| + |a|. \end{cases}$$

(2) *For all* $(a, b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ *and* $R > 0$,

$$|B_R^{\varrho}(a, b)| \sim R^{d_1 + d_2} \max\{R, |a|\}^{d_2}.$$

(3) *There is a constant* $C > 0$ *such that for all* $(a, b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$$B_R^{\varrho}(a, b) \subseteq B_R^{|\cdot|}(a) \times B_{CR^2}^{|\cdot|}(b) \quad \text{whenever } R \geq |a|/4.$$

(4) *Let* $\delta_t(x,y) := (tx, t^2 y)$ *for* $(x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. *Then*

$$\varrho(\delta_t(x,y), \delta_t(a,b)) = t\varrho((x,y),(a,b)).$$

(5) *L possesses the finite propagation speed property with respect to* ϱ, *i.e., whenever* $f, g \in L^2(\mathbb{R}^d)$ *are supported in open subsets* $U, V \subseteq \mathbb{R}^d$ *and* $|t| < \varrho(U,V)$, *then*

$$(\cos(t\sqrt{L})f, g) = 0.$$

Proof The estimates in (1) and (2) are part of Proposition 5.1 in [29]. The inclusion in (3) is a consequence of (1): Since the function ψ defined by $\psi(x,y) := |x-a|$ satisfies (2.2), the characterization (2.1) yields

$$|x-a| \le \varrho((x,y),(a,b)).$$

Thus, if we suppose $(x,y) \in B_R^{\varrho}(a,b)$ for $R \ge |a|/4$, then the inequality above implies $x \in B_R^{|\cdot|}(a)$, and $|x| \le |x-a| + |a| < 5R$. Moreover, if $|y-b|^{1/2} < |x| + |a|$, then (1) yields

$$|y-b| \lesssim (|x|+|a|)\varrho((x,y),(a,b)) < 9R^2,$$

and if $|y-b|^{1/2} \ge |x| + |a|$, then $|y-b| \lesssim \varrho((x,y),(a,b))^2 < R^2$, which proves (3). The scaling invariance in (4) is an immediate consequence of the characterization (2.1). For the finite propagation speed property, see Proposition 4.1 of [29], or alternatively the approach of Melrose in [24, Proposition 3.4]. □

The finite propagation speed property will be of fundamental importance in the proofs of Theorem 1.1 and Theorem 1.2. Moreover, note that the volume estimate in part (2) of Proposition 2.1 yields in particular that the metric measure space $(\mathbb{R}^d, \varrho, |\cdot|)$ (with $|\cdot|$ denoting the Lebesgue measure) is a space of homogeneous type with homogeneous dimension $Q = d_1 + 2d_2$.

3 Truncated restriction type estimates

In this section, we prove restriction type estimates where the multiplier is additionally truncated along the spectrum of the Laplacian on $\mathbb{R}^{d_2}$. As in [5], the idea is to apply a discrete restriction estimate in the variable $x \in \mathbb{R}^{d_1}$ and the classical Stein–Tomas restriction estimate in $y \in \mathbb{R}^{d_2}$. Due to the conditions $1 \le p \le 2d_1/(d_1+2)$ and $1 \le p \le 2(d_2+1)/(d_2+3)$ in the corresponding restriction (type) estimates, we have to assume $1 \le p \le p_{d_1,d_2}$ in Theorem 3.4 (with p_{d_1,d_2} being defined as in (1.1)).

We first discuss the spectral decomposition of the Grushin operator L. Let $\mathcal{F}_2 : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denote the Fourier transform in the second component, i.e.,

$$\mathcal{F}_2 f(x,\eta) = \int_{\mathbb{R}^{d_2}} f(x,y) e^{-i\eta y}\, dy, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

We will also write $f^\eta(x) = \mathcal{F}_2 f(x,\eta)$ in the following. Then

$$(Lf)^\eta = (-\Delta_x + |x|^2|\eta|^2) f^\eta.$$

For fixed $\eta \in \mathbb{R}^{d_2} \setminus \{0\}$, the operator

$$L^\eta := -\Delta_x + |x|^2|\eta|^2 \quad \text{on } L^2(\mathbb{R}^{d_1})$$

is a rescaled version of the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^{d_1}$. It is well-known [33, Section 1.1] that H has discrete spectrum consisting of the eigenvalues

$$[k] := 2k + d_1, \quad k \in \mathbb{N}.$$

For a multiindex $\nu \in \mathbb{N}^{d_1}$, let Φ_ν denote the ν-th Hermite function on $\mathbb{R}^{d_1}$, i.e.,

$$\Phi_\nu(x) := \prod_{j=1}^{d_1} h_{\nu_j}(x_j), \quad x \in \mathbb{R}^{d_1},$$

where, for $\ell \in \mathbb{N}$, h_ℓ shall denote the ℓ-th Hermite function on $\mathbb{R}$, i.e.,

$$h_\ell(u) := (-1)^\ell (2^\ell \ell! \sqrt{\pi})^{-1/2} e^{u^2/2} \Big(\frac{d}{du}\Big)^\ell (e^{-u^2}), \quad u \in \mathbb{R}.$$

The Hermite functions Φ_ν form an orthonormal basis of $L^2(\mathbb{R}^{d_1})$ and are eigenfunctions of the Hermite operator H since $H\Phi_\nu = (2|\nu|_1 + d_1)\Phi_\nu$, where $|\nu|_1 = \nu_1 + \ldots + \nu_{d_1}$ denotes the length of the multiindex $\nu \in \mathbb{N}^{d_1}$.

Furthermore, for $\eta \in \mathbb{R}^{d_2} \setminus \{0\}$, let Φ_ν^η be given by

$$\Phi_\nu^\eta(x) := |\eta|^{d_1/4} \Phi_\nu(|\eta|^{1/2} x), \quad x \in \mathbb{R}^{d_1}.$$

Then the functions Φ_ν^η form an orthonormal basis of $L^2(\mathbb{R}^{d_1})$ and are eigenfunctions of the operator L^η since $L^\eta \Phi_\nu^\eta = (2|\nu|_1 + d_1)|\eta|\Phi_\nu^\eta$. Thus the projection P_k^η onto the eigenspace associated to the eigenvalue $[k]|\eta|$ of L^η is given by

$$P_k^\eta g = \sum_{|\nu|_1 = k} (g, \Phi_\nu^\eta)\Phi_\nu^\eta, \quad g \in L^2(\mathbb{R}^{d_1}).$$

In particular, the projection P_k^η possesses an integral kernel $\mathcal{K}_k^\eta$ which is given by

$$\mathcal{K}_k^\eta(x, a) = \sum_{|\nu|_1 = k} \Phi_\nu^\eta(x)\Phi_\nu^\eta(a), \quad x, a \in \mathbb{R}^{d_1}. \tag{3.1}$$

Moreover, let L_j and T_k be the differential operators given by

$$L_j = (-i\partial_{x_j})^2 + |x_j|^2 \sum_{k=1}^{d_2} (-i\partial_{y_k})^2, \qquad T_k = -i\partial_{y_k}.$$

Then the Grushin operator L is equal to the sum $L_1 + \cdots + L_{d_1}$. As shown in [22], the operators $L_1, \ldots, L_{d_1}, T_1, \ldots, T_{d_2}$ have a joint functional calculus which can be explicitly written down in terms of the Fourier transform and Hermite function expansion. In particular, the operators L and $T = (|T_1|^2 + \cdots + |T_{d_2}|^2)^{1/2} = (-\Delta_y)^{1/2}$ have a joint functional calculus, so we can define the operators $G(L, T)$ for every Borel function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.

Lemma 3.1 *For all bounded Borel functions $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,*

$$(G(L, T) f)^\eta(x) = G(L^\eta, |\eta|) f^\eta(x)$$

for all $f \in L^2(\mathbb{R}^d)$ and almost all $(x, \eta) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Moreover, if G is additionally compactly supported in $\mathbb{R} \times (\mathbb{R}\setminus\{0\})$, the operator $G(L, T)$ possesses an integral kernel $\mathcal{K}_{G(L,T)}$, which is given by

$$\mathcal{K}_{G(L,T)}((x, y), (a, b)) = (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} \sum_{k=0}^{\infty} G\big([k]|\eta|, |\eta|\big) \mathcal{K}_k^\eta(x, a) e^{i(y-b)\eta} \, d\eta$$

for almost all $(x, y), (a, b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Proof See Proposition 5 of [22], and its proof. □

The proof of the truncated restriction type estimates for the Grushin operator relies on the following restriction type estimate for L^η.

Proposition 3.2 *Let $1 \le p \le 2d_1/(d_1+2)$. Then, for all $g \in D(\mathbb{R}^{d_1})$ and $\eta \in \mathbb{R}^{d_2} \setminus \{0\}$,*

$$\|P_k^\eta g\|_2 \le C_p |\eta|^{\frac{d_1}{2}(1/p-1/2)} [k]^{\frac{d_1}{2}(1/p-1/2)-1/2} \|g\|_p.$$

Proof Via substitution, the proof of the estimate can be reduced to the case where $|\eta| = 1$ (cf. Proposition 3.2 of [5]). For the case $|\eta| = 1$, see Corollary 3.2 of [16]. Alternatively, for $1 \le p < 2d_1/(d_1+2)$, this result can also be found in [15, Theorem 3] and [6, Proposition II.8] (in conjunction with Mehler's formula). □

Another ingredient for the proof of the restriction type estimates are pointwise estimates for Hermite functions. In the following, we let

$$H_k^\eta(x) := \mathcal{K}_k^\eta(x,x), \quad x \in \mathbb{R}^{d_1}. \tag{3.2}$$

Lemma 3.3 *If $d_1 \ge 2$, then, for all $k \in \mathbb{N}$ and $\eta \in \mathbb{R}^{d_2} \setminus \{0\}$,*

$$H_k^\eta(x) \le \begin{cases} C|\eta|^{d_1/2}[k]^{d_1/2-1} & \text{for all } x \in \mathbb{R}^{d_1}, \\ C|\eta|^{d_1/2}\exp(-c|\eta||x|_\infty^2) & \text{when } |\eta||x|_\infty^2 \ge 2[k]. \end{cases}$$

Proof See [22, Lemma 8] and the references therein. □

Now we state the restriction type estimates of the Grushin operator L. The new feature in comparison to [5] is the truncation along the spectrum of T instead of employing weights in the restriction type estimates. Let ϱ denote again the Carnot–Carathéodory distance associated to L.

Theorem 3.4 *Let $1 \le p \le p_{d_1,d_2}$. Suppose that $F : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function supported in $[1/8, 8]$. For $\ell \in \mathbb{N}$, let $G_\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be given by*

$$G_\ell(\lambda, r) = F(\sqrt{\lambda})\chi_\ell(\lambda/r) \quad \text{for } \lambda \ge 0, r \ne 0$$

and $G_\ell(\lambda, r) = 0$ else, where χ_ℓ is defined via (1.2). Then

$$\|G_\ell(L,T)\|_{p\to 2} \le C_p 2^{-\ell d_2(1/p-1/2)} \|F\|_2. \tag{3.3}$$

In particular, for $\iota \in \mathbb{N}$,

$$\Big\| \sum_{\ell > \iota} G_\ell(L,T) \Big\|_{p\to 2} \le C_p 2^{-\iota d_2(1/p-1/2)} \|F\|_2. \tag{3.4}$$

Moreover, for $(a,b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $0 < R < |a|/4$,

$$\|F(\sqrt{L})\chi_{B_R^\varrho(a,b)}\|_{p\to 2} \le C_p |a|^{-d_2(1/p-1/2)} \|F\|_2. \tag{3.5}$$

Remark 3.5 By Lemma 3.1, we have

$$(G_\ell(L,T)f)^\eta = \sum_{k=0}^\infty F(\sqrt{[k]|\eta|})\chi_\ell([k]) P_k^\eta f^\eta$$

for almost all $\eta \in \mathbb{R}^{d_2}$. Note that $d_1 \geq 2$ due to the assumption on the range of p. Thus $\chi_j([k]) = 0$ for all $j \leq 0$ and $k \in \mathbb{N}$, whence

$$\sum_{\ell=1}^{\infty} G_\ell(L, T) f = F(\sqrt{L}) f.$$

Proof We first prove (3.3). Note that (3.4) is a direct consequence of (3.3) since

$$\Big\| \sum_{\ell > \iota} G_\ell(L, T) \Big\|_{p \to 2} \leq \sum_{\ell > \iota} \|G_\ell(L, T)\|_{p \to 2}.$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$. In the following, let $g_k^\eta := F(\sqrt{[k]|\eta|}) f^\eta$ for $\eta \in \mathbb{R}^{d_2}$ and $k \in \mathbb{N}$. Using Plancherel's theorem, Lemma 3.1, and orthogonality in $L^2(\mathbb{R}^{d_1})$, we obtain

$$\begin{aligned}
\|G_\ell(L, T) f\|_{L^2(\mathbb{R}^d)}^2 &\sim \|G_\ell(L^\eta, |\eta|) f^\eta\|_{L^2(\mathbb{R}^{d_1} \times \mathbb{R}_\eta^{d_2})}^2 \\
&= \Big\| \sum_{k=0}^{\infty} F(\sqrt{[k]|\eta|}) \chi_\ell([k]) P_k^\eta f^\eta \Big\|_{L^2(\mathbb{R}^{d_1} \times \mathbb{R}_\eta^{d_2})}^2 \\
&= \sum_{k=0}^{\infty} \chi_\ell([k])^2 \|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1} \times \mathbb{R}_\eta^{d_2})}^2. \qquad (3.6)
\end{aligned}$$

The restriction type estimate of Proposition 3.2 provides the estimate

$$\begin{aligned}
\|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1})} &\lesssim |\eta|^{\frac{d_1}{2}(1/p - 1/2)} [k]^{\frac{d_1}{2}(1/p-1/2)-1/2} \|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})} \\
&\sim [k]^{-1/2} \|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})} \qquad (3.7)
\end{aligned}$$

since $[k]|\eta| \sim 1$ whenever $[k]|\eta| \in \operatorname{supp} F$. Moreover, Minkowski's integral inequality yields

$$\| \|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})} \|_{L^2(\mathbb{R}_\eta^{d_2})} \leq \| \|g_k^\eta(x)\|_{L^2(\mathbb{R}_\eta^{d_2})} \|_{L^p(\mathbb{R}_x^{d_1})}. \qquad (3.8)$$

Let $f_x := f(x, \cdot)$ and $\widehat{\cdot}$ denote the Fourier transform on $\mathbb{R}^{d_2}$. Using polar coordinates and applying the classical Stein–Tomas restriction estimate [34] yields

$$\begin{aligned}
\|g_k^\eta(x)\|_{L^2(\mathbb{R}_\eta^{d_2})}^2 &= \int_0^\infty \int_{S^{d_2-1}} |F(\sqrt{[k]r}) \widehat{f_x}(r\omega)|^2 r^{d_2-1} \, d\sigma(\omega) \, dr \\
&= \int_0^\infty |F(\sqrt{[k]r})|^2 r^{-d_2-1} \int_{S^{d_2-1}} \big| \big(f_x(r^{-1} \cdot)\big)^\wedge(\omega) \big|^2 \, d\sigma(\omega) \, dr \\
&\lesssim \int_0^\infty |F(\sqrt{[k]r})|^2 r^{-d_2-1} \|f_x(r^{-1} \cdot)\|_{L^p(\mathbb{R}^{d_2})}^2 \, dr \\
&= \int_0^\infty |F(\sqrt{[k]r})|^2 r^{2d_2(1/p-1/2)-1} \, dr \, \|f_x\|_{L^p(\mathbb{R}^{d_2})}^2 \\
&\sim [k]^{-2d_2(1/p-1/2)} \int_0^\infty |F(\sqrt{r})|^2 \, dr \, \|f_x\|_{L^p(\mathbb{R}^{d_2})}^2.
\end{aligned}$$

Substituting $r \mapsto r^2$, we obtain, together with (3.8),

$$\| \|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})} \|_{L^2(\mathbb{R}_\eta^{d_2})} \lesssim [k]^{-d_2(1/p-1/2)} \|F\|_2 \|f\|_p. \qquad (3.9)$$

Together with (3.7), we get

$$\|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}_\eta)} \lesssim [k]^{-d_2(1/p-1/2)-1/2}\|F\|_2\|f\|_p. \tag{3.10}$$

Hence, in conjunction with (3.6), we finally get

$$\begin{aligned}\|G_\ell(L,T)f\|^2_{L^2(\mathbb{R}^d)} &\lesssim \sum_{[k]\sim 2^\ell}[k]^{-2d_2(1/p-1/2)-1}\|F\|_2^2\|f\|_p^2\\ &\lesssim 2^{-2\ell d_2(1/p-1/2)}\|F\|_2^2\|f\|_p^2.\end{aligned}$$

This proves (3.3).

Now we prove (3.5). Suppose that f is supported in $B_R^\varrho(a,b)$. Applying (3.4) for $\iota=0$, we obtain

$$\|F(\sqrt{L})f\|_2 \lesssim \|F\|_2\|f\|_p.$$

Hence we can assume $|a|>1$ without loss of generality. As before, let $g_k^\eta = F(\sqrt{[k]|\eta|})f^\eta$. The same arguments as in (3.6) show that

$$\|F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^d)} \sim \sum_{k=0}^\infty \|P_k^\eta g_k^\eta\|^2_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}_\eta)}.$$

We split the sum over k in two parts, one part where $[k]\ge\gamma|a|$, and another part where $[k]<\gamma|a|$. The constant $\gamma>0$ will be chosen later sufficiently small.

Case 1: For those $k\in\mathbb{N}$ satisfying $[k]\ge\gamma|a|$, we use estimate (3.10) from before, and we are done since

$$\sum_{[k]\ge\gamma|a|}[k]^{-2d_2(1/p-1/2)-1} \lesssim_\gamma |a|^{-2d_2(1/p-1/2)}.$$

Case 2: For $[k]<\gamma|a|$, we replace the restriction type estimate of Proposition 3.2 by an estimation that uses Hölder's inequality and the pointwise estimates for Hermite functions provided by Lemma 3.3. (Note that we have assumed $d_1\ge 2$ by choosing $1\le p\le p_{d_1,d_2}$.) For the component $y\in\mathbb{R}^{d_2}$, we use the Stein–Tomas restriction estimate in the same way as before.

Fix $k\in\mathbb{N}$ with $[k]<\gamma|a|$. By Proposition 2.1 (3), g_k^η is supported in $B_R^{|\cdot|}(a)$ since f is supported in $B_R^\varrho(a,b)$. Recall that the projection P_k^η onto the eigenspace associated to the eigenvalue $[k]|\eta|$ possesses the integral kernel $\mathcal{K}_k^\eta$ given by (3.1). Using Hölder's inequality, we obtain

$$|P_k^\eta g_k^\eta(x)| \le \|\mathcal{K}_k^\eta(x,\cdot)\|_{L^{p'}(B_R^{|\cdot|}(a))}\|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})},$$

where p' is the dual exponent of p. Hence

$$\|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}_\eta)} \le \Big\|\underbrace{\big\|\|\mathcal{K}_k^\eta(x,\cdot)\|_{L^{p'}(B_R^{|\cdot|}(a))}\big\|_{L^2(\mathbb{R}^{d_1}_x)}}_{=:\beta_{k,\eta}}\|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})}\Big\|_{L^2(\mathbb{R}^{d_2}_\eta)}. \tag{3.11}$$

Since $|\mathcal{K}_k^\eta(x,\xi)|\le H_k^\eta(x)^{1/2}H_k^\eta(\xi)^{1/2}$ (with H_k^η being defined as in (3.2)), we get

$$\beta_{k,\eta} \le \|(H_k^\eta)^{1/2}\|_{L^2(\mathbb{R}^{d_1})}\|(H_k^\eta)^{1/2}\|_{L^{p'}(B_R^{|\cdot|}(a))}. \tag{3.12}$$

The first factor can be estimated by

$$\|(H_k^\eta)^{1/2}\|_{L^2(\mathbb{R}^{d_1})} = \Big(\sum_{|\nu|_1=k} \|\Phi_\nu^\eta\|_2^2\Big)^{1/2}$$
$$= |\{\nu \in \mathbb{N}^{d_1} : |\nu|_1 = k\}|^{1/2} \leq k^{d_1/2}. \tag{3.13}$$

Let $x \in B_R(a)$. Since $P_k^\eta g_k^\eta = 0$ for $[k]|\eta| \notin \operatorname{supp} F$, we may assume $[k]|\eta| \sim 1$. Thus, since $R < |a|/4$, we have

$$|\eta||x|_\infty^2 \sim |\eta||x|^2 \gtrsim |\eta||a|^2 \geq \frac{|\eta|[k]^2}{\gamma^2} \sim \frac{[k]}{\gamma^2}.$$

Choosing $\gamma > 0$ small enough absorbs all constants, so that $|\eta||x|_\infty^2 \geq 2[k]$. Thus, together with Lemma 3.3, we obtain

$$\|(H_k^\eta)^{1/2}\|_{L^{p'}(B_R^{|\cdot|}(a))} \lesssim \big\||\eta|^{d_1/4}\exp(-c|\eta||\cdot|^2)\big\|_{L^{p'}(B_R^{|\cdot|}(a))}$$
$$\leq |\eta|^{d_1/4}\exp(-\tilde{c}|\eta||a|^2)|B_R^{|\cdot|}(a)|^{1/p'}.$$

Recall that we have assumed $|a| > 1$, whence

$$|\eta|^{d_1/4}|B_R^{|\cdot|}(a)|^{1/p'} \lesssim (|\eta||a|^2)^{d_1/4}.$$

Moreover, since $[k]|\eta| \sim 1$ and $[k] < \gamma|a|$, we have $|\eta||a| \gtrsim 1/\gamma$. Hence

$$\|(H_k^\eta)^{1/2}\|_{L^{p'}(B_r(a))} \lesssim_N (|\eta||a|^2)^{-N} \lesssim_{N,\gamma} |a|^{-N} \tag{3.14}$$

for any $N \in \mathbb{N}$. Gathering the estimates (3.12), (3.13), (3.14) yields

$$\beta_{k,\eta} \lesssim_{N,\gamma} [k]^{d_1/2}|a|^{-N}. \tag{3.15}$$

Furthermore, recall that Minkowski's integral inequality and the Stein–Tomas restriction estimate gave us (3.9), which yields in particular

$$\|\|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})}\|_{L^2(\mathbb{R}_\eta^{d_2})} \lesssim \|F\|_2\|f\|_p. \tag{3.16}$$

Altogether, (3.11), (3.15) and (3.16) provide

$$\|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}_\eta^{d_2})} \leq \big\|\beta_{k,\eta}\|g_k^\eta\|_{L^p(\mathbb{R}^{d_1})}\big\|_{L^2(\mathbb{R}_\eta^{d_2})}$$
$$\lesssim_{N,\gamma} [k]^{d_1/2}|a|^{-N}\|F\|_2\|f\|_p.$$

Finally, by choosing $N \in \mathbb{N}$ large enough, we obtain

$$\sum_{[k]<\gamma|a|} \|P_k^\eta g_k^\eta\|_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}_\eta^{d_2})}^2 \lesssim_{N,\gamma} \sum_{[k]<\gamma|a|} [k]^{d_1}|a|^{-2N}\|F\|_2^2\|f\|_p^2$$
$$\lesssim_\gamma |a|^{-2d_2(1/p-1/2)}\|F\|_2^2\|f\|_p^2.$$

This finishes the proof. □

4 Proofs of Theorem 1.1 and Theorem 1.2

Let again ϱ denote the Carnot–Carathéodory distance associated to the Grushin operator L, let $d = d_1 + d_2$ be the topological dimension, and $Q = d_1 + 2d_2$ be the homogeneous dimension of the metric measure space $(\mathbb{R}^d, \varrho, |\cdot|)$. Moreover, let p_{d_1,d_2} be defined as in (1.1). Given any bounded Borel function $G : \mathbb{R} \to \mathbb{C}$, let

$$G^{(j)} := (\hat{G}\chi_j)^\vee \quad \text{for } j \in \mathbb{Z},$$

where χ_j is defined by (1.2).

We will use the following result of [6, Proposition I.22], which we record here in a slightly modified version, see the remark below. The proof of the result in [6] relies on standard Calderón-Zygmund theory arguments.

Proposition 4.1 *Let L be a non-negative self-adjoint operator on a metric measure space (X, d, μ) of homogeneous type with homogeneous dimension Q. Let $1 \le p_0 < p < 2$. Suppose that L satisfies the following properties:*

(1) *L satisfies the finite propagation speed property.*
(2) *For all $t > 0$ and all bounded Borel functions $F : \mathbb{R} \to \mathbb{C}$ supported in $[0, 1]$,*

$$\|F(t\sqrt{L})\chi_{B_R}\|_{p_0\to 2} \le C_{p_0}\Big(\frac{(R/t)^Q}{\mu(B_R)}\Big)^{1/p_0-1/2}\|F\|_\infty \tag{4.1}$$

for all balls $B_R \subseteq X$ of radius $R > t$.

Then for any $s > 1/2$ and every bounded Borel function $F : \mathbb{R} \to \mathbb{C}$ satisfying $\|F\|_{\mathrm{sloc},s} < \infty$ and

$$\|(F\chi_i)^{(j)}(\sqrt{L})\|_{p\to p} \le \alpha(i + j)\|F\|_{\mathrm{sloc},s} \quad \text{for all } i, j \in \mathbb{Z}, \tag{4.2}$$

with $\sum_{\iota\ge 1} \iota\alpha(\iota) \le C_{p,s}$, the operator $F(\sqrt{L})$ is bounded on L^p, and

$$\|F(\sqrt{L})\|_{p\to p} \le C_{p,s}\|F\|_{\mathrm{sloc},s}. \tag{4.3}$$

Remark 4.2 Proposition I.22 of [6] requires the condition $(\mathrm{E}_{p_0,2})$ in place of the Stein–Tomas restriction type condition (4.1), which is however an equivalent property by Proposition I.3 of the same paper. The additionally required condition (I.3.12) in [6] is automatically fulfilled by Theorem I.5. Furthermore, in [6] it is only stated that the operator $F(\sqrt{L})$ is of weak type (p, p), but L^p-boundedness can easily be recovered via interpolation, while the estimate (4.3) follows by the closed graph theorem. The assumption $s > 1/2$ in Proposition 4.1 ensures that $\|F|_{(0,\infty)}\|_\infty \lesssim \|F\|_{\mathrm{sloc},s}$.

With Proposition 4.1 at hand, the proofs of Theorem 1.1 and Theorem 1.2 boil down to proving the following statement.

Proposition 4.3 *Let $1 \le p \le p_{d_1,d_2}$ and $G : \mathbb{R} \to \mathbb{C}$ be an even bounded Borel function supported in $[-2, -1/2] \cup [1/2, 2]$ such that $G \in L^2_s(\mathbb{R})$ for some $s > d(1/p - 1/2)$. Then there exists $\varepsilon > 0$ such that*

$$\|G^{(\iota)}(\sqrt{L})\|_{p\to p} \le C_{p,s}2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s} \quad \text{for all } \iota \ge 0.$$

Before we prove Proposition 4.3, we briefly show how Theorem 1.1 and Theorem 1.2 follow. The Bochner–Riesz summability of Theorem 1.2 (for $p > 1$) might be seen as a consequence of Theorem 1.1, but it is however a direct consequence of Proposition 4.3, without any Calderón–Zygmund theory involved.

Proof of Theorem 1.2 Let $G(\lambda) := (1-\lambda^2)_+^\delta$. As in Proposition 2.1 (4), define the dilations δ_t via $\delta_t(x, y) := (tx, t^2y)$ for $t > 0$ and $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Since L is homogenous with respect to δ_t, we have

$$G(\sqrt{L})(f \circ \delta_t) = (G(t\sqrt{L})f) \circ \delta_t.$$

Hence

$$\|(1-t^2L)_+^\delta\|_{p\to p} = \|(1-L)_+^\delta\|_{p\to p} \quad \text{for all } t > 0.$$

Thus we may assume $t = 1$. Choose $s > 0$ such that $d(1/p - 1/2) < s < \delta + 1/2$. Let J_α be the Bessel function of the first kind of order $\alpha > -1/2$, i.e.,

$$J_\alpha(r) = \frac{(r/2)^\alpha}{\Gamma(\alpha+1/2)\pi^{1/2}} \int_{-1}^{1} e^{ir\lambda}(1-\lambda^2)^{\alpha-1/2}\, d\lambda, \quad r > 0.$$

Since $|J_\alpha(r)| \lesssim r^{-1/2}$ (see Lemma 3.11 in Chapter IV of [31] for instance),

$$|\hat{G}(\xi)| \sim |\xi|^{-\delta-1/2}|J_{\delta+1/2}(|\xi|)| \lesssim |\xi|^{-\delta-1} \quad \text{for } \xi \in \mathbb{R}\backslash\{0\}.$$

Hence $|\xi^s\hat{G}(\xi)| \lesssim |\xi|^{s-\delta-1}$ and therefore $G \in L_s^2(\mathbb{R})$ since $s - \delta - 1 < -1/2$. We may decompose $G = G\psi + G(1-\psi)$ where $\psi : \mathbb{R} \to \mathbb{C}$ is a bump function supported in $[-3/4, 3/4]$ with $\psi(\lambda) = 1$ for $|\lambda| \le 1/2$. Then $G\psi$ is a bump function that may be treated for instance by the Mikhlin–Hörmander type result of [19, Theorem 1]. Moreover, applying Proposition 4.3 for $G(1-\psi)$, we obtain

$$\|(G(1-\psi))^{(\iota)}(\sqrt{L})\|_{p\to p} \lesssim 2^{-\varepsilon\iota}\|G\|_{L_s^2(\mathbb{R})} \quad \text{for } \iota \ge 0.$$

Furthermore, $\sum_{\iota<0}(G(1-\psi))^{(\iota)} = (G(1-\psi)) * (\sum_{\iota<0}\chi_\iota)^\vee$ is a Schwartz function that may again be treated by Theorem 1 of [19]. Taking the sum over all $\iota \ge 0$ finishes the proof. □

Proof of Theorem 1.1 Since $\|F\|_{\mathrm{sloc},s} \sim \|\tilde{F}\|_{\mathrm{sloc},s}$ where $F(\lambda) = \tilde{F}(\sqrt{\lambda})$ for $\lambda \ge 0$, we may replace $F(L)$ by $F(\sqrt{L})$ in the proof. Moreover, we may assume without loss of generality that F is an even function since L is a positive operator. To show L^p-boundedness of $F(\sqrt{L})$, we verify the assumptions of Proposition 4.1. Note that $s > 1/2$ since $p \le p_{d_1,d_2}$. The required condition (4.1) is a consequence of (3.4) and (3.5). Indeed, in our setting, since $|B_R(a,b)| \sim R^d \max\{R, |a|\}^{d_2}$ by Proposition 2.1 (2), the first factor of the right-hand side of (4.1) is given by

$$\left(\frac{(R/t)^Q}{|B_R(a,b)|}\right)^{1/p_0-1/2} \sim t^{-Q(1/p_0-1/2)} \quad \text{if } R \ge |a|/4,$$

and, since $R > t$,

$$\begin{aligned}\left(\frac{(R/t)^Q}{|B_R(a,b)|}\right)^{1/p_0-1/2} &\sim (|a|^{-d_2}t^{-d}(R/t)^{d_2})^{1/p_0-1/2}\\ &\ge (|a|^{d_2}t^d)^{-(1/p_0-1/2)} \quad \text{if } R < |a|/4.\end{aligned}$$

Let δ_t be again the dilation from Proposition 2.1 (4). Then

$$F(\sqrt{L})(f \circ \delta_t) = (F(t\sqrt{L})f) \circ \delta_t. \tag{4.4}$$

Let $t > 0$ and F be supported in $[1/2, 2]$. Since ϱ is homogeneous with respect to δ_t by Proposition 2.1 (4), (3.5) yields for $R < |a|/4$

$$\begin{aligned}\|F(t\sqrt{L})(\chi_{B_R^\varrho(a,b)} f)\|_2 &= t^{Q/2}\|F(\sqrt{L})(\chi_{B^\varrho_{R/t}(a/t,b/t^2)}(f\circ\delta_t))\|_2 \\ &\lesssim t^{Q/2}(|a|/t)^{-d_2(1/p_0-1/2)}\|F\|_2\|f\circ\delta_t\|_{p_0} \\ &= (t^d|a|^{d_2})^{-(1/p_0-1/2)}\|F\|_2\|f\|_{p_0}. \end{aligned} \tag{4.5}$$

Given a bounded Borel function $F:\mathbb{R}\to\mathbb{C}$ supported in $[0,1]$, we decompose F as

$$F = \sum_{i\le 0} F\chi_i.$$

Applying (4.5) for $\tilde{t} = t/2^i$ and $\tilde{F} = F(2^i\,\cdot)\chi$ and using $\|\tilde{F}\|_2 \lesssim \|F\|_\infty$, we obtain

$$\begin{aligned}\|F(t\sqrt{L})f\|_2 &\le \sum_{i\le 0}\|(F\chi_i)(t\sqrt{L})f\|_2 \\ &\lesssim \sum_{i\le 0}((t/2^i)^d|a|^{d_2})^{-(1/p_0-1/2)}\|F\|_\infty\|f\|_{p_0} \\ &\sim (t^d|a|^{d_2})^{-(1/p_0-1/2)}\|F\|_\infty\|f\|_{p_0}.\end{aligned}$$

The computation for the case $R \ge |a|/4$ is similar. This establishes condition (4.1).

Now we verify (4.2). For $i\in\mathbb{Z}$, let $F_i := F\chi_i$. Given $i,j\in\mathbb{Z}$, let $\iota := i+j$ and

$$G(\lambda) := F(2^i\lambda)\chi(\lambda),\quad \lambda\in\mathbb{R},$$

where χ is given by (1.2). Then G is an even function, and

$$\begin{aligned}(F_i)^{(j)}(\lambda) &= (\widehat{F}_i\chi_j)^\vee(\lambda) = (2^i\hat{G}(2^i\cdot)\chi_j)^\vee(\lambda) \\ &= (\hat{G}\chi_\iota)^\vee(2^{-i}\lambda) = G^{(\iota)}(2^{-i}\lambda).\end{aligned}$$

Moreover, by the homogeneity (4.4),

$$\|G^{(\iota)}(2^{-i}\sqrt{L})\|_{p\to p} = \|G^{(\iota)}(\sqrt{L})\|_{p\to p}.$$

Hence, for $\iota\ge 0$, Proposition 4.3 provides

$$\begin{aligned}\|(F\chi_i)^{(j)}(\sqrt{L})\|_{p\to p} &= \|G^{(\iota)}(\sqrt{L})\|_{p\to p} \\ &\lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s} \lesssim 2^{-\varepsilon\iota}\|F\|_{\mathrm{sloc},s}.\end{aligned}$$

The case $\iota < 0$ will be treated by the Mikhlin–Hörmander type result of [19]. Suppose $\iota < 0$. Let $\psi := \sum_{i\le 2}\chi_i$. Then ψ is supported in $[-8,8]$. We decompose $G^{(\iota)}$ as $G^{(\iota)} = G^{(\iota)}\psi + G^{(\iota)}(1-\psi)$. Since $G^{(\iota)} = G * \check{\chi}_\iota$, $\operatorname{supp} G \subseteq [-2,2]$ and $\chi\in\mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}\left|\left(\frac{d}{d\lambda}\right)^\alpha G^{(\iota)}(\lambda)\right| &= \left|\left(\frac{d}{d\lambda}\right)^\alpha\int_{-2}^2 2^\iota G(\tau)\check{\chi}(2^\iota(\lambda-\tau))\,d\tau\right| \\ &\lesssim_N 2^{\iota(\alpha+1)}\int_{-2}^2\frac{|G(\tau)|}{(1+2^\iota|\lambda-\tau|)^N}\,d\tau,\quad \alpha\in\mathbb{N}.\end{aligned} \tag{4.6}$$

Choosing $N := 0$ in (4.6) and using $2^{\iota(\alpha+1)}\le 1$, we obtain

$$\|G^{(\iota)}\psi\|_{\mathrm{sloc},\lceil d/2\rceil} \lesssim_\psi \|G\|_2 \lesssim \|F\|_{\mathrm{sloc},s}. \tag{4.7}$$

On the other hand, choosing $N := \alpha + 1$ in (4.6) yields in particular

$$\left|\left(\frac{d}{d\lambda}\right)^{\alpha} G^{(\iota)}(\lambda)\right| \lesssim |\lambda|^{-\alpha}\|G\|_2 \quad \text{for } |\lambda| \geq 4.$$

Since all derivatives of $1-\psi$ are Schwartz functions, Leibniz rule yields

$$\|G^{(\iota)}(1-\psi)\|_{\mathrm{sloc},\lceil d/2\rceil} \lesssim_{\psi} \|G\|_2 \lesssim \|F\|_{\mathrm{sloc},s}. \tag{4.8}$$

Hence applying Theorem 1 of [19] provides

$$\begin{aligned}\|(F_i)^{(j)}(\sqrt{L})\|_{p\to p} &= \|G^{(\iota)}(2^{-i}\sqrt{L})\|_{p\to p}\\ &= \|G^{(\iota)}(\sqrt{L})\|_{p\to p} \lesssim \|F\|_{\mathrm{sloc},s}.\end{aligned}$$

This establishes (4.2). Hence we may apply Proposition 4.1. □

The rest of this section is devoted to the proof of Proposition 4.3. The approach of our proof is essentially the same as in the proofs of Lemma 4.1 and Theorem 4.2 in [5]. The new feature is the decomposition into eigenvalues of the rescaled Hermite operator L^{η} via the truncation along the spectrum of T afforded by the operators $\chi_\ell(L/T)$. This truncation corresponds to a subtler analysis of the sub-Riemannian geometry regarding the finite propagation speed property. A central ingredient of this analysis is the following weighted Plancherel estimate from [19, Lemma 11], which we can fortunately use out of the box.

Lemma 4.4 *Let $H : \mathbb{R} \to \mathbb{C}$ be a bounded Borel function supported in $[1/8, 8]$, and, for $\ell \in \mathbb{N}$, let $H_\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by*

$$H_\ell(\lambda, r) = H(\sqrt{\lambda})\chi_\ell(\lambda/r) \quad \text{for } \lambda \geq 0, r \neq 0$$

and $H_\ell(\lambda, r) = 0$ else. Then, for all $N \in \mathbb{N}$ and almost all $(a,b) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$$\int_{\mathbb{R}^d} \left||y-b|^N \mathcal{K}_{H_\ell(L,T)}((x,y),(a,b))\right|^2 d(x,y) \leq C_{\chi,N} 2^{\ell(2N-d_2)}\|H\|^2_{L^2_N},$$

where $\mathcal{K}_{H_\ell(L,T)}$ denotes the integral kernel of the operator $H_\ell(L,T)$.

Proof of Proposition 4.3 Let $\iota \in \mathbb{N}$ and $R := 2^{\iota}$. We proceed in several steps.

(1) *Reduction to compactly supported functions.* Let $f \in D(\mathbb{R}^d)$. We will first show that we may restrict to functions supported in balls of radius R with respect to the Carnot–Carathéodory distance ϱ. Recall that ϱ induces the Euclidean topology on $\mathbb{R}^d$, which implies in particular that the metric space $(\mathbb{R}^d, \varrho)$ is separable. Since the metric measure space $(\mathbb{R}^d, \varrho, |\cdot|)$ is a space of homogeneous type, we may thus choose a decomposition into disjoint sets $B_n \subseteq B^{\varrho}_R(a_n, b_n)$, $n \in \mathbb{N}$, such that for every $\lambda \geq 1$, the number of overlapping dilated balls $B^{\varrho}_{\lambda R}(a_n, b_n)$ may be bounded by a constant $C(\lambda)$, which is independent of ι. We decompose f as

$$f = \sum_{n=0}^{\infty} f_n \quad \text{where } f_n := f\chi_{B_n}.$$

Since G is even, so is $\hat{G}$. As χ_ι is even as well, the Fourier inversion formula provides

$$G^{(\iota)}(\sqrt{L})f_n = \frac{1}{2\pi}\int_{2^{\iota-1}\leq|\tau|\leq 2^{\iota+1}} \chi_\iota(\tau)\hat{G}(\tau)\cos(\tau\sqrt{L})f_n\,d\tau.$$

By Proposition 2.1 (5), L satisfies the finite propagation speed property, whence $G^{(\iota)}(\sqrt{L})f_n$ is supported in $B^{\varrho}_{3R}(a_n, b_n)$ by the formula above. Since the balls $B^{\varrho}_{3R}(a_n, b_n)$ have only a bounded overlap, we obtain

$$\|G^{(\iota)}(\sqrt{L})f_n\|_p^p \lesssim \sum_{n=0}^{\infty} \|G^{(\iota)}(\sqrt{L})f_n\|_p^p.$$

Thus, since the functions f_n have disjoint support, it suffices to show

$$\|G^{(\iota)}(\sqrt{L})f_n\|_p \lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s}\|f_n\|_p, \tag{4.9}$$

with a constant independent of $n \in \mathbb{N}$.

(2) *Localizing the multiplier.* Next we show that only the part of the multiplier $G^{(\iota)}$ located at $|\lambda| \sim 1$ is relevant. Let $\psi := \sum_{|i|\le 2} \chi_i$. Then ψ is supported in $\{\lambda \in \mathbb{R} : 1/8 \le |\lambda| \le 8\}$, while $1-\psi$ is supported in $\{\lambda \in \mathbb{R} : |\lambda| \notin (1/4, 4)\}$. We decompose $G^{(\iota)}$ as $G^{(\iota)} = G^{(\iota)}\psi + G^{(\iota)}(1-\psi)$. The second part of this decomposition can be treated by the Mikhlin–Hörmander type result of [19]. As in (4.6), we observe

$$\Big|\Big(\frac{d}{d\lambda}\Big)^{\alpha} G^{(\iota)}(\lambda)\Big| \lesssim_N 2^{\iota(\alpha+1)} \int_{-2}^{2} \frac{|G(\tau)|}{(1+2^{\iota}|\lambda-\tau|)^N}\, d\tau, \quad \alpha \in \mathbb{N}. \tag{4.10}$$

Recall that G is supported in $[-2,-1/2] \cup [1/2, 2]$. Thus, choosing $N := \alpha + 2$ in (4.10), we obtain

$$\Big|\Big(\frac{d}{d\lambda}\Big)^{\alpha} G^{(\iota)}(\lambda)\Big| \lesssim 2^{-\iota}\min\{|\lambda|^{-\alpha}, 1\}\|G\|_2 \quad \text{whenever } |\lambda| \notin [1/4, 4].$$

Similar as in (4.7) and (4.8), we obtain

$$\|G^{(\iota)}(1-\psi)\|_{\mathrm{sloc},\lceil d/2\rceil} \lesssim_{\psi} 2^{-\iota}\|G\|_2.$$

Hence applying Theorem 1 of [19] provides

$$\|(G^{(\iota)}(1-\psi))(\sqrt{L})\|_{p\to p} \lesssim 2^{-\iota}\|G\|_2.$$

Thus, in place of (4.9), it suffices to show

$$\|\chi_{B^{\varrho}_{3R}(a_n,b_n)}(G^{(\iota)}\psi)(\sqrt{L})f_n\|_p \lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s}\|f_n\|_p. \tag{4.11}$$

To that end, we distinguish the cases $|a_n| > 4R$ and $|a_n| \le 4R$.

(3) *The elliptic region.* Suppose $|a_n| > 4R$. Then, by Proposition 2.1 (2),

$$|B^{\varrho}_{3R}(a_n, b_n)| \sim R^d \max\{R, |a_n|\}^{d_2} = R^d|a_n|^{d_2}.$$

Let $2 \le q \le \infty$ such that $1/q = 1/p - 1/2$. Applying Hölder's inequality together with the restriction type estimate (3.5) for the multiplier $G^{(\iota)}\psi|_{[0,\infty)}$ (recall that L is a positive operator) yields

$$\begin{aligned}\|\chi_{B^{\varrho}_{3R}(a_n,b_n)}(G^{(\iota)}\psi)(\sqrt{L})f_n\|_p &\lesssim (R^d|a_n|^{d_2})^{1/q}\|(G^{(\iota)}\psi)(\sqrt{L})f_n\|_2\\ &\lesssim 2^{\iota d/q}\|G^{(\iota)}\psi\|_2\|f_n\|_p\\ &\lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s}\|f_n\|_p\end{aligned}$$

if we choose $0 < \varepsilon < s - d/q$. This shows (4.11) in the case $|a_n| > 4R$.

(4) *The non-elliptic region: Truncation along the spectrum of T.* Suppose $|a_n| \le 4R$. Let $G_\ell^{(\iota)} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be given by

$$G_\ell^{(\iota)}(\lambda, r) = (G^{(\iota)}\psi)(\sqrt{\lambda})\chi_\ell(\lambda/r) \quad \text{for } \lambda \ge 0, r \ne 0$$

and $G_\ell^{(\iota)}(\lambda, r) = 0$ else. We decompose the function on the left-hand side of (4.11) as

$$\begin{aligned}
&\chi_{B^\varrho_{3R}(a_n,b_n)}(G^{(\iota)}\psi)(\sqrt{L}) f_n \\
&= \chi_{B^\varrho_{3R}(a_n,b_n)}\left(\sum_{\ell=0}^{\iota} + \sum_{\ell=\iota+1}^{\infty}\right) G_\ell^{(\iota)}(L, T) f_n =: g_{n,\le\iota} + g_{n,>\iota}.
\end{aligned}$$

The second summand $g_{n,>\iota}$ can be directly treated by Theorem 3.4. Indeed, Proposition 2.1 (2), Hölder's inequality and the restriction type estimate (3.4) imply

$$\begin{aligned}
\|g_{n,>\iota}\|_p &\lesssim R^{Q/q}\|g_{n,>\iota}\|_2 \\
&\lesssim 2^{\iota(Q-d_2)/q}\|G^{(\iota)}\psi\|_2\|f_n\|_p \\
&\lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s}\|f_n\|_p
\end{aligned}$$

if we choose $0 < \varepsilon < s - d/q$. Hence we are done once we have shown

$$\|g_{n,\le\iota}\|_p \lesssim 2^{-\varepsilon\iota}\|G^{(\iota)}\|_{L^2_s}\|f_n\|_p. \tag{4.12}$$

(5) *(Almost) finite propagation speed on Euclidean scales in the non-elliptic region.* The key idea is as follows: Since $T = (-\Delta_y)^{1/2}$, we have

$$(G^{(\iota)}\psi)(\sqrt{L})\chi_{\{2k+d_1\}}(L/T) = H_\iota\big([k]\sqrt{-\Delta_y}\big)\chi_{\{2k+d_1\}}(L/T),$$

where $H_\iota(\lambda) := (G^{(\iota)}\psi)\big(\sqrt{\lambda}\big)$. Thus one might expect that the operator $G_\ell^{(\iota)}(L, T)$ behaves roughly like $H_\iota(2^\ell\sqrt{-\Delta_y})$ regarding the finite propagation property. Since $|a_n| \le 4R$, Proposition 2.1 (3) yields

$$B^\varrho_R(a_n, b_n) \subseteq B^{|\cdot|}_R(a_n) \times B^{|\cdot|}_{CR^2}(b_n).$$

Hence, for every $0 \le \ell \le \iota$, we find a decomposition of $B_n \subseteq B^\varrho_R(a_n, b_n)$ such that

$$B_n = \bigcup_{m=1}^{M_{n,\ell}} B^{(\ell)}_{n,m},$$

where $B^{(\ell)}_{n,m} \subseteq B^{|\cdot|}_R(a_n) \times B^{|\cdot|}_{CR_\ell}(b^{(\ell)}_{n,m})$ with $R_\ell := 2^\ell R$ are disjoint subsets, and

$$|b^{(\ell)}_{n,m} - b^{(\ell)}_{n,m'}| > R_\ell/2 \quad \text{for } m \ne m'.$$

The number of subsets in this decomposition is bounded by

$$M_{n,\ell} \lesssim (R^2/R_\ell)^{d_2} = 2^{(\iota-\ell)d_2}. \tag{4.13}$$

Moreover, given $\gamma > 0$, the number N_γ of overlapping balls

$$\tilde{B}^{(\ell)}_{n,m} := B^{|\cdot|}_{3R}(a_n) \times B^{|\cdot|}_{2^{\gamma\iota+1}CR_\ell}(b^{(\ell)}_{n,m}), \quad 1 \le m \le M_{n,\ell}$$

can be bounded by $N_\gamma \lesssim_\iota 1$, where

$$A \lesssim_\iota B$$

means $A \leq 2^{C(p,d_1,d_2)\iota\gamma} B$ for some constant $C(p, d_1, d_2) > 0$ depending only on the parameters p, d_1, d_2. (The parameter $\gamma > 0$ is necessary for having rapid decay for the negligible part of the propagation, see (4.19). This trick has also been used in a similar fashion in [25].) We decompose f_n as

$$f_n = \sum_{m=1}^{M_{n,\ell}} f_{n,m}^{(\ell)} \quad \text{where } f_{n,m}^{(\ell)} := f_n \chi_{B_{n,m}^{(\ell)}}.$$

In the next step, we show that the function

$$g_{n,m}^{(\ell)} := \chi_{B_{3R}^{\varrho}(a_n,b_n)} G_\ell^{(\iota)}(L,T) f_{n,m}^{(\ell)}$$

is essentially supported in the ball $\tilde{B}_{n,m}^{(\ell)}$. Let $\chi_{n,m}^{(\ell)}$ denote the indicator function of $\tilde{B}_{n,m}^{(\ell)}$. We decompose $g_{n,\leq\iota}$ as

$$g_{n,\leq\iota} = \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} \tilde{g}_{n,m}^{(\ell)} + \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} (1-\chi_{n,m}^{(\ell)}) g_{n,m}^{(\ell)},$$

where $\tilde{g}_{n,m}^{(\ell)} := \chi_{n,m}^{(\ell)} g_{n,m}^{(\ell)}$. The first summand represents the essential parts of the propagation, while the second one should be seen as an error term.

For the first summand, we observe that Hölder's inequality and the bounded overlapping property of the balls $\tilde{B}_{n,m}^{(\ell)}$ imply

$$\left\| \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} \tilde{g}_{n,m}^{(\ell)} \right\|_p^p \leq ((\iota+1)N_\gamma)^{p-1} \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} \|\tilde{g}_{n,m}^{(\ell)}\|_p^p. \tag{4.14}$$

Using Hölder's inequality together with (3.3) yields

$$\begin{aligned} \|\tilde{g}_{n,m}^{(\ell)}\|_p &\lesssim_\iota (R^{d_1} R_\ell^{d_2})^{1/q} \|g_{n,m}^{(\ell)}\|_2 \\ &= 2^{\iota d/q + \ell d_2/q} \|g_{n,m}^{(\ell)}\|_2 \\ &\lesssim 2^{\iota d/q} \|G^{(\iota)}\|_2 \|f_{n,m}^{(\ell)}\|_p \\ &\sim 2^{\iota(d/q-s)} \|G^{(\iota)}\|_{L_s^2} \|f_{n,m}^{(\ell)}\|_p. \end{aligned} \tag{4.15}$$

By (4.14) and (4.15), we obtain

$$\left\| \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} \tilde{g}_{n,m}^{(\ell)} \right\|_p^p \lesssim 2^{-\varepsilon\iota} \|G^{(\iota)}\|_{L_s^2}^p \|f_n\|_p^p$$

for some $\varepsilon > 0$ provided we choose $\gamma > 0$ small enough before. As an upshot, to verify (4.12) it remains to show

$$\left\| \sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}} (1-\chi_{n,m}^{(\ell)}) g_{n,m}^{(\ell)} \right\|_p \lesssim 2^{-\varepsilon\iota} \|G^{(\iota)}\|_{L_s^2} \|f_n\|_p. \tag{4.16}$$

(6) *The negligible part of the propagation.* For showing (4.16), we interpolate between L^1 and L^2 via the Riesz–Thorin interpolation theorem. The L^2-estimate is allowed to be quite rough, since the rapid decay in terms of 2^ι derives from the L^1-estimate. For the L^2-estimate, we employ the Sobolev embedding

$$\|G^{(\iota)}\|_\infty \lesssim \|G^{(\iota)}\|_{L^2_{1/2+\delta}} \sim 2^{\iota(1/2+\delta)} \|G^{(\iota)}\|_{L^2}, \quad \delta > 0,$$

which in conjunction with Hölder's inequality and (4.13) provides

$$\begin{aligned}\Big\|\sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}}(1-\chi_{n,m}^{(\ell)})g_{n,m}^{(\ell)}\Big\|_2 &\le \|G^{(\iota)}\psi\|_\infty\sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}}\|f_{n,m}^{(\ell)}\|_2\\ &\le \|G^{(\iota)}\psi\|_\infty(\iota+1)2^{\iota d_2/2}\|f_n\|_2\\ &\lesssim 2^{\iota(1+d_2/2)}\|G^{(\iota)}\|_2\|f_n\|_2. \qquad (4.17)\end{aligned}$$

The L^1-estimate is derived from an L^∞ integral kernel estimate. Let $\mathcal{K}_\ell^{(\iota)}$ denote the integral kernel of $G_\ell^{(\iota)}(L,T)$. Then

$$G_\ell^{(\iota)}(L,T)f_{n,m}^{(\ell)}(x,y)=\int_{\mathbb{R}^d}\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))f_{n,m}^{(\ell)}(a,b)\,d(a,b).$$

For $b\in\mathbb{R}^{d_2}$, define the set

$$B_n^{(b)}:=\{(x,y)\in B_{3R}^{\varrho}(a_n,b_n):|y-b|\ge 2^{\gamma\iota}CR_\ell\}.$$

Note that $(x,y)\in\operatorname{supp}((1-\chi_{n,m}^{(\ell)})\chi_{B_{3R}^{\varrho}(a_n,b_n)})$ and $(a,b)\in\operatorname{supp}f_{n,m}^{(\ell)}$ imply

$$|y-b_{n,m}^{(\ell)}|\ge 2^{\gamma\iota+1}CR_\ell \quad\text{and}\quad |b-b_{n,m}^{(\ell)}|<CR_\ell,$$

and thus in particular $(x,y)\in B_n^{(b)}$. Hence

$$\begin{aligned}&\Big\|\sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}}(1-\chi_{n,m}^{(\ell)})g_{n,m}^{(\ell)}\Big\|_1\\ &\le\int_{\mathbb{R}^d}\sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}}(1-\chi_{n,m}^{(\ell)}(x,y))\chi_{B_{3R}^{\varrho}(a_n,b_n)}(x,y)\\ &\qquad\times\int_{\mathbb{R}^d}|\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))f_{n,m}^{(\ell)}(a,b)|\,d(a,b)\,d(x,y)\\ &\le\int_{B_R^{\varrho}(a_n,b_n)}\int_{B_n^{(b)}}\sum_{\ell=0}^{\iota}\sum_{m=1}^{M_{n,\ell}}|\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))f_{n,m}^{(\ell)}(a,b)|\,d(x,y)\,d(a,b)\\ &=\int_{B_R^{\varrho}(a_n,b_n)}\kappa_\gamma(a,b)|f_n(a,b)|\,d(a,b), \qquad (4.18)\end{aligned}$$

where

$$\kappa_\gamma(a,b):=\sum_{\ell=0}^{\iota}\int_{B_n^{(b)}}|\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))|\,d(x,y).$$

Given $N\in\mathbb{N}$, the Cauchy–Schwarz inequality yields

$$\begin{aligned}\kappa_\gamma(a,b)&\lesssim\sum_{\ell=0}^{\iota}(2^{\gamma\iota}R_\ell)^{-N}\int_{B_{3R}^{\varrho}(a_n,b_n)}\big||y-b|^N\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))\big|\,d(x,y)\\ &\le\sum_{\ell=0}^{\iota}(2^{\gamma\iota}R_\ell)^{-N}|B_{3R}^{\varrho}(a_n,b_n)|^{1/2}\Big(\int_{\mathbb{R}^d}\big||y-b|^N\mathcal{K}_\ell^{(\iota)}((x,y),(a,b))\big|^2\,d(x,y)\Big)^{1/2}.\end{aligned}$$

Recall that $R_\ell = 2^{\iota+\ell}$ and $R = 2^\iota$, and $|a_n| \le 4R$. By Proposition 2.1 (2), we have

$$|B^{\varrho}_{3R}(a_n, b_n)| \sim R^Q = 2^{\iota Q}.$$

Now, applying Lemma 4.4 for $H = G^{(\iota)}\psi|_{[0,\infty)}$, and using the fact

$$2^{-\iota N}\|G^{(\iota)}\psi|_{[0,\infty)}\|_{L^2_N} \lesssim_\psi \|G^{(\iota)}\|_2,$$

we get

$$\begin{aligned}\kappa_\gamma(a,b) &\lesssim \sum_{\ell=0}^{\iota} 2^{-(\gamma\iota+\iota+\ell)N} 2^{\iota Q/2} 2^{\ell(N-d_2/2)} \|G^{(\iota)}\psi|_{[0,\infty)}\|_{L^2_N} \\ &\lesssim 2^{-\gamma\iota N} 2^{\iota Q/2} \|G^{(\iota)}\|_2.\end{aligned}$$

Hence, plugging this estimate into (4.18), we obtain

$$\Big\| \sum_{\ell=0}^{\iota} \sum_{m=1}^{M_{n,\ell}} (1-\chi^{(\ell)}_{n,m}) g^{(\ell)}_{n,m} \Big\|_1 \lesssim 2^{-\gamma\iota N} 2^{\iota Q/2} \|G^{(\iota)}\|_2 \|f_n\|_1. \tag{4.19}$$

Via (4.17) and (4.19), the Riesz–Thorin interpolation theorem provides

$$\Big\| \sum_{\ell=0}^{\iota} \sum_{m=1}^{M_{n,\ell}} (1-\chi^{(\ell)}_{n,m}) g^{(\ell)}_{n,m} \Big\|_p \lesssim (2^{-\gamma\iota N} 2^{\iota Q/2})^{1-\theta} (2^{\iota(1+d_2/2)})^{\theta} \|G^{(\iota)}\|_2 \|f_n\|_p,$$

where $\theta := 2(1-1/p) < 1$. Choosing $N \in \mathbb{N}$ large enough yields (4.16), whence we are done with the proof. □

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