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The index of some mixed order Dirac type operators and generalised Dirichlet–Neumann tensor fields

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Abstract

We revisit a construction principle of Fredholm operators using Hilbert complexes of densely defined, closed linear operators and apply this to particular choices of differential operators. The resulting index is then computed using an explicit description of the cohomology groups of generalised ('harmonic') Dirichlet and Neumann tensor fields. The main results of this contribution are the computation of the indices of Dirac type operators associated to the elasticity complex and the newly found biharmonic complex, relevant for the biharmonic equation, elasticity, and for the theory of general relativity. The differential operators are of mixed order and cannot be seen as leading order type with relatively compact perturbation. As a by-product we present a comprehensive description of the underlying generalised Dirichlet–Neumann vector and tensor fields defining the respective cohomology groups, including an explicit construction of bases in terms of topological invariants, which are of both analytical and numerical interest. Though being defined by certain projection mechanisms, we shall present a way of computing these basis functions by solving certain PDEs given in variational form. For all of this we rephrase core arguments in the work of Rainer Picard [42] applied to the de Rham complex and use them as a blueprint for the more involved cases presented here. In passing, we also provide new vector-analytical estimates of generalised Poincaré–Friedrichs type useful for elasticity or the theory of general relativity.

Keywords Dirac operator · Picard's extended Maxwell system · Fredholm index · Cohomology · Hilbert complex · Elasticity complex · biharmonic complex · Harmonic Dirichlet and Neumann tensors

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1 Introduction

It is one of the greatest mathematical achievements of the twentieth century to relate the analytic notion of the Fredholm index for operators defined on Hilbert spaces to particular elliptic operators and their corresponding geometric properties of the underlying compact manifold the operators are defined on. Here, a closed, densely defined, linear operator $\mathcal{D}\colon \operatorname{dom}\mathcal{D} \subseteq \mathcal{K}_0 \to \mathcal{K}_1$ between Hilbert spaces $\mathcal{K}_0$ and $\mathcal{K}_1$ is called a *Fredholm operator*, if the range, $\operatorname{ran}\mathcal{D} \subseteq \mathcal{K}_1$, is closed and both the kernel, $\ker\mathcal{D}$, and the co-kernel, $(\operatorname{ran}\mathcal{D})^{\perp}$, are finite-dimensional. In this case, the *index of* $\mathcal{D}$, $\operatorname{ind}\mathcal{D}$, is given by

$$\operatorname{ind}\mathcal{D} = \dim\ker\mathcal{D} - \dim(\operatorname{ran}\mathcal{D})^{\perp}.$$

We refer to the concluding parts in [17, Chapter 3] for a brief round up and some standard references to the theory of unbounded Fredholm operators in Hilbert spaces. Generally spoken it is often very difficult—if not impossible—to compute either $\dim\ker\mathcal{D}$ or $\dim(\operatorname{ran}\mathcal{D})^{\perp}$ directly. However, due to invariance under homotopies and compact perturbations, it is sometimes possible to have a better understanding of $\operatorname{ind}\mathcal{D}$ instead.

Indeed, one of the corner stones of results hinted at above is the celebrated Atyiah–Singer index theorem, see e.g. [25], where the (Fredholm) index for some elliptic operators defined on a manifold can be represented solely in terms of the topological properties of this manifold. The methods of proof led to the invention of K-theory, which has evolved ever since and is an active field of research. Albeit being a breakthrough in mathematics, K-theory is a rather difficult tool to work with when it comes to explicitly compute the index for particular examples. In any case there is a need to provide many examples, where it is possible to obtain an index formula, which is as explicit as possible. In fact, the Fredholm index for a perturbed Dirac operator represents physical quantities, see the concluding example in [13] and the references therein. The Witten index, a generalised version of the Fredholm index, is interesting for both physics and mathematics. Indeed, it has been shown that in particular situations the Witten index corresponds to the electromagnetic spin of a particle as well as to the spectral flow of an operator family, see the seminal paper [16].

The results in [16] are based on—among other things—a deeper understanding of the one-dimensional situation of [13], which addresses the Fredholm case. The higher-dimensional cases treated in [13] (with an index formula properly justified in [10]) were generalised in [17]. The transition from the Fredholm situation to the Witten index has been performed in [17, Chapter 14]. Again, a thorough understanding of the Fredholm case has led to further examples for the Witten index, which in turn might prove useful for both mathematics and physics.

The main contribution of the present study is to enlarge the variety of examples, where it is possible to explicitly compute the Fredholm index in terms of the topological properties of the underlying (bounded) domain $\Omega \subseteq \mathbb{R}^3$ the differential (Fredholm) operators are defined on. The list of examples treated here is even more particular as it is possible to compute not only the index but also the dimension of the kernel and the co-dimension of the range in terms of topological invariants.

Moreover, this article is concerned with the explicit computation of the Fredholm index if a differential operator is 'apparently' of mixed order. We shall establish a collection of theorems like the following:

Theorem 1.1 *Let $\Omega \subseteq \mathbb{R}^3$ be open, bounded with strong*[1] *Lipschitz boundary. Then there exists a subspace $\mathcal{V} \subseteq L^{2,3\times 3}_{\mathbb{T}}(\Omega) \times L^2(\Omega)$ such that*

$$\mathcal{D} := \begin{pmatrix} \operatorname{Div} & 0 \\ \operatorname{symCurl} & \operatorname{Gradgrad} \end{pmatrix} : \mathcal{V} \subseteq L^{2,3\times 3}_{\mathbb{T}}(\Omega) \times L^2(\Omega) \to L^{2,3}(\Omega) \times L^{2,3\times 3}_{\mathbb{S}}(\Omega)$$

and $\mathcal{D}^*$ are densely defined and closed Fredholm operators, where $L^{2,3\times 3}_{\mathbb{T}}(\Omega)$ and $L^{2,3\times 3}_{\mathbb{S}}(\Omega)$ denote the sets of trace free and symmetric 3×3 matrices with entries in $L^2(\Omega)$, respectively. Moreover,

$$\operatorname{ind} \mathcal{D} = 4(p - m - n + 1), \qquad \operatorname{ind} \mathcal{D}^* = - \operatorname{ind} \mathcal{D}$$

where n is the number of connected components of Ω, m is the number of connected components of its complement $\mathbb{R}^3 \setminus \overline{\Omega}$, and p is the number of handles (see Sect. 10).

A closer inspection of the operator $\mathcal{D}$ also shows the following estimate; see also Corollary 7.7. Note that the subspace $\mathcal{W}$ asserted to exist in the following result—and this is the catch of the corollary—is constructed explicitly by providing a basis, see Sect. 12.2.

Corollary 1.2 *There exists a finite-dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$ and $c > 0$ such that for all $(T, u) \in \mathcal{V} \cap \mathcal{W}^{\perp_{L^2}}$ we have*

$$c \left|(T, u)\right|_{L^{2,3\times 3}_{\mathbb{T}}(\Omega) \times L^2(\Omega)} \leq |\operatorname{Gradgrad} u|_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} + |\operatorname{Div} T|_{L^{2,3}(\Omega)} + |\operatorname{symCurl} T|_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}.$$

In the course of the manuscript, we shall, too, describe the subspace $\mathcal{V} = \operatorname{dom} \mathcal{D}$ explicitly, see Theorem 7.4 and Remark 7.5. A refined notation will indicate (full) natural boundary conditions by $\circ$ and algebraic properties of the tensor fields belonging to the domain of definition

[1] The boundary of a strong Lipschitz domain is locally a graph of some Lipschitz function.

of the respective operators by $\mathbb{S}$ and $\mathbb{T}$ (symmetric and trace free), e.g., the aforementioned operators read

$$\mathcal{D} = \mathcal{D}^{\mathsf{bih},1} := \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \operatorname{Grad}\mathring{\operatorname{grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{bih},1})^* = \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix}.$$

These operators are related to the (primal and dual) first biharmonic complex, also called Gradgrad or divDiv complex, i.e.,

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\operatorname{Grad}\mathring{\operatorname{grad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{-\operatorname{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}},$$

relevant for the biharmonic equation, elasticity, and in the theory of general relativity. Here and in the following ι_V and π_V denote the canonical embedding from a closed subspace V of a Hilbert space H into H and the orthogonal projection from H onto V; the space of piecewise Raviart–Thomas vector-fields, $\mathsf{RT}_{\mathsf{pw}}$, is defined in (7.1).

We discuss another example, which is based on the second biharmonic complex where the boundary conditions are interchanged, i.e.,

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{dev}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}} L^2(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{Gradgrad}} L^2(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},$$

leading to the operators (for the space of piecewise first order polynomials, $\mathsf{P}^1_{\mathsf{pw}}$, we refer to (8.2))

$$\mathcal{D}^{\mathsf{bih},2} := \begin{pmatrix} \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{bih},2})^* = \begin{pmatrix} \operatorname{Gradgrad} & \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}} \end{pmatrix}.$$

The corresponding index results can be found in Theorem 8.5 and Remark 8.6.

Finally, we address the elasticity complex, also called CurlCurl complex, i.e., (the space of piecewise rigid motions, $\mathsf{RM}_{\mathsf{pw}}$, is defined in (9.3))

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{Curl}\mathring{\operatorname{Curl}}^{\top}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{CurlCurl}^{\top}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}.$$

Here, we shall discuss the Fredholm index of the operators

$$\mathcal{D}^{\mathsf{ela}} := \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{CurlCurl}^{\top}_{\mathbb{S}} & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{ela}})^* = \begin{pmatrix} -\operatorname{symGrad} & \operatorname{Curl}\mathring{\operatorname{Curl}}^{\top}_{\mathbb{S}} \\ 0 & -\operatorname{Div}_{\mathbb{S}} \end{pmatrix}.$$

The solution to the corresponding index problem is provided in Theorem 9.4 and Remark 9.5. Note that in a distributional setting results concerning the computation of the dimension of the generalised Neumann fields have been obtained in [14], where a variational setting is preferred.

Here and throughout this paper, we denote by grad, curl, and div the classical operators from vector analysis. Moreover, Grad acts componentwise as $\operatorname{grad}^{\top}$ mapping vector fields to tensor fields. Curl and Div act row-wise as $\operatorname{curl}^{\top}$ and div mapping tensor fields to tensor and vector fields, respectively. L^2-spaces with k components (or $k \times k$-many components) are denoted by $L^{2,k}$ (or $L^{2,k\times k}$). A similar notation is used for C^{∞} and similar sets.

The approach to compute the index in situations as in Theorem 1.1 is rooted in a construction principle for Fredholm operators provided in [11]. The fundamental observation given in [11] is that it is possible to construct a Fredholm operator with the help of Hilbert complexes of densely defined and closed (possibly unbounded) linear operators, i.e,

$$\cdots \xrightarrow{\cdots} H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2 \xrightarrow{A_2} H_3 \xrightarrow{\cdots} \cdots,$$
$$\cdots \xleftarrow{\cdots} H_0 \xleftarrow{A_0^*} H_1 \xleftarrow{A_1^*} H_2 \xleftarrow{A_2^*} H_3 \xleftarrow{\cdots} \cdots.$$

More precisely, if A_0, A_1, and A_2 are densely defined, closed linear operators defined on suitable Hilbert spaces H_l such that

$$\operatorname{ran} A_0 \subseteq \ker A_1, \qquad \operatorname{ran} A_1 \subseteq \ker A_2,$$

then the block matrix operator

$$\mathcal{D} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} \tag{1.1}$$

with its natural domain of definition is closed and densely defined. It is Fredholm, if the ranges $\operatorname{ran} A_0$, $\operatorname{ran} A_1$, and $\operatorname{ran} A_2$ are closed and if both kernels

$$N_0 := \ker A_0, \qquad N_{2,*} := \ker A_2^*$$

and both cohomology groups

$$K_1 := \ker A_1 \cap \ker A_0^*, \qquad K_2 := \ker A_2 \cap \ker A_1^*$$

are finite-dimensional. In this case, its index is then given by

$$\operatorname{ind} \mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \tag{1.2}$$

cf. Theorem 3.8. For its adjoint, which is then Fredholm as well, we simply have

$$\mathcal{D}^* := \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix}, \qquad \operatorname{ind} \mathcal{D}^* = -\operatorname{ind} \mathcal{D}.$$

In a first application of this observation presented in this article, we look at the classical de Rham complex

$$\{0\} \xrightarrow{A_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{A_0=\mathring{\operatorname{grad}}} L^{2,3}(\Omega) \xrightarrow{A_1=\mathring{\operatorname{curl}}} L^{2,3}(\Omega) \xrightarrow{A_2=\mathring{\operatorname{div}}} L^2(\Omega) \xrightarrow{A_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{A_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{A_0^*=-\operatorname{div}} L^{2,3}(\Omega) \xleftarrow{A_1^*=\operatorname{curl}} L^{2,3}(\Omega) \xleftarrow{A_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{A_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}},$$

where again the super index $^\circ$ signifies homogeneous Dirichlet boundary conditions, see Theorem 6.8 and $\mathbb{R}_{\mathsf{pw}}$ denotes the space of piecewise constants on Ω, see (6.3). By (1.2) in order to compute the index it is necessary to calculate the dimension of the cohomology groups, i.e., the dimension of the harmonic Dirichlet and Neumann fields

$$\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) := K_1 = \ker(\mathring{\operatorname{curl}}) \cap \ker(\operatorname{div}),$$
$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) := K_2 = \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{curl}),$$

respectively. In [42], this has been done by Picard. Explicit constructions of harmonic Dirichlet and Neumann fields can also be found in [3,15] the latter of the two references providing substantial detail to the geometrical setting. As it turns out these dimensions are related to topological properties of the underlying domain the differential operators are defined on, that is,

$$\dim \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = m - 1, \qquad \dim \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = p,$$

see Theorem 6.6. In consequence, it is possible to compute the indices for the block de Rham operators

$$\mathcal{D}^{\mathsf{Rhm}} := \begin{pmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{Rhm}})^* := \begin{pmatrix} -\operatorname{grad} & \mathring{\operatorname{curl}} \\ 0 & -\operatorname{div} \end{pmatrix}$$

by (1.2) in terms of m, p, and n, i.e.,

$$\operatorname{ind} \mathcal{D}^{\mathsf{Rhm}} = p - m - n + 1, \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{Rhm}})^* = -\operatorname{ind} \mathcal{D}^{\mathsf{Rhm}},$$

see Theorem 6.8. It is noteworthy that this index theorem provides an index theorem for the Dirac operator on open manifolds with boundary endowed with a particular boundary condition, see [45] and Sect. 6.3 below.

The proofs of the index theorems discussed here combine the structural viewpoint outlined by [11] and ideas taken from the explicit computation of the dimension of the cohomology groups in [42]. More precisely, we shall rephrase the proofs in [42] and use these reformulations as a blueprint for the proofs for other complexes. We emphasise that the construction of the generalised Neumann fields is based on subtle interactions of matrix algebra and differential operators (see Lemma 12.10) and a suitable application of so-called Poincaré maps yielding (for instance) a representation of vector fields by curve integrals over tensor fields, see e.g. Lemma 12.11. The foundation for all of this to be applicable, however, is the newly found biharmonic complex, see [37,38], and the more familiar elasticity complex, see [39–41]. In [37,38] the crucial properties and compact embedding results have been found for the biharmonic Hilbert complex underlying the computation of the index in Theorem 1.1. In [39–41] the corresponding results are presented for the elasticity complex. These results also stress that the mixed order differential operators discussed here *cannot* be viewed as a leading order term subject to a relatively compact perturbation.

As the technique computing a basis for the classical de Rham harmonic Dirichlet and Neumann fields developed in [42], we shall comment on other developments particularly concerning the computation of these basis functions in different and/or more general topological settings. Given a suitable formulation of the Hilbert complexes also in these settings, it may be possible to provide an analogous formulation of harmonic Dirichlet/Neumann tensor fields to eventually more substantially address Problem 2.2 below. A topological setting is used in [8] helping to avoid Assumption 10.3 and still being able to construct the harmonic vector fields. With a more partial differential equations oriented way a detour of Assumption 10.3 in conjunction with mixed boundary conditions is possible as demonstrated in [2, Appendix A.4]. This might provide a hint on how to address Problem 6.10. Generalising the topological setting from open subsets of $\mathbb{R}^3$ to situations involving a more geometric point of view, the authors of [9,12,18,24] discussed properties of the de Rham complex in general and the corresponding harmonic Dirchlet and Neumann fields in particular.

Before we come to a more in depth description of the structure of the paper, we emphasise the importance of a deeper understanding of Hilbert complexes for index theory and other areas of partial differential equations.

Next to [11] (and others), the notion of Hilbert complexes in connection with (Fredholm) index theory has also been addressed in [46] and references therein. The work in [46] is particularly interesting as the authors address manifolds with boundary. The focus is on characterising the Fredholm property (i.e., the finiteness of the cohomology groups) for certain complexes with boundary in terms of the principal symbol. Here, the Fredholm property of the Hilbert complex (i.e., with the terminology of [46], that a Hilbert complex is a Fredholm complex) follows from suitable compactness criteria (see e.g. [38,55]) and all the dimensions of the cohomology groups and not just the index of the complex are addressed here explicitly.

An understanding of Hilbert complexes in connection with partial differential equations involving the classical vector analytic operators div, curl, and grad led to [43], where the kernel of the classical Maxwell operator is written by means of other differential operators. The resulting Picard's extended Maxwell system is useful for numerical studies [49] as well as for the study of the low frequency asymptotics of Maxwell's equations [43]. More involved low frequency asymptotics for Maxwell's equations can be found in the series [29–33] based on the series [56–59] for the reduced scalar and elasticity wave equations. The connections of the extended Maxwell system and the Dirac operator are drawn in [45] and shortly commented in this manuscript below.

Rather recently, the notion of Hilbert complexes (reusing the idea of writing the kernel of differential operators by means of other operators) has found applications in the context of homogenisation theory of partial differential equations. More precisely, it was possible to derive a certain operator-theoretic version of the so-called div-curl lemma (see [26,48]), which implied a whole family of div-curl lemma-type results, see [35,50].

Furthermore, the abstract div-curl result together with theory from Hilbert complexes are used to define and study the notion of nonlocal H-convergence, [51]. The applications presented in [51,52] as well as in [28] use the assumption of exactness of the Hilbert complex, that is, triviality of certain cohomology groups. It is one corollary of the present study to describe the topological properties of the domains the differential operators are defined on to yield exact Hilbert complexes making the theory of nonlocal H-convergence applicable, see also [51, Section 8]. This then results in new homogenisation and compactness theorems for nonlocal homogenisation problems. We postpone further results in this direction to future studies.

We shortly outline the plan of the paper next. The main results, that is, the dimensions of the cohomology groups and the indices of the operators involved, are summarised in Sect. 2. In Sect. 3, we briefly recall the notion of Hilbert complexes of densely defined and closed linear operators. Also, we provide a small introduction to the construction principle for Fredholm operators provided in [11]. As we slightly deviate from the approach presented there we recall some of the proofs for convenience of the reader. As an addendum to Sect. 3, we provide an abstract set of Poincaré–Friedrichs inequalities in Sect. 4 and outline an abstract perspective to variable coefficents in Sect. 5. In order to have a first non-trivial yet rather elementary example at hand, we present the so-called Picard's extended Maxwell system in Sect. 6. This sets the stage for the index theorem for the Dirac operator provided in Sect. 6.3. In Sect. 7, we recall the first biharmonic complex and provide a more explicit formulation of Theorem 1.1 (see Theorem 7.4). Similar results will be presented in Sect. 8 for the second biharmonic complex and in Sect. 9 for the elasticity complex. Section 10 is concerned with the topological setting introduced in [42] forming our main assumption on Ω. The Sects. 11 and 12 address the computation of bases and hence the dimensions of the generalised Dirichlet and Neumann vector and tensor fields for the different complexes, respectively, and thus concluding the proofs of our main results. In passing, we also provide partial differential equations whose unique solutions will correspond to the basis functions under consideration. This is particularly important for numerically computing these basis functions. Amongst these PDEs we recover the one in [14], when the generalised Neumann fields for the elasticity complex are concerned (see in particular Remark 12.37; the regularity assumptions on Ω are the same here). In Sect. 13 we provide a small conclusion.

Note that unlike to many research topics in the analysis of partial differential equations (and other topics), we shall use Ω being 'open' and a 'domain' as synonymous terms. In particular, we shall not imply Ω to satisfy any connectivity properties, when calling Ω a domain.

2 A brief overview of the main results

We employ the notations and assumptions of Sect. 1. In particular, we shall assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded, strong Lipschitz domain. The number of connected components of Ω is n, the number of connected components of $\mathbb{R}^3\setminus\overline{\Omega}$ is m, and p denotes the number of handles (see Assumption 10.3 below). We introduce the cohomology groups

$$K_1 = \mathcal{H}_{D}^{\cdots}(\Omega), \quad K_2 = \mathcal{H}_{N}^{\cdots}(\Omega),$$

i.e., the Dirichlet and Neumann fields

$$\begin{aligned}
\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) &= \ker(\mathring{\operatorname{curl}}) \cap \ker(\operatorname{div}), & \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) &= \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{curl}),\\
\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) &= \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}}), & \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &= \ker(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \ker(\operatorname{symCurl}_{\mathbb{T}}),\\
\mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) &= \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}) \cap \ker(\operatorname{Div}_{\mathbb{T}}), & \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) &= \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{Curl}_{\mathbb{S}}),\\
\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= \ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \ker(\operatorname{Div}_{\mathbb{S}}), & \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}).
\end{aligned}$$

We will compute the dimensions of the kernels N_0, $N_{2,*}$, i.e.,

$$\begin{aligned}
\dim\ker(\mathring{\operatorname{grad}}) &= 0, & \dim\ker(\operatorname{grad}) &= n,\\
\dim\ker(\operatorname{Grad}\mathring{\operatorname{grad}}) &= 0, & \dim\ker(\operatorname{devGrad}) &= 4n,\\
\dim\ker(\operatorname{dev}\mathring{\operatorname{Grad}}) &= 0, & \dim\ker(\operatorname{Gradgrad}) &= 4n,\\
\dim\ker(\operatorname{sym}\mathring{\operatorname{Grad}}) &= 0, & \dim\ker(\operatorname{symGrad}) &= 6n,
\end{aligned}$$

and the dimensions of the cohomology groups K_1, K_2, i.e.,

$$\begin{aligned}
\dim\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) &= m-1, & \dim\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) &= p,\\
\dim\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) &= 4(m-1), & \dim\mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &= 4p,\\
\dim\mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) &= 4(m-1), & \dim\mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) &= 4p,\\
\dim\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= 6(m-1), & \dim\mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= 6p,
\end{aligned}$$

and the indices $\operatorname{ind}\mathcal{D}$, $\operatorname{ind}\mathcal{D}^*$ of the involved Fredholm operators, i.e.,

$$\begin{aligned}
\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} &= p-m-n+1, & \operatorname{ind}(\mathcal{D}^{\mathsf{Rhm}})^* &= -\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}},\\
\operatorname{ind}\mathcal{D}^{\mathsf{bih},1} &= 4(p-m-n+1), & \operatorname{ind}(\mathcal{D}^{\mathsf{bih},1})^* &= -\operatorname{ind}\mathcal{D}^{\mathsf{bih},1},\\
\operatorname{ind}\mathcal{D}^{\mathsf{bih},2} &= 4(p-m-n+1), & \operatorname{ind}(\mathcal{D}^{\mathsf{bih},2})^* &= -\operatorname{ind}\mathcal{D}^{\mathsf{bih},2},\\
\operatorname{ind}\mathcal{D}^{\mathsf{ela}} &= 6(p-m-n+1), & \operatorname{ind}(\mathcal{D}^{\mathsf{ela}})^* &= -\operatorname{ind}\mathcal{D}^{\mathsf{ela}}.
\end{aligned}$$

Remark 2.1 We observe that in all of our examples, where generally the operators A_j carry the boundary condition and the adjoints A_j^* do not have any boundary condition, the dimensions of the first and second cohomology groups K_1 and K_2 (Dirichlet fields and Neumann fields) are given by

$$\dim K_1 = \frac{\dim N_{2,*}}{n} \cdot (m-1), \qquad \dim K_2 = \frac{\dim N_{2,*}}{n} \cdot p,$$

respectively. The indices of $\mathcal{D}$ (see (1.1)) and $\mathcal{D}^*$ are

$$-\operatorname{ind} \mathcal{D}^* = \operatorname{ind} \mathcal{D} = \frac{\dim N_{2,*}}{n} \cdot (p-m-n+1). \tag{2.1}$$

Remark 2.1 leads to the following problem that seems to be open:

Problem 2.2 *Is it possible to find differential operators on $\Omega \subseteq \mathbb{R}^3$ (bounded, strong Lipschitz domain) of the form* (1.1) *as discussed in Remark 2.1 that violate the general index formula for $\mathcal{D}$ in* (2.1)*?*

3 The construction principle and the index theorem

In this section, we provide the basic construction principle, which is the basis for the operators in question. The theory in more general terms has been developed already in [11]. Here, we rephrase the situation with a slightly more particular viewpoint. For the convenience of the reader, we carry out the necessary proofs here.

Throughout this section, we let H_0, H_1, H_2, H_3 be Hilbert spaces, and

$$\begin{aligned} A_0 &: \operatorname{dom} A_0 \subseteq H_0 \longrightarrow H_1, \\ A_1 &: \operatorname{dom} A_1 \subseteq H_1 \longrightarrow H_2, \\ A_2 &: \operatorname{dom} A_2 \subseteq H_2 \longrightarrow H_3 \end{aligned}$$

be densely defined and <u>closed</u> linear operators.

Definition 3.1 Let A_0, A_1, A_2 be defined as above.

- We call a pair (A_0, A_1) a *complex* (*Hilbert complex*), if $\operatorname{ran} A_0 \subseteq \ker A_1$. In this situation we also write

$$H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2.$$

- We say a complex (A_0, A_1) is *closed*, if $\operatorname{ran} A_0$ and $\operatorname{ran} A_1$ are closed.
- A complex (A_0, A_1) is said to be *compact*, if the embedding $\operatorname{dom} A_1 \cap \operatorname{dom} A_0^* \hookrightarrow H_1$ is compact.
- The triple (A_0, A_1, A_2) is called a *(closed/compact) complex*, if both (A_0, A_1) and (A_1, A_2) are (closed/compact) complexes.
- We say that a complex (A_0, A_1, A_2) is *maximal compact*, if (A_0, A_1, A_2) is a compact complex and both embeddings $\operatorname{dom} A_0 \hookrightarrow H_0$ and $\operatorname{dom} A_2^* \hookrightarrow H_3$ are compact as well.

Remark 3.2 The 'FA-ToolBox' ('Functional-Analysis-Tool Box') from [34–36,38–41] shows that

$$(A_0, A_1) \text{ (closed/compact) complex} \iff (A_1^*, A_0^*) \text{ (closed/compact) complex.}$$

As a consequence, we obtain (A_0, A_1, A_2) is a (closed/compact/maximal compact) complex if and only if (A_2^*, A_1^*, A_0^*) is a (closed/compact/maximal compact) complex.

Throughout this section, we assume that (A_0, A_1, A_2) is a complex, i.e.,

$$H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2 \xrightarrow{A_2} H_3,$$
$$H_0 \xleftarrow{A_0^*} H_1 \xleftarrow{A_1^*} H_2 \xleftarrow{A_2^*} H_3.$$

We define the operator

$$\mathcal{D} : (\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*) \times \operatorname{dom} A_0 \subseteq H_2 \times H_0 \longrightarrow H_3 \times H_1$$
$$(x, y) \longmapsto (A_2 x, A_1^* x + A_0 y).$$

In block operator matrix notation, we have

$$\mathcal{D} = \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix}.$$

From the introduction, we recall the notation

$$N_0 := \ker A_0, \qquad N_{2,*} := \ker A_2^* \tag{3.1}$$

and

$$K_1 := \ker A_1 \cap \ker A_0^*, \qquad K_2 := \ker A_2 \cap \ker A_1^*. \tag{3.2}$$

The aim of this section is to provide a proof of Theorem 3.8 below. As a standard tool for this and related results, we recall the standard orthogonal decompositions

$$\begin{aligned} H_2 &= \overline{\operatorname{ran} A_2^*} \oplus_{H_2} \ker A_2, & H_2 &= \ker A_1^* \oplus_{H_2} \overline{\operatorname{ran} A_1}, \\ \operatorname{dom} A_2 &= (\operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*}) \oplus_{H_2} \ker A_2, & \operatorname{dom} A_1^* &= \ker A_1^* \oplus_{H_2} (\operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1}). \end{aligned} \tag{3.3}$$

Using (3.2), by the complex property we get

$$\ker A_2 = K_2 \oplus_{H_2} \overline{\operatorname{ran} A_1} \tag{3.4}$$

and hence we obtain the following (abstract) Helmholtz type decomposition

$$\begin{aligned} H_2 &= \overline{\operatorname{ran} A_2^*} \oplus_{H_2} K_2 \oplus_{H_2} \overline{\operatorname{ran} A_1}, \\ \operatorname{dom} A_2 \cap \operatorname{dom} A_1^* &= (\operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*}) \oplus_{H_2} K_2 \oplus_{H_2} (\operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1}). \end{aligned} \tag{3.5}$$

We gather some elementary facts about $\mathcal{D}$.

Proposition 3.3 *$\mathcal{D}$ is a densely defined and closed linear operator.*

Proof For the closedness of $\mathcal{D}$, we let $\big((x_k, y_k)\big)_k$ be a sequence in $\operatorname{dom} \mathcal{D}$ with $\big((x_k, y_k)\big)_k$ converging to some (x, y) in $H_2 \times H_0$ and $(\mathcal{D}(x_k, y_k))_k$ converging to (w, z) in $H_3 \times H_1$. One readily sees using the closedness of A_2 that $x \in \operatorname{dom} A_2$ and $A_2 x = w$. Next, we observe that $\operatorname{ran} A_0 \subseteq \ker A_1 \perp_{H_1} \operatorname{ran} A_1^*$. Hence, $(A_1^* x_k)_k$ and $(A_0 y_k)$ are both convergent to some $z_1 \in H_1$ and $z_2 \in H_1$, respectively. By the closedness of both A_1^* and A_0, we thus deduce that $x \in \operatorname{dom} A_1^*$ and $y \in \operatorname{dom} A_0$ with $z_1 = A_1^* x$ and $z_2 = A_0 y$ as well as $z = z_1 + z_2 = A_1^* x + A_0 y$.

For $\mathcal{D}$ being densely defined, we see that by assumption, $\operatorname{dom} A_0$ is dense in H_0. Hence, it suffices to show that $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ is dense in H_2. Thus, as $\operatorname{dom} A_2$ and $\operatorname{dom} A_1^*$ are dense in H_2, we deduce by (3.3) that $\operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*}$ and $\operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1}$ are dense in $\overline{\operatorname{ran} A_2^*}$ and $\overline{\operatorname{ran} A_1}$, respectively. Thus, the decomposition in (3.5) implies that $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ is dense in H_2, which yields the assertion. □

Theorem 3.4 $\mathcal{D}^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix}$. *More precisely,*

$$\begin{aligned}\mathcal{D}^* : \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*) \subseteq H_3 \times H_1 &\longrightarrow H_2 \times H_0 \\ (w, z) &\longmapsto (A_2^* w + A_1 z, A_0^* z).\end{aligned}$$

Proof Note that

$$\begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} \subseteq \mathcal{D}^*$$

holds by definition since for all $(x, y) \in \operatorname{dom} \mathcal{D} = (\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*) \times \operatorname{dom} A_0$ and for all $(w, z) \in \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*)$

$$\begin{aligned}\big\langle \mathcal{D}(x, y), (w, z)\big\rangle_{H_3 \times H_1} &= \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x + A_0 y, z\rangle_{H_1} \\ &= \langle x, A_2^* w + A_1 z\rangle_{H_2} + \langle y, A_0^* z\rangle_{H_0} = \big\langle (x, y), \mathcal{D}^*(w, z)\big\rangle_{H_2 \times H_0}.\end{aligned}$$

Let $(w, z) \in \operatorname{dom} \mathcal{D}^*$ and set $(u, v) := \mathcal{D}^*(w, z)$. For $y \in \operatorname{dom} A_0$ we have $(0, y) \in \operatorname{dom} \mathcal{D}$. We obtain

$$\langle A_0 y, z\rangle_{H_1} = \big\langle \mathcal{D}(0, y), (w, z)\big\rangle_{H_3 \times H_1} = \big\langle (0, y), \mathcal{D}^*(w, z)\big\rangle_{H_2 \times H_0} = \langle y, v\rangle_{H_0}.$$

Hence, $z \in \operatorname{dom} A_0^*$ and $A_0^* z = v$.

For all $x \in \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ we see $(x, 0) \in \operatorname{dom} \mathcal{D}$ and deduce that

$$\begin{aligned}\langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} &= \big\langle \mathcal{D}(x, 0), (w, z)\big\rangle_{H_3 \times H_1} \\ &= \big\langle (x, 0), \mathcal{D}^*(w, z)\big\rangle_{H_2 \times H_0} = \langle x, u\rangle_{H_2}.\end{aligned} \tag{3.6}$$

Let π_2 denote the orthonormal projector onto $\overline{\operatorname{ran} A_2^*}$ in (3.3). Then for $\widetilde{x} \in \operatorname{dom} A_2$ we have

$$x := \pi_2 \widetilde{x} \in \operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*} \subseteq \operatorname{dom} A_2 \cap \ker A_1^* \subseteq \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*, \quad A_2 x = A_2 \widetilde{x}$$

and by (3.6)

$$\langle A_2 \widetilde{x}, w\rangle_{H_3} = \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} = \langle x, u\rangle_{H_2} = \langle \widetilde{x}, \pi_2 u\rangle_{H_2}.$$

Thus $w \in \operatorname{dom} A_2^*$ and $A_2^* w = \pi_2 u$. Analogously, let π_1 denote the orthonormal projector onto $\overline{\operatorname{ran} A_1}$ in (3.3). Then for $\widetilde{x} \in \operatorname{dom} A_1^*$ we have

$$x := \pi_1 \widetilde{x} \in \operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1} \subseteq \operatorname{dom} A_1^* \cap \ker A_2 \subseteq \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*, \quad A_1^* x = A_1^* \widetilde{x}$$

and by (3.6)

$$\langle A_1^* \widetilde{x}, z\rangle_{H_1} = \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} = \langle x, u\rangle_{H_2} = \langle \widetilde{x}, \pi_1 u\rangle_{H_2}.$$

Thus $z \in \operatorname{dom} A_1$ and $A_1 z = \pi_1 u$. Therefore, $(w, z) \in \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*)$. Moreover, using the orthonormal projector π_0 onto K_2 in (3.5) we see for $x \in K_2$ by (3.6)

$$\langle x, \pi_0 u\rangle_{H_2} = \langle \pi_0 x, u\rangle_{H_2} = \langle x, u\rangle_{H_2} = \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} = 0,$$

yielding $\pi_0 u = 0$. Finally, by (3.5) we arrive at

$$\mathcal{D}^*(w, z) = (u, v) = (\pi_0 u + \pi_1 u + \pi_2 u, A_0^* z) = (A_1 z + A_2^* w, A_0^* z),$$

completing the proof. □

Lemma 3.5 *With the settings (3.1) and (3.2), the kernels of $\mathcal{D}$ and $\mathcal{D}^*$ read*

$$\ker\mathcal{D} = K_2 \times N_0 = (\ker A_2 \cap \ker A_1^*) \times \ker A_0,$$
$$\ker\mathcal{D}^* = N_{2,*} \times K_1 = \ker A_2^* \times (\ker A_1 \cap \ker A_0^*).$$

Proof For $(x, y) \in \ker\mathcal{D}$ we have $A_2 x = 0$ and $A_1^* x + A_0 y = 0$. By orthogonality and the complex property, i.e., $\operatorname{ran} A_0 \subseteq \ker A_1 \perp_{H_1} \operatorname{ran} A_1^*$, we see $A_1^* x = A_0 y = 0$. The assertion about $\ker\mathcal{D}^*$ (use Theorem 3.4 and Remark 3.2) follows analogously. □

With Lemma 3.5 at hand, the following result is immediate.

Corollary 3.6 *The closures of the ranges of $\mathcal{D}$ and $\mathcal{D}^*$ are given by*

$$\overline{\operatorname{ran}\mathcal{D}} = (\ker\mathcal{D}^*)^{\perp_{H_3\times H_1}} = N_{2,*}^{\perp_{H_3}} \times K_1^{\perp_{H_1}},$$
$$\overline{\operatorname{ran}\mathcal{D}^*} = (\ker\mathcal{D})^{\perp_{H_2\times H_0}} = K_2^{\perp_{H_2}} \times N_0^{\perp_{H_0}}.$$

Lemma 3.7 *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then the embedding $\operatorname{dom}\mathcal{D} \hookrightarrow H_2 \times H_0$ is compact, and so is the embedding $\operatorname{dom}\mathcal{D}^* \hookrightarrow H_3 \times H_1$.*

Proof Let $\big((x_k, y_k)\big)_k$ be a $(\operatorname{dom}\mathcal{D})$-bounded sequence in $\operatorname{dom}\mathcal{D}$. Then, as in the proof of Lemma 3.5, by orthogonality and the complex property $(x_k)_k$ is a $(\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*)$-bounded sequence in $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ and $(y_k)_k$ is a $(\operatorname{dom} A_0)$-bounded sequence in $\operatorname{dom} A_0$. Since (A_0, A_1, A_2) is maximal compact, we can extract converging subsequences of $(x_k)_k$ and $(y_k)_k$. Analogously, using Theorem 3.4 and Remark 3.2, we see that also $\operatorname{dom}\mathcal{D}^* \hookrightarrow H_3 \times H_1$ is compact, finishing the proof. □

We now recall the abstract index theorem taken from [11] formulated for the present situation.

Theorem 3.8 *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then $\mathcal{D}$ and $\mathcal{D}^*$ are Fredholm operators with indices*

$$\operatorname{ind}\mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \qquad \operatorname{ind}\mathcal{D}^* = -\operatorname{ind}\mathcal{D}.$$

Proof Utilising the 'FA-ToolBox' from, e.g., [34–36,38–41], and Lemma 3.7 we observe that both ranges $\operatorname{ran}\mathcal{D}$ and $\operatorname{ran}\mathcal{D}^*$ are closed and that both kernels $\ker\mathcal{D}$ and $\ker\mathcal{D}^*$ are finite-dimensional. Therefore, both $\mathcal{D}$ and $\mathcal{D}^*$ are Fredholm operators. The index $\operatorname{ind}\mathcal{D} = \dim\ker\mathcal{D} - \dim\ker\mathcal{D}^*$ is then easily computed with the help of Lemma 3.5. □

4 Abstract Poincaré–Friedrichs type inequalities

Let us mention some additional features of the 'FA-ToolBox' from [34–36,38–41]. Lemma 3.7 and Theorem 3.8 imply some additional results for the reduced operators

$$\mathcal{D}_{\mathsf{red}} := \mathcal{D}|_{\operatorname{ran}\mathcal{D}^*} = \mathcal{D}|_{(\ker\mathcal{D})^{\perp_{H_2\times H_0}}}, \qquad \mathcal{D}^*_{\mathsf{red}} := \mathcal{D}^*|_{\operatorname{ran}\mathcal{D}} = \mathcal{D}^*|_{(\ker\mathcal{D}^*)^{\perp_{H_3\times H_1}}}.$$

Corollary 4.1 *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then the inverse operators $\mathcal{D}_{red}^{-1} : \operatorname{ran}\mathcal{D} \to \operatorname{ran}\mathcal{D}^*$ and $(\mathcal{D}^*_{red})^{-1} : \operatorname{ran}\mathcal{D}^* \to \operatorname{ran}\mathcal{D}$ are compact. Moreover, $\mathcal{D}_{red}^{-1} : \operatorname{ran}\mathcal{D} \to \operatorname{dom}\mathcal{D}_{red}$ and $(\mathcal{D}^*_{red})^{-1} : \operatorname{ran}\mathcal{D}^* \to \operatorname{dom}\mathcal{D}^*_{red}$ are continuous and, equivalently, the Friedrichs–Poincaré type estimates*

$$\big|(x, y)\big|_{H_2\times H_0} \leq c_{\mathcal{D}}\big|\mathcal{D}(x, y)\big|_{H_3\times H_1} = c_{\mathcal{D}}\big(|A_2 x|_{H_3}^2 + |A_1^* x|_{H_1}^2 + |A_0 y|_{H_1}^2\big)^{1/2},$$

$$\big|(w,z)\big|_{H_3\times H_1} \le c_{\mathcal{D}}\big|\mathcal{D}^*(w,z)\big|_{H_2\times H_0} = c_{\mathcal{D}}\big(|A_2^* w|_{H_2}^2 + |A_1 z|_{H_2}^2 + |A_0^* z|_{H_0}^2\big)^{1/2}$$

hold for all $(x, y) \in \operatorname{dom}\mathcal{D}_{red}$ *and for all* $(w, z) \in \operatorname{dom}\mathcal{D}^*_{red}$ *with the same optimal constant* $c_{\mathcal{D}} > 0$.

The latter estimates are additive combinations of the corresponding estimates for A_0 and (A_2, A_1^*) as well as A_2^* and (A_1, A_0^*), respectively.

Remark 4.2 The compactness assumptions (maximal compact) are not needed to render $\mathcal{D}$ and $\mathcal{D}^*$ Fredholm operators. It suffices to assume that (A_0, A_1, A_2) is a closed Hilbert complex with finite-dimensional kernels N_0 and $N_{2,*}$ and finite-dimensional cohomology groups K_1 and K_2. In this case, the latter Friedrichs–Poincaré type estimates still hold and $\mathcal{D}_{\mathsf{red}}^{-1}$ and $(\mathcal{D}^*_{\mathsf{red}})^{-1}$ are still continuous.

Remark 4.3 There are simple relations between the primal, dual, and adjoint complexes, when $\mathcal{D}$ is considered. More precisely, let us denote the latter primal operators $\mathcal{D}$ and $\mathcal{D}^*$ of the primal complex (A_0, A_1, A_2) by

$$\mathcal{D} = \mathcal{D}^p = \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix}, \qquad \mathcal{D}^* = (\mathcal{D}^p)^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix},$$

and the dual operators corresponding to the dual complex (A_2^*, A_1^*, A_0^*) by

$$\mathcal{D}^d = \begin{pmatrix} A_0^* & 0 \\ A_1 & A_2^* \end{pmatrix}, \qquad (\mathcal{D}^d)^* = \begin{pmatrix} A_0 & A_1^* \\ 0 & A_2 \end{pmatrix}.$$

By Remark 3.2 (A_0, A_1, A_2) is a maximal compact complex, if and only if (A_2^*, A_1^*, A_0^*) is a maximal compact complex. Note that we may weaken the assumptions along the lines sketched in Remark 4.2. Theorem 3.8 shows that $\mathcal{D}^p$, $(\mathcal{D}^p)^*$, $\mathcal{D}^d$, $(\mathcal{D}^d)^*$ are Fredholm operators with indices

$$\operatorname{ind}\mathcal{D}^p = \dim N_0^p - \dim K_1^p + \dim K_2^p - \dim N_{2,*}^p, \qquad \operatorname{ind}(\mathcal{D}^p)^* = -\operatorname{ind}\mathcal{D}^p,$$
$$\operatorname{ind}\mathcal{D}^d = \dim N_0^d - \dim K_1^d + \dim K_2^d - \dim N_{2,*}^d, \qquad \operatorname{ind}(\mathcal{D}^d)^* = -\operatorname{ind}\mathcal{D}^d.$$

Next we observe

$$N_0^d = \ker A_2^* = N_{2,*}^p, \qquad N_{2,*}^d = \ker A_0 = N_0^p,$$
$$K_1^d = \ker A_1^* \cap \ker A_2 = K_2^p, \qquad K_2^d = \ker A_0^* \cap \ker A_1 = K_1^p.$$

Hence

$$-\operatorname{ind}(\mathcal{D}^d)^* = \operatorname{ind}\mathcal{D}^d = -\operatorname{ind}\mathcal{D}^p = \operatorname{ind}(\mathcal{D}^p)^*.$$

Note that basically $\mathcal{D}^d$ and $(\mathcal{D}^p)^*$ as well as $\mathcal{D}^p$ and $(\mathcal{D}^d)^*$ are the ‘same’ operators.

5 The case of variable coefficients

Note that the Hilbert space adjoints A_l^* depend on the particular choice of the inner products (metrics) of the underlying Hilbert spaces H_l. A typical example is simply given by ‘weighted’ inner products induced by ‘weights’ λ_l, $l \in \{0, 1, 2, 3\}$, i.e., symmetric and positive topological isomorphisms (symmetric and positive bijective bounded linear operators) $\lambda_l : H_l \to H_l$ inducing inner products

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\widetilde{H}_l} := \langle\lambda_l\,\cdot\,,\,\cdot\,\rangle_{H_l} : \widetilde{H}_l \times \widetilde{H}_l \to \mathbb{C},$$

where $\widetilde{H}_l := H_l$ (as linear space) equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\widetilde{H}_l}$. A sufficiently general situation is defined by $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, and λ_1, λ_2 being symmetric and positive topological isomorphisms, as well as $\widetilde{H}_l := \big(H_l, \langle\lambda_l\,\cdot\,,\,\cdot\,\rangle_{H_l}\big)$, $l \in \{0,1,2,3\}$. Then the modified operators[2]

$$
\begin{aligned}
\widetilde{A}_0 : \operatorname{dom}\widetilde{A}_0 &:= \operatorname{dom}A_0 \subseteq \widetilde{H}_0 \longrightarrow \widetilde{H}_1; & x &\longmapsto A_0x,\\
\widetilde{A}_1 : \operatorname{dom}\widetilde{A}_1 &:= \operatorname{dom}A_1 \subseteq \widetilde{H}_1 \longrightarrow \widetilde{H}_2; & y &\longmapsto \lambda_2^{-1}A_1y,\\
\widetilde{A}_2 : \operatorname{dom}\widetilde{A}_2 &:= \lambda_2^{-1}\operatorname{dom}A_2 \subseteq \widetilde{H}_2 \longrightarrow \widetilde{H}_3; & z &\longmapsto A_2\lambda_2z,\\
\widetilde{A}_0^* : \operatorname{dom}\widetilde{A}_0^* &= \lambda_1^{-1}\operatorname{dom}A_0^* \subseteq \widetilde{H}_1 \longrightarrow \widetilde{H}_0; & y &\longmapsto A_0^*\lambda_1y,\\
\widetilde{A}_1^* : \operatorname{dom}\widetilde{A}_1^* &= \operatorname{dom}A_1^* \subseteq \widetilde{H}_2 \longrightarrow \widetilde{H}_1; & z &\longmapsto \lambda_1^{-1}A_1^*z,\\
\widetilde{A}_2^* : \operatorname{dom}\widetilde{A}_2^* &= \operatorname{dom}A_2^* \subseteq \widetilde{H}_3 \longrightarrow \widetilde{H}_2; & x &\longmapsto A_2^*x
\end{aligned}
$$

form again a primal and dual Hilbert complex, i.e.,

$$
\widetilde{H}_0 \xrightarrow{\widetilde{A}_0} \widetilde{H}_1 \xrightarrow{\widetilde{A}_1} \widetilde{H}_2 \xrightarrow{\widetilde{A}_2} \widetilde{H}_3,
$$
$$
\widetilde{H}_0 \xleftarrow{\widetilde{A}_0^*} \widetilde{H}_1 \xleftarrow{\widetilde{A}_1^*} \widetilde{H}_2 \xleftarrow{\widetilde{A}_2^*} \widetilde{H}_3,
$$

and we can define

$$
\widetilde{\mathcal{D}} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix}, \qquad \widetilde{\mathcal{D}}^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix}.
$$

The closedness of the operators $\widetilde{A}_l$ and the complex properties are easily checked. Moreover, it is not hard to see that the closedness of $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$ is implied by the closedness of (A_0, A_1, A_2). Remark 3.2, Proposition 3.3, Theorem 3.4, Lemma 3.5, and Corollary 3.6 can be applied to $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$ as well. In particular,

$$
\begin{aligned}
\ker\widetilde{\mathcal{D}} &= \widetilde{K}_2 \times \widetilde{N}_0 = (\ker\widetilde{A}_2 \cap \ker\widetilde{A}_1^*) \times \ker\widetilde{A}_0\\
&= \big((\lambda_2^{-1}\ker A_2) \cap \ker A_1^*\big) \times \ker A_0,\\
\ker\widetilde{\mathcal{D}}^* &= \widetilde{N}_{2,*} \times \widetilde{K}_1 = \ker\widetilde{A}_2^* \times (\ker\widetilde{A}_1 \cap \ker\widetilde{A}_0^*)\\
&= \ker A_2^* \times \big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big),\\
\overline{\operatorname{ran}\widetilde{\mathcal{D}}} &= (\ker\widetilde{\mathcal{D}}^*)^{\perp_{\widetilde{H}_3\times\widetilde{H}_1}} = \widetilde{N}_{2,*}^{\perp_{\widetilde{H}_3}} \times \widetilde{K}_1^{\perp_{\widetilde{H}_1}},\\
\overline{\operatorname{ran}\widetilde{\mathcal{D}}^*} &= (\ker\widetilde{\mathcal{D}})^{\perp_{\widetilde{H}_2\times\widetilde{H}_0}} = \widetilde{K}_2^{\perp_{\widetilde{H}_2}} \times \widetilde{N}_0^{\perp_{\widetilde{H}_0}}.
\end{aligned}
$$

It is possible to relate the statements in Lemma 3.7 and Theorem 3.8 to the corresponding ones of the original complex (A_0, A_1, A_2). This will be done next.

Lemma 5.1 *The compactness properties and the dimensions of the kernels and cohomology groups of the latter complexes are independent of the weights λ_l. More precisely,*

(i) $\widetilde{N}_0 = N_0$ *and* $\widetilde{N}_{2,*} = N_{2,*}$, *as* $\operatorname{dom}\widetilde{A}_0 = \operatorname{dom}A_0$ *and* $\operatorname{dom}\widetilde{A}_{2,*} = \operatorname{dom}A_{2,*}$,

(ii₁) $\dim\big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big) = \dim\widetilde{K}_1 = \dim K_1 = \dim(\ker A_1 \cap \ker A_0^*)$,

[2] E.g., we compute $\widetilde{A}_0^*$. Let $y \in \operatorname{dom}\widetilde{A}_0^*$. Then for $x \in \operatorname{dom}\widetilde{A}_0 = \operatorname{dom}A_0$

$$
\langle x, \widetilde{A}_0^*y\rangle_{H_0} = \langle x, \widetilde{A}_0^*y\rangle_{\widetilde{H}_0} = \langle\widetilde{A}_0x, y\rangle_{\widetilde{H}_1} = \langle A_0x, \lambda_1y\rangle_{H_1},
$$

showing that $\lambda_1y \in \operatorname{dom}A_0^*$ and $A_0^*\lambda_1y = \widetilde{A}_0^*y$.

(ii₂) $\dim\big(\ker A_2 \cap (\lambda_2^{-1}\ker A_1^*)\big) = \dim \widetilde{K}_2 = \dim K_2 = \dim(\ker A_2 \cap \ker A_1^*)$,

(iii₁) $\operatorname{dom} \widetilde{A}_1 \cap \operatorname{dom} \widetilde{A}_0^* = \operatorname{dom} A_1 \cap (\lambda_1^{-1} \operatorname{dom} A_0^*) \hookrightarrow \widetilde{H}_1$ *compactly* $\Leftrightarrow$ $\operatorname{dom} A_1 \cap \operatorname{dom} A_0^* \hookrightarrow H_1$ *compactly,*

(iii₂) $\operatorname{dom} \widetilde{A}_2 \cap \operatorname{dom} \widetilde{A}_1^* = \operatorname{dom} A_2 \cap (\lambda_2^{-1} \operatorname{dom} A_1^*) \hookrightarrow \widetilde{H}_2$ *compactly* $\Leftrightarrow$ $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^* \hookrightarrow H_2$ *compactly.*

Proof For the proof we follow in close lines the ideas of [6, Theorem 6.1], where [6] is the extended version of [7]. (i) is trivial and it is sufficient to show only (ii$_1$) and (iii$_1$).

For (ii$_1$), let μ be another weight having the same properties as λ_1. Similar to (3.3), (3.5) we have by orthogonality in $\widetilde{H}_1$ and by the complex property

$$\begin{aligned}
\widetilde{H}_1 &= \overline{\operatorname{ran} \widetilde{A}_0} \oplus_{\widetilde{H}_1} \ker \widetilde{A}_0^* = \overline{\operatorname{ran} A_0} \oplus_{\widetilde{H}_1} \lambda_1^{-1} \ker A_0^*,\\
\ker \widetilde{A}_1 &= \overline{\operatorname{ran} \widetilde{A}_0} \oplus_{\widetilde{H}_1} (\ker \widetilde{A}_1 \cap \ker \widetilde{A}_0^*)\\
&= \overline{\operatorname{ran} A_0} \oplus_{\widetilde{H}_1} \big(\ker A_1 \cap (\lambda_1^{-1} \ker A_0^*)\big),
\end{aligned} \tag{5.1}$$

and we note that $\widetilde{H}_1 = H_1$ and $\ker \widetilde{A}_1 = \ker A_1$ as sets. We denote the $\widetilde{H}_1$-orthonormal projector along $\overline{\operatorname{ran} A_0}$ onto $\lambda_1^{-1} \ker A_0^*$ by π. Then, by (5.1), we deduce

$$\pi(\ker A_1) = \pi(\ker \widetilde{A}_1) = \ker A_1 \cap (\lambda_1^{-1} \ker A_0^*).$$

We consider the linear mapping

$$\widehat{\pi} : \ker A_1 \cap (\mu^{-1} \ker A_0^*) \longrightarrow \ker A_1 \cap (\lambda_1^{-1} \ker A_0^*); \qquad y \longrightarrow \pi y.$$

Then $\widehat{\pi}$ is injective. Indeed, let $y \in \ker A_1 \cap (\mu^{-1} \ker A_0^*)$ with $\widehat{\pi} y = \pi y = 0$. Then $y \in \overline{\operatorname{ran} A_0}$ and $\mu y \in \ker A_0^*$. Since $\overline{\operatorname{ran} A_0} \perp_{H_1} \ker A_0^*$, using that $\mu \geq \mu_0$ in the sense of positive definiteness for some $d > 0$, we infer $\mu_0 |y|_{H_1}^2 \leq \langle \mu y, y \rangle_{H_1} = 0$. Thus

$$\dim\big(\ker A_1 \cap (\mu^{-1} \ker A_0^*)\big) \leq \dim\big(\ker A_1 \cap (\lambda_1^{-1} \ker A_0^*)\big).$$

The other inequality $\geq$ is deduced by symmetry (in μ and λ_1) and hence equality holds.

For (iii$_1$), we use a similar decomposition strategy. Let μ be as before and let

$$\operatorname{dom} A_1 \cap (\lambda_1^{-1} \operatorname{dom} A_0^*) \hookrightarrow H_1 \tag{5.2}$$

be compact. Moreover, let us consider a bounded sequence

$$(y_k)_k \text{ in } \operatorname{dom} A_1 \cap (\mu^{-1} \operatorname{dom} A_0^*),$$

i.e., $(y_k)_k$, $(A_1 y_k)_k$, $(A_0^* \mu\, y_k)_k$ are bounded. Similar to (5.1) we get

$$\begin{aligned}
\operatorname{dom} \widetilde{A}_1 &= \overline{\operatorname{ran} \widetilde{A}_0} \oplus_{\widetilde{H}_1} (\operatorname{dom} \widetilde{A}_1 \cap \ker \widetilde{A}_0^*) = \overline{\operatorname{ran} A_0} \oplus_{\widetilde{H}_1} \big(\operatorname{dom} A_1 \cap (\lambda_1^{-1} \ker A_0^*)\big),\\
\operatorname{dom} \widetilde{A}_0^* &= (\overline{\operatorname{ran} \widetilde{A}_0} \cap \operatorname{dom} \widetilde{A}_0^*) \oplus_{\widetilde{H}_1} \ker \widetilde{A}_0^* = \big(\overline{\operatorname{ran} A_0} \cap (\lambda_1^{-1} \operatorname{dom} A_0^*)\big) \oplus_{\widetilde{H}_1} \lambda_1^{-1} \ker A_0^*,
\end{aligned}$$

and $\operatorname{dom} \widetilde{A}_1 = \operatorname{dom} A_1$ and $\operatorname{dom} \widetilde{A}_0^* = \lambda_1^{-1} \operatorname{dom} A_0^*$ as sets. Now, we apply these decompositions to $(y_k)_k$. First, we $\widetilde{H}_1$-orthogonally decompose $y_k \in \operatorname{dom} A_1$ into

$$y_k = u_k + v_k$$

with

$$u_k \in \overline{\operatorname{ran} A_0} \subseteq \ker A_1, \quad v_k \in \operatorname{dom} A_1 \cap (\lambda_1^{-1} \ker A_0^*), \quad A_1 y_k = A_1 v_k.$$

Therefore $(v_k)_k$ is bounded in $\operatorname{dom} A_1 \cap (\lambda_1^{-1} \ker A_0^*)$ and by (5.2) we can extract an H_1-converging subsequence, again denoted by $(v_k)_k$. Second, we $\widetilde{H}_1$-orthogonally decompose $\lambda_1^{-1}\mu y_k \in \lambda_1^{-1} \operatorname{dom} A_0^*$ into

$$\lambda_1^{-1}\mu y_k = w_k + z_k$$

with

$$\begin{aligned} &w_k \in \overline{\operatorname{ran} A_0} \cap (\lambda_1^{-1} \operatorname{dom} A_0^*) \subseteq \ker A_1 \cap (\lambda_1^{-1} \operatorname{dom} A_0^*), \ z_k \in \lambda_1^{-1} \ker A_0^*, \\ &A_0^*\mu y_k = A_0^*\lambda_1 w_k. \end{aligned}$$

Hence $(w_k)_k$ is bounded in $\ker A_1 \cap (\lambda_1^{-1} \operatorname{dom} A_0^*)$ and by (5.2) we can extract an H_1-converging subsequence, again denoted by $(w_k)_k$. Finally, again by H_1-orthogonality, i.e., $u_k \in \overline{\operatorname{ran} A_0} \perp_{H_1} \ker A_0^* \ni \lambda_1 z_k$,

$$\begin{aligned} \langle \mu(y_k - y_l), y_k - y_l \rangle_{H_1} &= \langle \mu(y_k - y_l), u_k - u_l \rangle_{H_1} + \langle \mu(y_k - y_l), v_k - v_l \rangle_{H_1} \\ &= \langle \lambda_1(w_k - w_l), u_k - u_l \rangle_{H_1} + \langle \mu(y_k - y_l), v_k - v_l \rangle_{H_1} \\ &\le c\big(|w_k - w_l|_{H_1} + |v_k - v_l|_{H_1}\big) \end{aligned}$$

for some $c > 0$ independently of k, l, which shows that $(y_k)_k$ is an H_1-Cauchy sequence in H_1. Thus $\operatorname{dom} A_1 \cap (\mu^{-1} \operatorname{dom} A_0^*) \hookrightarrow H_1$ is compact. □

Now we can formulate the counterparts of Lemma 3.7 and Theorem 3.8. The proofs follow immediately by Lemma 5.1.

Lemma 5.2 *Maximal compactness does not depend on the weights λ_l. More precisely: (A_0, A_1, A_2) is a maximal compact Hilbert complex, if and only if the Hilbert complex $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$ is maximal compact. In either case,* $\operatorname{dom} \widetilde{\mathcal{D}} \hookrightarrow \widetilde{H}_2 \times \widetilde{H}_0$ *and* $\operatorname{dom} \widetilde{\mathcal{D}}^* \hookrightarrow \widetilde{H}_3 \times \widetilde{H}_1$ *are compact.*

Theorem 5.3 *The Fredholm indices do not depend on the weights λ_l. More precisely: Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then $\mathcal{D}$, $\widetilde{\mathcal{D}}$, $\mathcal{D}^*$, and $\widetilde{\mathcal{D}}^*$ are Fredholm operators with indices*

$$\operatorname{ind} \widetilde{\mathcal{D}} = \operatorname{ind} \mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \quad \operatorname{ind} \widetilde{\mathcal{D}}^* = \operatorname{ind} \mathcal{D}^* = -\operatorname{ind} \mathcal{D}.$$

6 The De Rham complex and its indices

As a first application of our abstract findings, in this section, we specialise to a particular choice of the operators A_0, A_1, A_2. Also, we will show that the assumptions of Theorem 3.8 are satisfied for this particular choice of operators. We will, thus, obtain an index formula. The computations of the dimensions of the occurring cohomology groups date back to [42].

Definition 6.1 Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We put

$$\begin{aligned} \operatorname{grad}_c : C_c^\infty(\Omega) \subseteq L^2(\Omega) &\longrightarrow L^{2,3}(\Omega), & \phi &\longmapsto \operatorname{grad} \phi, \\ \operatorname{curl}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\longrightarrow L^{2,3}(\Omega), & \Phi &\longmapsto \operatorname{curl}\Phi, \\ \operatorname{div}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\longrightarrow L^2(\Omega), & \Phi &\longmapsto \operatorname{div} \Phi, \end{aligned}$$

and further define the densely defined and closed linear operators

$$\operatorname{grad} := -\operatorname{div}_c^*, \qquad \operatorname{curl} := \operatorname{curl}_c^*, \qquad \operatorname{div} := -\operatorname{grad}_c^*,$$

$$\mathring{\text{grad}} := -\operatorname{div}^* = \overline{\text{grad}_c}, \qquad \mathring{\text{curl}} := \text{curl}^* = \overline{\text{curl}_c}, \qquad \mathring{\text{div}} := -\operatorname{grad}^* = \overline{\text{div}_c}.$$

In terms of classical definitions and notions, we record the following equalities (that are easily seen):

$$\begin{aligned}
\operatorname{dom}(\text{grad}) &= H^1(\Omega), & \operatorname{dom}(\mathring{\text{grad}}) &= \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega),\\
\operatorname{dom}(\text{curl}) &= H(\text{curl}, \Omega), & \operatorname{dom}(\mathring{\text{curl}}) &= \overline{C_c^{\infty,3}(\Omega)}^{H(\text{curl},\Omega)} = H_0(\text{curl}, \Omega),\\
\operatorname{dom}(\text{div}) &= H(\text{div}, \Omega), & \operatorname{dom}(\mathring{\text{div}}) &= \overline{C_c^{\infty,3}(\Omega)}^{H(\text{div},\Omega)} = H_0(\text{div}, \Omega).
\end{aligned}$$

6.1 Picard's extended Maxwell system

We want to apply the index theorem in the following situation of the classical de Rham complex:

$$\begin{aligned}
A_0 &:= \mathring{\text{grad}}, & A_1 &:= \mathring{\text{curl}}, & A_2 &:= \mathring{\text{div}},\\
A_0^* &= -\operatorname{div}, & A_1^* &= \text{curl}, & A_2^* &= -\operatorname{grad},
\end{aligned}$$

$$\mathcal{D}^{\mathsf{Rhm}} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\text{div}} & 0 \\ \text{curl} & \mathring{\text{grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{Rhm}})^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{grad} & \mathring{\text{curl}} \\ 0 & -\operatorname{div} \end{pmatrix},$$

$$\begin{aligned}
&\{0\} \xrightarrow{A_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{A_0=\mathring{\text{grad}}} L^{2,3}(\Omega) \xrightarrow{A_1=\mathring{\text{curl}}} L^{2,3}(\Omega) \xrightarrow{A_2=\mathring{\text{div}}} L^2(\Omega) \xrightarrow{A_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}},\\
&\{0\} \xleftarrow{A_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{A_0^*=-\operatorname{div}} L^{2,3}(\Omega) \xleftarrow{A_1^*=\text{curl}} L^{2,3}(\Omega) \xleftarrow{A_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{A_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}.
\end{aligned} \tag{6.1}$$

We note

$$\begin{aligned}
\operatorname{dom}\mathcal{D}^{\mathsf{Rhm}} &= (\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*) \times \operatorname{dom} A_0 = \big(H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega)\big) \times H_0^1(\Omega),\\
\operatorname{dom}(\mathcal{D}^{\mathsf{Rhm}})^* &= \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*) = H^1(\Omega) \times \big(H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)\big).
\end{aligned}$$

The complex properties, i.e., $A_1 A_0 \subseteq 0$ and $A_2 A_1 \subseteq 0$, are based on Schwarz's lemma ensuring that $\text{curl}_c \, \text{grad}_c = 0$ and $\text{div}_c \, \text{curl}_c = 0$.

Proposition 6.2 *Let $\Omega \subseteq \mathbb{R}^3$ be open. Then*

$$\begin{aligned}
\operatorname{ran} A_0 &= \operatorname{ran}(\mathring{\text{grad}}) \subseteq \ker(\mathring{\text{curl}}) = \ker A_1,\\
\operatorname{ran} A_1 &= \operatorname{ran}(\mathring{\text{curl}}) \subseteq \ker(\mathring{\text{div}}) = \ker A_2
\end{aligned}$$

and by Remark 3.2 the same holds for the adjoints (operators without homogeneous boundary conditions).

Proof See, e.g., [47, Proposition 6.1.5]. □

Theorem 6.3 (Picard–Weber–Weck selection theorem, [44,53,55]) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded weak*[3] *Lipschitz domain. Then*

$$\operatorname{dom} A_1 \cap \operatorname{dom} A_0^* = \operatorname{dom}(\mathring{\text{curl}}) \cap \operatorname{dom}(\text{div}),$$

[3] The boundary of a weak Lipschitz domain is a 2-dimensional submanifold of the 3-dimensional Lipschitz manifold $\overline{\Omega}$ with boundary.

$$\operatorname{dom} A_2 \cap \operatorname{dom} A_1^* = \operatorname{dom}(\mathring{\operatorname{div}}) \cap \operatorname{dom}(\operatorname{curl})$$

are both compactly embedded into $H_1 = H_2 = L^{2,3}(\Omega)$.

Remark 6.4 Proposition 6.2 in conjunction with Theorem 6.3 and Rellich's selection theorems show that $(\mathring{\operatorname{grad}}, \mathring{\operatorname{curl}}, \mathring{\operatorname{div}})$ is a maximal compact complex. By Remark 3.2 so is the dual complex $(-\operatorname{grad}, \operatorname{curl}, -\operatorname{div})$.

Note that

$$\begin{aligned} N_0^{\mathsf{Rhm}} &= \ker A_0 = \ker(\mathring{\operatorname{grad}}), \\ N_{2,*}^{\mathsf{Rhm}} &= \ker A_2^* = \ker(\operatorname{grad}), \\ K_1^{\mathsf{Rhm}} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{curl}}) \cap \ker(\operatorname{div}) =: \mathcal{H}_D^{\mathsf{Rhm}}(\Omega), \\ K_2^{\mathsf{Rhm}} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{curl}) =: \mathcal{H}_N^{\mathsf{Rhm}}(\Omega), \end{aligned} \tag{6.2}$$

where we recall from the introduction the classical harmonic Dirichlet and Neumann fields $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$ and $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$, respectively.

Definition 6.5 Let $\Omega \subseteq \mathbb{R}^3$ be bounded and open. Then we denote by

- n the number of connected components of Ω,
- m the number of connected components of the complement $\mathbb{R}^3 \setminus \overline{\Omega}$,
- p the number of handles of Ω, see Assumption 10.3.

For p to be well-defined we suppose Assumption 10.3 to hold.

The dimensions of the cohomology groups are given as follows.

Theorem 6.6 ([42, Theorem 1]) *Let* $\Omega \subseteq \mathbb{R}^3$ *be open and bounded with continuous boundary. Moreover, suppose Assumption 10.3. Then*

$$\dim \mathcal{H}_D^{Rhm}(\Omega) = m - 1, \qquad \dim \mathcal{H}_N^{Rhm}(\Omega) = p.$$

In comparison to [42, Theorem 1] a modified proof of Theorem 6.6 is provided in the Sects. 11.1 and 12.1. Note that in [42] unbounded domains where considered as well, which necessitates a slightly different rationale.

Remark 6.7 Note that for Ω to have a continuous boundary[4] is equivalent for it to have the segment property, see, e.g., [4, Remark 7.8 (a)].

Let us introduce the space of piecewise constants by

$$\mathbb{R}_{\mathsf{pw}} := \left\{u \in L^2(\Omega) : \forall\, C \in \mathsf{cc}(\Omega)\, \exists\, \alpha_C \in \mathbb{R} : u|_C = \alpha_C\right\}, \tag{6.3}$$

where

$$\mathsf{cc}(\Omega) := \{C \subseteq \Omega : \text{ C is a connected component of } \Omega\}. \tag{6.4}$$

Theorem 6.8 *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded weak Lipschitz domain. Then* $\mathcal{D}^{Rhm}$ *is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{D}^{Rhm} = \dim N_0^{Rhm} - \dim K_1^{Rhm} + \dim K_2^{Rhm} - \dim N_{2,*}^{Rhm}.$$

If additionally $\Gamma = \partial\Omega$ *is continuous and Assumption 10.3 holds, then*

$$\operatorname{ind} \mathcal{D}^{Rhm} = p - m - n + 1.$$

[4] A boundary being locally representable as the graph of a continuous function.

Proof We recall Remark 6.4 and apply Theorem 3.8 together with (6.2), the observations

$$N_0^{\mathsf{Rhm}} = \ker(\mathring{\mathrm{grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{Rhm}} = \ker(\mathrm{grad}) = \mathbb{R}_{\mathsf{pw}}, \tag{6.5}$$

and Theorem 6.6. □

Remark 6.9 By Theorem 3.8 the adjoint of the de Rham operator $(\mathcal{D}^{\mathsf{Rhm}})^*$ is Fredholm as well with index $\mathrm{ind}(\mathcal{D}^{\mathsf{Rhm}})^* = -\,\mathrm{ind}\,\mathcal{D}^{\mathsf{Rhm}}$. Moreover, Picard's extended Maxwell system is given by

$$\mathcal{M}^{\mathsf{Rhm}} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{Rhm}} \\ -(\mathcal{D}^{\mathsf{Rhm}})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & A_2 & 0 \\ 0 & 0 & A_1^* & A_0 \\ -A_2^* & -A_1 & 0 & 0 \\ 0 & -A_0^* & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\mathrm{div}} & 0 \\ 0 & 0 & \mathrm{curl} & \mathring{\mathrm{grad}} \\ \mathrm{grad} & -\mathring{\mathrm{curl}} & 0 & 0 \\ 0 & \mathrm{div} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{Rhm}})^* = -\mathcal{M}^{\mathsf{Rhm}}$ and $\mathrm{ind}\,\mathcal{M}^{\mathsf{Rhm}} = \dim\ker\mathcal{M}^{\mathsf{Rhm}} - \dim\ker(\mathcal{M}^{\mathsf{Rhm}})^* = 0$. Moreover, $\dim\ker\mathcal{M}^{\mathsf{Rhm}} = n + m + p - 1$ as

$$\begin{aligned} \ker\mathcal{M}^{\mathsf{Rhm}} &= \ker(\mathcal{D}^{\mathsf{Rhm}})^* \times \ker\mathcal{D}^{\mathsf{Rhm}} \\ &= N_{2,*}^{\mathsf{Rhm}} \times K_1^{\mathsf{Rhm}} \times K_2^{\mathsf{Rhm}} \times N_0^{\mathsf{Rhm}} \\ &= \ker A_2^* \times (\ker A_1 \cap \ker A_0^*) \times (\ker A_2 \cap \ker A_1^*) \times \ker A_0 \\ &= \ker(\mathrm{grad}) \times \big(\ker(\mathring{\mathrm{curl}}) \cap \ker(\mathrm{div})\big) \times \big(\ker(\mathring{\mathrm{div}}) \cap \ker(\mathrm{curl})\big) \times \ker(\mathring{\mathrm{grad}}) \\ &= \mathbb{R}_{\mathsf{pw}} \times \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \times \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \times \{0\}. \end{aligned}$$

6.2 Variable coefficients and Poincaré–Friedrichs type inequalities

The construction of a maximal compact Hilbert complex is also possible for mixed boundary conditions as well as for inhomogeneous and anisotropic media, such as constitutive material laws, see, e.g., [5,35,36]. For mixed boundary conditions we note the following:

Problem 6.10 *In order to provide a greater variety of index theorems, it would be interesting to compute the dimensions of the harmonic Dirichlet and Neumann fields also in the situation of mixed boundary conditions. This would be particularly interesting for the more involved situations described below. At least for the authors of this article it is completely beyond their expertise in geometry and topology and it appears to be an open problem as to which index formulas could be expected in terms of subcohomologies and related concepts. In the de Rham situation discussed in this section, note that Fredholmness is guaranteed by the compactness result in [5] in conjunction with Theorem 3.8 for a suitably large class of underlying sets and boundaries.*

For inhomogeneous and anisotropic media (constitutive material laws) we have:

Remark 6.11 As mentioned before, a maximal compact Hilbert complex can also be constructed for inhomogeneous and anisotropic media. These may be considered as weights λ_l as presented in Theorem 5.3. For Maxwell's equations a typical situation is given by the choices $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, and $\lambda_1 := \varepsilon, \lambda_2 := \mu : \Omega \to \mathbb{R}^{3\times3}$ being symmetric and uniformly positive definite $L^\infty(\Omega)$-matrix (tensor) fields. Let us introduce the Hilbert spaces $L_\varepsilon^{2,3}(\Omega) := \widetilde{H}_1 := \big(L^{2,3}(\Omega), \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{L^{2,3}(\Omega)}\big)$ and similarly $L_\mu^{2,3}(\Omega) := \widetilde{H}_2$ as well as $\widetilde{H}_0 = \widetilde{H}_3 = H_0 = H_3 = L^2(\Omega)$. We look at

$$\widetilde{A}_0 := \mathring{\mathrm{grad}}, \qquad \widetilde{A}_1 := \mu^{-1}\mathring{\mathrm{curl}}, \qquad \widetilde{A}_2 := \mathring{\mathrm{div}}\mu,$$

$$\widetilde{A}_0^* = -\operatorname{div}\varepsilon, \qquad \widetilde{A}_1^* = \varepsilon^{-1}\operatorname{curl}, \qquad \widetilde{A}_2^* = -\operatorname{grad},$$

$$\widetilde{\mathcal{D}}^{\mathsf{Rhm}} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{div}}\mu & 0 \\ \varepsilon^{-1}\operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix},$$

$$(\widetilde{\mathcal{D}}^{\mathsf{Rhm}})^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{grad} & \mu^{-1}\mathring{\operatorname{curl}} \\ 0 & -\operatorname{div}\varepsilon \end{pmatrix},$$

i.e., the de Rham complex, cf. (6.1),

$$\begin{aligned}
&\{0\} \xrightarrow{\widetilde{A}_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\widetilde{A}_0=\mathring{\operatorname{grad}}} L^{2,3}_{\varepsilon}(\Omega) \xrightarrow{\widetilde{A}_1=\mu^{-1}\mathring{\operatorname{curl}}} L^{2,3}_{\mu}(\Omega) \\
&\qquad \xrightarrow{\widetilde{A}_2=\mathring{\operatorname{div}}\mu} L^2(\Omega) \xrightarrow{\widetilde{A}_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}, \\
&\{0\} \xleftarrow{\widetilde{A}_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\widetilde{A}_0^*=-\operatorname{div}\varepsilon} L^{2,3}_{\varepsilon}(\Omega) \xleftarrow{\widetilde{A}_1^*=\varepsilon^{-1}\operatorname{curl}} L^{2,3}_{\mu}(\Omega) \\
&\qquad \xleftarrow{\widetilde{A}_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{\widetilde{A}_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}.
\end{aligned} \tag{6.6}$$

Lemmas 5.1, 5.2, and Theorem 5.3 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the de Rham complex do not depend on the material weights ε and μ. More precisely,

- $\dim\big(\ker(\mathring{\operatorname{curl}})\cap(\varepsilon^{-1}\ker(\operatorname{div}))\big) = \dim\big(\ker(\mathring{\operatorname{curl}})\cap\ker(\operatorname{div})\big) = \dim\mathcal{H}^{\mathsf{Rhm}}_D(\Omega) = m-1$,
- $\dim\big((\mu^{-1}\ker(\mathring{\operatorname{div}}))\cap\ker(\operatorname{curl})\big) = \dim\big(\ker(\mathring{\operatorname{div}})\cap\ker(\operatorname{curl})\big) = \dim\mathcal{H}^{\mathsf{Rhm}}_N(\Omega) = p$,
- $\operatorname{dom}(\mathring{\operatorname{curl}})\cap\big(\varepsilon^{-1}\operatorname{dom}(\operatorname{div})\big) \hookrightarrow L^{2,3}_{\varepsilon}(\Omega)$ compactly
  $\Leftrightarrow\ \operatorname{dom}(\mathring{\operatorname{curl}})\cap\operatorname{dom}(\operatorname{div}) \hookrightarrow L^{2,3}(\Omega)$ compactly,
- $\big(\mu^{-1}\operatorname{dom}(\mathring{\operatorname{div}})\big)\cap\operatorname{dom}(\operatorname{curl}) \hookrightarrow L^{2,3}_{\mu}(\Omega)$ compactly
  $\Leftrightarrow\ \operatorname{dom}(\mathring{\operatorname{div}})\cap\operatorname{dom}(\operatorname{curl}) \hookrightarrow L^{2,3}(\Omega)$ compactly,
- $(\mathring{\operatorname{grad}}, \mu^{-1}\mathring{\operatorname{curl}}, \mathring{\operatorname{div}}\mu)$ is maximal compact iff $(\mathring{\operatorname{grad}}, \mathring{\operatorname{curl}}, \mathring{\operatorname{div}})$ is maximal compact,
- $-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{Rhm}})^* = \operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{Rhm}} = \operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} = p-m-n+1$.

At this point, see Lemma 3.5, Corollary 3.6, and (6.5), we note that the kernels and ranges are given by

$$\begin{aligned}
\ker\mathcal{D}^{\mathsf{Rhm}} &= K_2^{\mathsf{Rhm}} \times N_0^{\mathsf{Rhm}} = \mathcal{H}^{\mathsf{Rhm}}_N(\Omega)\times\{0\},\\
\ker(\mathcal{D}^{\mathsf{Rhm}})^* &= N_{2,*}^{\mathsf{Rhm}} \times K_1^{\mathsf{Rhm}} = \mathbb{R}_{\mathsf{pw}}\times\mathcal{H}^{\mathsf{Rhm}}_D(\Omega),\\
\operatorname{ran}\mathcal{D}^{\mathsf{Rhm}} &= (\ker(\mathcal{D}^{\mathsf{Rhm}})^*)^{\perp_{L^2(\Omega)\times L^{2,3}(\Omega)}} = \mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}} \times \mathcal{H}^{\mathsf{Rhm}}_D(\Omega)^{\perp_{L^{2,3}(\Omega)}},\\
\operatorname{ran}(\mathcal{D}^{\mathsf{Rhm}})^* &= (\ker\mathcal{D}^{\mathsf{Rhm}})^{\perp_{L^{2,3}(\Omega)\times L^2(\Omega)}} = \mathcal{H}^{\mathsf{Rhm}}_N(\Omega)^{\perp_{L^{2,3}(\Omega)}} \times L^2(\Omega).
\end{aligned}$$

Finally, Corollary 4.1 yields additional results for the corresponding reduced operators

$$\begin{aligned}
\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}} &= \mathcal{D}^{\mathsf{Rhm}}|_{(\ker\mathcal{D}^{\mathsf{Rhm}})^{\perp_{H_2\times H_0}}} = \begin{pmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{Rhm}}_N(\Omega)^{\perp_{L^{2,3}(\Omega)}}\times L^2(\Omega)},\\
(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^* &= (\mathcal{D}^{\mathsf{Rhm}})^*|_{(\ker(\mathcal{D}^{\mathsf{Rhm}})^*)^{\perp_{H_3\times H_1}}} = \begin{pmatrix} -\operatorname{grad} & \mathring{\operatorname{curl}} \\ 0 & -\operatorname{div} \end{pmatrix}\Big|_{\mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{Rhm}}_D(\Omega)^{\perp_{L^{2,3}(\Omega)}}}.
\end{aligned}$$

Corollary 6.12 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded weak Lipschitz domain with continuous boundary. Then*

$$(\mathcal{D}^{\mathit{Rhm}}_{\mathit{red}})^{-1} : \operatorname{ran}\mathcal{D}^{\mathit{Rhm}} \to \operatorname{ran}(\mathcal{D}^{\mathit{Rhm}})^*,$$

$$((\mathcal{D}_{red}^{Rhm})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{Rhm})^* \to \operatorname{ran}\mathcal{D}^{Rhm}$$

are compact. Furthermore,

$$(\mathcal{D}_{red}^{Rhm})^{-1} : \operatorname{ran}\mathcal{D}^{Rhm} \to \operatorname{dom}\mathcal{D}_{red}^{Rhm},$$
$$((\mathcal{D}_{red}^{Rhm})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{Rhm})^* \to \operatorname{dom}(\mathcal{D}_{red}^{Rhm})^*$$

are continuous and, equivalently, the Friedrichs–Poincaré type estimate

$$\left|(E,u)\right|_{L^{2,3}(\Omega)\times L^2(\Omega)} \le c_{\mathcal{D}^{Rhm}}\left(|\operatorname{grad} u|^2_{L^{2,3}(\Omega)} + |\operatorname{div} E|^2_{L^2(\Omega)} + |\operatorname{curl} E|^2_{L^{2,3}(\Omega)}\right)^{1/2}$$

holds for all (E,u) *in*

$$\operatorname{dom}\mathcal{D}_{red}^{Rhm} = \left(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl},\Omega)\cap \mathcal{H}_N^{Rhm}(\Omega)^{\perp_{L^{2,3}(\Omega)}}\right)\times H_0^1(\Omega)$$

or (u,E) *in*

$$\operatorname{dom}(\mathcal{D}_{red}^{Rhm})^* = \left(H^1(\Omega)\cap \mathbb{R}_{pw}^{\perp_{L^2(\Omega)}}\right)\times\left(H_0(\operatorname{curl},\Omega)\cap H(\operatorname{div},\Omega)\cap \mathcal{H}_D^{Rhm}(\Omega)^{\perp_{L^{2,3}(\Omega)}}\right)$$

with some optimal constant $c_{\mathcal{D}^{Rhm}} > 0$.

Note that the latter estimate is an additive combination of the well-known Friedrichs–Poincaré estimates for grad and the well-known Maxwell estimates for (curl, div).

6.3 The Dirac operator

In this section, we flag up a relationship of the Dirac operator and Picard's extended Maxwell system. Let the assumptions of Theorem 6.8 be satisfied. The extended Maxwell operator is an operator that is surprisingly close to the Dirac operator, see [45]. We shall carry out this construction in the following. Recall from Remark 6.9 that Picard's extended Maxwell system is given by the operator

$$\mathcal{M} := \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^* & 0 \end{pmatrix}, \quad \mathcal{D} := \mathcal{D}^{\mathsf{Rhm}}.$$

Next, we introduce the Dirac operator. For this, we define the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Setting

$$\begin{aligned} \mathcal{Q}\colon \operatorname{dom}\mathcal{Q} \subseteq L^{2,2}(\Omega) &\longrightarrow L^{2,2}(\Omega) \\ \psi &\longmapsto \sum_{j=1}^{3}\partial_j\sigma_j\psi = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}\psi, \end{aligned}$$

we define the Dirac operator

$$\mathcal{L} := \begin{pmatrix} 0 & \mathcal{Q} \\ -\mathcal{Q}^* & 0 \end{pmatrix}.$$

We have not specified the domain of definition of $\mathcal{Q}$, yet. For now, we only record that $C_c^{\infty,2}(\Omega) \subseteq \operatorname{dom}\mathcal{Q}$, and the domain of definition of $\mathcal{Q}$ corresponding to $\mathcal{M}$ is provided below, see also Proposition 6.13. We introduce the unitary operators from $L^{2,4}(\Omega)$ into itself

$$W := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad U := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the operators $\mathcal{L}$ (Dirac operator) and $\mathcal{M}$ (Picard's extended Maxwell operator) are unitarily equivalent. More precisely, we have with V from Proposition 6.13

$$\mathcal{M} = \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \mathcal{L} \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix},$$

$$\operatorname{dom} \mathcal{Q}^* \times \operatorname{dom} \mathcal{Q} := \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix} \big(\operatorname{dom} \mathcal{D}^* \times \operatorname{dom} \mathcal{D}\big) \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$$

and, consequently, $\mathcal{Q}$ with domain $\operatorname{dom}(V^*U^*\mathcal{D}WV) = \operatorname{dom}(\mathcal{D}WV)$ is a Fredholm operator. Moreover, we have $\operatorname{ind} \mathcal{L} = 0$ and

$$\operatorname{ind} \mathcal{Q} = \operatorname{ind} \mathcal{D} = p - m - n + 1.$$

We conclude this section by stating the missing proposition used above. The proofs of which are straightforward and will therefore be omitted. In a similar fashion, they can be found in [45]. For the next result we use $L^2_{\mathbb{R}}(\Omega)$ and $L^2_{\mathbb{C}}(\Omega)$ to denote the Hilbert space $L^2(\Omega)$ with the reals and the complex numbers as respective underlying field.

Proposition 6.13 (Realification of $\mathcal{L}$) *It holds:*

(i) $V : L^2_{\mathbb{C}}(\Omega) \to L^{2,2}_{\mathbb{R}}(\Omega)$ *with* $Vf := (\Re f, \Im f)$ *is unitary.*

(ii) $ViV^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(iii) $\widetilde{\mathcal{Q}} := V\mathcal{Q}V^* = \partial_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \partial_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
with $\operatorname{dom} \widetilde{\mathcal{Q}} = V \operatorname{dom} \mathcal{Q}\, V^*$.

7 The first biharmonic complex and its indices

In this section, we focus on our first main result and properly define the operators involved in the formulation of Theorem 1.1. Thus, we introduce the first biharmonic complex (see [37,38]) constructed for biharmonic problems and general relativity, but also relevant in problems for elasticity. It will be interesting to see that the differential operator is apparently of mixed order rather than just of first order. It is worth noting that the apparently leading order term is *not* dominating the lower order differential operators.

Definition 7.1 Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We put

$$\begin{aligned} \operatorname{Gradgrad}_c : C^\infty_c(\Omega) \subseteq L^2(\Omega) &\longrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega), & \phi &\longmapsto \operatorname{Gradgrad} \phi, \\ \operatorname{Curl}_c : C^{\infty,3\times3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times3}_{\mathbb{S}}(\Omega) &\longrightarrow L^{2,3\times3}_{\mathbb{T}}(\Omega), & \Phi &\longmapsto \operatorname{Curl} \Phi, \\ \operatorname{Div}_c : C^{\infty,3\times3}_{c,\mathbb{T}}(\Omega) \subseteq L^{2,3\times3}_{\mathbb{T}}(\Omega) &\longrightarrow L^{2,3}(\Omega), & \Phi &\longmapsto \operatorname{Div} \Phi, \end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned} \operatorname{divDiv}_{\mathbb{S}} &:= \operatorname{Gradgrad}_c^*, & \mathring{\operatorname{Gradgrad}} &:= \operatorname{divDiv}_{\mathbb{S}}^* = \overline{\operatorname{Gradgrad}_c}, \\ \operatorname{symCurl}_{\mathbb{T}} &:= \operatorname{Curl}_c^*, & \mathring{\operatorname{Curl}}_{\mathbb{S}} &:= \operatorname{symCurl}_{\mathbb{T}}^* = \overline{\operatorname{Curl}_c}, \\ \operatorname{devGrad} &:= -\operatorname{Div}_c^*, & \mathring{\operatorname{Div}}_{\mathbb{T}} &:= -\operatorname{devGrad}^* = \overline{\operatorname{Div}_c}. \end{aligned}$$

We shall apply the index theorem in the following situation of the first biharmonic complex:

$$\begin{aligned} A_0 &:= \mathring{\operatorname{Gradgrad}}, & A_1 &:= \mathring{\operatorname{Curl}}_{\mathbb{S}}, & A_2 &:= \mathring{\operatorname{Div}}_{\mathbb{T}}, \\ A_0^* &= \operatorname{divDiv}_{\mathbb{S}}, & A_1^* &= \operatorname{symCurl}_{\mathbb{T}}, & A_2^* &= -\operatorname{devGrad}, \end{aligned}$$

$$\begin{aligned} \mathcal{D}^{\mathsf{bih},1} &:= \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \end{pmatrix}, \\ (\mathcal{D}^{\mathsf{bih},1})^* &= \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix}. \end{aligned}$$

Introducing the space of piecewise Raviart–Thomas fields by

$$\mathsf{RT}_{\mathsf{pw}} := \big\{ v \in L^{2,3}(\Omega) : \forall\, C \in \mathsf{cc}(\Omega) \quad \exists\, \alpha_C \in \mathbb{R},\ \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C x + \beta_C \big\}, \tag{7.1}$$

for $\mathsf{cc}(\Omega)$ see (6.4), we can write the first biharmonic complex as

$$\begin{aligned} &\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \\ &\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}, \\ &\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \\ &\xleftarrow{-\operatorname{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}. \end{aligned} \tag{7.2}$$

The foundation of the index theorem to hold is the following compactness result established by Pauly and Zulehner. Note that it holds $\operatorname{dom}(\mathring{\operatorname{Gradgrad}}) = H_0^2(\Omega)$ and $\operatorname{dom}(\operatorname{devGrad}) = H^{1,3}(\Omega)$.

Theorem 7.2 ([38, Lemma 3.22, Theorem 3.23]) *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded strong Lipschitz domain. Then* $(\mathring{\operatorname{Gradgrad}}, \mathring{\operatorname{Curl}}_{\mathbb{S}}, \mathring{\operatorname{Div}}_{\mathbb{T}})$ *is a maximal compact Hilbert complex.*

We observe and define

$$\begin{aligned} N_0^{\mathsf{bih},1} &= \ker A_0 = \ker(\mathring{\operatorname{Gradgrad}}), \\ N_{2,*}^{\mathsf{bih},1} &= \ker A_2^* = \ker(\operatorname{devGrad}), \\ K_1^{\mathsf{bih},1} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}}) =: \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega), \\ K_2^{\mathsf{bih},1} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \ker(\operatorname{symCurl}_{\mathbb{T}}) =: \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega). \end{aligned} \tag{7.3}$$

The dimensions of the cohomology groups are given as follows.

Theorem 7.3 *Let* $\Omega \subseteq \mathbb{R}^3$ *be open and bounded with continuous boundary. Moreover, suppose Assumption 10.3. Then*

$$\dim \mathcal{H}_{D,\mathbb{S}}^{bih,1}(\Omega) = 4(m-1), \qquad \dim \mathcal{H}_{N,\mathbb{T}}^{bih,1}(\Omega) = 4p.$$

Proof We postpone the proof to Sects. 11.2 and 12.2. □

The proper formulation of the first main result, Theorem 1.1, reads as follows.

Theorem 7.4 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{bih,1}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{D}^{bih,1} = \dim N_0^{bih,1} - \dim K_1^{bih,1} + \dim K_2^{bih,1} - \dim N_{2,*}^{bih,1}.$$

If additionally Assumption 10.3 holds, then

$$\operatorname{ind} \mathcal{D}^{bih,1} = 4(p - m - n + 1).$$

Proof Using Theorem 7.2, we apply Theorem 3.8 together with (7.3) and the observations

$$N_0^{\mathsf{bih},1} = \ker(\operatorname{Grad}\mathring{\operatorname{grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{bih},1} = \ker(\operatorname{devGrad}) = \mathsf{RT}_{\mathsf{pw}}, \tag{7.4}$$

see [38, Lemma 3.2, Lemma 3.3], and Theorem 7.3. □

Remark 7.5 By Theorem 3.8 the adjoint $(\mathcal{D}^{\mathsf{bih},1})^*$ is Fredholm as well with index simply given by $\operatorname{ind}(\mathcal{D}^{\mathsf{bih},1})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{bih},1}$. Similar to Remark 6.9 we define the extended first biharmonic operator

$$\mathcal{M}^{\mathsf{bih},1} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{bih},1} \\ -(\mathcal{D}^{\mathsf{bih},1})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ 0 & 0 & \operatorname{symCurl}_{\mathbb{T}} & \operatorname{Grad}\mathring{\operatorname{grad}} \\ \operatorname{devGrad} & -\mathring{\operatorname{Curl}}_{\mathbb{S}} & 0 & 0 \\ 0 & -\operatorname{divDiv}_{\mathbb{S}} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{bih},1})^* = -\mathcal{M}^{\mathsf{bih},1}$ and $\operatorname{ind}\mathcal{M}^{\mathsf{bih},1} = 0$. Moreover, $\dim\ker\mathcal{M}^{\mathsf{bih},1} = 4(n + m + p - 1)$ as $\ker\mathcal{M}^{\mathsf{bih},1} = \mathsf{RT}_{\mathsf{pw}} \times \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \times \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) \times \{0\}$.

Variable Coefficients and Poincaré–Friedrichs Type Inequalities. Inhomogeneous and anisotropic media may also be considered for the first biharmonic complex, cf. Remark 6.11.

Remark 7.6 Let $\lambda_0 := \operatorname{Id}$, $\lambda_3 := \operatorname{Id}$, $\lambda_1 := \varepsilon : \Omega \to \mathbb{R}^{3\times3\times3\times3}$, and $\lambda_2 := \mu : \Omega \to \mathbb{R}^{3\times3\times3\times3}$ be $L^\infty(\Omega)$-tensor fields such that the induced respective operators on $L^{2,3\times3}_{\mathbb{S}}(\Omega)$ and $L^{2,3\times3}_{\mathbb{T}}(\Omega)$ are symmetric and strictly positive definite. Moreover, let us introduce

$$L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) := \widetilde{H}_1 := \big(L^{2,3\times3}_{\mathbb{S}}(\Omega), \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\big)$$

and similarly $L^{2,3\times3}_{\mathbb{T},\mu}(\Omega) := \widetilde{H}_2$ as well as $\widetilde{H}_0 = H_0 = L^2(\Omega)$, $\widetilde{H}_3 = H_3 = L^{2,3}(\Omega)$. We look at

$$\begin{aligned} \widetilde{A}_0 &:= \operatorname{Grad}\mathring{\operatorname{grad}}, & \widetilde{A}_1 &:= \mu^{-1}\mathring{\operatorname{Curl}}_{\mathbb{S}}, & \widetilde{A}_2 &:= \mathring{\operatorname{Div}}_{\mathbb{T}}\mu, \\ \widetilde{A}_0^* &= \operatorname{divDiv}_{\mathbb{S}}\varepsilon, & \widetilde{A}_1^* &= \varepsilon^{-1}\operatorname{symCurl}_{\mathbb{T}}, & \widetilde{A}_2^* &= -\operatorname{devGrad}, \end{aligned}$$

$$\begin{aligned} \widetilde{\mathcal{D}}^{\mathsf{bih},1} &:= \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}}\mu & 0 \\ \varepsilon^{-1}\operatorname{symCurl}_{\mathbb{T}} & \operatorname{Grad}\mathring{\operatorname{grad}} \end{pmatrix}, \\ (\widetilde{\mathcal{D}}^{\mathsf{bih},1})^* &= \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{devGrad} & \mu^{-1}\mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}}\varepsilon \end{pmatrix}, \end{aligned}$$

i.e., the first biharmonic complex, cf. (7.2),

$$\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} L^{2,3\times 3}_{\mathbb{S},\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\mathring{\operatorname{Curl}}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{T},\mu}(\Omega) \\
&\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}\mu} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}, \\
&\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}\varepsilon} L^{2,3\times 3}_{\mathbb{S},\varepsilon}(\Omega) \xleftarrow{\varepsilon^{-1}\operatorname{symCurl}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T},\mu}(\Omega) \\
&\xleftarrow{-\operatorname{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}.
\end{aligned} \tag{7.5}$$

Lemma 5.1, Lemma 5.2, and Theorem 5.3 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the first biharmonic complex do not depend on the material weights ε and μ. More precisely,

- $$\begin{aligned}\dim\big(\ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap (\varepsilon^{-1}\ker(\operatorname{divDiv}_{\mathbb{S}}))\big) &= \dim\big(\ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}})\big) \\ &= \dim \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega) = 4(m-1),\end{aligned}$$
- $$\begin{aligned}\dim\big((\mu^{-1}\ker(\mathring{\operatorname{Div}}_{\mathbb{T}})) \cap \ker(\operatorname{symCurl}_{\mathbb{T}})\big) &= \dim\big(\ker(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \ker(\operatorname{symCurl}_{\mathbb{T}})\big) \\ &= \dim \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) = 4p,\end{aligned}$$
- $$\begin{aligned}&\operatorname{dom}(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap (\varepsilon^{-1}\operatorname{dom}(\operatorname{divDiv}_{\mathbb{S}})) \hookrightarrow L^{2,3\times 3}_{\mathbb{S},\varepsilon}(\Omega) \text{ compactly} \\ \Leftrightarrow\quad &\operatorname{dom}(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{divDiv}_{\mathbb{S}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{S}}(\Omega) \text{ compactly},\end{aligned}$$
- $$\begin{aligned}&(\mu^{-1}\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{T}})) \cap \operatorname{dom}(\operatorname{symCurl}_{\mathbb{T}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{T},\mu}(\Omega) \text{ compactly} \\ \Leftrightarrow\quad &\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \operatorname{dom}(\operatorname{symCurl}_{\mathbb{T}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{T}}(\Omega) \text{ compactly},\end{aligned}$$
- $$\begin{aligned}&(\mathring{\operatorname{Gradgrad}}, \mu^{-1}\mathring{\operatorname{Curl}}_{\mathbb{S}}, \mathring{\operatorname{Div}}_{\mathbb{T}}\mu) \text{ maximal compact} \\ \Leftrightarrow\quad &(\mathring{\operatorname{Gradgrad}}, \mathring{\operatorname{Curl}}_{\mathbb{S}}, \mathring{\operatorname{Div}}_{\mathbb{T}}) \text{ maximal compact},\end{aligned}$$
- $$-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{bih},1})^* = \operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{bih},1} = \operatorname{ind}\mathcal{D}^{\mathsf{bih},1} = 4(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\begin{aligned}
\ker \mathcal{D}^{\mathsf{bih},1} &= K^{\mathsf{bih},1}_2 \times N^{\mathsf{bih},1}_0 = \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \times \{0\}, \\
\ker(\mathcal{D}^{\mathsf{bih},1})^* &= N^{\mathsf{bih},1}_{2,*} \times K^{\mathsf{bih},1}_1 = \mathsf{RT}_{\mathsf{pw}} \times \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega), \\
\operatorname{ran}\mathcal{D}^{\mathsf{bih},1} &= (\ker(\mathcal{D}^{\mathsf{bih},1})^*)^{\perp_{L^{2,3}(\Omega)\times L^{2,3\times 3}_{\mathbb{S}}(\Omega)}} = \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \times \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}, \\
\operatorname{ran}(\mathcal{D}^{\mathsf{bih},1})^* &= (\ker \mathcal{D}^{\mathsf{bih},1})^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)\times L^2(\Omega)}} = \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}} \times L^2(\Omega),
\end{aligned}$$

see Lemma 3.5, Corollary 3.6, and (7.4). Corollary 4.1 shows additional results for the corresponding reduced operators

$$\begin{aligned}
\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}} &= \mathcal{D}^{\mathsf{bih},1}|_{(\ker \mathcal{D}^{\mathsf{bih},1})^{\perp_{H_2\times H_0}}} \\
&= \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}\times L^2(\Omega)}, \\
(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^* &= (\mathcal{D}^{\mathsf{bih},1})^*|_{(\ker(\mathcal{D}^{\mathsf{bih},1})^*)^{\perp_{H_3\times H_1}}} \\
&= \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix}\Big|_{\mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\times \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}}.
\end{aligned}$$

Corollary 7.7 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$(\mathcal{D}_{red}^{bih,1})^{-1} : \operatorname{ran} \mathcal{D}^{bih,1} \to \operatorname{ran}(\mathcal{D}^{bih,1})^*,$$
$$((\mathcal{D}_{red}^{bih,1})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{bih,1})^* \to \operatorname{ran} \mathcal{D}^{bih,1}$$

are compact. Furthermore,

$$(\mathcal{D}_{red}^{bih,1})^{-1} : \operatorname{ran} \mathcal{D}^{bih,1} \to \operatorname{dom} \mathcal{D}_{red}^{bih,1},$$
$$((\mathcal{D}_{red}^{bih,1})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{bih,1})^* \to \operatorname{dom}(\mathcal{D}_{red}^{bih,1})^*$$

are continuous and, equivalently, the Friedrichs–Poincaré type estimates

$$\begin{aligned}\big|(T,u)\big|_{L^{2,3\times3}_{\mathbb{T}}(\Omega)\times L^2(\Omega)} &\leq c_{\mathcal{D}^{bih,1}}\big(|\operatorname{Gradgrad} u|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \\ &\qquad + |\operatorname{Div} T|^2_{L^{2,3}(\Omega)} + |\operatorname{symCurl} T|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\big)^{1/2}, \\ \big|(v,S)\big|_{L^{2,3}(\Omega)\times L^{2,3\times3}_{\mathbb{S}}(\Omega)} &\leq c_{\mathcal{D}^{bih,1}}\big(|\operatorname{devGrad} v|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} \\ &\qquad + |\operatorname{divDiv} S|^2_{L^2(\Omega)} + |\operatorname{Curl} S|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}\big)^{1/2}\end{aligned}$$

hold for all (T,u) in

$$\operatorname{dom} \mathcal{D}_{red}^{bih,1} = \big(\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \operatorname{dom}(\operatorname{symCurl}_{\mathbb{T}}) \cap \mathcal{H}^{bih,1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}}\big) \times H^2_0(\Omega)$$

for all (v,S) in

$$\begin{aligned}\operatorname{dom}(\mathcal{D}_{red}^{bih,1})^* = &\big(H^{1,3}(\Omega) \cap RT_{pw}^{\perp_{L^{2,3}(\Omega)}}\big) \\ &\times \big(\operatorname{dom}(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{divDiv}_{\mathbb{S}}) \cap \mathcal{H}^{bih,1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\big)\end{aligned}$$

with some optimal constant $c_{\mathcal{D}^{bih,1}} > 0$.

8 The second biharmonic complex and its indices

The second major application of the abstract findings in the Sects. 3, 4, and 5 is concerned with the second biharmonic complex. The needed operators are provided next. It is worth recalling the definitions of the operators devGrad, $\operatorname{symCurl}_{\mathbb{T}}$, and $\operatorname{divDiv}_{\mathbb{S}}$ from Definition 7.1.

Definition 8.1 Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We put

$$\begin{aligned}\operatorname{devGrad}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\longrightarrow L^{2,3\times3}_{\mathbb{T}}(\Omega), && \phi \longmapsto \operatorname{devGrad}\phi, \\ \operatorname{symCurl}_c : C_{c,\mathbb{T}}^{\infty,3\times3}(\Omega) \subseteq L^{2,3\times3}_{\mathbb{T}}(\Omega) &\longrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega), && \Phi \longmapsto \operatorname{symCurl}_{\mathbb{T}}\Phi, \\ \operatorname{divDiv}_c : C_{c,\mathbb{S}}^{\infty,3\times3}(\Omega) \subseteq L^{2,3\times3}_{\mathbb{S}}(\Omega) &\longrightarrow L^2(\Omega), && \Phi \longmapsto \operatorname{divDiv}_{\mathbb{S}}\Phi,\end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned}\operatorname{Div}_{\mathbb{T}} &:= -\operatorname{devGrad}_c^*, & \mathring{\operatorname{devGrad}} &:= -\operatorname{Div}_{\mathbb{T}}^* = \overline{\operatorname{devGrad}_c}, \\ \operatorname{Curl}_{\mathbb{S}} &:= \operatorname{symCurl}_c^*, & \mathring{\operatorname{symCurl}}_{\mathbb{T}} &:= \operatorname{Curl}_{\mathbb{S}}^* = \overline{\operatorname{symCurl}_c}, \\ \operatorname{Gradgrad} &:= \operatorname{divDiv}_c^*, & \mathring{\operatorname{divDiv}}_{\mathbb{S}} &:= \operatorname{Gradgrad}^* = \overline{\operatorname{divDiv}_c}.\end{aligned}$$

We shall apply the index theorem in the following situation of the second biharmonic complex:

$$A_0 := \operatorname{dev}\mathring{\operatorname{Grad}}, \qquad A_1 := \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \qquad A_2 := \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},$$
$$A_0^* = -\operatorname{Div}_{\mathbb{T}}, \qquad A_1^* = \operatorname{Curl}_{\mathbb{S}}, \qquad A_2^* = \operatorname{Gradgrad},$$

$$\mathcal{D}^{\mathsf{bih},2} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \end{pmatrix},$$
$$(\mathcal{D}^{\mathsf{bih},2})^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} \operatorname{Gradgrad} & \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}} \end{pmatrix},$$

$$\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{dev}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \\
&\xrightarrow{\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}} L^2(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}, \\
&\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \\
&\xleftarrow{\operatorname{Gradgrad}} L^2(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},
\end{aligned} \tag{8.1}$$

where we used the space of piecewise first order polynomials (for $\mathsf{cc}(\Omega)$ see (6.4))

$$\mathsf{P}^1_{\mathsf{pw}} := \left\{ v \in L^2(\Omega) : \forall\, C \in \mathsf{cc}(\Omega) \quad \exists\, \alpha_C \in \mathbb{R},\ \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C + \beta_C \cdot x \right\}. \tag{8.2}$$

Note that $\operatorname{dom}(\operatorname{dev}\mathring{\operatorname{Grad}}) = H^{1,3}_0(\Omega)$ by [38, Lemma 3.2].

Lemma 8.2 *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded strong Lipschitz domain. Then the regularity* $\operatorname{dom}(\operatorname{Gradgrad}) = H^2(\Omega)$ *holds and there exists* $c > 0$ *such that for all* $u \in H^2(\Omega)$

$$c\, |u|_{H^2(\Omega)} \leq |u|_{L^2(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{L^{2,3\times 3}(\Omega)}.$$

Proof Let $u \in \operatorname{dom}(\operatorname{Gradgrad})$. Then $\operatorname{grad} u \in H^{-1,3}(\Omega)$ and $\operatorname{Grad}\operatorname{grad} u \in L^{2,3\times 3}(\Omega)$. Nečas' regularity yields $\operatorname{grad} u \in L^{2,3}(\Omega)$ and thus $u \in H^1(\Omega)$ and $\operatorname{grad} u \in H^{1,3}(\Omega)$. Hence $u \in H^2(\Omega)$ and by Nečas' inequality (see [27]) we have

$$\begin{aligned}
|\operatorname{grad} u|_{L^{2,3}(\Omega)} &\leq c\big(|\operatorname{grad} u|_{H^{-1,3}(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{H^{-1,3\times 3}(\Omega)}\big) \\
&\leq c\big(|u|_{L^2(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{L^{2,3\times 3}(\Omega)}\big),
\end{aligned}$$

showing the desired estimate. □

Theorem 8.3 *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded strong Lipschitz domain. Then the second biharmonic complex* $(\operatorname{dev}\mathring{\operatorname{Grad}}, \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})$ *is a maximal compact Hilbert complex.*

Proof The assertions can be shown by using the 'FA-ToolBox' from [34–36,38–41]. In particular, the crucial compact embeddings can be shown by the same techniques used in the proof of [40, Theorem 4.7]. □

We observe and define

$$\begin{aligned}
N^{\mathsf{bih},2}_0 &= \ker A_0 = \ker(\operatorname{dev}\mathring{\operatorname{Grad}}), \\
N^{\mathsf{bih},2}_{2,*} &= \ker A_2^* = \ker(\operatorname{Gradgrad}), \\
K^{\mathsf{bih},2}_1 &= \ker A_1 \cap \ker A_0^* = \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}) \cap \ker(\operatorname{Div}_{\mathbb{T}}) =: \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega), \\
K^{\mathsf{bih},2}_2 &= \ker A_2 \cap \ker A_1^* = \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{Curl}_{\mathbb{S}}) =: \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega).
\end{aligned} \tag{8.3}$$

Theorem 8.4 *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 10.3. Then*

$$\dim \mathcal{H}^{bih,2}_{D,\mathbb{T}}(\Omega) = 4(m-1), \qquad \dim \mathcal{H}^{bih,2}_{N,\mathbb{S}}(\Omega) = 4p.$$

Proof We postpone the proof to Sects. 11.3 and 12.3. □

Theorem 8.5 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{bih,2}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{D}^{bih,2} = \dim N_0^{bih,2} - \dim K_1^{bih,2} + \dim K_2^{bih,2} - \dim N_{2,*}^{bih,2}.$$

If additionally Assumption 10.3 holds, then

$$\operatorname{ind} \mathcal{D}^{bih,2} = 4(p-m-n+1).$$

Proof Using Theorem 8.3 apply Theorem 3.8 together with (8.3), the observations

$$N_0^{\mathsf{bih},2} = \ker(\operatorname{dev}\mathring{\operatorname{Grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{bih},2} = \ker(\operatorname{Gradgrad}) = \mathsf{P}^1_{\mathsf{pw}} \tag{8.4}$$

by using [38, Lemma 3.2 (i)], and Theorem 8.4. □

Remark 8.6 By Theorem 3.8 the adjoint $(\mathcal{D}^{\mathsf{bih},2})^*$ is Fredholm as well with index simply given by $\operatorname{ind}(\mathcal{D}^{\mathsf{bih},2})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{bih},2}$. Similar to Remark 6.9 and Remark 7.5 we define the extended second biharmonic operator

$$\mathcal{M}^{\mathsf{bih},2} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{bih},2} \\ -(\mathcal{D}^{\mathsf{bih},2})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \\ -\operatorname{Gradgrad} & -\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} & 0 & 0 \\ 0 & \operatorname{Div}_{\mathbb{T}} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{bih},2})^* = -\mathcal{M}^{\mathsf{bih},2}$ and $\operatorname{ind}\mathcal{M}^{\mathsf{bih},2} = 0$. Moreover, $\dim\ker\mathcal{M}^{\mathsf{bih},2} = 4(n+m+p-1)$ as $\ker\mathcal{M}^{\mathsf{bih},2} = \mathsf{P}^1_{\mathsf{pw}} \times \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \times \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \times \{0\}$.

Variable Coefficients and Poincaré–Friedrichs Type Inequalities. Inhomogeneous and anisotropic media may also be considered for the second biharmonic complex, cf. Remark 6.11 and Remark 7.6.

Remark 8.7 Recall the notations from Remark 7.6 and set $\lambda_0 := \operatorname{Id}$, $\lambda_3 := \operatorname{Id}$, $\lambda_1 := \varepsilon$, $\lambda_2 := \mu$, and $\widetilde{H}_1 := L^{2,3\times3}_{\mathbb{T},\varepsilon}(\Omega)$, $\widetilde{H}_2 := L^{2,3\times3}_{\mathbb{S},\mu}(\Omega)$, $\widetilde{H}_0 = H_0 = L^{2,3}(\Omega)$, $\widetilde{H}_3 = H_3 = L^2(\Omega)$. We look at

$$\widetilde{A}_0 := \operatorname{dev}\mathring{\operatorname{Grad}}, \qquad \widetilde{A}_1 := \mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \qquad \widetilde{A}_2 := \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu,$$
$$\widetilde{A}_0^* = -\operatorname{Div}_{\mathbb{T}}\varepsilon, \qquad \widetilde{A}_1^* = \varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}}, \qquad \widetilde{A}_2^* = \operatorname{Gradgrad},$$

$$\widetilde{\mathcal{D}}^{\mathsf{bih},2} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu & 0 \\ \varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \end{pmatrix},$$
$$(\widetilde{\mathcal{D}}^{\mathsf{bih},2})^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} \operatorname{Gradgrad} & \mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}}\varepsilon \end{pmatrix},$$

i.e., the second biharmonic complex, cf. (8.1),

$$\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{dev}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega) \\
&\xrightarrow{\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu} L^{2}(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}, \\
&\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}\varepsilon} L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega) \xleftarrow{\varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega) \\
&\xleftarrow{\operatorname{Gradgrad}} L^{2}(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}.
\end{aligned} \tag{8.5}$$

Lemmas 5.1, 5.2, and Theorem 5.3 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the second biharmonic complex do not depend on the material weights ε and μ. More precisely,

- $$\begin{aligned}\dim\big(\ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\big(\varepsilon^{-1}\ker(\operatorname{Div}_{\mathbb{T}})\big)\big) &= \dim\big(\ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\ker(\operatorname{Div}_{\mathbb{T}})\big)\\ &= \dim\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = 4(m-1),\end{aligned}$$
- $$\begin{aligned}\dim\big(\big(\mu^{-1}\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\big)\cap\ker(\operatorname{Curl}_{\mathbb{S}})\big) &= \dim\big(\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\cap\ker(\operatorname{Curl}_{\mathbb{S}})\big)\\ &= \dim\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) = 4p,\end{aligned}$$
- $$\begin{aligned}&\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\big(\varepsilon^{-1}\operatorname{dom}(\operatorname{Div}_{\mathbb{T}})\big) \hookrightarrow L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega) \text{ compactly}\\ \Leftrightarrow\quad &\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\operatorname{dom}(\operatorname{Div}_{\mathbb{T}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{T}}(\Omega) \text{ compactly},\end{aligned}$$
- $$\begin{aligned}&\big(\mu^{-1}\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\big)\cap\operatorname{dom}(\operatorname{Curl}_{\mathbb{S}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega) \text{ compactly}\\ \Leftrightarrow\quad &\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\cap\operatorname{dom}(\operatorname{Curl}_{\mathbb{S}}) \hookrightarrow L^{2,3\times 3}_{\mathbb{S}}(\Omega) \text{ compactly},\end{aligned}$$
- $$\begin{aligned}&(\operatorname{dev}\mathring{\operatorname{Grad}},\ \mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\ \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu) \text{ maximal compact}\\ \Leftrightarrow\quad &(\operatorname{dev}\mathring{\operatorname{Grad}},\ \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\ \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \text{ maximal compact},\end{aligned}$$
- $$-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{bih},2})^* = \operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{bih},2} = \operatorname{ind}\mathcal{D}^{\mathsf{bih},2} = 4(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\begin{aligned}
\ker\mathcal{D}^{\mathsf{bih},2} &= K^{\mathsf{bih},2}_2 \times N^{\mathsf{bih},2}_0 = \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)\times\{0\},\\
\ker(\mathcal{D}^{\mathsf{bih},2})^* &= N^{\mathsf{bih},2}_{2,*} \times K^{\mathsf{bih},2}_1 = \mathsf{P}^1_{\mathsf{pw}} \times \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega),\\
\operatorname{ran}\mathcal{D}^{\mathsf{bih},2} &= (\ker(\mathcal{D}^{\mathsf{bih},2})^*)^{\perp_{L^2(\Omega)\times L^{2,3\times 3}_{\mathbb{T}}(\Omega)}} = (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}} \times \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}},\\
\operatorname{ran}(\mathcal{D}^{\mathsf{bih},2})^* &= (\ker\mathcal{D}^{\mathsf{bih},2})^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}} = \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}} \times L^{2,3}(\Omega),
\end{aligned}$$

see Lemma 3.5, Corollary 3.6, and (8.4). Corollary 4.1 shows additional results for the corresponding reduced operators

$$\begin{aligned}
\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}} = \mathcal{D}^{\mathsf{bih},2}|_{(\ker\mathcal{D}^{\mathsf{bih},2})^{\perp_{H_2\times H_0}}} &= \begin{pmatrix}\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0\\ \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}}\end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega)},\\
(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^* = (\mathcal{D}^{\mathsf{bih},2})^*|_{(\ker(\mathcal{D}^{\mathsf{bih},2})^*)^{\perp_{H_3\times H_1}}}&\\
&= \begin{pmatrix}\operatorname{Gradgrad} & \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}\\ 0 & -\operatorname{Div}_{\mathbb{T}}\end{pmatrix}\Big|_{(\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}}.
\end{aligned}$$

Corollary 8.8 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$(\mathcal{D}_{red}^{bih,2})^{-1} : \operatorname{ran} \mathcal{D}^{bih,2} \to \operatorname{ran}(\mathcal{D}^{bih,2})^*,$$
$$((\mathcal{D}_{red}^{bih,2})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{bih,2})^* \to \operatorname{ran} \mathcal{D}^{bih,2}$$

are compact. Furthermore,

$$(\mathcal{D}_{red}^{bih,2})^{-1} : \operatorname{ran} \mathcal{D}^{bih,2} \to \operatorname{dom} \mathcal{D}_{red}^{bih,2},$$
$$((\mathcal{D}_{red}^{bih,2})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{bih,2})^* \to \operatorname{dom}(\mathcal{D}_{red}^{bih,2})^*$$

are continuous and, equivalently, the Friedrichs-Poincaré type estimates

$$\begin{aligned}\big|(S, v)\big|_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)} \leq c_{\mathcal{D}^{bih,2}}\big(&|\operatorname{devGrad} v|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\\ &+ |\operatorname{divDiv} S|^2_{L^2(\Omega)} + |\operatorname{Curl} S|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\big)^{1/2},\\ \big|(u, T)\big|_{L^2(\Omega)\times L^{2,3\times 3}_{\mathbb{T}}(\Omega)} \leq c_{\mathcal{D}^{bih,2}}\big(&|\operatorname{Gradgrad} u|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}\\ &+ |\operatorname{Div} T|^2_{L^{2,3}(\Omega)} + |\operatorname{symCurl} T|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}\big)^{1/2}\end{aligned}$$

hold for all (S, v) in

$$\operatorname{dom} \mathcal{D}_{red}^{bih,2} = \big(\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{Curl}_{\mathbb{S}}) \cap \mathcal{H}_{N,\mathbb{S}}^{bih,2}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\big) \times H_0^{1,3}(\Omega)$$

for all (u, T) in

$$\begin{aligned}\operatorname{dom}(\mathcal{D}_{red}^{bih,2})^* = &\big(H^2(\Omega) \cap (P^1_{pw})^{\perp_{L^2(\Omega)}}\big)\\ &\times \big(\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}) \cap \operatorname{dom}(\operatorname{Div}_{\mathbb{T}}) \cap \mathcal{H}_{D,\mathbb{T}}^{bih,2}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}\big)\end{aligned}$$

with some optimal constant $c_{\mathcal{D}^{bih,2}} > 0$.

9 The elasticity complex and its indices

This section is devoted to adapt our main results Theorems 1.1, 7.4, and 8.5, to the elasticity complex, see [39–41] for details. Its elasticity differential operator is of mixed order as well, this time in the center of the complex. As before for the biharmonic operators, the leading order term is *not* dominating the lower order differential operators.

Definition 9.1 Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We put

$$\begin{aligned}\operatorname{symGrad}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\to L^{2,3\times 3}_{\mathbb{S}}(\Omega), & \phi &\mapsto \operatorname{sym}\operatorname{Grad}\phi,\\ \operatorname{CurlCurl}_c^\top : C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega) &\to L^{2,3\times 3}_{\mathbb{S}}(\Omega), & \Phi &\mapsto \operatorname{CurlCurl}^\top \Phi := \operatorname{Curl}(\operatorname{Curl}\Phi)^\top,\\ \operatorname{Div}_c : C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega) &\to L^{2,3}(\Omega), & \Phi &\mapsto \operatorname{Div}\Phi,\end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned}\operatorname{Div}_{\mathbb{S}} &:= -\operatorname{symGrad}_c^*, & \operatorname{sym}\mathring{\operatorname{Grad}} &:= -\operatorname{Div}_{\mathbb{S}}^* = \overline{\operatorname{symGrad}_c},\\ \operatorname{CurlCurl}_{\mathbb{S}}^\top &:= (\operatorname{CurlCurl}_c^\top)^*, & \operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^\top &:= (\operatorname{CurlCurl}_{\mathbb{S}}^\top)^* = \overline{\operatorname{CurlCurl}_c^\top},\\ \operatorname{symGrad} &:= -\operatorname{Div}_c^*, & \mathring{\operatorname{Div}}_{\mathbb{S}} &:= -\operatorname{symGrad}^* = \overline{\operatorname{Div}_c}.\end{aligned}$$

We want to apply the index theorem in the following situation of the elasticity complex:

$$A_0 := \operatorname{sym}\mathring{\operatorname{Grad}}, \qquad A_1 := \mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}, \qquad A_2 := \mathring{\operatorname{Div}}_{\mathbb{S}},$$
$$A_0^* = -\operatorname{Div}_{\mathbb{S}}, \qquad A_1^* = \operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \qquad A_2^* = -\operatorname{sym}\operatorname{Grad},$$

$$\mathcal{D}^{\mathsf{ela}} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{CurlCurl}_{\mathbb{S}}^{\top} & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix},$$
$$(\mathcal{D}^{\mathsf{ela}})^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{symGrad} & \mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top} \\ 0 & -\operatorname{Div}_{\mathbb{S}} \end{pmatrix},$$

$$\begin{aligned} &\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \\ &\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}, \\ &\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \\ &\xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}. \end{aligned} \tag{9.1}$$

The foundation of the index theorem to follow is the following compactness result established in [39–41]. Note that we have by Korn's inequalities $\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Grad}}) = H_0^{1,3}(\Omega)$ and $\operatorname{dom}(\operatorname{symGrad}) = H^{1,3}(\Omega)$.

Theorem 9.2 ([40, Theorem 4.7]) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then* $(\operatorname{sym}\mathring{\operatorname{Grad}}, \mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}, \mathring{\operatorname{Div}}_{\mathbb{S}})$ *is a maximal compact Hilbert complex.*

We observe and define

$$\begin{aligned} N_0^{\mathsf{ela}} &= \ker A_0 = \ker(\operatorname{sym}\mathring{\operatorname{Grad}}), \\ N_{2,*}^{\mathsf{ela}} &= \ker A_2^* = \ker(\operatorname{symGrad}), \\ K_1^{\mathsf{ela}} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}) \cap \ker(\operatorname{Div}_{\mathbb{S}}) =: \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega), \\ K_2^{\mathsf{ela}} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) =: \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega). \end{aligned} \tag{9.2}$$

The dimensions of the cohomology groups are given as follows.

Theorem 9.3 *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 10.3. Then*

$$\dim \mathcal{H}_{D,\mathbb{S}}^{ela}(\Omega) = 6(m-1), \qquad \dim \mathcal{H}_{N,\mathbb{S}}^{ela}(\Omega) = 6p.$$

Proof We postpone the proof to the Sects. 11.4 and 12.4. □

Let us introduce the space of piecewise rigid motions by (for $\mathsf{cc}(\Omega)$ see (6.4))

$$\mathsf{RM}_{\mathsf{pw}} := \left\{ v \in L^{2,3}(\Omega) : \forall\, C \in \mathsf{cc}(\Omega) \;\; \exists\, \alpha_C, \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C \times x + \beta_C \right\}. \tag{9.3}$$

Theorem 9.4 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{ela}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{D}^{ela} = \dim N_0^{ela} - \dim K_1^{ela} + \dim K_2^{ela} - \dim N_{2,*}^{ela}.$$

If additionally Assumption 10.3 holds, then

$$\operatorname{ind} \mathcal{D}^{ela} = 6(p-m-n+1).$$

Proof Using Theorem 9.2 apply Theorem 3.8 together with (9.2), the observations

$$N_0^{\mathsf{ela}} = \ker(\operatorname{sym}\mathring{\operatorname{Grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{ela}} = \ker(\operatorname{symGrad}) = \mathsf{RM}_{\mathsf{pw}}, \tag{9.4}$$

see [39, Lemma 3.2], and Theorem 9.3. □

Remark 9.5 By Theorem 3.8 the adjoint $(\mathcal{D}^{\mathsf{ela}})^*$ is Fredholm as well with index simply given by $\operatorname{ind}(\mathcal{D}^{\mathsf{ela}})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{ela}}$. Similar to Remark 6.9, Remark 7.5, and Remark 8.6 we define the extended elasticity operator

$$\mathcal{M}^{\mathsf{ela}} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{ela}} \\ -(\mathcal{D}^{\mathsf{ela}})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & \operatorname{CurlCurl}_{\mathbb{S}}^{\top} & \operatorname{sym}\mathring{\operatorname{Grad}} \\ \operatorname{symGrad} & -\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top} & 0 & 0 \\ 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{ela}})^* = -\mathcal{M}^{\mathsf{ela}}$ and $\operatorname{ind}\mathcal{M}^{\mathsf{ela}} = 0$. Moreover, $\dim\ker\mathcal{M}^{\mathsf{ela}} = 6(n+m+p-1)$ as $\ker\mathcal{M}^{\mathsf{ela}} = \mathsf{RM}_{\mathsf{pw}} \times \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \times \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) \times \{0\}$.

Variable coefficients and Poincaré–Friedrichs type inequalities. Inhomogeneous and anisotropic media may also be considered for the elasticity complex, cf. Remarks 6.11, 7.6, and 8.7.

Remark 9.6 Recall the notations from Remark 7.6 and Remark 8.7 and set $\lambda_0 := \operatorname{Id}$, $\lambda_3 := \operatorname{Id}$, $\lambda_1 := \varepsilon$, $\lambda_2 := \mu$, and $\widetilde{H}_3 = \widetilde{H}_0 = H_3 = H_0 = L^{2,3}(\Omega)$, $\widetilde{H}_1 := L_{\mathbb{S},\varepsilon}^{2,3\times3}(\Omega)$, $\widetilde{H}_2 := L_{\mathbb{S},\mu}^{2,3\times3}(\Omega)$. We look at

$$\begin{aligned} \widetilde{A}_0 &:= \operatorname{sym}\mathring{\operatorname{Grad}}, & \widetilde{A}_1 &:= \mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, & \widetilde{A}_2 &:= \mathring{\operatorname{Div}}_{\mathbb{S}}\mu, \\ \widetilde{A}_0^* &= -\operatorname{Div}_{\mathbb{S}}\varepsilon, & \widetilde{A}_1^* &= \varepsilon^{-1}\operatorname{CurlCurl}_{\mathbb{S}}^{\top}, & \widetilde{A}_2^* &= -\operatorname{symGrad}, \end{aligned}$$

$$\begin{aligned} \widetilde{\mathcal{D}}^{\mathsf{ela}} &:= \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}}\mu & 0 \\ \varepsilon^{-1}\operatorname{CurlCurl}_{\mathbb{S}}^{\top} & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix}, \\ (\widetilde{\mathcal{D}}^{\mathsf{ela}})^* &= \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{symGrad} & \mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top} \\ 0 & -\operatorname{Div}_{\mathbb{S}}\varepsilon \end{pmatrix}, \end{aligned}$$

i.e., the elasticity complex, cf. (9.1),

$$\begin{aligned} &\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} L_{\mathbb{S},\varepsilon}^{2,3\times3}(\Omega) \xrightarrow{\mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}} L_{\mathbb{S},\mu}^{2,3\times3}(\Omega) \\ &\xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}\mu} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}, \\ &\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}\varepsilon} L_{\mathbb{S},\varepsilon}^{2,3\times3}(\Omega) \xleftarrow{\varepsilon^{-1}\operatorname{CurlCurl}_{\mathbb{S}}^{\top}} L_{\mathbb{S},\mu}^{2,3\times3}(\Omega) \\ &\xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}. \end{aligned} \tag{9.5}$$

Lemmas 5.1, 5.2, and Theorem 5.3 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the elasticity complex do not depend on the material weights ε and μ. More precisely,

- $\dim\big(\ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap (\varepsilon^{-1}\ker(\operatorname{Div}_{\mathbb{S}}))\big) = \dim\big(\ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \ker(\operatorname{Div}_{\mathbb{S}})\big)$
$\qquad = \dim\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) = 6(m-1),$

- $$\dim\big((\mu^{-1}\ker(\mathring{\mathrm{Div}}_{\mathbb{S}}))\cap\ker(\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top})\big)=\dim\big(\ker(\mathring{\mathrm{Div}}_{\mathbb{S}})\cap\ker(\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top})\big)$$
$$=\dim\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)=6p,$$
- $$\mathrm{dom}(\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top})\cap\big(\varepsilon^{-1}\,\mathrm{dom}(\mathrm{Div}_{\mathbb{S}})\big)\hookrightarrow L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega)\text{ compactly}$$
$$\Leftrightarrow\quad\mathrm{dom}(\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top})\cap\mathrm{dom}(\mathrm{Div}_{\mathbb{S}})\hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega)\text{ compactly},$$
- $$\big(\mu^{-1}\,\mathrm{dom}(\mathring{\mathrm{Div}}_{\mathbb{S}})\big)\cap\mathrm{dom}(\mathrm{Curl}\mathrm{Curl}_{\mathbb{S}}^{\top})\hookrightarrow L^{2,3\times3}_{\mathbb{S},\mu}(\Omega)\text{ compactly}$$
$$\Leftrightarrow\quad\mathrm{dom}(\mathring{\mathrm{Div}}_{\mathbb{S}})\cap\mathrm{dom}(\mathrm{Curl}\mathrm{Curl}_{\mathbb{S}}^{\top})\hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega)\text{ compactly},$$
- $$(\mathrm{sym}\mathring{\mathrm{Grad}},\,\mu^{-1}\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top},\,\mathring{\mathrm{Div}}_{\mathbb{S}}\mu)\text{ maximal compact}$$
$$\Leftrightarrow\quad(\mathrm{sym}\mathring{\mathrm{Grad}},\,\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top},\,\mathring{\mathrm{Div}}_{\mathbb{S}})\text{ maximal compact},$$
- $$-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{ela}})^{*}=\operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{ela}}=\operatorname{ind}\mathcal{D}^{\mathsf{ela}}=6(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\begin{aligned}
\ker\mathcal{D}^{\mathsf{ela}}&=K_2^{\mathsf{ela}}\times N_0^{\mathsf{ela}}=\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)\times\{0\},\\
\ker(\mathcal{D}^{\mathsf{ela}})^{*}&=N_{2,*}^{\mathsf{ela}}\times K_1^{\mathsf{ela}}=\mathsf{RM}_{\mathsf{pw}}\times\mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega),\\
\operatorname{ran}\mathcal{D}^{\mathsf{ela}}&=(\ker(\mathcal{D}^{\mathsf{ela}})^{*})^{\perp_{L^{2,3}(\Omega)\times L^{2,3\times3}_{\mathbb{S}}(\Omega)}}=\mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\times\mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}},\\
\operatorname{ran}(\mathcal{D}^{\mathsf{ela}})^{*}&=(\ker\mathcal{D}^{\mathsf{ela}})^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}}=\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega),
\end{aligned}$$

see Lemma 3.5, Corollary 3.6, and (9.4). Corollary 4.1 shows additional results for the corresponding reduced operators

$$\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}}=\mathcal{D}^{\mathsf{ela}}|_{(\ker\mathcal{D}^{\mathsf{ela}})^{\perp_{H_2\times H_0}}}=\begin{pmatrix}\mathring{\mathrm{Div}}_{\mathbb{S}}&0\\ \mathrm{Curl}\mathrm{Curl}_{\mathbb{S}}^{\top}&\mathrm{sym}\mathring{\mathrm{Grad}}\end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega)},$$

$$\begin{aligned}
(\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^{*}&=(\mathcal{D}^{\mathsf{ela}})^{*}|_{(\ker(\mathcal{D}^{\mathsf{ela}})^{*})^{\perp_{H_3\times H_1}}}\\
&=\begin{pmatrix}-\,\mathrm{sym}\mathrm{Grad}&\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top}\\ 0&-\,\mathrm{Div}_{\mathbb{S}}\end{pmatrix}\Big|_{\mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\times\mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}}.
\end{aligned}$$

Corollary 9.7 *Let $\Omega\subseteq\mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$\begin{aligned}
(\mathcal{D}^{ela}_{red})^{-1}&:\operatorname{ran}\mathcal{D}^{ela}\to\operatorname{ran}(\mathcal{D}^{ela})^{*},\\
((\mathcal{D}^{ela}_{red})^{*})^{-1}&:\operatorname{ran}(\mathcal{D}^{ela})^{*}\to\operatorname{ran}\mathcal{D}^{ela}
\end{aligned}$$

are compact. Furthermore,

$$\begin{aligned}
(\mathcal{D}^{ela}_{red})^{-1}&:\operatorname{ran}\mathcal{D}^{ela}\to\operatorname{dom}\mathcal{D}^{ela}_{red},\\
((\mathcal{D}^{ela}_{red})^{*})^{-1}&:\operatorname{ran}(\mathcal{D}^{ela})^{*}\to\operatorname{dom}(\mathcal{D}^{ela}_{red})^{*}
\end{aligned}$$

are continuous and, equivalently, the Friedrichs–Poincaré type estimate

$$\begin{aligned}
\big|(S,v)\big|_{L^{2,3\times3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}\le c_{\mathcal{D}^{ela}}\big(&|\,\mathrm{sym}\mathrm{Grad}\,v|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\\
&+|\,\mathrm{Div}\,S|^2_{L^{2,3}(\Omega)}+|\,\mathrm{CurlCurl}^{\top}\,S|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\big)^{1/2}
\end{aligned}$$

holds for all (S,v) in

$$\operatorname{dom}\mathcal{D}^{ela}_{red}=\big(\mathrm{dom}(\mathring{\mathrm{Div}}_{\mathbb{S}})\cap\mathrm{dom}(\mathrm{Curl}\mathrm{Curl}_{\mathbb{S}}^{\top})\cap\mathcal{H}^{ela}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\big)\times H^{1,3}_0(\Omega)$$

or (v, S) *in*

$$\operatorname{dom}(\mathcal{D}^{ela}_{red})^* = \big(H^{1,3}(\Omega) \cap RM_{pw}^{\perp_{L^{2,3}(\Omega)}}\big) \\ \times \big(\operatorname{dom}(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \operatorname{dom}(\operatorname{Div}_{\mathbb{S}}) \cap \mathcal{H}^{ela}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\big)$$

with some optimal constant $c_{\mathcal{D}^{ela}} > 0$.

10 The main topological assumptions

In Theorems 6.6, 7.3, 8.4, and 9.3 we have seen that the dimensions of the harmonic Dirichlet and Neumann fields are given by the topological invariants of the open and bounded set Ω and its complement

$$\Xi := \mathbb{R}^3 \setminus \overline{\Omega},$$

i.e., by

- n, the number of connected components Ω_k of Ω, i.e., $\Omega = \dot{\bigcup}_{k=1}^{n} \Omega_k$,
- m, the number of connected components Ξ_ℓ of Ξ, i.e., $\Xi = \dot{\bigcup}_{\ell=0}^{m-1} \Xi_\ell$,
- p, the number of handles of Ω, see Assumption 10.3.

Note that $\mathsf{cc}(\Omega) = \{\Omega_1, \ldots, \Omega_n\}$ and $\mathsf{cc}(\Xi) = \{\Xi_0, \ldots, \Xi_{m-1}\}$. We have claimed

$$\begin{aligned} \dim \mathcal{H}^{\mathsf{Rhm}}_{D}(\Omega) &= m-1, & \dim \mathcal{H}^{\mathsf{Rhm}}_{N}(\Omega) &= p, \\ \dim \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega) &= 4(m-1), & \dim \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) &= 4p, \\ \dim \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) &= 4(m-1), & \dim \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) &= 4p, \\ \dim \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) &= 6(m-1), & \dim \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) &= 6p. \end{aligned}$$

The concluding sections of this manuscript are devoted to provide the corresponding proofs in detail. For the de Rham complex we follow in close lines the arguments of Picard in [42] introducing some simplifications for bounded domains and trivial material tensors ε and μ. These ideas will be adapted and modified for the proofs of the corresponding results of the other Hilbert complexes.

Assumption 10.1 $\Omega \subseteq \mathbb{R}^3$ is open and bounded with segment property, i.e., Ω has a continuous boundary $\Gamma := \partial\Omega$, see Remark 6.7.

Assumption 10.2 $\Omega \subseteq \mathbb{R}^3$ is open, bounded, and Γ is strong Lipschitz.

In view of Assumption 10.1 and Assumption 10.2 we note:

- Assumption 10.1 guarantees that $m, n \in \mathbb{N}$ are well-defined. In particular, we have $\operatorname{int} \Xi_\ell \neq \emptyset$ for all $\ell \in \{0, \ldots, m-1\}$.
- Assumption 10.2 implies Assumption 10.1.
- Assumption 10.2 simplifies some arguments, in particular, all ranges in the crucial Helmholtz type decompositions used in our proofs are closed, cf. Remarks 12.4, 12.16, 12.26, and 12.36. We emphasise that all our results presented in the following still hold with Assumption 10.2 replaced by the weaker Assumption 10.1. In this case, however, the computation (and verification of the existence of) the Fredholm index in the sections above is more involved. In fact, it is not clear if the mentioned ranges are closed and

in some of our arguments we need to use some additional density and approximation arguments.
- Our results concerning the bases and dimensions of the generalised Dirichlet and Neumann fields extend naturally to exterior domains, i.e., domains with bounded complement Ξ. For simplicity and to avoid even longer and more technical proofs we restrict ourselves to the case of bounded domains Ω here.

The key topological assumptions to be satisfied by Ω to compute a basis for the Neumann fields and for p to be well-defined, is described in detail next. For this, we recall the construction in [42].

Assumption 10.3 ([42, Section 1]) Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded. There are $p \in \mathbb{N}_0$ piecewise C^1-curves ζ_j and p C^2-surfaces F_j, $j \in \{1, \dots, p\}$, with the following properties:

(**A1**) The curves ζ_j, $j \in \{1, \dots, p\}$, are pairwise disjoint and given any closed piecewise C^1-curve ζ in Ω there exists uniquely determined $\alpha_j \in \mathbb{Z}$, $j \in \{1, \dots, p\}$, such that for all $\Phi \in \ker(\operatorname{curl})$ being continuously differentiable we have

$$\int_\zeta \langle \Phi, \mathrm{d}\lambda \rangle = \sum_{j=1}^{p} \alpha_j \int_{\zeta_j} \langle \Phi, \mathrm{d}\lambda \rangle.$$

(**A2**) F_j, $j \in \{1, \dots, p\}$, are pairwise disjoint and $F_j \cap \zeta_k$ is a singleton, if $j = k$, and empty, if $j \neq k$.

(**A3**) If $\Omega_c \in \mathsf{cc}(\Omega)$, i.e., Ω_c is a connected component of Ω, then $\Omega_c \setminus \bigcup_{j=1}^{p} F_j$ is simply connected.

p is called the topological genus of Ω and the curves ζ_j are said to represent a basis of the respective homology group of Ω.

It is worth mentioning the following local regularity results for the Dirichlet and Neumann fields (see Lemma 12.2 below), which are crucial for the construction of the Neumann fields,

$$\begin{aligned}
\mathcal{H}_D^{\mathsf{Rhm}}(\Omega), \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) &\subseteq C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega), \\
\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega), \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega), \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega), \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &\subseteq C^{\infty,3\times 3}(\Omega) \cap L_{\mathbb{S}}^{2,3\times 3}(\Omega), \\
\mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega), \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &\subseteq C^{\infty,3\times 3}(\Omega) \cap L_{\mathbb{T}}^{2,3\times 3}(\Omega).
\end{aligned} \tag{10.1}$$

In particular, all Dirichlet and Neumann fields of the respective cohomology groups are continuous and square integrable.

11 The construction of the Dirichlet fields

Let us denote the unbounded connected component of Ξ by Ξ_0 and its boundary by $\Gamma_0 := \partial\Xi_0$. The remaining connected components of Ξ are $\Xi_1, \dots, \Xi_{m-1}$ with boundaries $\Gamma_\ell := \partial\Xi_\ell$. Note that none of $\Gamma_0, \dots, \Gamma_{m-1}$ need to be connected. Furthermore, let us introduce an open (and bounded) ball $B \supset \overline{\Omega}$ and set $\widetilde{\Xi}_0 := B \cap \Xi_0$. Then the connected components of $B \setminus \overline{\Omega}$ are $\widetilde{\Xi}_0$ and $\Xi_1, \dots, \Xi_{m-1}$. Moreover, let

$$\xi_\ell \in C_c^\infty(\mathbb{R}^3), \quad \ell \in \{1, \dots, m-1\}, \tag{11.1}$$

with disjoint supports such that $\xi_\ell = 0$ in a neighbourhood of Ξ_0 and in a neighbourhood of Ξ_k for all $k \in \{1, \dots, m-1\}, k \neq \ell$, as well as $\xi_\ell = 1$ in a neighbourhood of Ξ_ℓ. In particular,

$\xi_\ell = 0$ in a neighbourhood of Γ_0 and in a neighbourhood of Γ_k for all $k \in \{1, \dots, m-1\}$, $k \neq \ell$, and $\xi_\ell = 1$ in a neighbourhood of Γ_ℓ. Theses indicator type functions ξ_ℓ will be used to construct a basis for the respective Dirichlet fields.

11.1 Dirichlet vector fields of the classical de Rham complex

In this section, we rephrase the core arguments of [42] in the simplified setting of bounded domains and trivial materials ε and μ. In order to highlight the apparent similarities and to motivate our rationale carried out for more involved situations later on, we shall present the construction for Dirichlet fields (and similarly for Neumann fields) in a seemingly great detail.

For the de Rham complex, see also (3.3) and (3.4), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3}(\Omega) &= H_1 = \operatorname{ran} A_0 \oplus_{H_1} \ker A_0^* = \operatorname{ran}(\mathring{\operatorname{grad}}, \Omega) \oplus_{L^{2,3}(\Omega)} \ker(\operatorname{div}, \Omega), \\ \ker(\mathring{\operatorname{curl}}, \Omega) &= \ker(A_1) = \operatorname{ran} A_0 \oplus_{H_1} K_1 = \operatorname{ran}(\mathring{\operatorname{grad}}, \Omega) \oplus_{L^{2,3}(\Omega)} \mathcal{H}_D^{\mathsf{Rhm}}(\Omega). \end{aligned} \tag{11.2}$$

Remark 11.1 We have $\operatorname{dom}(\mathring{\operatorname{grad}}, \Omega) = H_0^1(\Omega)$. Moreover, the range in (11.2) is closed due to the Friedrichs estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^1(\Omega) \qquad |\phi|_{L^2(\Omega)} \leq c |\operatorname{grad} \phi|_{L^{2,3}(\Omega)},$$

which follows from Assumption 10.1. We recall that for the Friedrichs estimate to hold it suffices to assume that Ω is open and bounded only.

Let us denote by $\pi : L^{2,3}(\Omega) \to \ker(\operatorname{div}, \Omega)$ the orthogonal projector along $\operatorname{ran}(\mathring{\operatorname{grad}}, \Omega)$ onto $\ker(\operatorname{div}, \Omega)$, which is well-defined according to (11.2). Moreover, we observe by (11.2) that $\pi(\ker(\mathring{\operatorname{curl}}, \Omega)) = \mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$. Recall ξ_ℓ from (11.1). Then for $\ell \in \{1, \dots, m-1\}$

$$\operatorname{grad} \xi_\ell \in C_c^{\infty,3}(\Omega) \cap \ker(\operatorname{curl}, \Omega) \subseteq \ker(\mathring{\operatorname{curl}}, \Omega).$$

Again relying on (11.2) (and Remark 11.1) for all $\ell \in \{1, \dots, m-1\}$, we find uniquely determined $\psi_\ell \in H_0^1(\Omega)$ such that

$$\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \ni \pi \operatorname{grad} \xi_\ell = \operatorname{grad}(\xi_\ell - \psi_\ell) = \operatorname{grad} u_\ell, \qquad u_\ell := \xi_\ell - \psi_\ell \in H^1(\Omega). \tag{11.3}$$

We will show that

$$\mathcal{B}_D^{\mathsf{Rhm}} := \{\operatorname{grad} u_1, \dots, \operatorname{grad} u_{m-1}\} \subseteq \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \tag{11.4}$$

defines a basis of $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$. The first step for showing this statement is the next lemma.

Lemma 11.2 *Let Assumption 10.1 be satisfied. Then* $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = \operatorname{lin} \mathcal{B}_D^{\mathsf{Rhm}}$.

Proof Let $H \in \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = \ker(\mathring{\operatorname{curl}}, \Omega) \cap \ker(\operatorname{div}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero, $\widetilde{H}$, to B belongs to $\ker(\mathring{\operatorname{curl}}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique) $u \in H_0^1(B)$ such that $\operatorname{grad} u = \widetilde{H}$ in B, see, e.g., [38, Lemma 2.24]. As $\operatorname{grad} u = \widetilde{H} = 0$ in $B \setminus \overline{\Omega}$, u must be constant in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, u vanishes in $\widetilde{\Xi}_0$. Therefore, $H = \operatorname{grad} u$ in Ω and $u \in H_0^1(B)$ such that $u|_{\widetilde{\Xi}_0} = 0$ and $u|_{\Xi_\ell} =: \alpha_\ell \in \mathbb{R}$ for all $\ell \in \{1, \dots, m-1\}$. Let us consider

$$\widehat{H} := H - \sum_{\ell=1}^{m-1} \alpha_\ell \operatorname{grad} u_\ell = \operatorname{grad} \widehat{u} \in \mathcal{H}_D^{\mathsf{Rhm}}(\Omega), \qquad \widehat{u} := u - \sum_{\ell=1}^{m-1} \alpha_\ell u_\ell \in H^1(\Omega)$$

with u_ℓ from (11.3). The extension by zero of ψ_ℓ, $\widetilde{\psi}_\ell$, to the whole of B belongs to $H_0^1(B)$. Hence as an element of $H^1(B)$ we see that

$$\widehat{u}_B := u - \sum_{\ell=1}^{m-1} \alpha_\ell \xi_\ell + \sum_{\ell=1}^{m-1} \alpha_\ell \widetilde{\psi}_\ell \in H_0^1(B)$$

vanishes in Ξ_ℓ for all $\ell \in \{0, \dots, m-1\}$. Thus $\widehat{u} = \widehat{u}_B|_\Omega \in H_0^1(\Omega)$ by Assumption 10.1, and we compute

$$|\widehat{H}|^2_{L^{2,3}(\Omega)} = \langle \operatorname{grad} \widehat{u}, \widehat{H} \rangle_{L^{2,3}(\Omega)} = 0,$$

finishing the proof. □

Before we show linear independence of the set $\mathcal{B}_D^{\mathsf{Rhm}}$, we highlight the possibility of determining the functions constructed here by solving certain PDEs. This can be used for numerically determining a basis for $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$.

Remark 11.3 (Characterisation by PDEs)

(i) It is not difficult to see that $\psi_\ell \in H_0^1(\Omega)$ as in (11.3) can be found as the solution of the standard variational formulation

$$\forall \phi \in H_0^1(\Omega) \qquad \langle \operatorname{grad} \psi_\ell, \operatorname{grad} \phi \rangle_{L^{2,3}(\Omega)} = \langle \operatorname{grad} \xi_\ell, \operatorname{grad} \phi \rangle_{L^{2,3}(\Omega)},$$

i.e., $\psi_\ell = \Delta_D^{-1} \Delta \xi_\ell$, where $\Delta_D = \operatorname{div} \mathring{\operatorname{grad}}$ denotes the Laplacian with standard homogeneous Dirichlet boundary conditions on Ω.

(ii) As a consequence of (i) and (11.3), we obtain $u_\ell = \xi_\ell - \psi_\ell = (1 - \Delta_D^{-1} \Delta) \xi_\ell \in H^1(\Omega)$ and

$$\operatorname{grad} u_\ell = (1 - \operatorname{grad} \Delta_D^{-1} \operatorname{div}) \operatorname{grad} \xi_\ell.$$

Let us also mention that for $\ell \in \{1, \dots, m-1\}$, u_ℓ solves in classical terms the Dirichlet Laplace problem

$$\begin{aligned} -\Delta u_\ell = -\operatorname{div} \operatorname{grad} u_\ell &= 0 \quad \text{in } \Omega, \\ u_\ell &= 1 \quad \text{on } \Gamma_\ell, \\ u_\ell &= 0 \quad \text{on } \Gamma_k,\ k \in \{0, \dots, m-1\} \setminus \{\ell\}, \end{aligned} \tag{11.5}$$

which is uniquely solvable. In particular, for all $\ell \in \{1, \dots, m-1\}$, $u_\ell = 0$ on Γ_0.

(iii) u (representing $H = \operatorname{grad} u$) constructed in the proof of Lemma 11.2 solves the linear Dirichlet Laplace problem

$$\begin{aligned} -\Delta u = -\operatorname{div} \operatorname{grad} u = -\operatorname{div} H &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ u &= \alpha_\ell \in \mathbb{R} && \text{on } \Gamma_\ell,\ \ell \in \{1, \dots, m-1\}, \end{aligned}$$

which is uniquely solvable as long as the constants, α_ℓ, are prescribed.

Lemma 11.4 *Let Assumption 10.1 be satisfied. Then $\mathcal{B}_D^{\mathsf{Rhm}}$ is linearly independent.*

Proof Let $\alpha_\ell \in \mathbb{R}$ for all $\ell \in \{1, \dots, m-1\}$ such that

$$\sum_{\ell=1}^{m-1} \alpha_\ell \operatorname{grad} u_\ell = 0; \qquad \text{set } u := \sum_{\ell=1}^{m-1} \alpha_\ell u_\ell.$$

Then $\operatorname{grad} u = 0$ in Ω, i.e., u is constant in each connected component of Ω. We show $u = 0$. Since $\psi_\ell \in H_0^1(\Omega)$ and $\xi_\ell \in H_0^1(B)$ we can extend $u_\ell = \xi_\ell - \psi_\ell$ from (11.3) to B by setting

$$\widetilde{u}_\ell := \begin{cases} u_\ell & \text{in } \Omega, \\ \xi_\ell & \text{in } B\setminus\overline{\Omega}, \end{cases} \qquad \operatorname{grad} \widetilde{u}_\ell = \begin{cases} \operatorname{grad} u_\ell & \text{in } \Omega, \\ \operatorname{grad} \xi_\ell = 0 & \text{in } B\setminus\overline{\Omega}. \end{cases}$$

Note $\widetilde{u}_\ell \in H_0^1(B)$. Moreover, for all $\ell \in \{1, \dots, m-1\}$, we have $\widetilde{u}_\ell = \xi_\ell = 1$ in Ξ_ℓ and $\widetilde{u}_\ell = \xi_\ell = 0$ in $\widetilde{\Xi}_0 \cup \bigcup_{k\in\{1,\dots,m-1\}\setminus\{\ell\}} \Xi_k$. Then

$$\widetilde{u} := \sum_{\ell=1}^{m-1} \alpha_\ell \widetilde{u}_\ell \in H_0^1(B)$$

with $\widetilde{u} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{grad} \widetilde{u} = 0$ in $B\setminus\overline{\Omega}$ as well as $\operatorname{grad} \widetilde{u} = \operatorname{grad} u = 0$ in Ω by assumption. Hence, $\operatorname{grad} \widetilde{u} = 0$ in B, showing $\widetilde{u} = 0$ in B. In particular, $u = 0$ in Ω, and $\alpha_\ell = \widetilde{u}|_{\Xi_\ell} = 0$ for all $\ell \in \{1, \dots, m-1\}$, finishing the proof. □

Theorem 11.5 *Let Assumption 10.1 be satisfied. Then* $\dim \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = m - 1$ *and a basis of* $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$ *is given by* (11.4).

Proof Use Lemmas 11.2 and 11.4. □

11.2 Dirichlet tensor fields of the first biharmonic complex

For the first biharmonic complex, see also (3.3), (3.5), and (11.2), we have the orthogonal decompositions

$$\begin{aligned} L_{\mathbb{S}}^{2,3\times3}(\Omega) &= \operatorname{ran}(\mathring{\operatorname{Gradgrad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega), \\ \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega) &= \operatorname{ran}(\mathring{\operatorname{Gradgrad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega). \end{aligned} \tag{11.6}$$

Remark 11.6 By [38, Lemma 3.3] we have $\operatorname{dom}(\mathring{\operatorname{Gradgrad}}, \Omega) = H_0^2(\Omega)$. Moreover, the range in (11.6) is closed by the Friedrichs type estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^2(\Omega) \qquad |\phi|_{H^1(\Omega)} \leq c |\operatorname{Gradgrad} \phi|_{L^{2,3\times3}(\Omega)},$$

which holds by Assumption 10.1. Similar to Remark 11.1 it suffices to have Ω to be open and bounded.

We define $\pi : L_{\mathbb{S}}^{2,3\times3}(\Omega) \to \ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega)$ to be the projector onto $\ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega)$ along $\operatorname{ran}(\mathring{\operatorname{Gradgrad}}, \Omega)$. By (11.6) we obtain $\pi\big(\ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega)]\big) = \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$. We recall the functions ξ_ℓ from (11.1). In contrast to the derivation for the de Rham complex, here the second order nature of Gradgrad necessitates the introduction of polynomials $\widehat{p}_j$ given by

$$\widehat{p}_0(x) := 1, \quad \widehat{p}_j(x) := x_j \qquad \big(x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3\big)$$

for $j \in \{1, 2, 3\}$. We define $\xi_{\ell,j} := \xi_\ell \widehat{p}_j$ for all $\ell \in \{1, \dots, m-1\}$ and $j \in \{0, \dots, 3\}$. In particular, for all $j \in \{0, \dots, 3\}$ and $\ell \in \{1, \dots, m-1\}$ we have $\xi_{\ell,j} = 0$ in a neighbourhood of Ξ_k for all $k \in \{0, \dots, m-1\}\setminus\{\ell\}$ and $\xi_{\ell,j} = \widehat{p}_j$ in a neighbourhood of Ξ_ℓ. Then

$$\operatorname{Gradgrad} \xi_{\ell,j} \in C_{c,\mathbb{S}}^{\infty,3\times3}(\Omega) \cap \ker(\operatorname{Curl}_{\mathbb{S}}, \Omega) \subseteq \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega).$$

By (11.6) (and the Friedrichs type estimate for $\mathring{\text{Gradgrad}}$, see Remark 11.6) there exists a unique $\psi_{\ell,j} \in H_0^2(\Omega)$ such that

$$\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \ni \pi \,\text{Gradgrad}\, \xi_{\ell,j} = \text{Gradgrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \text{Gradgrad}\, u_{\ell,j},$$

where

$$u_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^2(\Omega). \tag{11.7}$$

We shall show that

$$\mathcal{B}_D^{\mathsf{bih},1} := \Big\{ \text{Gradgrad}\, u_{\ell,j} : \ell \in \{1, \dots, m-1\},\ j \in \{0, \dots, 3\} \Big\} \subseteq \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \tag{11.8}$$

defines a basis of $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$. In order to show that the linear hull of $\mathcal{B}_D^{\mathsf{bih},1}$ generates $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$, we cite the following prerequisite.

Lemma 11.7 ([38, Theorem 3.10 (i) and Remark 3.11 (i)]) *Let $D \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Assume D is* topologically trivial, *i.e., D is simply connected and $\mathbb{R}^3 \setminus D$ is connected. Then*

$$\ker(\mathring{\text{Curl}}_{\mathbb{S}}, D) = \text{ran}(\mathring{\text{Gradgrad}}, D), \qquad \ker(\text{Curl}_{\mathbb{S}}, D) = \text{ran}(\text{Gradgrad}, D).$$

Lemma 11.8 *Let Assumption 10.1 be satisfied. Then $\mathcal{H}_{D,\mathbb{S}}^{bih,1}(\Omega) = \text{lin}\, \mathcal{B}_D^{bih,1}$.*

Proof We follow in close lines the arguments used in the proof of Lemma 11.2. For this, let $S \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) = \ker(\mathring{\text{Curl}}_{\mathbb{S}}, \Omega) \cap \ker(\text{divDiv}_{\mathbb{S}}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero, $\widetilde{S}$, to B belongs to $\ker(\mathring{\text{Curl}}_{\mathbb{S}}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique) $u \in H_0^2(B)$ such that $\text{Gradgrad}\, u = \widetilde{S}$ in B, see Lemma 11.7 and (11.6) applied to $\Omega = D = B$. Since $\text{Gradgrad}\, u = \widetilde{S} = 0$ in $B \setminus \overline{\Omega}$, u must belong to P^1, the polynomials of order at most 1, in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, u vanishes in $\widetilde{\Xi}_0$. Therefore, $S = \text{Gradgrad}\, u$ in Ω and $u \in H_0^2(B)$ is such that $u|_{\widetilde{\Xi}_0} = 0$ and $u|_{\Xi_\ell} =: p_\ell =: \sum_{j=0}^3 \alpha_{\ell,j} \widehat{p}_j \in \mathsf{P}^1$, for some unique $\alpha_{\ell,j} \in \mathbb{R}$, for all $\ell \in \{1, \dots, m-1\}$ and $j \in \{0, \dots, 3\}$. Let us consider

$$\widehat{S} := S - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j}\, \text{Gradgrad}\, u_{\ell,j} = \text{Gradgrad}\, \widehat{u} \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega),$$

$$\widehat{u} := u - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} u_{\ell,j} \in H^2(\Omega)$$

with $u_{\ell,j}$ from (11.7). The extension $\widetilde{\psi}_{\ell,j}$ of $\psi_{\ell,j}$ by zero to B belongs to $H_0^2(B)$. Hence as an element of $H^2(B)$ we see that

$$\widehat{u}_B := u - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \xi_{\ell,j} + \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \widetilde{\psi}_{\ell,j} \in H_0^2(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{u} = \widehat{u}_B|_\Omega \in H_0^2(\Omega)$ by Assumption 10.1, and we compute

$$|\widehat{S}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \text{Gradgrad}\, \widehat{u}, \widehat{S} \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0,$$

finishing the proof. $\square$

Similar to the case of the de Rham complex, we have a look at a possible numerical implementation for the computation of the basis functions. Naturally, the PDEs in question differ from one another quite substantially.

Remark 11.9 (Characterisation by PDEs)

(i) The functions $\psi_{\ell,j} \in H_0^2(\Omega)$ introduced just above (11.7) can be characterised as solutions by the standard variational formulation

$$\forall\, \phi \in H_0^1(\Omega) \qquad \langle \operatorname{grad} \psi_\ell,\ \operatorname{grad} \phi\rangle_{L^{2,3}(\Omega)} = \langle \operatorname{grad} \xi_\ell,\ \operatorname{grad} \phi\rangle_{L^{2,3}(\Omega)},$$

i.e., $\psi_{\ell,j} = (\Delta_{DD}^2)^{-1}\Delta^2\xi_{\ell,j}$, where $\Delta_{DD}^2 = \operatorname{divDiv}_{\mathbb{S}} \mathring{\operatorname{Gradgrad}}$ is the bi-Laplacian with both the functions as well as the derivatives satisfying homogeneous Dirichlet boundary conditions.

(ii) With the statement in (i) together with (11.7), we deduce for all $\ell \in \{1,\dots,m-1\}$ and $j \in \{0,\dots,3\}$

$$u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = \big(1 - (\Delta_{DD}^2)^{-1}\Delta^2\big)\xi_{\ell,j} \in H^2(\Omega).$$

Hence,

$$\operatorname{Gradgrad} u_{\ell,j} = \big(1 - \operatorname{Gradgrad}(\Delta_{DD}^2)^{-1} \operatorname{divDiv}_{\mathbb{S}}\big) \operatorname{Gradgrad} \xi_{\ell,j}.$$

For all $\ell \in \{1,\dots,m-1\}$ and $j \in \{0,\dots,3\}$, $u_{\ell,j}$ solve in classical terms the biharmonic Dirichlet problem

$$\begin{aligned} \Delta^2 u_{\ell,j} = \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u_{\ell,j} &= 0 && \text{in } \Omega,\\ u_{\ell,j} = \widehat{p}_j, \quad \operatorname{grad} u_{\ell,j} = \operatorname{grad} \widehat{p}_j &= e^j && \text{on } \Gamma_\ell,\\ u_{\ell,j} = 0, \quad \operatorname{grad} u_{\ell,j} &= 0 && \text{on } \Gamma_k,\ k \in \{0,\dots,m-1\}\setminus\{\ell\}, \end{aligned} \tag{11.9}$$

which is uniquely solvable. In particular, we have for all $\ell \in \{1,\dots,m-1\}$ and all $j \in \{0,\dots,3\}$ that $u_{\ell,j} = 0$ and $\operatorname{grad} u_{\ell,j} = 0$ on Γ_0, where we denote by e^j, $j \in \{1,2,3\}$, the Euclidean unit vectors in $\mathbb{R}^3$ and set $e^0 := 0 \in \mathbb{R}^3$.

(iii) In classical terms, u (representing $S = \operatorname{Gradgrad} u$) derived in the proof of Lemma 11.8 solves the linear biharmonic Dirichlet problem

$$\begin{aligned} \Delta^2 u = \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u = \operatorname{divDiv}_{\mathbb{S}} S &= 0 && \text{in } \Omega,\\ u = 0, \quad \operatorname{grad} u &= 0 && \text{on } \Gamma_0,\\ u = p_\ell \in \mathsf{P}^1, \quad \operatorname{grad} u = \operatorname{grad} p_\ell &\in \mathbb{R}^3 && \text{on } \Gamma_\ell,\ \ell \in \{1,\dots,m-1\}, \end{aligned}$$

which is uniquely solvable as long as the polynomials, p_ℓ in P^1, are prescribed.

Lemma 11.10 *Let Assumption 10.1 be satisfied. Then $\mathcal{B}_D^{bih,1}$ is linearly independent.*

Proof For $\ell \in \{1,\dots,m-1\}$ and $j \in \{0,\dots,3\}$ we take $\alpha_{\ell,j} \in \mathbb{R}$ such that

$$\sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{Gradgrad} u_{\ell,j} = 0; \qquad \text{we put } u := \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} u_{\ell,j}.$$

Then $\operatorname{Gradgrad} u = 0$ in Ω, i.e., u belongs to $\mathsf{P}^1_{\mathsf{pw}}$, see (8.2). We will show $u = 0$. For this we extend $u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ (see (11.7)) to B via (note that $\xi_{\ell,j} \in H_0^2(B)$ and $\psi_{\ell,j} \in H_0^2(\Omega)$)

$$\widetilde{u}_{\ell,j} := \begin{cases} u_{\ell,j} & \text{in } \Omega,\\ \xi_{\ell,j} & \text{in } B\setminus\overline{\Omega}, \end{cases} \qquad \operatorname{Gradgrad} \widetilde{u}_{\ell,j} = \begin{cases} \operatorname{Gradgrad} u_{\ell,j} & \text{in } \Omega,\\ \operatorname{Gradgrad} \xi_{\ell,j} = 0 & \text{in } B\setminus\overline{\Omega}. \end{cases}$$

Note that $\widetilde{u}_{\ell,j} \in H_0^2(B)$. For all $\ell \in \{1, \dots, m-1\}$, $j \in \{0, \dots, 3\}$ we have $\widetilde{u}_{\ell,j} = \xi_{\ell,j} = \widehat{p}_j$ in Ξ_ℓ and $\widetilde{u}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0 \cup \bigcup_{k\in\{1,\dots,m-1\}\setminus\{\ell\}} \Xi_k$. Then

$$\widetilde{u} := \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \widetilde{u}_{\ell,j} \in H_0^2(B)$$

with $\widetilde{u} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{Gradgrad} \widetilde{u} = 0$ in $B \setminus \overline{\Omega}$ as well as $\operatorname{Gradgrad} \widetilde{u} = \operatorname{Gradgrad} u = 0$ in Ω by assumption. Hence, $\operatorname{Gradgrad} \widetilde{u} = 0$ in B, showing $\widetilde{u} = 0$ in B. In particular, $u = 0$ in Ω, and $\sum_{j=0}^{3} \alpha_{\ell,j} \widehat{p}_j = \widetilde{u}|_{\Xi_\ell} = 0$ for all $\ell \in \{1, \dots, m-1\}$. We conclude $\alpha_{\ell,j} = 0$ for all $j \in \{0, \dots, 3\}$ and all $\ell \in \{1, \dots, m-1\}$, finishing the proof. □

Theorem 11.11 *Let Assumption 10.1 be satisfied. Then* $\dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) = 4(m-1)$ *and a basis of* $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$ *is given by* (11.8).

Proof Use Lemmas 11.8 and 11.10. □

11.3 Dirichlet tensor fields of the second biharmonic complex

The rationale to derive a set of basis functions for the second biharmonic complex is somewhat similar to the first one. For the second biharmonic complex, similar to (3.3), (3.5), and (11.2), (11.6), we have the orthogonal decompositions

$$\begin{aligned} L_{\mathbb{T}}^{2,3\times3}(\Omega) &= \operatorname{ran}(\operatorname{dev}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{T}}^{2,3\times3}(\Omega)} \ker(\operatorname{Div}_{\mathbb{T}}, \Omega), \\ \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \Omega) &= \operatorname{ran}(\operatorname{dev}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{T}}^{2,3\times3}(\Omega)} \mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega). \end{aligned} \tag{11.10}$$

Remark 11.12 [38, Lemma 3.2] yields $\operatorname{dom}(\operatorname{dev}\mathring{\operatorname{Grad}}, \Omega) = H_0^{1,3}(\Omega)$. Moreover, the range in (11.10) is closed by the Friedrichs type estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^{1,3}(\Omega) \qquad |\phi|_{L^{2,3}(\Omega)} \le c |\operatorname{devGrad} \phi|_{L^{2,3\times3}(\Omega)}, \tag{11.11}$$

which holds by Assumption 10.1. Again, Ω being open and bounded would be sufficient already. Indeed, the estimate mentioned here is based on the Friedrichs estimate provided in Remark 11.1 and the following observations similar to the proof of Korn's inequality, cf. Remark 11.18: From $\operatorname{dom}(\operatorname{dev}\mathring{\operatorname{Grad}}, \Omega) = H_0^{1,3}(\Omega)$ it suffices to show (11.11) for smooth vector fields v with compact support in Ω. It is elementary to see that for matrices T in $\mathbb{R}^{3\times3}$ and the Frobenius norm $|T|_{\mathbb{R}^{3\times3}}$ we have $|T|_{\mathbb{R}^{3\times3}}^2 = |\operatorname{dev} T|_{\mathbb{R}^{3\times3}}^2 + \frac{1}{3}|\operatorname{tr} T|_{\mathbb{R}}^2$, where $\operatorname{dev} T = T - \frac{1}{3}(\operatorname{tr} T)\operatorname{Id}$ is the *deviatoric* ('trace-free') part of T and $\operatorname{tr} T$ is the trace of T. Integration by parts shows $|\operatorname{Grad} v|_{L^{2,3\times3}(\Omega)}^2 = |\operatorname{curl} v|_{L^{2,3}(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \ge |\operatorname{div} v|_{L^2(\Omega)}^2$ for all $v \in C_c^{\infty,3}(\Omega)$. Thus, from $\operatorname{tr} \operatorname{Grad} v = \operatorname{div} v$ we infer

$$|\operatorname{Grad} v|^2_{L^{2,3\times 3}(\Omega)} = |\operatorname{dev}\operatorname{Grad} v|^2_{L^{2,3\times 3}(\Omega)} + \frac{1}{3}|\operatorname{div} v|^2_{L^2(\Omega)}$$
$$\leq |\operatorname{dev}\operatorname{Grad} v|^2_{L^{2,3\times 3}(\Omega)} + \frac{1}{3}|\operatorname{Grad} v|^2_{L^{2,3\times 3}(\Omega)}.$$

Hence, $2|\operatorname{Grad} v|^2_{L^{2,3\times 3}(\Omega)} \leq 3|\operatorname{devGrad} v|^2_{L^{2,3\times 3}(\Omega)}$, and inequality (11.11) follows from Remark 11.1.

Using (11.10), we define the orthogonal projector $\pi : L^{2,3\times 3}_{\mathbb{T}}(\Omega) \to \ker(\operatorname{Div}_{\mathbb{T}}, \Omega)$ along $\operatorname{ran}(\operatorname{dev}\mathring{\operatorname{G}}\mathrm{rad}, \Omega)$ and we have $\pi(\ker(\operatorname{sym}\mathring{\operatorname{C}}\mathrm{url}_{\mathbb{T}}, \Omega)) = \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)$. Recalling $\xi_\ell \in C^\infty_c(\mathbb{R}^3)$ from (11.1) and introducing the Raviart–Thomas fields $\widehat{r}_j$ given by

$$\widehat{r}_0(x) := x, \qquad \widehat{r}_j(x) := e^j$$

for $j \in \{1, 2, 3\}$, we define $\xi_{\ell,j} := \xi_\ell \widehat{r}_j$ for all $\ell \in \{1, \dots, m-1\}$ and all $j \in \{0, \dots, 3\}$. It is easy to see that

$$\operatorname{devGrad} \xi_{\ell,j} \in C^{\infty,3\times 3}_{c,\mathbb{T}}(\Omega) \cap \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega) \subseteq \ker(\operatorname{sym}\mathring{\operatorname{C}}\mathrm{url}_{\mathbb{T}}, \Omega).$$

Due to Remark 11.12 in conjunction with (11.10), we find unique $\psi_{\ell,j} \in H^{1,3}_0(\Omega)$ such that

$$\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \ni \pi \operatorname{devGrad} \xi_{\ell,j} = \operatorname{devGrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \operatorname{devGrad} v_{\ell,j}$$

with

$$v_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^{1,3}(\Omega). \tag{11.12}$$

We shall show that

$$\mathcal{B}^{\mathsf{bih},2}_D := \left\{ \operatorname{devGrad} v_{\ell,j} : \ell \in \{1, \dots, m-1\},\ j \in \{0, \dots, 3\} \right\} \subseteq \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \tag{11.13}$$

defines a basis of $\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)$.

Lemma 11.13 ([38, Theorem 3.10 (iv) and Remark 3.11 (i)]) *Let $D \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Assume that D is topologically trivial. Then*

$$\ker(\operatorname{sym}\mathring{\operatorname{C}}\mathrm{url}_{\mathbb{T}}, D) = \operatorname{ran}(\operatorname{dev}\mathring{\operatorname{G}}\mathrm{rad}, D), \qquad \ker(\operatorname{symCurl}_{\mathbb{T}}, D) = \operatorname{ran}(\operatorname{devGrad}, D).$$

Lemma 11.14 *Let Assumption 10.1 be satisfied. Then $\mathcal{H}^{bih,2}_{D,\mathbb{T}}(\Omega) = \operatorname{lin} \mathcal{B}^{bih,2}_D$.*

Proof Let $T \in \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = \ker(\operatorname{sym}\mathring{\operatorname{C}}\mathrm{url}_{\mathbb{T}}, \Omega) \cap \ker(\operatorname{Div}_{\mathbb{T}}, \Omega)$ and let $\widetilde{T}$ be the extension of T by zero onto B. Then $\widetilde{T} \in \ker(\operatorname{sym}\mathring{\operatorname{C}}\mathrm{url}_{\mathbb{T}}, B)$. As B is topologically trivial (and smooth and bounded), by Lemma 11.13 there exists (a unique vector field) $v \in H^{1,3}_0(B)$ such that $\operatorname{devGrad} v = \widetilde{T}$ in B. Since $\operatorname{devGrad} v = \widetilde{T} = 0$ in $B\setminus\overline{\Omega}$, v is a Raviart–Thomas vector field, $v \in \mathsf{RT}$, in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B\setminus\overline{\Omega}$. Due to the boundary condition of $v \in H^{1,3}_0(B)$, v vanishes in $\widetilde{\Xi}_0$. Therefore, $T = \operatorname{devGrad} v$ in Ω and $v \in H^{1,3}_0(B)$ is such that $v|_{\widetilde{\Xi}_0} = 0$ and $v|_{\Xi_\ell} =: r_\ell =: \sum_{j=0}^3 \alpha_{\ell,j}\widehat{r}_j \in \mathsf{RT}$, for some $\alpha_{\ell,j} \in \mathbb{R}$, for all $\ell \in \{1, \dots, m-1\}$ and $j \in \{0, \dots, 3\}$. Define

$$\widehat{T} := T - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{devGrad} v_{\ell,j} = \operatorname{devGrad} \widehat{v} \in \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega),$$
$$\widehat{v} := v - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} v_{\ell,j} \in H^{1,3}(\Omega)$$

with $v_{\ell,j}$ from (11.12). Since $\widetilde{\psi}_{\ell,j} \in H_0^{1,3}(B)$, where $\widetilde{\psi}_{\ell,j}$ is the extension of $\psi_{\ell,j}$ by zero to B, as an element of $H^{1,3}(B)$ we see that

$$\widehat{v}_B := v - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3}\alpha_{\ell,j}\xi_{\ell,j} + \sum_{\ell=1}^{m-1}\sum_{j=0}^{3}\alpha_{\ell,j}\widetilde{\psi}_{\ell,j} \in H_0^{1,3}(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{v} = \widehat{v}_B|_\Omega \in H_0^{1,3}(\Omega)$ by Assumption 10.1, and

$$|\widehat{T}|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} = \langle \operatorname{devGrad}\widehat{v}, \widehat{T}\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} = 0$$

yields the assertion. □

Remark 11.15 (Characterisation by PDEs)

(i) Denoting $\Delta_{\mathbb{T},D} := \operatorname{Div}_{\mathbb{T}} \operatorname{dev}\mathring{\operatorname{Grad}}$ the 'deviatoric' Laplacian with homogeneous Dirichlet boundary conditions, we see that $\psi_{\ell,j} = \Delta_{\mathbb{T},D}^{-1}\Delta_{\mathbb{T}}\xi_{\ell,j}$ with $\Delta_{\mathbb{T}} := \operatorname{Div}_{\mathbb{T}} \operatorname{devGrad}$, which corresponds to the variational formulation

$$\begin{aligned}\forall\, \phi \in H_0^{1,3}(\Omega) \quad &\langle \operatorname{devGrad}\psi_{\ell,j}, \operatorname{devGrad}\phi\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} \\ &= \langle \operatorname{devGrad}\xi_{\ell,j}, \operatorname{devGrad}\phi\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}.\end{aligned}$$

(ii) For all $\ell \in \{1, \dots, m-1\}$ and all $j \in \{0, \dots, 3\}$ we have

$$v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = (1 - \Delta_{\mathbb{T},D}^{-1}\Delta_{\mathbb{T}})\xi_{\ell,j} \in H^{1,3}(\Omega)$$

and deduce

$$\operatorname{devGrad} v_{\ell,j} = (1 - \operatorname{devGrad}\Delta_{\mathbb{T},D}^{-1}\operatorname{Div}_{\mathbb{T}})\operatorname{devGrad}\xi_{\ell,j}.$$

In classical terms, this reads

$$\begin{aligned} -\Delta_{\mathbb{T}} v_{\ell,j} &= 0 && \text{in } \Omega, \\ v_{\ell,j} &= \widehat{r}_j && \text{on } \Gamma_\ell, \\ v_{\ell,j} &= 0 && \text{on } \Gamma_k,\ k \in \{0, \dots, m-1\}\setminus\{\ell\}, \end{aligned} \tag{11.14}$$

which is uniquely solvable.

(iii) In classical terms, v (representing $T = \operatorname{devGrad} v$) from the proof of Lemma 11.14 solves the linear elasticity type Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{T}} v = -\operatorname{Div}_{\mathbb{T}}\operatorname{devGrad} v = -\operatorname{Div}_{\mathbb{T}} T &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_0, \\ v &= r_\ell \in \mathsf{RT} && \text{on } \Gamma_\ell,\ \ell \in \{1, \dots, m-1\}, \end{aligned}$$

which is uniquely solvable given the knowledge of r_ℓ in RT.

Lemma 11.16 *Let Assumption 10.1 be satisfied. Then $\mathcal{B}_D^{bih,2}$ is linearly independent.*

Proof Let $\alpha_{\ell,j} \in \mathbb{R}$ with $\ell \in \{1, \dots, m-1\}$ and $j \in \{0, \dots, 3\}$ be such that

$$\sum_{\ell=1}^{m-1}\sum_{j=0}^{3}\alpha_{\ell,j}\operatorname{devGrad} v_{\ell,j} = 0; \qquad \text{set } v := \sum_{\ell=1}^{m-1}\sum_{j=0}^{3}\alpha_{\ell,j} v_{\ell,j}.$$

Then $\operatorname{devGrad} v = 0$ in Ω, i.e., $v \in \mathsf{RT}$ in each connected component of Ω. We show $v = 0$. Recalling $v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ in Ω from (11.7) and using $\xi_{\ell,j} \in H_0^1(B)$ and $\psi_{\ell,j} \in H_0^1(\Omega)$, we define

$$\widetilde{v}_{\ell,j} := \begin{cases} v_{\ell,j} & \text{in } \Omega, \\ \xi_{\ell,j} & \text{in } B\setminus\overline{\Omega}, \end{cases} \qquad \operatorname{devGrad} \widetilde{v}_{\ell,j} = \begin{cases} \operatorname{devGrad} v_{\ell,j} & \text{in } \Omega, \\ \operatorname{devGrad} \xi_{\ell,j} = 0 & \text{in } B\setminus\overline{\Omega}. \end{cases}$$

Note that $\widetilde{v}_{\ell,j} \in H_0^{1,3}(B)$. For all $\ell \in \{1,\dots,m-1\}$ and $j \in \{0,\dots,3\}$, we obtain $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = \widehat{r}_j$ in Ξ_ℓ and $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0 \cup \bigcup_{k\in\{1,\dots,m-1\}\setminus\{\ell\}} \Xi_k$. Then

$$\widetilde{v} := \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j}\widetilde{v}_{\ell,j} \in H_0^{1,3}(B)$$

with $\widetilde{v} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{devGrad}\widetilde{v} = 0$ in $B\setminus\overline{\Omega}$ as well as $\operatorname{devGrad}\widetilde{v} = \operatorname{devGrad} v = 0$ in Ω by assumption. Hence, $\operatorname{devGrad}\widetilde{v} = 0$ in B, showing $\widetilde{v} = 0$ in B. In particular, $v = 0$ in Ω, and $\sum_{j=0}^{3} \alpha_{\ell,j}\widehat{r}_j = \widetilde{v}|_{\Xi_\ell} = 0$ for all $\ell \in \{1,\dots,m-1\}$. We conclude $\alpha_{\ell,j} = 0$ for all $\ell \in \{1,\dots,m-1\}$ and $j \in \{0,\dots,3\}$. □

Theorem 11.17 *Let Assumption 10.1 be satisfied. Then* $\dim \mathcal{H}_{D,\mathbb{T}}^{bih,2}(\Omega) = 4(m-1)$ *and a basis of* $\mathcal{H}_{D,\mathbb{T}}^{bih,2}(\Omega)$ *is given by* (11.13).

Proof Use Lemmas 11.14 and 11.16. □

11.4 Dirichlet tensor fields of the elasticity complex

For the elasticity complex, similar to (3.3), (3.5), and (11.2), (11.6), (11.10), we have the orthogonal decompositions

$$\begin{aligned} L_{\mathbb{S}}^{2,3\times3}(\Omega) &= \operatorname{ran}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \ker(\operatorname{Div}_{\mathbb{S}}, \Omega), \\ \ker(\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}, \Omega) &= \operatorname{ran}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \mathcal{H}_{D,\mathbb{S}}^{ela}(\Omega). \end{aligned} \tag{11.15}$$

Remark 11.18 [39, Lemma 3.2] implies $\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) = H_0^{1,3}(\Omega)$. Moreover, the range in (11.15) is closed by the Friedrichs type estimate (and follows from the standard first Korn's inequality and Remark 11.1)

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^{1,3}(\Omega) \qquad |\phi|_{L^{2,3}(\Omega)} \le c|\operatorname{symGrad}\phi|_{L^{2,3\times3}(\Omega)}, \tag{11.16}$$

which holds by Assumption 10.1. Again, Ω open and bounded is sufficient for (11.16). Indeed, Korn's first inequality is easy to see as follows: For a tensor $T \in \mathbb{R}^{3\times3}$ we have $|T|_{\mathbb{R}^{3\times3}}^2 = |\operatorname{sym} T|_{\mathbb{R}^{3\times3}}^2 + |\operatorname{skw} T|_{\mathbb{R}^{3\times3}}^2$. Hence,

$$|\operatorname{Grad} v|_{\mathbb{R}^{3\times3}}^2 = |\operatorname{symGrad} v|_{\mathbb{R}^{3\times3}}^2 + |\operatorname{skw}\operatorname{Grad} v|_{\mathbb{R}^{3\times3}}^2 = |\operatorname{symGrad} v|_{\mathbb{R}^{3\times3}}^2 + \frac{1}{2}|\operatorname{curl} v|_{\mathbb{R}^3}^2.$$

By $|\operatorname{Grad} v|_{L^{2,3\times3}(\Omega)}^2 = |\operatorname{curl} v|_{L^{2,3}(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \ge |\operatorname{curl} v|_{L^{2,3}(\Omega)}^2$ for all $v \in H_0^{1,3}(\Omega)$, we get Korn's first inequality $|\operatorname{Grad} v|_{L^{2,3\times3}(\Omega)}^2 \le 2|\operatorname{symGrad} v|_{L^{2,3\times3}(\Omega)}^2$.

The orthogonal projector from $L_{\mathbb{S}}^{2,3\times3}(\Omega)$ onto $\ker(\operatorname{Div}_{\mathbb{S}}, \Omega)$ along $\operatorname{ran}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega)$ is denoted by π. From (11.15), we deduce $\pi(\ker(\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^{\top}, \Omega)) = \mathcal{H}_{D,\mathbb{S}}^{ela}(\Omega)$. Recall the functions $\xi_\ell \in C_c^\infty(\mathbb{R}^3)$ from (11.1) and introduce rigid motions $\widehat{r}_j$ given by

$$\widehat{r}_j(x) := e^j \times x, \qquad \widehat{r}_{j+3}(x) := e^j$$

for $j \in \{1, 2, 3\}$. We define $\xi_{\ell,j} := \xi_\ell \widehat{r}_j$ for all $\ell \in \{1, \ldots, m-1\}$ and for all $j \in \{1, \ldots, 6\}$. Then

$$\operatorname{symGrad} \xi_{\ell,j} \in C_{c,\mathbb{S}}^{\infty,3\times 3}(\Omega) \cap \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}, \Omega) \subseteq \ker(\operatorname{Curl}\mathring{\operatorname{C}}\operatorname{url}_{\mathbb{S}}^{\top}, \Omega).$$

We find unique $\psi_{\ell,j} \in H_0^{1,3}(\Omega)$ such that

$$\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \ni \pi \operatorname{symGrad} \xi_{\ell,j} = \operatorname{symGrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \operatorname{symGrad} v_{\ell,j}$$

with

$$v_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^{1,3}(\Omega). \tag{11.17}$$

We shall show that

$$\mathcal{B}_D^{\mathsf{ela}} := \Big\{ \operatorname{symGrad} v_{\ell,j} : \ell \in \{1, \ldots, m-1\},\ j \in \{1, \ldots, 6\} \Big\} \subseteq \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \tag{11.18}$$

defines a basis of $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega)$.

Lemma 11.19 ([39, Theorem 3.5]) *Let $D \subseteq \mathbb{R}^3$ a bounded, topologically trivial, strong Lipschitz domain. Then*

$$\ker(\operatorname{Curl}\mathring{\operatorname{C}}\operatorname{url}_{\mathbb{S}}^{\top}, D) = \operatorname{ran}(\operatorname{sym}\mathring{\operatorname{G}}\operatorname{rad}, D), \qquad \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}, D) = \operatorname{ran}(\operatorname{symGrad}, D).$$

Proof The result follows by [38, Corollary 2.29] for $m = 1$ in conjunction with the formulas in [38, Appendix]; see [39, Theorem 3.5] and [40] for the details. □

Lemma 11.20 *Let Assumption 10.1 be satisfied. Then $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) = \operatorname{lin} \mathcal{B}_D^{\mathsf{ela}}$.*

Proof We follow in close lines the arguments used in the proofs of Lemmas 11.2, 11.8, and 11.14. Let $S \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) = \ker(\operatorname{Curl}\mathring{\operatorname{C}}\operatorname{url}_{\mathbb{S}}^{\top}, \Omega) \cap \ker(\operatorname{Div}_{\mathbb{S}}, \Omega)$. and $\widetilde{S}$ its extension to B by zero. Then $\widetilde{S} \in \ker(\operatorname{Curl}\mathring{\operatorname{C}}\operatorname{url}_{\mathbb{S}}^{\top}, B)$. By Lemma 11.19, as B is topologically trivial (and smooth and bounded), there exists (a unique) $v \in H_0^{1,3}(B)$ such that $\operatorname{symGrad} v = \widetilde{S}$ in B. Since $\operatorname{symGrad} v = \widetilde{S} = 0$ in $B\setminus\overline{\Omega}$, v is a rigid motion, i.e., $v \in \mathsf{RM}$, in each connected component $\widetilde{\Xi}_0, \Xi_1, \ldots, \Xi_{m-1}$ of $B\setminus\overline{\Omega}$. Since $v \in H_0^{1,3}(B)$, v vanishes in $\widetilde{\Xi}_0$. Thus, $S = \operatorname{symGrad} v$ in Ω with some $v \in H_0^{1,3}(B)$ and we have $v|_{\Xi_\ell} =: r_\ell =: \sum_{j=1}^{6} \alpha_{\ell,j} \widehat{r}_j \in \mathsf{RM}$ for $\alpha_{\ell,j} \in \mathbb{R}$ and all $\ell \in \{1, \ldots, m-1\}$, $j \in \{1, \ldots, 6\}$. Let

$$\widehat{S} := S - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \operatorname{symGrad} v_{\ell,j} = \operatorname{symGrad} \widehat{v} \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega),$$

$$\widehat{v} := v - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} v_{\ell,j} \in H^{1,3}(\Omega)$$

with $v_{\ell,j}$ from (11.17). With $\widetilde{\psi}_{\ell,j} \in H_0^{1,3}(B)$, the extension of $\psi_{\ell,j}$ by zero, we see, as an element of $H^{1,3}(B)$, that

$$\widehat{v}_B := v - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \xi_{\ell,j} + \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \widetilde{\psi}_{\ell,j} \in H_0^{1,3}(B)$$

vanishes in Ξ_ℓ for all ℓ. Thus $\widehat{v} = \widehat{v}_B|_\Omega \in H_0^{1,3}(\Omega)$ by Assumption 10.1, and we conclude

$$|\widehat{S}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{symGrad} \widehat{v}, \widehat{S} \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0,$$

finishing the proof. □

For numerical purposes, we again highlight the partial differential equations satisfied by the functions constructed here.

Remark 11.21 (Characterisation by PDEs)

(i) For all $\ell \in \{1, \dots, m-1\}$, $j \in \{1, \dots, 6\}$, the vector field $\psi_{\ell,j} \in H_0^{1,3}(\Omega)$ can be found with the help of the standard variational formulation

$$\begin{aligned} \forall\, \phi \in H_0^{1,3}(\Omega) \quad & \langle \operatorname{symGrad} \psi_{\ell,j}, \operatorname{symGrad} \phi \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} \\ & = \langle \operatorname{symGrad} \xi_{\ell,j}, \operatorname{symGrad} \phi \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}, \end{aligned}$$

i.e., $\psi_{\ell,j} = \Delta_{\mathbb{S},D}^{-1} \Delta_{\mathbb{S}} \xi_{\ell,j}$, where $\Delta_{\mathbb{S},D} = \operatorname{Div}_{\mathbb{S}} \operatorname{sym}\mathring{\operatorname{Grad}}$ and $\Delta_{\mathbb{S}} = \operatorname{Div}_{\mathbb{S}} \operatorname{symGrad}$ are the 'symmetric' Laplacians with homogeneous Dirichlet boundary conditions and no boundary conditions, respectively.

(ii) For all $\ell \in \{1, \dots, m-1\}$, $j \in \{1, \dots, 6\}$ we have

$$v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = (1 - \Delta_{\mathbb{S},D}^{-1} \Delta_{\mathbb{S}}) \xi_{\ell,j} \in H^{1,3}(\Omega)$$

and thus

$$\operatorname{symGrad} v_{\ell,j} = (1 - \operatorname{symGrad} \Delta_{\mathbb{S},D}^{-1} \operatorname{Div}_{\mathbb{S}}) \operatorname{symGrad} \xi_{\ell,j}.$$

In classical terms, $v_{\ell,j}$ solves the linear elasticity Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{S}} v_{\ell,j} &= 0 && \text{in } \Omega, \\ v_{\ell,j} &= \widehat{r}_j && \text{on } \Gamma_\ell, \\ v_{\ell,j} &= 0 && \text{on } \Gamma_k,\ k \in \{0, \dots, m-1\} \setminus \{\ell\}. \end{aligned} \tag{11.19}$$

which is uniquely solvable.

(iii) In classical terms, v (representing $S = \operatorname{symGrad} v$) from the proof of Lemma 11.20 solves the linear elasticity Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{S}} v = -\operatorname{Div}_{\mathbb{S}} S &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_0, \\ v &= r_\ell \in \mathsf{RM} && \text{on } \Gamma_\ell,\ \ell \in \{1, \dots, m-1\}, \end{aligned}$$

which is uniquely solvable as long as the rigid motions r_ℓ in RM are prescribed.

Lemma 11.22 *Let Assumption 10.1 be satisfied. Then $\mathcal{B}_D^{ela}$ is linearly independent.*

Proof Let $\alpha_{\ell,j} \in \mathbb{R}$ for all $\ell \in \{1, \dots, m-1\}$, $j \in \{1, \dots, 6\}$ such that

$$\sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \operatorname{symGrad} v_{\ell,j} = 0; \qquad \text{set } v := \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} v_{\ell,j}.$$

Then $\operatorname{symGrad} v = 0$ in Ω, i.e., $v \in \mathsf{RM}$ in each connected component of Ω. We show $v = 0$. Recall $v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ in Ω. Using $\psi_{\ell,j} \in H^1_0(\Omega)$ and $\xi_{\ell,j} \in H^1_0(B)$ we extend $v_{\ell,j}$ to B via

$$\widetilde{v}_{\ell,j} := \begin{cases} v_{\ell,j} & \text{in } \Omega, \\ \xi_{\ell,j} & \text{in } B\setminus\overline{\Omega}, \end{cases} \qquad \operatorname{symGrad} \widetilde{v}_{\ell,j} = \begin{cases} \operatorname{symGrad} v_{\ell,j} & \text{in } \Omega, \\ \operatorname{symGrad} \xi_{\ell,j} = 0 & \text{in } B\setminus\overline{\Omega}. \end{cases}$$

Note that $\widetilde{v}_{\ell,j} \in H^1_0(B)$. Then, for all $\ell \in \{1, \ldots, m-1\}$ and all $j \in \{1, \ldots, 6\}$ we have $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0 \cup \bigcup_{k\in\{1,\ldots,m-1\}\setminus\{\ell\}} \Xi_k$ and $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = \widehat{r}_j$ in Ξ_ℓ. Thus

$$\widetilde{v} := \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \widetilde{v}_{\ell,j} \in H^{1,3}_0(B)$$

with $\widetilde{v} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{symGrad} \widetilde{v} = 0$ in $B\setminus\overline{\Omega}$ as well as $\operatorname{symGrad} \widetilde{v} = \operatorname{symGrad} v = 0$ in Ω by assumption. Hence, $\operatorname{symGrad} \widetilde{v} = 0$ in B, showing $\widetilde{v} = 0$ in B. In particular, $v = 0$ in Ω, and $\sum_{j=1}^{6} \alpha_{\ell,j} \widehat{r}_j = \widetilde{v}|_{\Xi_\ell} = 0$ for all $\ell \in \{1, \ldots, m-1\}$. We conclude $\alpha_{\ell,j} = 0$ for all $\ell \in \{1, \ldots, m-1\}$ and all $j \in \{1, \ldots, 6\}$, finishing the proof. □

Theorem 11.23 *Let Assumption 10.1 be satisfied. Then* $\dim \mathcal{H}^{ela}_{D,\mathbb{S}}(\Omega) = 6(m-1)$ *and a basis of* $\mathcal{H}^{ela}_{D,\mathbb{S}}(\Omega)$ *is given by* (11.18).

Proof Use Lemmas 11.20 and 11.22. □

12 The construction of the Neumann fields

The construction of the Neumann fields is more involved than the one for the generalised harmonic Dirichlet fields. We start off with some general definitions and remarks all the basis constructions have in common.

Since Ω consists of the connected components Ω_k, i.e., $\mathsf{cc}(\Omega) = \{\Omega_1, \ldots, \Omega_n\}$, we have by Assumption 10.3 (A3) for all $k \in \{1, \ldots, n\}$ that $\Omega_k \setminus \bigcup_{j=1}^{p} F_j$ is simply connected. We define

$$\Omega_F := \Omega \setminus \bigcup_{j=1}^{p} F_j.$$

For $j \in \{1, \ldots, p\}$, let $\widehat{F}_j \subseteq \widetilde{F}_j$ be two stacked, open, and simply connected neighbourhoods of F_j, i.e,

$$\overline{F_j} \subseteq \widehat{F}_j \subseteq \overline{\widehat{F}_j} \subseteq \widetilde{F}_j,$$

let

$$\Upsilon_j := \widehat{F}_j \cap \Omega, \qquad \widetilde{\Upsilon}_j := \widetilde{F}_j \cap \Omega,$$

and let $\theta_j \in C^\infty(\Omega_F)$ be a bounded (together with all derivatives) function with the following properties:

- $F_j \subseteq \Upsilon_j \subseteq \widetilde{\Upsilon}_j$.
- Υ_j and $\widetilde{\Upsilon}_j$ are (nonempty, open, and) simply connected.
- $\widetilde{F}_j$ are pairwise disjoint.

- $\Upsilon_j \setminus F_j = \Upsilon_{j,0} \dot{\cup} \Upsilon_{j,1}$ and $\widetilde{\Upsilon}_j \setminus F_j = \widetilde{\Upsilon}_{j,0} \dot{\cup} \widetilde{\Upsilon}_{j,1}$ with $\Upsilon_{j,0} \subseteq \widetilde{\Upsilon}_{j,0}$ and $\Upsilon_{j,1} \subseteq \widetilde{\Upsilon}_{j,1}$ (which are all nonempty, open, and simply connected).
- $\overline{\Upsilon}_{j,0} \cap \overline{\Upsilon}_{j,1} = \overline{F}_j$.
- $\operatorname{supp} \theta_j \subseteq \overline{\widetilde{\Upsilon}_{j,1}}$.
- $\theta_j|_{\Upsilon_{j,0}} = 0$ and $\theta_j|_{\Upsilon_{j,1}} = 1$.

Additionally, for all $l \in \{1, \dots, p\}$ we pick curves

- $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l$ with fixed starting points $x_{l,0} \in \Upsilon_{l,0}$ and fixed endpoints $x_{l,1} \in \Upsilon_{l,1}$.

Remark 12.1 Roughly speaking, $\widetilde{\Upsilon}_j \setminus F_j$ consists of exactly two open, nonempty, and simply connected components $\widetilde{\Upsilon}_{j,0}$ and $\widetilde{\Upsilon}_{j,1}$, on which subsets $\Upsilon_{j,0}$ (one side) and $\Upsilon_{j,1}$ (other side) the indicator function θ_j is 0 and 1, respectively. Note that $\Upsilon_{j,0}$ and $\Upsilon_{j,1}$ touch each other at the whole surface F_j, i.e., $\overline{\Upsilon}_{j,0} \cap \overline{\Upsilon}_{j,1} = \overline{F}_j$. Moreover, $\widetilde{\Upsilon}_j$ are pairwise disjoint and θ_j is supported in $\overline{\widetilde{\Upsilon}_{j,1}}$. As a consequence, θ_j *cannot* be continuously extended to Ω. On the other hand, $\operatorname{grad} \theta_j = 0$ in $\Upsilon_j \setminus F_j$, and hence $\operatorname{grad} \theta_j$ *can* be continuously extended to Υ_j and thus to Ω. Note that for all $l, j \in \{1, \dots, p\}$ it holds $\theta_j(x_{l,0}) = 0$ and $\theta_j(x_{l,1}) = \delta_{l,j}$.

For the construction of bases and to compute the dimensions of the Neumann fields it is crucial that these fields are sufficiently regular, e.g., continuous in Ω. We even have the following local regularity results.

Lemma 12.2 (local regularity of the cohomology groups) *Let* $\Omega \subseteq \mathbb{R}^3$ *be open. Then*

$$\begin{aligned} \mathcal{H}_D^{Rhm}(\Omega),\ \mathcal{H}_N^{Rhm}(\Omega) &\subseteq C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega), \\ \mathcal{H}_{D,\mathbb{S}}^{bih,1}(\Omega),\ \mathcal{H}_{D,\mathbb{S}}^{ela}(\Omega),\ \mathcal{H}_{N,\mathbb{S}}^{bih,2}(\Omega),\ \mathcal{H}_{N,\mathbb{S}}^{ela}(\Omega) &\subseteq C^{\infty,3\times3}(\Omega) \cap L_{\mathbb{S}}^{2,3\times3}(\Omega), \\ \mathcal{H}_{D,\mathbb{T}}^{bih,2}(\Omega),\ \mathcal{H}_{N,\mathbb{T}}^{bih,1}(\Omega) &\subseteq C^{\infty,3\times3}(\Omega) \cap L_{\mathbb{T}}^{2,3\times3}(\Omega). \end{aligned}$$

Proof Vector fields in $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \cup \mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$ are harmonic and thus belong to $C^{\infty,3}(\Omega)$.

Let

$$S \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \cup \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) \subseteq \ker(\operatorname{Curl}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}}).$$

Then S can be represented locally, e.g., in any topologically trivial and smooth subdomain $D \subseteq \Omega$, by $S = \operatorname{Gradgrad} u$ with some $u \in H^2(D)$, see Lemma 11.7. Therefore, $\operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u = 0$ in D. Local regularity for the biharmonic equation shows $u \in C^{\infty}(D)$ and hence $S = \operatorname{Gradgrad} u \in C^{\infty,3\times3}(D)$, i.e., $S \in C^{\infty,3\times3}(\Omega)$.

Next, let

$$T \in \mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) \cup \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) \subseteq \ker(\operatorname{symCurl}_{\mathbb{T}}) \cap \ker(\operatorname{Div}_{\mathbb{T}}).$$

Then, for any topologically trivial and smooth subdomain $D \subseteq \Omega$ we find $v \in H^{1,3}(D)$ such that $T = \operatorname{devGrad} v$, see Lemma 11.13. Thus $\operatorname{Div}_{\mathbb{T}} \operatorname{devGrad} v = 0$ in D. Local elliptic regularity shows $v \in C^{\infty,3}(D)$ and hence $T = \operatorname{devGrad} v \in C^{\infty,3\times3}(D)$, i.e., $T \in C^{\infty,3\times3}(\Omega)$.

Finally, let

$$S \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \cup \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) \subseteq \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}) \cap \ker(\operatorname{Div}_{\mathbb{S}}).$$

For $D \subseteq \Omega$ smooth, bounded, and topologically trivial, we find $v \in H^{1,3}(D)$ representing $S = \operatorname{symGrad} v$, see Lemma 11.19. Thus, $\operatorname{Div}_{\mathbb{S}} \operatorname{symGrad} v = 0$ in D. Local elliptic regularity shows $v \in C^{\infty,3}(D)$ and thus $S = \operatorname{symGrad} v \in C^{\infty,3\times3}(D)$, i.e., $S \in C^{\infty,3\times3}(\Omega)$. □

12.1 Neumann vector fields of the classical de Rham complex

Similar to our reasoning for the generalised harmonic Dirichlet fields, we start off with the arguably easiest case of the de Rham complex. Since we rely on the rephrasing of Picard's ideas in the forthcoming sections, we carry out the full construction of the harmonic Neumann fields. Note that we heavily use the functions and sets introduced at the beginning of Sect. 12, cf. Remark 12.1. Let $j \in \{1, \dots, p\}$. By definition $\theta_j = 0$ outside of $\widetilde{\Upsilon}_{j,1}$ and θ_j is constant on each connected component $\Upsilon_{j,0}$ and $\Upsilon_{j,1}$ of $\Upsilon_j \backslash F_j$. Hence $\operatorname{grad} \theta_j = 0$ in $\Upsilon_j \backslash F_j$ and—due to the support condition—also $\theta_j = 0$ in $\dot{\bigcup}_{l \in \{1,\dots,p\} \backslash \{j\}} \Upsilon_l \backslash F_l$. Thus, $\operatorname{grad} \theta_j$ can be continuously extended by zero to $\Theta_j \in C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega)$ with $\Theta_j = 0$ in $\dot{\bigcup}_{l \in \{1,\dots,p\}} \Upsilon_l$.

Lemma 12.3 *Let Assumption 10.3 be satisfied. Then* $\Theta_j \in \ker(\mathit{curl}, \Omega)$.

Proof Let $\Phi \in C_c^{\infty,3}(\Omega)$. As $\operatorname{supp} \Theta_j \subseteq \widetilde{\Upsilon}_j \backslash \Upsilon_j$ we can pick another cut-off function $\varphi \in C_c^{\infty}(\Omega_F)$ with $\varphi|_{\operatorname{supp} \Theta_j \cap \operatorname{supp} \Phi} = 1$. Then

$$\langle \Theta_j, \operatorname{curl} \Phi \rangle_{L^{2,3}(\Omega)} = \langle \Theta_j, \operatorname{curl} \Phi \rangle_{L^{2,3}(\operatorname{supp} \Theta_j \cap \operatorname{supp} \Phi)} = \big\langle \operatorname{grad} \theta_j, \operatorname{curl}(\varphi \Phi) \big\rangle_{L^{2,3}(\Omega_F)} = 0,$$

as $\varphi \Phi \in C_c^{\infty,3}(\Omega_F)$. □

Let $l, j \in \{1, \dots, p\}$. We recall from the latter proof and from Remark 12.1 that $\operatorname{supp} \Theta_j \subseteq \widetilde{\Upsilon}_j \backslash \Upsilon_j$ and thus

$$\begin{aligned} \int_{\zeta_l} \langle \Theta_j, \mathrm{d}\,\lambda \rangle &= \int_{\zeta_l \backslash \Upsilon_j} \langle \operatorname{grad} \theta_j, \mathrm{d}\,\lambda \rangle \\ &= \int_{\zeta_{x_{l,0}, x_{l,1}}} \langle \operatorname{grad} \theta_j, \mathrm{d}\,\lambda \rangle = \theta_j(x_{l,1}) - \theta_j(x_{l,0}) = \delta_{l,j}, \end{aligned}$$

where we recall $\zeta_{x_{l,0}, x_{l,1}} \subseteq \zeta_l$ with chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$. Hence we define functionals β_l in the way that

$$\beta_l(\Theta_j) := \int_{\zeta_l} \langle \Theta_j, \mathrm{d}\,\lambda \rangle = \delta_{l,j}, \qquad l, j \in \{1, \dots, p\}. \tag{12.1}$$

Let Assumption 10.1 be satisfied. For the de Rham complex, similar to (3.3), (3.5), and (11.2), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3}(\Omega) = H_2 &= \operatorname{ran} A_2^* \oplus_{H_2} \ker A_2 = \operatorname{ran}(\operatorname{grad}, \Omega) \oplus_{L^{2,3}(\Omega)} \ker(\mathring{\operatorname{div}}, \Omega), \\ \ker(\operatorname{curl}, \Omega) = \ker(A_1^*) &= \operatorname{ran} A_2^* \oplus_{H_2} K_2 = \operatorname{ran}(\operatorname{grad}, \Omega) \oplus_{L^{2,3}(\Omega)} \mathcal{H}_N^{\mathsf{Rhm}}(\Omega). \end{aligned} \tag{12.2}$$

Remark 12.4 By definition $\operatorname{dom}(\operatorname{grad}, \Omega) = H^1(\Omega)$, and the range in (12.2) is closed by the Poincaré estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H^1(\Omega) \cap \mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}} \qquad |\phi|_{L^2(\Omega)} \le c |\operatorname{grad} \phi|_{L^{2,3}(\Omega)},$$

which is implied by a contradiction argument using Rellich's selection theorem as Assumption 10.1 holds.

Similar to the case of harmonic Dirichlet fields, we denote in (12.2) the orthogonal projector along $\operatorname{ran}(\operatorname{grad}, \Omega)$ onto $\ker(\mathring{\operatorname{div}}, \Omega)$ by π. By Lemma 12.3 for all $j \in \{1, \dots, p\}$ there exists some $\psi_j \in H^1(\Omega)$ (unique up to $\mathbb{R}_{\mathsf{pw}}$) such that

$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \ni \pi \Theta_j = \Theta_j - \operatorname{grad} \psi_j, \qquad (\Theta_j - \operatorname{grad} \psi_j)\big|_{\Omega_F} = \operatorname{grad}(\theta_j - \psi_j).$$

By Lemma 12.2 we have $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \subseteq C^{\infty,3}(\Omega)$. Therefore, $\Theta_j \in C^{\infty,3}(\Omega)$ and we obtain $\operatorname{grad} \psi_j \in C^{\infty,3}(\Omega)$. Hence, $\psi_j \in H^1(\Omega) \cap C^\infty(\Omega)$ and the following path integrals are well-defined and can be computed by (12.1), i.e., for all $l, j \in \{1, \dots, p\}$

$$\beta_l(\pi\Theta_j) = \int_{\zeta_l} \langle \pi\Theta_j, \mathrm{d}\lambda \rangle = \int_{\zeta_l} \langle \Theta_j, \mathrm{d}\lambda \rangle - \int_{\zeta_l} \langle \operatorname{grad} \psi_j, \mathrm{d}\lambda \rangle = \delta_{l,j} + 0 = \delta_{l,j}. \tag{12.3}$$

We will show that

$$\mathcal{B}_N^{\mathsf{Rhm}} := \{\pi\Theta_1, \dots, \pi\Theta_p\} \subseteq \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \tag{12.4}$$

defines a basis of $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$.

Also for the harmonic Neumann fields, we provide a possible variational formulation for obtaining ψ_j constructed here:

Remark 12.5 (Characterisation by PDEs) Let $j \in \{1, \dots, p\}$. Then $\psi_j \in H^1(\Omega) \cap \mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}}$ satisfies

$$\forall \phi \in H^1(\Omega) \qquad \langle \operatorname{grad} \psi_j, \operatorname{grad} \phi \rangle_{L^{2,3}(\Omega)} = \langle \Theta_j, \operatorname{grad} \phi \rangle_{L^{2,3}(\Omega)},$$

i.e., $\psi_j = \Delta_N^{-1}\big(\operatorname{div}\Theta_j, \nu \cdot \Theta_j|_\Gamma\big)$, where $\Delta_N \subseteq \operatorname{div}\operatorname{grad}$ is the Laplacian with inhomogeneous Neumann boundary conditions restricted to a subset of $H^1(\Omega) \cap \mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}}$. Therefore,

$$\pi\Theta_j = \Theta_j - \operatorname{grad} \psi_j = \Theta_j - \operatorname{grad} \Delta_N^{-1}\big(\operatorname{div}\Theta_j, \nu \cdot \Theta_j|_\Gamma\big).$$

In classical terms, ψ_j solves the Neumann Laplace problem

$$\begin{aligned} -\Delta\psi_j &= -\operatorname{div}\Theta_j && \text{in } \Omega, \\ \nu \cdot \operatorname{grad} \psi_j &= \nu \cdot \Theta_j && \text{on } \Gamma, \\ \int_{\Omega_k} \psi_j &= 0 && \text{for } k \in \{1, \dots, n\}, \end{aligned} \tag{12.5}$$

which is uniquely solvable.

Lemma 12.6 *Let Assumptions 10.1 and 10.3 be satisfied. Then* $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = \operatorname{lin} \mathcal{B}_N^{\mathsf{Rhm}}$.

Proof Let $H \in \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = \ker(\mathring{\operatorname{div}}, \Omega) \cap \ker(\operatorname{curl}, \Omega) \subseteq C^{\infty,3}(\Omega)$ (see Lemma 12.2), and define the numbers

$$\gamma_j := \beta_j(H) = \int_{\zeta_j} \langle H, \mathrm{d}\lambda \rangle \in \mathbb{R}, \qquad j \in \{1, \dots, p\}.$$

We shall show that

$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^p \gamma_j \pi\Theta_j = 0 \quad \text{in } \Omega.$$

The aim is to prove that there exists $u \in H^1(\Omega)$ such that $\operatorname{grad} u = \widehat{H}$, since then

$$|\widehat{H}|_{L^{2,3}(\Omega)}^2 = \langle \operatorname{grad} u, \widehat{H} \rangle_{L^{2,3}(\Omega)} = 0.$$

Using (12.3), we obtain

$$\int_{\zeta_l} \langle \widehat{H}, \mathrm{d}\lambda \rangle = \int_{\zeta_l} \langle H, \mathrm{d}\lambda \rangle - \sum_{j=1}^p \gamma_j \int_{\zeta_l} \langle \pi\Theta_j, \mathrm{d}\lambda \rangle$$

$$= \gamma_l - \sum_{j=1}^{p} \gamma_j \beta_l(\pi\Theta_j) = \gamma_l - \sum_{j=1}^{p} \gamma_l \delta_{l,j} = 0.$$

Note that $\widehat{H} \in \ker(\operatorname{curl}, \Omega) \cap C^{\infty,3}(\Omega)$. Hence, by Assumption 10.3 (A.1) we have for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} \langle \widehat{H}, \mathrm{d}\,\lambda \rangle = 0. \tag{12.6}$$

Recall the connected components $\Omega_1, \ldots, \Omega_n$ of Ω. For $1 \leq k \leq n$ let Ω_k and some $x_0 \in \Omega_k$ be fixed. By (12.6) and the fundamental theorem of calculus the function $u : \Omega \to \mathbb{R}$ given by

$$u(x) := \int_{\zeta(x_0,x)} \langle \widehat{H}, \mathrm{d}\,\lambda \rangle, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the choice of the respective curve $\zeta(x_0, x)$, and belongs to $C^{\infty}(\Omega_k)$ with $\operatorname{grad} u = \widehat{H} \in L^{2,3}(\Omega_k)$. Thus[5] $u \in L^2(\Omega_k)$, see, e.g., [22, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)], and hence $u \in H^1(\Omega_k)$, showing $u \in H^1(\Omega)$. □

Remark 12.7 Note that in the latter proof the existence of $u \in H^1(\Omega_k)$ with $\operatorname{grad} u = \widehat{H}$ in Ω_k is well-known, if the connected component Ω_k of Ω is even simply connected. Indeed, in this case $\ker(\operatorname{curl}, \Omega_k) = \operatorname{ran}(\operatorname{grad}, \Omega_k)$.

Lemma 12.8 *Let Assumptions 10.1 and 10.3 be satisfied. Then $\mathcal{B}_N^{\mathsf{Rhm}}$ is linearly independent.*

Proof Let $\sum_{j=1}^{p} \gamma_j \pi\Theta_j = 0$ for some $\gamma_j \in \mathbb{R}$. Then (12.3) implies

$$\begin{aligned} 0 &= \sum_{j=1}^{p} \gamma_j \int_{\zeta_l} \langle \pi\Theta_j, \mathrm{d}\,\lambda \rangle \\ &= \sum_{j=1}^{p} \gamma_j \beta_l(\pi\Theta_j) = \sum_{j=1}^{p} \gamma_j \delta_{l,j} = \gamma_l \end{aligned}$$

for all $l \in \{1, \ldots, p\}$. □

Theorem 12.9 *Let Assumptions 10.1 and 10.3 be satisfied. Then* $\dim \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = p$ *and a basis of* $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$ *is given by* (12.4).

Proof Use Lemmas 12.6 and 12.8. □

12.2 Neumann tensor fields of the first biharmonic complex

The main difference of the constructions to come to the one in the previous section is the introduction of β: a suitable collection of functionals that very easily allows for testing of linear independence and for a straightforward application of Assumption 10.3 (A1). As a preparation for this, we need the next results. The first one—also important for the sections to come—is rather combinatorical and analyses the interplay between vector analysis and

[5] Indeed, it is sufficient to assume $u \in L^2_{\mathsf{loc}}(\Omega_k)$, see, e.g., [23, Satz 6.6.26, Beweis; Folgerung 6.3.2] or [60, Theorem 7.4].

matrix calculus; the second and third one deal with so-called Poincaré maps, which form the foundation of the construction of the desired set of functionals. Note that for the subsequent sections Lemma 12.10 is of independent interest. For this, we introduce for $v \in \mathbb{R}^3$ the skew-symmetric matrix

$$\operatorname{spn} v := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

and the corresponding isometric mapping $\operatorname{spn} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\mathrm{skw}}$.

Lemma 12.10 *Let $u \in C^\infty_c(\mathbb{R}^3)$, $v, w \in C^{\infty,3}_c(\mathbb{R}^3)$, and $S \in C^{\infty,3\times 3}_c(\mathbb{R}^3)$. Then:*

- $(\operatorname{spn} v)\, w = v \times w = -(\operatorname{spn} w)\, v$ *and* $(\operatorname{spn} v)(\operatorname{spn}^{-1} S) = -Sv$, *if* $\operatorname{sym} S = 0$
- $\operatorname{sym} \operatorname{spn} v = 0$ *and* $\operatorname{dev}(u \operatorname{Id}) = 0$
- $\operatorname{tr} \operatorname{Grad} v = \operatorname{div} v$ *and* $2 \operatorname{skw} \operatorname{Grad} v = \operatorname{spn} \operatorname{curl} v$
- $\operatorname{Div}(u \operatorname{Id}) = \operatorname{grad} u$ *and* $\operatorname{Curl}(u \operatorname{Id}) = -\operatorname{spn} \operatorname{grad} u$,
  in particular, $\operatorname{curl} \operatorname{Div}(u \operatorname{Id}) = 0$ *and* $\operatorname{curl} \operatorname{spn}^{-1} \operatorname{Curl}(u \operatorname{Id}) = 0$
  and $\operatorname{sym} \operatorname{Curl}(u \operatorname{Id}) = 0$
- $\operatorname{Div} \operatorname{spn} v = -\operatorname{curl} v$ *and* $\operatorname{Div} \operatorname{skw} S = -\operatorname{curl} \operatorname{spn}^{-1} \operatorname{skw} S$,
  in particular, $\operatorname{div} \operatorname{Div} \operatorname{skw} S = 0$
- $\operatorname{Curl} \operatorname{spn} v = (\operatorname{div} v) \operatorname{Id} - (\operatorname{Grad} v)^\top$
  and $\operatorname{Curl} \operatorname{skw} S = (\operatorname{div} \operatorname{spn}^{-1} \operatorname{skw} S) \operatorname{Id} - (\operatorname{Grad} \operatorname{spn}^{-1} \operatorname{skw} S)^\top$
- $\operatorname{dev} \operatorname{Curl} \operatorname{spn} v = -(\operatorname{dev} \operatorname{Grad} v)^\top$
- $-2 \operatorname{Curl} \operatorname{sym} \operatorname{Grad} v = 2 \operatorname{Curl} \operatorname{skw} \operatorname{Grad} v = -(\operatorname{Grad} \operatorname{curl} v)^\top$
- $2 \operatorname{spn}^{-1} \operatorname{skw} \operatorname{Curl} S = \operatorname{Div} S^\top - \operatorname{grad} \operatorname{tr} S = \operatorname{Div}\big(S - (\operatorname{tr} S) \operatorname{Id}\big)^\top$,
  in particular, $\operatorname{curl} \operatorname{Div} S^\top = 2 \operatorname{curl} \operatorname{spn}^{-1} \operatorname{skw} \operatorname{Curl} S$
  and $2 \operatorname{skw} \operatorname{Curl} S = \operatorname{spn} \operatorname{Div} S^\top$, *if* $\operatorname{tr} S = 0$
- $\operatorname{tr} \operatorname{Curl} S = 2 \operatorname{div} \operatorname{spn}^{-1} \operatorname{skw} S$, *in particular,* $\operatorname{tr} \operatorname{Curl} S = 0$, *if* $\operatorname{skw} S = 0$,
  and $\operatorname{tr} \operatorname{Curl} \operatorname{sym} S = 0$ *and* $\operatorname{tr} \operatorname{Curl} \operatorname{skw} S = \operatorname{tr} \operatorname{Curl} S$
- $2(\operatorname{Grad} \operatorname{spn}^{-1} \operatorname{skw} S)^\top = (\operatorname{tr} \operatorname{Curl} \operatorname{skw} S) \operatorname{Id} - 2 \operatorname{Curl} \operatorname{skw} S$
- $3 \operatorname{Div}(\operatorname{dev} \operatorname{Grad} v)^\top = 2 \operatorname{grad} \operatorname{div} v$
- $2 \operatorname{Curl} \operatorname{sym} \operatorname{Grad} v = -2 \operatorname{Curl} \operatorname{skw} \operatorname{Grad} v = -\operatorname{Curl} \operatorname{spn} \operatorname{curl} v = (\operatorname{Grad} \operatorname{curl} v)^\top$
- $2 \operatorname{Div} \operatorname{sym} \operatorname{Curl} S = -2 \operatorname{Div} \operatorname{skw} \operatorname{Curl} S = \operatorname{curl} \operatorname{Div} S^\top$
- $\operatorname{Curl}(\operatorname{Curl} \operatorname{sym} S)^\top = \operatorname{sym} \operatorname{Curl}(\operatorname{Curl} S)^\top$
- $\operatorname{Curl}(\operatorname{Curl} \operatorname{skw} S)^\top = \operatorname{skw} \operatorname{Curl}(\operatorname{Curl} S)^\top$

All formulas extend to distributions as well.

Proof Almost all formulas can be found in [38, Lemma 3.9] and [38, Lemma A.1]. It is elementary to show that $\operatorname{skw} T = 0$ implies $\operatorname{skw} \operatorname{Curl}(\operatorname{Curl} T)^\top = 0$, and that $\operatorname{sym} T = 0$ implies $\operatorname{sym} \operatorname{Curl}(\operatorname{Curl} T)^\top = 0$. Note that the needed (straightforward-to-prove) formulas for this are provided in [39, Appendix B]. Hence sym commutes with $\operatorname{Curl} \operatorname{Curl}^\top$ as

$$\operatorname{Curl}(\operatorname{Curl} \operatorname{sym} T)^\top = \operatorname{sym} \operatorname{Curl}(\operatorname{Curl} \operatorname{sym} T)^\top = \operatorname{sym} \operatorname{Curl}(\operatorname{Curl} T)^\top,$$

and so does skw. □

In Lemma 12.11 below for a tensor field T the operation $T \,\mathrm{d}\lambda := \big(\langle \operatorname{row}_\ell T, \mathrm{d}\lambda\rangle\big)_{\ell=1,2,3}$ has to be understood row-wise, i.e., the transpose of the ℓth row of T is denoted by $\operatorname{row}_\ell T$, yielding the vector object $T \,\mathrm{d}\lambda$. More precisely,

$$\Big(\int_{\zeta_{x_0,x}} T\,\mathrm{d}\,\lambda\Big)_\ell = \int_{\zeta_{x_0,x}} \langle \mathrm{row}_\ell T, \mathrm{d}\,\lambda\rangle = \int_0^1 \Big\langle (\mathrm{row}_\ell T)\big(\varphi(t)\big), \varphi'(t)\Big\rangle \,\mathrm{d}\,t, \quad \ell \in \{1,2,3\},$$

with some parametrisation $\varphi \in C^{1,3}_{\mathsf{pw}}\big([0,1]\big)$ of a piecewise C^1-curve $\zeta_{x_0,x}$ connecting $x_0 \in \Omega$ and $x \in \Omega$. Furthermore, we define

$$\int_{\zeta_{x_0,x}} (x-y)\big\langle(\mathrm{Div}\, T^\top)(y), \mathrm{d}\,\lambda_y\big\rangle := \int_0^1 \big(x-\varphi(t)\big)\Big\langle(\mathrm{Div}\, T^\top)\big(\varphi(t)\big), \varphi'(t)\Big\rangle\,\mathrm{d}\,t.$$

The first statement concerned with describing vector fields and their divergence by means of curve integrals over tensor fields reads as follows.

Lemma 12.11 *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subseteq \Omega$ be a piecewise C^1-curve connecting x_0 and x.*

(i) *Let $v \in C^{\infty,3}(\Omega)$. Then v and its divergence $\mathrm{div}\, v$ can be represented by*

$$v(x) - v(x_0) - \frac{1}{3}\big(\mathrm{div}\, v(x_0)\big)(x-x_0) = \int_{\zeta_{x_0,x}} \mathrm{devGrad}\, v\,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \big\langle \mathrm{Div}(\mathrm{devGrad}\, v)^\top, \mathrm{d}\,\lambda\big\rangle\Big)\,\mathrm{Id}\,\mathrm{d}\,\lambda_y$$

and

$$\mathrm{div}\, v(x) - \mathrm{div}\, v(x_0) = \frac{3}{2}\int_{\zeta_{x_0,x}} \big\langle \mathrm{Div}(\mathrm{devGrad}\, v)^\top, \mathrm{d}\,\lambda\big\rangle.$$

(ii) *Let $T \in C^{\infty,3\times 3}(\Omega)$. Then*

$$\int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \langle \mathrm{Div}\, T^\top, \mathrm{d}\,\lambda\rangle\Big)\,\mathrm{Id}\,\mathrm{d}\,\lambda_y = \int_{\zeta_{x_0,x}} (x-y)\big\langle(\mathrm{Div}\, T^\top)(y), \mathrm{d}\,\lambda_y\big\rangle.$$

Proof For (i), let

$$T := \mathrm{devGrad}\, v = \mathrm{Grad}\, v - \frac{1}{3}(\mathrm{tr}\,\mathrm{Grad}\, v)\,\mathrm{Id} = \mathrm{Grad}\, v - \frac{1}{3}(\mathrm{div}\, v)\,\mathrm{Id}$$

and observe $3\,\mathrm{Div}\, T^\top = 2\,\mathrm{grad}\,\mathrm{div}\, v$ by Lemma 12.10. Thus

$$v_k(x) - v_k(x_0) = \int_{\zeta_{x_0,x}} \langle \mathrm{grad}\, v_k, \mathrm{d}\,\lambda\rangle, \qquad k \in \{1,2,3\},$$

$$\mathrm{div}\, v(x) - \mathrm{div}\, v(x_0) = \int_{\zeta_{x_0,x}} \langle \mathrm{grad}\,\mathrm{div}\, v, \mathrm{d}\,\lambda\rangle = \frac{3}{2}\int_{\zeta_{x_0,x}} \langle \mathrm{Div}\, T^\top, \mathrm{d}\,\lambda\rangle.$$

Therefore,

$$v(x) - v(x_0) = \int_{\zeta_{x_0,x}} \mathrm{Grad}\, v\,\mathrm{d}\,\lambda = \int_{\zeta_{x_0,x}} \mathrm{devGrad}\, v\,\mathrm{d}\,\lambda + \frac{1}{3}\int_{\zeta_{x_0,x}} \mathrm{div}\, v\,\mathrm{Id}\,\mathrm{d}\,\lambda$$

$$
\begin{aligned}
&= \int_{\zeta_{x_0,x}} T \, \mathrm{d}\,\lambda + \frac{1}{3} \int_{\zeta_{x_0,x}} \operatorname{div} v(y) \, \mathrm{Id} \, \mathrm{d}\,\lambda_y \\
&= \int_{\zeta_{x_0,x}} T \, \mathrm{d}\,\lambda + \frac{1}{3} \operatorname{div} v(x_0) \int_{\zeta_{x_0,x}} \mathrm{Id} \, \mathrm{d}\,\lambda_y \\
&\quad + \frac{1}{2} \int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \rangle \Big) \mathrm{Id} \, \mathrm{d}\,\lambda_y .
\end{aligned}
$$

Moreover, [6] $\int_{\zeta_{x_0,x}} \mathrm{Id} \, \mathrm{d}\,\lambda_y = \int_{\zeta_{x_0,x}} \operatorname{Grad} y \, \mathrm{d}\,\lambda_y = x - x_0$. For (ii) we compute

$$
\begin{aligned}
&\int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \rangle \Big) \mathrm{Id} \, \mathrm{d}\,\lambda_y \\
&\quad = \int_0^1 \Big(\int_{\zeta_{x_0,\varphi(s)}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \rangle \Big) \mathrm{Id} \, \varphi'(s) \, \mathrm{d}\,s \\
&\quad = \int_0^1 \Big(\int_0^s \Big\langle (\operatorname{Div} T^\top)\big(\varphi(t)\big), \varphi'(t) \Big\rangle \mathrm{d}\,t \Big) \varphi'(s) \, \mathrm{d}\,s \\
&\quad = \int_0^1 \int_t^1 \varphi'(s) \, \mathrm{d}\,s \Big\langle (\operatorname{Div} T^\top)\big(\varphi(t)\big), \varphi'(t) \Big\rangle \mathrm{d}\,t \\
&\quad = \int_0^1 (x - \varphi(t)) \Big\langle (\operatorname{Div} T^\top)\big(\varphi(t)\big), \varphi'(t) \Big\rangle \mathrm{d}\,t \\
&\quad = \int_{\zeta_{x_0,x}} (x - y) \big\langle (\operatorname{Div} T^\top)(y), \mathrm{d}\,\lambda_y \big\rangle
\end{aligned}
$$

with φ parametrising $\zeta_{x_0,x}$ as above. □

Proposition 12.12 *Let $x_0 \in \Omega_0 \Subset \mathsf{cc}(\Omega)$ and let $S, T \in C^{\infty,3\times 3}(\Omega_0)$.*

(a) *The following conditions are equivalent:*

(i) *For all $\zeta \subseteq \Omega_0$ closed, piecewise C^1-curves*

$$
\int_\zeta \big\langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \big\rangle = 0.
$$

(ii) *For all $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ piecewise C^1-curves connecting x_0 with x*

$$
\int_{\zeta_{x_0,x}} \big\langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \big\rangle = \int_{\widetilde{\zeta}_{x_0,x}} \big\langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \big\rangle.
$$

(iii) *There exists $u \in C^\infty(\Omega_0)$ such that* $\operatorname{grad} u = \operatorname{Div} T^\top$.

In the case one of the above conditions is true the function

$$
x \mapsto u(x) = \int_{\zeta_{x_0,x}} \big\langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda \big\rangle \tag{12.7}
$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ piecewise C^1-curve connecting x_0 with x is a (well-defined) possible choice for u in (iii).

---

[6] Alternatively, note $\int_{\zeta_{x_0 x}} \mathrm{Id} \, \mathrm{d}\,\lambda_y = \int_0^1 \mathrm{Id} \, \varphi'(s) \, \mathrm{d}\,s = \int_0^1 \varphi'(s) \, \mathrm{d}\,s = x - x_0$ with the parametrisation φ of $\zeta_{x_0,x}$ from above.

*(**b**) The following conditions are equivalent:*

*(**i**) For all $\zeta \subseteq \Omega_0$ closed, piecewise C^1 curves*

$$\int_\zeta S \,\mathrm{d}\,\lambda = 0.$$

*(**ii**) For all $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ piecewise C^1 curves connecting x_0 with x*

$$\int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda = \int_{\tilde{\zeta}_{x_0,x}} S \,\mathrm{d}\,\lambda.$$

*(**iii**) There exists $v \in C^{\infty,3}(\Omega_0)$ such that* $\operatorname{Grad} v = S$.

In the case one of the above conditions is true the vector field

$$x \mapsto v(x) = \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda \tag{12.8}$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ piecewise C^1-curve connecting x_0 with x is a (well-defined) possible choice for v in (iii).

*(**c**) Let $S := T + \frac{1}{2} u \operatorname{Id}$ with $u \in C^\infty(\Omega_0)$ and $\operatorname{grad} u = \operatorname{Div} T^\top$ as in (a), (iii). Moreover, let $v \in C^{\infty,3}(\Omega_0)$ such that $\operatorname{Grad} v = S$ as in (b), (iii). Then*

*(**i**)* $\operatorname{tr} T = 0$,
*(**ii**)* $\operatorname{devGrad} v = T$

are equivalent. In either case, we have

$$\operatorname{symCurl}_{\mathbb{T}} T = 0 \quad \textit{and} \quad \operatorname{grad} u = \frac{2}{3} \operatorname{grad} \operatorname{div} v. \tag{12.9}$$

Proof (a): The conditions (i) and (ii) are clearly equivalent. Assuming (ii), we obtain that the choice of u in (12.7) is well-defined. By the fundamental theorem of calculus it follows that this u satisfies the equation in (iii) and, consequently, $u \in C^\infty(\Omega_0)$. If, on the other hand, (iii) is true, then again using the fundamental theorem of calculus, we obtain (ii).

(b): The proof of the equivalence follows almost exactly the same way as for (a).

(c): We compute $\operatorname{devGrad} v = \operatorname{dev} S = \operatorname{dev}(T + \frac{1}{2} u \operatorname{Id}) = \operatorname{dev} T$. Hence (i) and (ii) are equivalent. Finally, if (i) or (ii) is true, then by the complex property

$$\operatorname{symCurl} T = \operatorname{symCurl} \operatorname{devGrad} v = 0,$$

and we conclude $\operatorname{grad} u = \operatorname{Div} T^\top = \operatorname{Div}(\operatorname{devGrad} v)^\top = \frac{2}{3} \operatorname{grad} \operatorname{div} v$ by Lemma 12.10. □

Remark 12.13 Related to Proposition 12.12 we note with Lemma 12.10 the following:

(**i**) For $T \in C^{\infty,3\times3}(\Omega)$ we have

$$\operatorname{curl} \operatorname{Div} T^\top = 2 \operatorname{Div} \operatorname{symCurl} T.$$

(**ii**) For $T \in C^{\infty,3\times3}_{\mathbb{T}}(\Omega)$ and $S := T + \frac{1}{2} u \operatorname{Id}$ with $\operatorname{grad} u = \operatorname{Div} T^\top$ it holds

$$\operatorname{Curl} S = \operatorname{Curl} T - \frac{1}{2} \operatorname{spn} \operatorname{grad} u = \operatorname{Curl} T - \operatorname{skw} \operatorname{Curl} T = \operatorname{symCurl} T.$$

(**iii**) If Ω_0 is simply connected, Proposition 12.12 (a), (iii) and (b), (iii) are equivalent to $\operatorname{curl} \operatorname{Div} T^\top = 0$ and $\operatorname{Curl} S = 0$, respectively.

Arguing for each connected component separately (and using formulas (12.7) and (12.8) on every connected component), we obtain the following more condensed version of Proposition 12.12.

Corollary 12.14 *Let* $S, T \in C^{\infty,3\times 3}(\Omega)$.

(a) The following conditions are equivalent:

- *(i) For all* $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_\zeta \big\langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\big\rangle = 0$.
- *(ii) There exists* $u \in C^\infty(\Omega)$ *such that* $\operatorname{grad} u = \operatorname{Div} T^\top$.

(b) The following conditions are equivalent:

- *(i) For all* $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_\zeta S\,\mathrm{d}\,\lambda = 0$.
- *(ii) There exists* $v \in C^{\infty,3}(\Omega)$ *such that* $\operatorname{Grad} v = S$.

(c) Let $S = T + \frac{1}{2} u \operatorname{Id}$ *with* $u \in C^\infty(\Omega)$ *and* $\operatorname{grad} u = \operatorname{Div} T^\top$ *as in (a), (ii). Moreover, let* $v \in C^{\infty,3}(\Omega)$ *with* $\operatorname{Grad} v = S$ *as in (b), (ii). Then* $\operatorname{tr} T = 0$ *in* Ω *if and only if* $\operatorname{devGrad} v = T$ *in* Ω.

The construction of the harmonic Neumann tensor fields for the first biharmonic complex forms a nontrivial adaptation of the rationale developed in the previous section for the de Rham complex. We shortly recall that for $j \in \{1,\dots,p\}$, by construction, $\theta_j = 0$ outside of $\widetilde{\Upsilon}_{j,1}$ and that θ_j is constant on each connected component $\Upsilon_{j,0}$ and $\Upsilon_{j,1}$ of $\Upsilon_j \setminus F_j$. Let $\widehat{r}_k$ be the Raviart–Thomas fields from Sect. 11.3 given by $\widehat{r}_0(x) := x$ and $\widehat{r}_k(x) := e^k$ for $k \in \{1,2,3\}$. We define the vector fields $\theta_{j,k} := \theta_j \widehat{r}_k$ and note $\operatorname{devGrad} \theta_{j,k} = 0$ in $\dot{\bigcup}_{l\in\{1,\dots,p\}} \Upsilon_l \setminus F_l$ for all $j \in \{1,\dots,p\}$ and all $k \in \{1,2,3\}$. Thus $\operatorname{devGrad} \theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times 3}(\Omega) \cap L^{2,3\times 3}_{\mathbb{T}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the neighbourhoods Υ_l of all the surfaces F_l, $l \in \{1,\dots,p\}$.

Lemma 12.15 *Let Assumption 10.3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega)$.

Proof Let $\Phi \in C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega)$. As $\operatorname{supp} \Theta_{j,k} \subseteq \widetilde{\Upsilon}_j \setminus \Upsilon_j$ we can pick another cut-off function $\varphi \in C^\infty_c(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi} = 1$. Then

$$
\begin{aligned}
&\langle \Theta_{j,k}, \operatorname{Curl}_{\mathbb{S}} \Phi\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\\
&\quad= \langle \Theta_{j,k}, \operatorname{Curl}_{\mathbb{S}} \Phi\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi)}\\
&\quad= \big\langle \operatorname{devGrad} \theta_{j,k}, \operatorname{Curl}_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega_F)}\\
&\quad= \big\langle \operatorname{Grad} \theta_{j,k}, \operatorname{dev}\operatorname{Curl}_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega_F)}\\
&\quad= \big\langle \operatorname{Grad} \theta_{j,k}, \operatorname{Curl}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}(\Omega_F)} = 0
\end{aligned}
$$

as $\varphi\Phi \in C^{\infty,3\times 3}_c(\Omega_F)$, where in the second to last equality sign, we used that the Curl applied to a symmetric tensor fields is trace-free, i.e., deviatoric, see Lemma 12.10. □

Next, we note that for $l, j \in \{1,\dots,p\}$ and $k \in \{0,\dots,3\}$ and for the curves $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l$ with the chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$ we can compute by Lemma 12.11

$$
\mathbb{R} \ni \beta_{l,0}(\Theta_{j,k}) := \frac{1}{2}\int_{\zeta_l} \langle \operatorname{Div} \Theta^\top_{j,k}, \mathrm{d}\,\lambda\rangle = \frac{1}{2}\int_{\zeta_{x_{l,0},x_{l,1}}} \big\langle \operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top, \mathrm{d}\,\lambda\big\rangle
$$

$$= \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,1}) - \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,0}) = \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,1})$$
$$= \frac{1}{3}\delta_{l,j}\operatorname{div}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } k\in\{1,2,3\},\end{cases}$$

and

$$\begin{aligned}
\mathbb{R}^3 \ni b_l(\Theta_{j,k}) &:= \int_{\zeta_l}\Theta_{j,k}\,\mathrm{d}\lambda + \frac{1}{2}\int_{\zeta_l}(x_{l,1}-y)\big\langle(\operatorname{Div}\Theta_{j,k}^\top)(y),\mathrm{d}\lambda_y\big\rangle\\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\operatorname{devGrad}\theta_{j,k}\,\mathrm{d}\lambda\\
&\quad + \frac{1}{2}\int_{\zeta_{x_{l,0},x_{l,1}}}(x_{l,1}-y)\Big\langle\big(\operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top\big)(y),\mathrm{d}\lambda_y\Big\rangle\\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\Bigg(\operatorname{devGrad}\theta_{j,k}(y)\\
&\quad + \frac{1}{2}\Big(\int_{\zeta_{x_{l,0},y}}\big\langle\operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top,\mathrm{d}\lambda\big\rangle\Big)\operatorname{Id}\Bigg)\mathrm{d}\lambda_y\\
&= \theta_{j,k}(x_{l,1}) - \theta_{j,k}(x_{l,0}) - \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,0})(x_{l,1}-x_{l,0}) = \theta_{j,k}(x_{l,1})\\
&= \delta_{l,j}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases}x_{l,1}, & \text{if } k=0,\\ e^k, & \text{if } k\in\{1,2,3\}.\end{cases}
\end{aligned}$$

Thus, for $l\in\{1,\dots,p\}$ and $\ell\in\{0,\dots,3\}$ we have functionals $\beta_{l,\ell}$, given by

$$\beta_{l,0}(\Theta_{j,k}) = \delta_{l,j}\delta_{0,k}$$

for $l, j\in\{1,\dots,p\}$ and $k\in\{0,\dots,3\}$, as well as

$$\beta_{l,\ell}(\Theta_{j,k}) := \big\langle b_l(\Theta_{j,k}), e^\ell\big\rangle = \delta_{l,j}\begin{cases}\langle x_{l,1}, e^\ell\rangle = (x_{l,1})_\ell, & \text{if } k=0,\\ \langle e^k, e^\ell\rangle = \delta_{\ell,k}, & \text{if } k\in\{1,2,3\},\end{cases}$$

for $l, j\in\{1,\dots,p\}$ and $\ell\in\{1,2,3\}$ and $k\in\{0,\dots,3\}$. Therefore, we have

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + (1-\delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_\ell,\quad l,j\in\{1,\dots,p\},\quad k,\ell\in\{0,\dots,3\}. \tag{12.10}$$

Let Assumption 10.2 be satisfied. For the first biharmonic complex, similar to (3.3), (3.5), we have the orthogonal decompositions

$$\begin{aligned}
L^{2,3\times 3}_{\mathbb{T}}(\Omega) &= \operatorname{ran}(\operatorname{devGrad},\Omega)\oplus_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\ker(\mathring{\operatorname{Div}}_{\mathbb{T}},\Omega),\\
\ker(\operatorname{symCurl}_{\mathbb{T}},\Omega) &= \operatorname{ran}(\operatorname{devGrad},\Omega)\oplus_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega).
\end{aligned} \tag{12.11}$$

Remark 12.16 By Assumption 10.2, [38, Lemma 3.2] yields $\operatorname{dom}(\operatorname{devGrad},\Omega) = H^{1,3}(\Omega)$. As a consequence using Rellich's selection theorem, the range in (12.11) is closed and the Poincaré type estimate

$$\exists\, c>0\quad \forall\,\phi\in H^{1,3}(\Omega)\cap \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\qquad |\phi|_{L^{2,3}(\Omega)} \le c\,|\operatorname{devGrad}\phi|_{L^{2,3\times 3}(\Omega)},$$

holds, see also [38, Lemma 3.2].

Let $\pi : L^{2,3\times 3}_{\mathbb{T}}(\Omega) \to \ker(\mathring{\mathrm{Div}}_{\mathbb{T}}, \Omega)$ denote the orthogonal projector onto $\ker(\mathring{\mathrm{Div}}_{\mathbb{T}}, \Omega)$ along $\mathrm{ran}(\mathrm{devGrad}, \Omega)$, see (12.11). We have $\pi\big(\ker(\mathrm{symCurl}_{\mathbb{T}}, \Omega)\big) = \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)$. By Lemma 12.15 there exists some $\psi_{j,k} \in H^{1,3}(\Omega)$ such that

$$\begin{aligned}\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \ni \pi\Theta_{j,k} &= \Theta_{j,k} - \mathrm{devGrad}\,\psi_{j,k},\\ (\Theta_{j,k} - \mathrm{devGrad}\,\psi_{j,k})\big|_{\Omega_F} &= \mathrm{devGrad}(\theta_{j,k} - \psi_{j,k}).\end{aligned}$$

As $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \subseteq C^{\infty,3\times 3}(\Omega)$, cf. (10.1), we conclude by $\pi\Theta_{j,k}, \Theta_{j,k} \in C^{\infty,3\times 3}(\Omega)$ that also $\mathrm{devGrad}\,\psi_{j,k} \in C^{\infty,3\times 3}(\Omega)$ and hence $\psi_{j,k} \in C^{\infty,3}(\Omega)$. Thus all path integrals over the closed curves ζ_l are well-defined. Furthermore, we observe by Lemma 12.11

$$\begin{aligned}\beta_{l,0}(\mathrm{devGrad}\,\psi_{j,k}) &= \frac{1}{2}\int_{\zeta_l}\big\langle \mathrm{Div}(\mathrm{devGrad}\,\psi_{j,k})^{\top}, \mathrm{d}\,\lambda\big\rangle\\ &= \frac{1}{3}\,\mathrm{div}\,\psi_{j,k}(x_{l,1}) - \frac{1}{3}\,\mathrm{div}\,\psi_{j,k}(x_{l,1}) = 0\end{aligned}$$

and

$$\begin{aligned}&b_l(\mathrm{devGrad}\,\psi_{j,k})\\ &= \int_{\zeta_l}\mathrm{devGrad}\,\psi_{j,k}\,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_l}(x_{l,1}-y)\Big\langle\big(\mathrm{Div}(\mathrm{devGrad}\,\psi_{j,k})^{\top}\big)(y), \mathrm{d}\,\lambda_y\Big\rangle\\ &= \int_{\zeta_{x_{l,1},x_{l,1}}}\left(\mathrm{devGrad}\,\psi_{j,k}(y) + \frac{1}{2}\Big(\int_{\zeta_{x_{l,1},y}}\big\langle \mathrm{Div}(\mathrm{devGrad}\,\psi_{j,k})^{\top}, \mathrm{d}\,\lambda\big\rangle\Big)\,\mathrm{Id}\right)\mathrm{d}\,\lambda_y\\ &= \psi_{j,k}(x_{l,1}) - \psi_{j,k}(x_{l,1}) - \frac{1}{3}\,\mathrm{div}\,\psi_{j,k}(x_{l,1})(x_{l,1} - x_{l,1}) = 0.\end{aligned}$$

Therefore, $\beta_{l,\ell}(\mathrm{devGrad}\,\psi_{j,k}) = 0$ and by (12.10) we have

$$\begin{aligned}\beta_{l,\ell}(\pi\Theta_{j,k}) &= \beta_{l,\ell}(\Theta_{j,k}) - \beta_{l,\ell}(\mathrm{devGrad}\,\psi_{j,k})\\ &= \delta_{l,j}\delta_{\ell,k} + (1-\delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_{\ell}\end{aligned} \tag{12.12}$$

for all $l, j \in \{1,\ldots,p\}$ and all $\ell, k \in \{0,1,2,3\}$. We shall show that

$$\mathcal{B}^{\mathsf{bih},1}_{N} := \big\{\pi\Theta_{j,k} : j \in \{1,\ldots,p\},\ k \in \{0,\ldots,3\}\big\} \subseteq \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \tag{12.13}$$

defines a basis of $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)$.

Remark 12.17 (Characterisation by PDEs) Note that $\psi_{j,k} \in H^{1,3}(\Omega) \cap \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$ can be found by the variational formulation

$$\forall\,\phi \in H^{1,3}(\Omega) \qquad \langle \mathrm{devGrad}\,\psi_{j,k}, \mathrm{devGrad}\,\phi\rangle_{L^{2,3\times 3}(\Omega)} = \langle \Theta_{j,k}, \mathrm{devGrad}\,\phi\rangle_{L^{2,3\times 3}(\Omega)},$$

i.e., $\psi_{j,k} = \Delta^{-1}_{\mathbb{T},N}\big(\mathrm{Div}_{\mathbb{T}}\,\Theta_{j,k}, \Theta_{j,k}\nu|_{\Gamma}\big)$, where $\Delta_{\mathbb{T},N} \subseteq \mathrm{Div}_{\mathbb{T}}\,\mathrm{devGrad}$ denotes the 'deviatoric' Laplacian with inhomogeneous Neumann boundary conditions restricted to a subset of $H^{1,3}(\Omega) \cap \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$. Therefore,

$$\pi\Theta_{j,k} = \Theta_{j,k} - \mathrm{devGrad}\,\psi_{j,k} = \Theta_{j,k} - \mathrm{devGrad}\,\Delta^{-1}_{\mathbb{T},N}\big(\mathrm{Div}_{\mathbb{T}}\,\Theta_{j,k}, \Theta_{j,k}\nu|_{\Gamma}\big).$$

In classical terms, $\psi_{j,k}$ solves the Neumann elasticity type problem

$$\begin{aligned} -\Delta_{\mathbb{T}}\psi_{j,k} &= -\operatorname{Div}_{\mathbb{T}}\Theta_{j,k} && \text{in } \Omega, \\ (\operatorname{Grad}\psi_{j,k})\nu &= \Theta_{j,k}\nu && \text{on } \Gamma, \\ \int_{\Omega_l}(\psi_{j,k})_\ell &= 0 && \text{for } l \in \{1,\dots,n\}, \quad \ell \in \{1,2,3\}, \\ \int_{\Omega_l} x\cdot\psi_{j,k}(x)\,\mathrm{d}\lambda_x &= 0 && \text{for } l \in \{1,\dots,n\}, \end{aligned} \tag{12.14}$$

which is uniquely solvable.

Lemma 12.18 *Let Assumption 10.2 as well as Assumption 10.3 be satisfied. Then we have* $\mathcal{H}^{bih,1}_{N,\mathbb{T}}(\Omega) = \operatorname{lin}\mathcal{B}^{bih,1}_N$.

Proof Let $H \in \mathcal{H}^{\mathrm{bih},1}_{N,\mathbb{T}}(\Omega) = \ker(\mathring{\operatorname{Div}}_{\mathbb{T}},\Omega)\cap\ker(\operatorname{symCurl}_{\mathbb{T}},\Omega) \subseteq C^{\infty,3\times3}_{\mathbb{T}}(\Omega)$, see Lemma 12.2. With the above introduced functionals $\beta_{l,0}$ and b_l, $l \in \{1,\dots,p\}$, we recall

$$\begin{aligned} \mathbb{R} \ni \beta_{l,0}(H) &= \frac{1}{2}\int_{\zeta_l}\langle\operatorname{Div} H^\top, \mathrm{d}\lambda\rangle, \\ \mathbb{R}^3 \ni b_l(H) &= \int_{\zeta_l} H\,\mathrm{d}\lambda + \frac{1}{2}\int_{\zeta_l}(x_{l,1}-y)\big\langle(\operatorname{Div} H^\top)(y), \mathrm{d}\lambda_y\big\rangle, \end{aligned}$$

and define for $l \in \{1,\dots,p\}$ and $\ell \in \{1,2,3\}$ the numbers

$$\begin{aligned} \gamma_{l,0} &:= \gamma_{l,0}(H) := \beta_{l,0}(H), \\ \gamma_{l,\ell} &:= \gamma_{l,\ell}(H) := \big\langle b_l(H) - \beta_{l,0}(H)x_{l,1}, e^\ell\big\rangle = \beta_{l,\ell}(H) - \beta_{l,0}(H)(x_{l,1})_\ell. \end{aligned} \tag{12.15}$$

We shall show that

$$\mathcal{H}^{\mathrm{bih},1}_{N,\mathbb{T}}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\pi\Theta_{j,k} = 0 \quad \text{in } \Omega.$$

Similar to the proof of Lemma 12.6, the aim is to prove the existence of $v \in H^{1,3}(\Omega)$ such that $\operatorname{devGrad} v = \widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = \langle \operatorname{devGrad} v, \widehat{H}\rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = 0.$$

For finding v, we will apply Corollary 12.14 and Remark 12.13 to $T = \widehat{H}$. By (12.12) we observe for all $l \in \{1,\dots,p\}$

$$\begin{aligned} \frac{1}{2}\int_{\zeta_l}\langle\operatorname{Div}\widehat{H}^\top, \mathrm{d}\lambda\rangle = \beta_{l,0}(\widehat{H}) &= \beta_{l,0}(H) - \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\beta_{l,0}(\pi\Theta_{j,k}) \\ &= \gamma_{l,0} - \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\delta_{l,j}\delta_{0,k} = 0. \end{aligned}$$

Note that $\operatorname{Div}\widehat{H}^\top \in \ker(\operatorname{curl},\Omega)\cap C^{\infty,3}(\Omega)$ by Remark 12.13 (i) as $\widehat{H}$ belongs to $\ker(\operatorname{symCurl}_{\mathbb{T}},\Omega)\cap C^{\infty,3\times3}(\Omega)$. Thus, by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta}\langle\operatorname{Div}\widehat{H}^\top, \mathrm{d}\lambda\rangle = 0. \tag{12.16}$$

Let $u \in C^\infty(\Omega)$ be as in Corollary 12.14 (a), (ii), i.e., $\operatorname{grad} u = \operatorname{Div} \widehat{H}^\top$, and define $S : \Omega \to \mathbb{R}^{3\times 3}$ by

$$S := \widehat{H} + \frac{1}{2}\, u \operatorname{Id}.$$

Our next aim is to show condition (b), (ii) of Corollary 12.14. For this, let $l \in \{1, \dots, p\}$. Note that $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l \subseteq \Omega_0$ for some $\Omega_0 \in \mathsf{cc}(\Omega)$. Then we have with $c := u(x_{l,1}) \in \mathbb{R}$ for all $x \in \zeta_l$

$$\begin{aligned} u(x) = u(x) - u(x_{l,1}) + c &= \int_{\zeta_{x_{l,1},x}} \langle \operatorname{grad} u, \mathrm{d}\,\lambda\rangle + c \\ &= \int_{\zeta_{x_{l,1},x}} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle + c, \end{aligned}$$

where $\zeta_{x_{l,1},x}$ denotes the path from $x_{l,1}$ to x along ζ_l. Moreover,

$$\int_{\zeta_l} (c \operatorname{Id})\, \mathrm{d}\,\lambda = c \int_{\zeta_l} \operatorname{Grad} x \, \mathrm{d}\,\lambda_x = 0.$$

Next, we consider the closed curve ζ_l as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. Then, using Lemma 12.11 and the definition of b_l, we compute

$$\begin{aligned} \int_{\zeta_l} S\, \mathrm{d}\,\lambda &= \int_{\zeta_l} \widehat{H}\, \mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_l} (u \operatorname{Id})\, \mathrm{d}\,\lambda \\ &= \int_{\zeta_l} \widehat{H}\, \mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_{x_{l,1},x_{l,1}}} \Big(\int_{\zeta_{x_{l,1},y}} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle \Big) \operatorname{Id} \mathrm{d}\,\lambda_y \\ &= \int_{\zeta_l} \widehat{H}\, \mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_l} (x_{l,1} - y) \big\langle (\operatorname{Div} \widehat{H}^\top)(y), \mathrm{d}\,\lambda \big\rangle \mathrm{d}\,\lambda_y = b_l(\widehat{H}). \end{aligned}$$

Hence, for $\ell \in \{1, 2, 3\}$ we get by (12.12) recalling (12.15)

$$\begin{aligned} \Big(\int_{\zeta_l} S\, \mathrm{d}\,\lambda\Big)_\ell &= \Big\langle \int_{\zeta_l} S\, \mathrm{d}\,\lambda, e^\ell \Big\rangle \\ &= \langle b_l(\widehat{H}), e^\ell\rangle = \beta_{l,\ell}(\widehat{H}) \\ &= \beta_{l,\ell}(H) - \sum_{j=1}^{p} \sum_{k=0}^{3} \gamma_{j,k} \beta_{l,\ell}(\pi \Theta_{j,k}) \\ &= \beta_{l,\ell}(H) - \sum_{j=1}^{p} \sum_{k=0}^{3} \gamma_{j,k} \big(\delta_{l,j}\delta_{\ell,k} \\ &\quad + (1 - \delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_\ell\big) \\ &= \beta_{l,\ell}(H) - \gamma_{l,0}(x_{l,1})_\ell - \gamma_{l,\ell} = \beta_{l,\ell}(H) \\ &\quad - \beta_{l,0}(H)(x_{l,1})_\ell - \gamma_{l,\ell} = 0. \end{aligned}$$

Therefore, $\int_{\zeta_l} S\, \mathrm{d}\,\lambda = 0$ for all $l \in \{1, \dots, p\}$. Note that $S \in \ker(\operatorname{Curl}, \Omega) \cap C^{\infty,3\times 3}(\Omega)$ by Remark 12.13 (ii) as $\widehat{H} \in \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega) \cap C^{\infty,3\times 3}_{\mathbb{T}}(\Omega)$. Thus, by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} S\, \mathrm{d}\,\lambda = 0. \tag{12.17}$$

Hence, Corollary 12.14 (b) and (c) (note $\operatorname{tr}\widehat{H} = 0$) imply the existence of a smooth vector field $v : \Omega \to \mathbb{R}^3$ such that $\operatorname{devGrad} v = \widehat{H}$. Finally, similar to the end of the proof of Lemma 12.6, elliptic regularity and, e.g., [22, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)], show that $v \in C^{\infty,3}(\Omega_0)$ and $\operatorname{devGrad} v \in L^{2,3\times3}_{\mathbb{T}}(\Omega_0)$ imply $v \in H^{1,3}(\Omega_0)$ for all $\Omega_0 \in \mathsf{cc}(\Omega)$ and thus $v \in H^{1,3}(\Omega)$, completing the proof. (Let us note that $v \in H^{1,3}(\Omega)$ implies also $S \in L^{2,3\times3}(\Omega)$ and hence $u \in L^2(\Omega)$.) □

Lemma 12.19 *Let Assumptions 10.2 and 10.3 be satisfied. Then $\mathcal{B}^{\mathsf{bih},1}_N$ is linearly independent.*

Proof Let $\gamma_{j,k} \in \mathbb{R}$, $j \in \{1, \dots, p\}$, $k \in \{0, \dots, 3\}$, be such that $\sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\pi\Theta_{j,k} = 0$. Then (12.12) implies for $l \in \{1, \dots, p\}$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,0}, \qquad \ell = 0,$$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell} + \gamma_{l,0}(x_{l,1})_\ell = \gamma_{l,\ell}, \qquad \ell \in \{1, 2, 3\},$$

finishing the proof. □

Theorem 12.20 *Let Assumptions 10.2 and 10.3 be satisfied. Then $\dim \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) = 4p$ and a basis of $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)$ is given by (12.13).*

Proof Use Lemmas 12.18 and 12.19. □

12.3 Neumann tensor fields of the second biharmonic complex

The rationale for the second biharmonic complex in comparison to the first one has to be changed appropriately. For this we also use Lemma 12.10, the Poincaré maps, however, differ from one another.

In Lemma 12.21 below for a tensor field S and a parametrisation $\varphi \in C^{1,3}_{\mathsf{pw}}\big([0,1]\big)$ of a curve ζ we define

$$\int_\zeta \langle x - y, S(y)\,\mathrm{d}\lambda_y\rangle := \int_0^1 \Big\langle x - \varphi(t), S\big(\varphi(t)\big)\varphi'(t)\Big\rangle\,\mathrm{d}t.$$

Lemma 12.21 *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subseteq \Omega$ be a piecewise C^1-curve connecting x_0 and x.*

(i) *Let $u \in C^\infty(\Omega)$. Then u and its gradient $\operatorname{grad} u$ can be represented by*

$$u(x) - u(x_0) - \big\langle \operatorname{grad} u(x_0), x - x_0\big\rangle = \int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} \operatorname{Gradgrad} u\,\mathrm{d}\lambda, \mathrm{d}\lambda_y\Big\rangle$$

and

$$\operatorname{grad} u(x) - \operatorname{grad} u(x_0) = \int_{\zeta_{x_0,x}} \operatorname{Gradgrad} u\,\mathrm{d}\lambda.$$

(ii) *For all $S \in C^{\infty,3\times3}(\Omega)$*

$$\int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} S\,\mathrm{d}\lambda, \mathrm{d}\lambda_y\Big\rangle = \int_{\zeta_{x_0,x}} \langle x - y, S(y)\,\mathrm{d}\lambda_y\rangle.$$

Proof For (i),

$$u(x) - u(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} u, \mathrm{d}\,\lambda \rangle,$$

$$\partial_k u(x) - \partial_k u(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} \partial_k u, \mathrm{d}\,\lambda \rangle, \qquad k \in \{1, 2, 3\},$$

i.e.,

$$\operatorname{grad} u(x) - \operatorname{grad} u(x_0) = \int_{\zeta_{x_0,x}} \operatorname{Grad} \operatorname{grad} u \,\mathrm{d}\,\lambda.$$

Therefore,

$$\begin{aligned} u(x) - u(x_0) &= \int_{\zeta_{x_0,x}} \big\langle \operatorname{grad} u(y), \mathrm{d}\,\lambda_y \big\rangle \\ &= \int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} \operatorname{Grad} \operatorname{grad} u \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle \\ &\quad + \int_{\zeta_{x_0,x}} \big\langle \operatorname{grad} u(x_0), \mathrm{d}\,\lambda_y \big\rangle. \end{aligned}$$

Using $\varphi \in C^{1,3}_{\mathsf{pw}}\big([0,1]\big)$ as a parametrisation of $\zeta_{x_0,x}$, we conclude the proof of (i) by

$$\int_{\zeta_{x_0,x}} \big\langle \operatorname{grad} u(x_0), \mathrm{d}\,\lambda_y \big\rangle = \int_0^1 \big\langle \operatorname{grad} u(x_0), \varphi'(t) \big\rangle \,\mathrm{d}\,t = \big\langle \operatorname{grad} u(x_0), x - x_0 \big\rangle.$$

For (ii), we compute

$$\begin{aligned} \int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} S \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle &= \int_0^1 \Big\langle \int_{\zeta_{x_0,\varphi(s)}} S \,\mathrm{d}\,\lambda, \varphi'(s) \Big\rangle \,\mathrm{d}\,s \\ &= \int_0^1 \Big\langle \int_0^s S\big(\varphi(t)\big)\varphi'(t) \,\mathrm{d}\,t, \varphi'(s) \Big\rangle \,\mathrm{d}\,s \\ &= \int_0^1 \Big\langle S\big(\varphi(t)\big)\varphi'(t), \int_t^1 \varphi'(s) \,\mathrm{d}\,s \Big\rangle \,\mathrm{d}\,t \\ &= \int_0^1 \Big\langle S\big(\varphi(t)\big)\varphi'(t), x - \varphi(t) \Big\rangle \,\mathrm{d}\,t \\ &= \int_{\zeta_{x_0,x}} \big\langle x - y, S(y) \,\mathrm{d}\,\lambda_y \big\rangle \end{aligned}$$

again with φ parametrising $\zeta_{x_0,x}$. □

Proposition 12.22 *Let $x_0 \in \Omega_0 \in \mathsf{cc}(\Omega)$ and let $w \in C^{\infty,3}(\Omega_0)$ and $S \in C^{\infty,3\times 3}(\Omega_0)$.*

(a) *The following conditions are equivalent:*

- ***(i)*** *For all $\zeta \subseteq \Omega_0$ closed, piecewise C^1-curves*

$$\int_\zeta S \,\mathrm{d}\,\lambda = 0.$$

(ii) For all $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curves connecting* x_0 *with* x

$$\int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda = \int_{\widetilde{\zeta}_{x_0,x}} S \,\mathrm{d}\,\lambda.$$

(iii) There exists $v \in C^{\infty,3}(\Omega_0)$ *such that* $\operatorname{Grad} v = S$.

In the case one of the above conditions is true the vector field

$$x \mapsto v(x) = \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda \tag{12.18}$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curve connecting* x_0 *with* x *is a (well-defined) possible choice for* v *in (iii).*

(b) The following conditions are equivalent:

(i) For all $\zeta \subseteq \Omega_0$ *closed, piecewise* C^1 *curves*

$$\int_{\zeta} \langle w, \mathrm{d}\,\lambda\rangle = 0.$$

(ii) For all $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1 *curves connecting* x_0 *with* x

$$\int_{\zeta_{x_0,x}} \langle w, \mathrm{d}\,\lambda\rangle = \int_{\widetilde{\zeta}_{x_0,x}} \langle w, \mathrm{d}\,\lambda\rangle.$$

(iii) There exists $u \in C^{\infty}(\Omega_0)$ *such that* $\operatorname{grad} u = w$.

In the case one of the above conditions is true the function

$$x \mapsto u(x) = \int_{\zeta_{x_0,x}} \big\langle w, \mathrm{d}\,\lambda\big\rangle \tag{12.19}$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curve connecting* x_0 *with* x *is a (well-defined) possible choice for* u *in (iii).*

(c) Let $v \in C^{\infty,3}(\Omega_0)$ *with* $\operatorname{Grad} v = S$ *as in (a), (iii) and let* $u \in C^{\infty}(\Omega_0)$ *with* $\operatorname{grad} u = v$ *as in (b), (iii). Then* $\operatorname{Gradgrad} u = S$, $\operatorname{skw} S = 0$, *and* $\operatorname{Curl}_{\mathbb{S}} S = 0$.

Proof The statements in (a) and (b) are straightforward consequences of the fundamental theorem of calculus and follow essentially the same lines as (a) and (b) of Proposition 12.12. Schwarz's Lemma and the complex property show (c). □

Remark 12.23 Related to Proposition 12.22 we note with Lemma 12.10 the following:

(i) For $v \in C^{\infty,3}(\Omega)$ we have $\operatorname{curl} v = 2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Grad} v$.
(ii) If Ω_0 is simply connected, Proposition 12.22 (a), (iii) and (b), (iii) are equivalent to $\operatorname{Curl} S = 0$ and $\operatorname{curl} w = 0$, respectively.

Similar to the first biharmonic complex, there exists an analogous version of Proposition 12.22 irrespective of the components. We only formulate the following slightly weaker statement, which is an easy consequence of Proposition 12.22.

Corollary 12.24 *Let* $w \in C^{\infty,3}(\Omega)$ *and* $S \in C^{\infty,3\times 3}(\Omega)$.

(a) The following conditions are equivalent:

(i) For all $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_{\zeta} S \,\mathrm{d}\,\lambda = 0$.

*(**ii**) There exists* $v \in C^{\infty,3}(\Omega)$ *such that* $\operatorname{Grad} v = S$.

*(**b**) The following conditions are equivalent:*

*(**i**) For all* $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_\zeta \langle w, \mathrm{d}\lambda \rangle = 0$.
*(**ii**) There exists* $u \in C^\infty(\Omega)$ *such that* $\operatorname{grad} u = w$.

*(**c**) Let* $v \in C^{\infty,3}(\Omega)$ *with* $\operatorname{Grad} v = S$ *as in (a), (ii) and let* $u \in C^\infty(\Omega)$ *with* $\operatorname{grad} u = v$ *as in (b), (ii). Then* $\operatorname{Gradgrad} u = S$, $\operatorname{skw} S = 0$, *and* $\operatorname{Curl}_{\mathbb{S}} S = 0$.

Next, we turn to the actual construction of the Neumann fields for the second biharmonic complex. Let $j \in \{1, \ldots, p\}$. For this, recall from the beginning of Sect. 12 that θ_j is constant on each connected component $\Upsilon_{j,0}$ and $\Upsilon_{j,1}$ of $\Upsilon_j \setminus F_j$ and vanishes outside of $\widetilde{\Upsilon}_{j,1}$. Moreover, let $\widehat{p}_k$ be the polynomials from Sect. 11.2 given by $\widehat{p}_0(x) := 1$ and $\widehat{p}_k(x) := x_k$ for $k \in \{1, 2, 3\}$. We define the functions $\theta_{j,k} := \theta_j \widehat{p}_k$ and note $\operatorname{Gradgrad} \theta_{j,k} = 0$ in $\dot{\bigcup}_{l\in\{1,\ldots,p\}} \Upsilon_l \setminus F_l$ for all $k \in \{1, 2, 3\}$. Thus $\operatorname{Gradgrad} \theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the neighbourhoods Υ_l of all the surfaces F_l, $l \in \{1, \ldots, p\}$.

Lemma 12.25 *Let Assumption 10.3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{Curl}_{\mathbb{S}}, \Omega)$.

Proof Let $\Phi \in C^{\infty,3\times3}_{c,\mathbb{T}}(\Omega)$. As $\operatorname{supp} \Theta_{j,k} \subseteq \widetilde{\Upsilon}_j \setminus \Upsilon_j$ we can pick another cut-off function $\varphi \in C^\infty_c(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi} = 1$. Then

$$\begin{aligned}\langle \Theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}} \Phi \rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} &= \langle \Theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}} \Phi \rangle_{L^{2,3\times3}_{\mathbb{S}}(\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi)} \\ &= \big\langle \operatorname{Gradgrad} \theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}}(\varphi\Phi) \big\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega_F)} \\ &= \big\langle \operatorname{Grad}(\operatorname{grad} \theta_{j,k}), \operatorname{Curl}(\varphi\Phi) \big\rangle_{L^{2,3\times3}(\Omega_F)} = 0\end{aligned}$$

as $\varphi\Phi, \operatorname{Curl}(\varphi\Phi) \in C^{\infty,3\times3}_c(\Omega_F)$. □

Similar to the first biharmonic complex, we introduce a set of functionals.

For $l, j \in \{1, \ldots, p\}$ and $k \in \{0, \ldots, 3\}$ and for the curves $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l$ with the chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$ we can compute by Lemma 12.21

$$\begin{aligned}\mathbb{R}^3 \ni b_l(\Theta_{j,k}) := \int_{\zeta_l} \Theta_{j,k} \,\mathrm{d}\lambda &= \int_{\zeta_{x_{l,0},x_{l,1}}} \operatorname{Gradgrad} \theta_{j,k} \,\mathrm{d}\lambda \\ &= \operatorname{grad} \theta_{j,k}(x_{l,1}) - \operatorname{grad} \theta_{j,k}(x_{l,0}) = \operatorname{grad} \theta_{j,k}(x_{l,1}) \\ &= \delta_{l,j} \operatorname{grad} \widehat{p}_k(x_{l,1}) = \delta_{l,j} \begin{cases} 0, & \text{if } k = 0, \\ e^k, & \text{if } k = 1, 2, 3, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\mathbb{R} \ni \beta_{l,0}(\Theta_{j,k}) &:= \int_{\zeta_l} \big\langle x_{l,1} - y, \Theta_{j,k}(y) \,\mathrm{d}\lambda_y \big\rangle \\ &= \int_{\zeta_{x_{l,0},x_{l,1}}} \big\langle x_{l,1} - y, \operatorname{Gradgrad} \theta_{j,k}(y) \,\mathrm{d}\lambda_y \big\rangle \\ &= \int_{\zeta_{x_{l,0},x_{l,1}}} \Big\langle \int_{\zeta_{x_{l,0},y}} \operatorname{Gradgrad} \theta_{j,k} \,\mathrm{d}\lambda, \mathrm{d}\lambda_y \Big\rangle\end{aligned}$$

$$= \theta_{j,k}(x_{l,1}) - \theta_{j,k}(x_{l,0}) - \langle \operatorname{grad}\theta_{j,k}(x_{l,0}), x_{l,1} - x_{l,0}\rangle = \theta_{j,k}(x_{l,1})$$
$$= \delta_{l,j}\widehat{p}_k(x_{l,1}) = \delta_{l,j}\begin{cases} 1, & \text{if } k = 0, \\ (x_{l,1})_k, & \text{if } k \in \{1,2,3\}. \end{cases}$$

Thus, for $l \in \{1,\dots,p\}$ and $\ell \in \{0,\dots,3\}$ we have functionals $\beta_{l,\ell}$ given by

$$\beta_{l,\ell}(\Theta_{j,k}) := \langle b_l(\Theta_{j,k}), e^\ell\rangle = \delta_{l,j}\begin{cases} 0, & \text{if } k = 0, \\ \delta_{\ell,k}, & \text{if } k \in \{1,2,3\}, \end{cases}$$

for $l, j \in \{1,\dots,p\}$ and $\ell \in \{1,2,3\}$ and $k \in \{0,1,2,3\}$, as well as

$$\beta_{l,0}(\Theta_{j,k}) = \delta_{l,j}\delta_{0,k} + \delta_{l,j}(1-\delta_{0,k})(x_{l,1})_k$$

for $l, j \in \{1,\dots,p\}$ and $k \in \{0,1,2,3\}$. Therefore, we have

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + (1-\delta_{0,k})\delta_{\ell,0}\delta_{l,j}(x_{l,1})_k, \quad l, j \in \{1,\dots,p\},\ k,\ell \in \{0,1,2,3\}. \tag{12.20}$$

Let Assumption 10.2 be satisfied. For the second biharmonic complex, similar to (3.3), (3.5), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3\times3}_{\mathbb{S}}(\Omega) &= \operatorname{ran}(\operatorname{Gradgrad},\Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},\Omega), \\ \ker(\operatorname{Curl}_{\mathbb{S}},\Omega) &= \operatorname{ran}(\operatorname{Gradgrad},\Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega). \end{aligned} \tag{12.21}$$

Remark 12.26 Lemma 8.2 shows $\operatorname{dom}(\operatorname{Gradgrad},\Omega) = H^2(\Omega)$. Thus, employing a contradiction argument together with Rellich's selection theorem, we obtain the Poincaré type estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H^2(\Omega) \cap (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}} \qquad |\phi|_{L^2(\Omega)} \le c\,|\operatorname{Grad}\operatorname{grad}\phi|_{L^{2,3\times3}(\Omega)},$$

as Assumption 10.2 holds. Thus, the range in (12.21) is closed.

Let $\pi : L^{2,3\times3}_{\mathbb{S}}(\Omega) \to \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},\Omega)$ denote the orthogonal projector onto $\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},\Omega)$ along $\operatorname{ran}(\operatorname{Gradgrad},\Omega)$, see (12.21). In particular, $\pi\big(\ker(\operatorname{Curl}_{\mathbb{S}},\Omega)\big) = \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)$. By Lemma 12.25 there exists some $\psi_{j,k} \in H^2(\Omega)$ such that

$$\begin{aligned} \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \ni \pi\Theta_{j,k} &= \Theta_{j,k} - \operatorname{Gradgrad}\psi_{j,k}, \\ (\Theta_{j,k} - \operatorname{Gradgrad}\psi_{j,k})\big|_{\Omega_F} &= \operatorname{Gradgrad}(\theta_{j,k} - \psi_{j,k}). \end{aligned}$$

As $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \subseteq C^{\infty,3\times3}(\Omega)$, see Lemma 12.2, we conclude by $\pi\Theta_{j,k}, \Theta_{j,k} \in C^{\infty,3\times3}(\Omega)$ that also $\operatorname{Gradgrad}\psi_{j,k} \in C^{\infty,3\times3}(\Omega)$ and hence $\psi_{j,k} \in C^\infty(\Omega)$. Hence all path integrals over the closed curves ζ_l are well-defined. Furthermore, we observe by Lemma 12.21

$$b_l(\operatorname{Gradgrad}\psi_{j,k}) = \int_{\zeta_l} \operatorname{Gradgrad}\psi_{j,k}\,\mathrm{d}\lambda = \operatorname{grad}\psi_{j,k}(x_{l,1}) - \operatorname{grad}\psi_{j,k}(x_{l,1}) = 0$$

and

$$\begin{aligned} \beta_{l,0}(\operatorname{Gradgrad}\psi_{j,k}) &= \int_{\zeta_l} \langle x_{l,1} - y, \operatorname{Gradgrad}\psi_{j,k}(y)\,\mathrm{d}\lambda_y\rangle \\ &= \int_{\zeta_{x_{l,1},x_{l,1}}} \Big\langle \int_{\zeta_{x_{l,1},y}} \operatorname{Gradgrad}\psi_{j,k}\,\mathrm{d}\lambda, \mathrm{d}\lambda_y\Big\rangle \end{aligned}$$

$$= \psi_{j,k}(x_{l,1}) - \psi_{j,k}(x_{l,1}) - \big\langle \operatorname{grad} \psi_{j,k}(x_{l,1}), x_{l,1} - x_{l,1} \big\rangle = 0.$$

Therefore, $\beta_{l,\ell}(\operatorname{Gradgrad} \psi_{j,k}) = 0$ and by (12.20) we get

$$\begin{aligned} \beta_{l,\ell}(\pi \Theta_{j,k}) =& \beta_{l,\ell}(\Theta_{j,k}) - \beta_{l,\ell}(\operatorname{Gradgrad} \psi_{j,k}) \\ =& \delta_{l,j}\delta_{\ell,k} + (1-\delta_{0,k})\delta_{\ell,0}\delta_{l,j}(x_{l,1})_k \end{aligned} \tag{12.22}$$

for all $l, j \in \{1, \dots, p\}$ and all $\ell, k \in \{0, 1, 2, 3\}$. We shall show that

$$\mathcal{B}_N^{\mathsf{bih},2} := \big\{\pi \Theta_{j,k} : j \in \{1, \dots, p\},\ k \in \{0, \dots, 3\}\big\} \subseteq \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) \tag{12.23}$$

defines a basis of $\mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega)$.

Remark 12.27 (Characterisation by PDEs) Let $j \in \{1, \dots, p\}$ and $k \in \{0, \dots, 3\}$. Then $\psi_{j,k} \in H^2(\Omega) \cap (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}$ can be found by the variational formulation

$$\forall \phi \in H^2(\Omega) \qquad \langle \operatorname{Gradgrad} \psi_{j,k}, \operatorname{Gradgrad} \phi\rangle_{L^{2,3\times3}(\Omega)} = \langle \Theta_{j,k}, \operatorname{Gradgrad} \phi\rangle_{L^{2,3\times3}(\Omega)},$$

i.e., $\psi_{j,k} = (\Delta_{NN}^2)^{-1}\big(\operatorname{divDiv}_{\mathbb{S}} \Theta_{j,k}, \Theta_{j,k}\nu|_\Gamma, \nu \cdot \operatorname{Div} \Theta_{j,k}|_\Gamma\big)$, where $\Delta_{NN}^2 \subseteq \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad}$ is the bi-Laplacian with inhomogeneous Neumann type boundary conditions restricted to a subset of $H^2(\Omega) \cap (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}$. Therefore,

$$\begin{aligned} \pi \Theta_{j,k} &= \Theta_{j,k} - \operatorname{Gradgrad} \psi_{j,k} \\ &= \Theta_{j,k} - \operatorname{Gradgrad}(\Delta_{NN}^2)^{-1}\big(\operatorname{divDiv}_{\mathbb{S}} \Theta_{j,k}, \Theta_{j,k}\nu|_\Gamma, \nu \cdot \operatorname{Div} \Theta_{j,k}|_\Gamma\big). \end{aligned}$$

In classical terms, $\psi_{j,k}$ solves the biharmonic Neumann problem

$$\begin{aligned} \Delta^2 \psi_{j,k} &= \operatorname{divDiv}_{\mathbb{S}} \Theta_{j,k} && \text{in } \Omega, \\ (\operatorname{Gradgrad} \psi_{j,k})\nu &= \Theta_{j,k}\nu && \text{on } \Gamma, \\ \nu \cdot \operatorname{Div} \operatorname{Gradgrad} \psi_{j,k} &= \nu \cdot \operatorname{Div} \Theta_{j,k} && \text{on } \Gamma, \\ \int_{\Omega_l} \psi_{j,k} &= 0 && \text{for } l \in \{1, \dots, n\}, \\ \int_{\Omega_l} x_\ell \psi_{j,k}(x) \, \mathrm{d}\lambda_x &= 0 && \text{for } l \in \{1, \dots, n\}, \quad \ell \in \{1, 2, 3\}, \end{aligned} \tag{12.24}$$

which is uniquely solvable.

Lemma 12.28 *Let Assumptions 10.2 and 10.3 be satisfied. Then* $\mathcal{H}_{N,\mathbb{S}}^{bih,2}(\Omega) = \operatorname{lin} \mathcal{B}_N^{bih,2}$.

Proof Let $H \in \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) = \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega) \cap \ker(\operatorname{Curl}_{\mathbb{S}}, \Omega) \subseteq C_{\mathbb{S}}^{\infty,3\times3}(\Omega)$, see Lemma 12.2. With the above introduced functions $\beta_{l,0}$ and b_l, $l \in \{1, \dots, p\}$, we recall

$$\begin{aligned} \mathbb{R}^3 \ni b_l(H) &= \int_{\zeta_l} H \, \mathrm{d}\lambda, \\ \mathbb{R} \ni \beta_{l,0}(H) &= \int_{\zeta_l} \big\langle x_{l,1} - y, H(y) \, \mathrm{d}\lambda_y \big\rangle, \end{aligned}$$

and define for $l \in \{1, \dots, p\}$ the numbers

$$\begin{aligned} \gamma_{l,\ell} &:= \gamma_{l,\ell}(H) := \big\langle b_l(H), e^\ell \big\rangle = \beta_{l,\ell}(H), \qquad \ell \in \{1, 2, 3\}, \\ \gamma_{l,0} &:= \gamma_{l,0}(H) := \beta_{l,0}(H) - \sum_{k=1}^{3} \beta_{l,k}(H)(x_{l,1})_k. \end{aligned}$$

We shall show that

$$\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\pi\Theta_{j,k} = 0 \quad \text{in } \Omega.$$

Similar to the proof of Lemma 12.6, the aim is to prove that there exists $u \in H^2(\Omega)$ such that $\operatorname{Gradgrad} u = \widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{Gradgrad} u, \widehat{H}\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0.$$

For this, we shall apply Corollary 12.24 and Remark 12.23 to $S = \widehat{H}$. By (12.22) we observe for $\ell \in \{1, 2, 3\}$ and $l \in \{1, \ldots, p\}$

$$\begin{aligned}
\Big(\int_{\zeta_l} \widehat{H}\,\mathrm{d}\lambda\Big)_\ell &= \Big\langle \int_{\zeta_l} \widehat{H}\,\mathrm{d}\lambda, e^\ell\Big\rangle \\
&= \big\langle b_l(\widehat{H}), e^\ell\big\rangle = \beta_{l,\ell}(\widehat{H}) \\
&= \beta_{l,\ell}(H) - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) \\
&= \gamma_{l,\ell} - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\delta_{l,j}\delta_{\ell,k} = 0.
\end{aligned}$$

Note that $\widehat{H} \in \ker(\operatorname{Curl}, \Omega) \cap C^{\infty,3\times 3}(\Omega)$. Thus by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} \widehat{H}\,\mathrm{d}\lambda = 0. \tag{12.25}$$

By Corollary 12.24 (a), we find $v \in C^{\infty,3}(\Omega)$ such that $\operatorname{Grad} v = \widehat{H}$. Next, let $l \in \{1, \ldots, p\}$. Then, with $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l \subseteq \Omega_0$ for some $\Omega_0 \in \mathsf{cc}(\Omega)$, we obtain with $c := v(x_{l,1}) \in \mathbb{R}^3$ for all $x \in \zeta_l$

$$v(x) = v(x) - v(x_{l,1}) + c = \int_{\zeta_{x_{l,1},x}} \operatorname{Grad} v\,\mathrm{d}\lambda + c = \int_{\zeta_{x_{l,1},x}} \widehat{H}\,\mathrm{d}\lambda + c$$

and

$$\int_{\zeta_l} \langle c, \mathrm{d}\lambda\rangle = \sum_{\ell=1}^{3} c_\ell \int_{\zeta_l} \langle \operatorname{grad} x_\ell, \mathrm{d}\lambda\rangle = 0.$$

We consider the closed curve ζ_l as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. By Lemma 12.21, the definition of $\beta_{l,0}$, and (12.22) we have

$$\begin{aligned}
\int_{\zeta_l} \langle v, \mathrm{d}\lambda\rangle &= \int_{\zeta_l} \Big\langle \int_{\zeta_{x_{l,1},y}} \widehat{H}\,\mathrm{d}\lambda, \mathrm{d}\lambda_y\Big\rangle \\
&= \int_{\zeta_{x_{l,1},x_{l,1}}} \Big\langle \int_{\zeta_{x_{l,1},y}} \widehat{H}\,\mathrm{d}\lambda, \mathrm{d}\lambda_y\Big\rangle \\
&= \int_{\zeta_l} \big\langle x_{l,1} - y, \widehat{H}(y)\,\mathrm{d}\lambda_y\big\rangle = \beta_{l,0}(\widehat{H})
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{l,0}(H) - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,0}(\pi\Theta_{j,k}) \\
&= \beta_{l,0}(H) - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\big(\delta_{l,j}\delta_{0,k} + (1-\delta_{0,k})\delta_{l,j}(x_{l,1})_k\big) \\
&= \beta_{l,0}(H) - \gamma_{l,0} - \sum_{k=1}^{3} \gamma_{l,k}(x_{l,1})_k \\
&= \beta_{l,0}(H) - \gamma_{l,0} - \sum_{k=1}^{3} \beta_{l,k}(H)(x_{l,1})_k = 0.
\end{aligned}
$$

Note that $v \in \ker(\operatorname{curl}, \Omega) \cap C^{\infty,3}(\Omega)$ by Remark 12.23 (i) as $\operatorname{Grad} v = \widehat{H} \in L^{2,3\times3}_{\mathbb{S}}(\Omega)$. Therefore, by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} \langle v, \mathrm{d}\,\lambda \rangle = 0. \tag{12.26}$$

Hence, by Corollary 12.24 (b), we find $u \in C^{\infty}(\Omega)$ with $\operatorname{grad} u = v$ and thus

$$\operatorname{Gradgrad} u = \operatorname{Grad} v = \widehat{H} \in C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega).$$

Similar to the end of the proof of Lemma 12.6, elliptic regularity and, e.g., [22, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)] show that $v \in C^{\infty,3}(\Omega)$ together with $\operatorname{Grad} v \in L^{2,3\times3}(\Omega)$ imply $v \in H^{1,3}(\Omega)$. Then, analogously, $u \in C^{\infty}(\Omega)$ and $\operatorname{grad} u = v \in L^{2,3}(\Omega)$ imply $u \in H^1(\Omega)$ and hence $u \in H^2(\Omega)$, completing the proof. □

Lemma 12.29 *Let Assumptions 10.2 and 10.3 be satisfied. Then $\mathcal{B}^{bih,2}_{N}$ is linearly independent.*

Proof Let $\gamma_{j,k} \in \mathbb{R}$, $j \in \{1, \dots, p\}$, and $k \in \{0, \dots, 3\}$ be such that $\sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\pi\Theta_{j,k} = 0$.

Then (12.22) implies for $l \in \{1, \dots, p\}$

$$
\begin{aligned}
0 &= \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell}, && \ell \in \{1, 2, 3\}, \\
0 &= \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,0} + \sum_{k=1}^{3} \gamma_{l,k}(x_{l,1})_k = \gamma_{l,0}, && \ell = 0,
\end{aligned}
$$

finishing the proof. □

Theorem 12.30 *Let Assumptions 10.2 and 10.3 be satisfied. Then $\dim \mathcal{H}^{bih,2}_{N,\mathbb{S}}(\Omega) = 4p$ and a basis of $\mathcal{H}^{bih,2}_{N,\mathbb{S}}(\Omega)$ is given by (12.23).*

Proof Use Lemmas 12.28 and 12.29. □

12.4 Neumann tensor fields of the elasticity complex

The concluding example for our general construction principle is the elasticity complex. Again, we require some preparations regarding integration along curves and the operators involved in the elasticity complex. We need the formulas providing the interaction of matrix and vector analytic operations as outlined in Lemma 12.10 (in particular, note we defined spn there.)

In Lemma 12.31 below for a tensor field S and a parametrisation $\varphi \in C^{1,3}_{\mathrm{pw}}\big([0,1]\big)$ of a curve ζ in Ω we define

$$\begin{aligned}\int_\zeta &\operatorname{spn}\big((\operatorname{Curl} S)^\top(y)\,\mathrm{d}\lambda_y\big)(x-y)\\ &:= \int_0^1 \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\big(x-\varphi(t)\big)\,\mathrm{d}t.\end{aligned}$$

Lemma 12.31 *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subseteq \Omega$ be a piecewise C^1-curve connecting x_0 to x.*

*(**i**) Let $v \in C^{\infty,3}(\Omega)$. Then v and its rotation curl$v$ can be represented by*

$$\begin{aligned}v(x) &- v(x_0) - \frac{1}{2}\big(\operatorname{curl} v(x_0)\big)\times(x-x_0)\\ &= \int_{\zeta_{x_0,x}} \operatorname{symGrad} v\,\mathrm{d}\lambda + \int_{\zeta_{x_0,x}}\int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl}\ \operatorname{symGrad} v)^\top\,\mathrm{d}\lambda\big)\,\mathrm{d}\lambda_y\end{aligned}$$

and

$$\operatorname{curl} v(x) - \operatorname{curl} v(x_0) = 2\int_{\zeta_{x_0,x}} (\operatorname{Curl}\ \operatorname{symGrad} v)^\top\,\mathrm{d}\lambda.$$

*(**ii**) Let $S \in C^{\infty,3\times 3}(\Omega)$. Then*

$$\begin{aligned}\int_{\zeta_{x_0,x}}\int_{\zeta_{x_0,y}} &\operatorname{spn}\big((\operatorname{Curl} S)^\top\,\mathrm{d}\lambda\big)\,\mathrm{d}\lambda_y\\ &= \int_{\zeta_{x_0,x}} \operatorname{spn}\big((\operatorname{Curl} S)^\top(y)\,\mathrm{d}\lambda_y\big)(x-y).\end{aligned}$$

Proof For (i), let

$$S := \operatorname{symGrad} v = \operatorname{Grad} v - \operatorname{skw}\operatorname{Grad} v$$

and observe $2\operatorname{Curl} S = -2\operatorname{Curl}\operatorname{skw}\operatorname{Grad} v = (\operatorname{Grad}\operatorname{curl} v)^\top$ by Lemma 12.10. Thus

$$\begin{aligned}v_k(x) - v_k(x_0) &= \int_{\zeta_{x_0,x}} \langle \operatorname{grad} v_k, \mathrm{d}\lambda\rangle, \qquad k\in\{1,2,3\},\\ v(x) - v(x_0) &= \int_{\zeta_{x_0,x}} \operatorname{Grad} v\,\mathrm{d}\lambda,\\ \operatorname{curl} v(x) - \operatorname{curl} v(x_0) &= \int_{\zeta_{x_0,x}} \operatorname{Grad}\operatorname{curl} v\,\mathrm{d}\lambda = 2\int_{\zeta_{x_0,x}} (\operatorname{Curl} S)^\top\,\mathrm{d}\lambda.\end{aligned}$$

Therefore, by Lemma 12.10

$$\begin{aligned}v(x) &- v(x_0)\\ &= \int_{\zeta_{x_0,x}} \operatorname{Grad} v\,\mathrm{d}\lambda = \int_{\zeta_{x_0,x}} \operatorname{symGrad} v\,\mathrm{d}\lambda\\ &\quad + \int_{\zeta_{x_0,x}} \operatorname{skw}\operatorname{Grad} v\,\mathrm{d}\lambda\end{aligned}$$

$$
\begin{aligned}
&= \int_{\zeta_{x_0,x}} \operatorname{symGrad} v \,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_{x_0,x}} \operatorname{spn}\operatorname{curl} v(y)\,\mathrm{d}\,\lambda_y \\
&= \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_{x_0,x}} \operatorname{curl} v(x_0)\,\mathrm{d}\,\lambda_y \\
&\quad + \int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} (\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\Big)\,\mathrm{d}\,\lambda_y \\
&= \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_{x_0,x}} \operatorname{curl} v(x_0)\,\mathrm{d}\,\lambda_y \\
&\quad + \int_{\zeta_{x_0,x}} \int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\big)\,\mathrm{d}\,\lambda_y .
\end{aligned}
$$

Moreover, with $\varphi \in C^{1,3}_{\mathsf{pw}}\big([0,1]\big)$ parametrising $\zeta_{x_0,x}$ [7]

$$
\begin{aligned}
\int_{\zeta_{x_0,x}} \operatorname{spn}\operatorname{curl} v(x_0)\,\mathrm{d}\,\lambda_y &= \int_0^1 \big(\operatorname{spn}\operatorname{curl} v(x_0)\big)\varphi'(s)\,\mathrm{d}\,s \\
&= \big(\operatorname{spn}\operatorname{curl} v(x_0)\big)(x-x_0) = \big(\operatorname{curl} v(x_0)\big)\times(x-x_0).
\end{aligned}
$$

For (ii), we compute

$$
\begin{aligned}
&\int_{\zeta_{x_0,x}} \int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\big)\,\mathrm{d}\,\lambda_y \\
&= \int_0^1 \Big(\int_{\zeta_{x_0,\varphi(s)}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\big)\Big)\varphi'(s)\,\mathrm{d}\,s \\
&= \int_0^1 \Big(\int_0^s \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\,\mathrm{d}\,t\Big)\varphi'(s)\,\mathrm{d}\,s \\
&= \int_0^1 \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\int_t^1 \varphi'(s)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&= \int_0^1 \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\big(x-\varphi(t)\big)\,\mathrm{d}\,t \\
&= \int_{\zeta_{x_0,x}} \operatorname{spn}\big((\operatorname{Curl} S)^\top(y)\,\mathrm{d}\,\lambda_y\big)(x-y)
\end{aligned}
$$

with φ from above. □

Proposition 12.32 *Let $x_0 \in \Omega_0 \Subset cc(\Omega)$ and let $S, T \in C^{\infty,3\times3}(\Omega_0)$.*

*(**a**) The following conditions are equivalent:*

[7] Alternatively, we can compute with $\mathrm{Id} = \operatorname{Grad} y$

$$
\begin{aligned}
&\int_{\zeta_{x_0,x}} \underbrace{\operatorname{spn}\operatorname{curl} v(x_0)}_{=(\operatorname{spn}\operatorname{curl} v(x_0))\,\mathrm{Id}}\,\mathrm{d}\,\lambda_y \\
&= \operatorname{spn}\operatorname{curl} v(x_0)\int_{\zeta_{x_0,x}} \operatorname{Grad} y\,\mathrm{d}\,\lambda_y = \big(\operatorname{spn}\operatorname{curl} v(x_0)\big)(x-x_0).
\end{aligned}
$$

(i) *For all* $\zeta \subseteq \Omega_0$ *closed, piecewise* C^1*-curves*

$$\int_\zeta (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda = 0.$$

(ii) *For all* $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curves connecting* x_0 *with* x

$$\int_{\zeta_{x_0,x}} (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda = \int_{\widetilde{\zeta}_{x_0,x}} (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda.$$

(iii) *There exists* $w \in C^{\infty,3}(\Omega_0)$ *such that* $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$.

In either of the above cases,

$$x \mapsto w(x) = \int_{\zeta_{x_0,x}} (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda \tag{12.27}$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curve connecting* x_0 *with* x *is a (well-defined) possible choice for* w *in (iii).*

(b) *The following conditions are equivalent:*

(i) *For all* $\zeta \subseteq \Omega_0$ *closed, piecewise* C^1*-curves*

$$\int_\zeta T \, \mathrm{d}\,\lambda = 0.$$

(ii) *For all* $\zeta_{x_0,x}, \widetilde{\zeta}_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curves connecting* x_0 *with* x

$$\int_{\zeta_{x_0,x}} T \, \mathrm{d}\,\lambda = \int_{\widetilde{\zeta}_{x_0,x}} T \, \mathrm{d}\,\lambda.$$

(iii) *There exists* $v \in C^{\infty,3}(\Omega_0)$ *such that* $\operatorname{Grad} v = T$.

In either of the above cases,

$$x \mapsto v(x) = \int_{\zeta_{x_0,x}} T \, \mathrm{d}\,\lambda \tag{12.28}$$

for some $\zeta_{x_0,x} \subseteq \Omega_0$ *piecewise* C^1*-curve connecting* x_0 *with* x *is a (well-defined) possible choice for* v *in (iii).*

(c) *Let* $T := S + \operatorname{spn} w$ *with* $w \in C^{\infty,3}(\Omega_0)$ *and* $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$ *as in (a), (iii). Moreover, let* $v \in C^{\infty,3}(\Omega_0)$ *with* $\operatorname{Grad} v = T$ *as in (b), (iii). Then*

(i) $\operatorname{skw} S = 0$,
(ii) $\operatorname{symGrad} v = S$

are equivalent. In either case, we have

$$\operatorname{CurlCurl}_{\mathbb{S}}^\top S = 0, \qquad \operatorname{Grad} w = \frac{1}{2} \operatorname{Grad} \mathit{curl} v. \tag{12.29}$$

Proof The proofs of (a) and (b) are easy (fundamental theorem of calculus) and follow in a similar way to Propositions 12.22. For (c), we compute $\operatorname{symGrad} v = \operatorname{sym} T = \operatorname{sym} S$. Hence (i) and (ii) are equivalent. Finally, if (i) or (ii) is true, then by the complex property

$$\operatorname{CurlCurl}^\top S = \operatorname{CurlCurl}^\top \operatorname{symGrad} v = 0,$$

and we conclude $\operatorname{Grad} w = (\operatorname{Curl} \operatorname{sym} \operatorname{Grad} v)^\top = \frac{1}{2} \operatorname{Grad} \operatorname{curl} v$ by Lemma 12.10. □

The respective result for the whole of Ω reads as follows.

Corollary 12.33 *Let* $S, T \in C^{\infty,3\times3}(\Omega)$.

(a) The following conditions are equivalent:

(i) For all $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_\zeta (\operatorname{Curl} S)^\top \,\mathrm{d}\lambda = 0$.

(ii) There exists $w \in C^{\infty,3}(\Omega)$ *such that* $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$.

(b) The following conditions are equivalent:

(i) For all $\zeta \subseteq \Omega$ *closed, piecewise* C^1*-curves* $\int_\zeta T \,\mathrm{d}\lambda = 0$.

(ii) There exists $v \in C^{\infty,3}(\Omega)$ *such that* $\operatorname{Grad} v = T$.

(c) Let $T = S + \operatorname{spn} w$ *with* $w \in C^{\infty,3}(\Omega)$ *and* $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$ *as in (a), (ii). Moreover, let* $v \in C^{\infty,3}(\Omega)$ *with* $\operatorname{Grad} v = T$ *as in (b), (ii). Then* $\operatorname{skw} S = 0$ *in* Ω *if and only if* $\operatorname{symGrad} v = S$ *in* Ω.

Remark 12.34 Related to Proposition 12.32 and Corollary 12.33 we note with Lemma 12.10 the following:

(i) For $S \in C^{\infty,3\times3}_{\mathbb{S}}(\Omega)$ and $T := S + \operatorname{spn} w$ with $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$ it holds

$$\begin{aligned}\operatorname{Curl} T &= \operatorname{Curl} S + (\operatorname{div} w)\operatorname{Id} - ((\operatorname{Curl} S)^\top)^\top\\ &= \operatorname{tr}\operatorname{Grad} w = \operatorname{tr}\operatorname{Curl} S = \operatorname{tr}\operatorname{Curl}\operatorname{skw} S = 0.\end{aligned}$$

(ii) If Ω_0 is simply connected, Proposition 12.32 (a), (iii) and (b), (iii) are equivalent to $\operatorname{Curl}(\operatorname{Curl} S)^\top = 0$ and $\operatorname{Curl} T = 0$, respectively.

Next, we provide the construction of the basis tensor fields for the Neumann fields for the elasticity complex. Let $j \in \{1, \dots, p\}$. From the beginning of Sect. 12 recall that θ_j is constant on each connected component $\Upsilon_{j,0}$ and $\Upsilon_{j,1}$ of $\Upsilon_j \setminus F_j$ and vanishes outside of $\widetilde{\Upsilon}_{j,1}$. Let $\widehat{r}_k$ be the rigid motions (Nedelec fields) from Sect. 11.4 given by $\widehat{r}_m(x) := e^m \times x = \operatorname{spn}(e^m)\,x$ and $\widehat{r}_{m+3}(x) := e^m$ for $m \in \{1, 2, 3\}$. We define the vector fields $\theta_{j,k} := \theta_j \widehat{r}_k$ and note $\operatorname{symGrad}\theta_{j,k} = 0$ in $\dot{\bigcup}_{l\in\{1,\dots,p\}}\Upsilon_l \setminus F_l$ for all $k \in \{1, \dots, 6\}$. Thus $\operatorname{symGrad}\theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the neighbourhoods Υ_l of all surfaces F_l, $l \in \{1, \dots, p\}$, $k \in \{1, \dots, 6\}$.

Lemma 12.35 *Let Assumption 10.3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{CurlCurl}^\top_{\mathbb{S}}, \Omega)$.

Proof Let $\Phi \in C^{\infty,3\times3}_{c,\mathbb{S}}(\Omega)$. As $\operatorname{supp}\Theta_{j,k} \subseteq \widetilde{\Upsilon}_j \setminus \Upsilon_j$ we can pick another cut-off function $\varphi \in C^\infty_c(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi} = 1$. Then

$$\begin{aligned}&\langle\Theta_{j,k}, \operatorname{CurlCurl}^\top_{\mathbb{S}}\Phi\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} = \langle\Theta_{j,k}, \operatorname{CurlCurl}^\top_{\mathbb{S}}\Phi\rangle_{L^{2,3\times3}_{\mathbb{S}}(\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi)}\\ &= \big\langle\operatorname{symGrad}\theta_{j,k}, \operatorname{CurlCurl}^\top_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega_F)} = \big\langle\operatorname{Grad}\theta_{j,k}, \operatorname{CurlCurl}^\top_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega_F)}\\ &= \Big\langle\operatorname{Grad}\theta_{j,k}, \operatorname{Curl}\big(\operatorname{Curl}(\varphi\Phi)\big)^\top\Big\rangle_{L^{2,3\times3}(\Omega_F)} = 0\end{aligned}$$

as $\varphi\Phi, \operatorname{CurlCurl}^\top_{\mathbb{S}}(\varphi\Phi) \in C^{\infty,3\times3}_{c,\mathbb{S}}(\Omega_F)$ by Lemma 12.10. □

Next, we construct functionals similar to the previous sections. Here, however, due to the complex structure, we need six times as many (instead of four) as for the de Rham complex. For starters, note that for $l, j \in \{1, \dots, p\}$ and $k \in \{1, \dots, 6\}$ and for the curves

$\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l$ with the chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$ we can compute[8] by Lemma 12.31

$$\begin{aligned}\mathbb{R}^3 \ni a_l(\Theta_{j,k}) &:= \int_{\zeta_l} (\operatorname{Curl} \Theta_{j,k})^\top \,\mathrm{d}\,\lambda = \int_{\zeta_{x_{l,0},x_{l,1}}} (\operatorname{Curl} \operatorname{symGrad} \theta_{j,k})^\top \,\mathrm{d}\,\lambda \\ &= \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,1}) - \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,0}) = \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,1}) \\ &= \frac{1}{2}\delta_{l,j}\operatorname{curl}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} e^k, & \text{if } k \in \{1,2,3\}, \\ 0, & \text{if } k \in \{4,5,6\}, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\mathbb{R}^3 \ni b_l(\Theta_{j,k}) &:= \int_{\zeta_l} \Theta_{j,k} \,\mathrm{d}\,\lambda + \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl} \Theta_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1} - y) \\ &= \int_{\zeta_{x_{l,0},x_{l,1}}} \operatorname{symGrad} \theta_{j,k} \,\mathrm{d}\,\lambda \\ &\quad + \int_{\zeta_{x_{l,0},x_{l,1}}} \operatorname{spn}\big((\operatorname{Curl} \operatorname{symGrad} \theta_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1} - y) \\ &= \int_{\zeta_{x_{l,0},x_{l,1}}} \Big(\operatorname{symGrad} \theta_{j,k}(y) \\ &\quad + \int_{\zeta_{x_{l,0},y}} \operatorname{spn}\big((\operatorname{Curl} \operatorname{symGrad} \theta_{j,k})^\top \,\mathrm{d}\,\lambda\big)\Big)\,\mathrm{d}\,\lambda_y \\ &= \theta_{j,k}(x_{l,1}) - \theta_{j,k}(x_{l,0}) - \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,0}) \times (x_{l,1} - x_{l,0}) = \theta_{j,k}(x_{l,1}) \\ &= \delta_{l,j}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} e^k \times x_{l,1}, & \text{if } k \in \{1,2,3\}, \\ e^{k-3}, & \text{if } k \in \{4,5,6\}. \end{cases}\end{aligned}$$

Thus, for $l \in \{1, \ldots, p\}$ and $\ell \in \{1, \ldots, 6\}$ we have functionals $\beta_{l,\ell}$ given by

$$\beta_{l,\ell}(\Theta_{j,k}) := \begin{cases} \big\langle a_l(\Theta_{j,k}), e^\ell \big\rangle, & \text{if } \ell \in \{1,2,3\}, \\ \big\langle b_l(\Theta_{j,k}), e^{\ell-3} \big\rangle, & \text{if } \ell \in \{4,5,6\}, \end{cases} \qquad j \in \{1, \ldots, p\}, \quad k \in \{1, \ldots, 6\}.$$

Then for $l, j \in \{1, \ldots, p\}$ and for $\ell \in \{1, 2, 3\}$

$$\beta_{l,\ell}(\Theta_{j,k}) = \big\langle a_l(\Theta_{j,k}), e^\ell \big\rangle = \delta_{l,j}\begin{cases} \langle e^k, e^\ell \rangle = \delta_{\ell,k}, & \text{if } k \in \{1,2,3\}, \\ \langle 0, e^\ell \rangle = 0, & \text{if } k \in \{4,5,6\}, \end{cases}$$

i.e.,

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k}, \qquad k \in \{1, \ldots, 6\},$$

[8] Note that $\operatorname{curl}\widehat{r}_k = 2e^k$ for $k \in \{1, 2, 3\}$, since, e.g.,

$$\begin{aligned}\operatorname{curl}\widehat{r}_1(x) &= \operatorname{curl}(e^1 \times x) = \operatorname{curl}(x_2\, e^1 \times e^2 + x_3\, e^1 \times e^3) \\ &= \operatorname{curl}(x_2 e^3 - x_3 e^2) \\ &= \operatorname{grad}(x_2) \times e^3 - \operatorname{grad}(x_3) \times e^2 = e^2 \times e^3 - e^3 \times e^2 = 2e^1.\end{aligned}$$

and for $\ell \in \{4, 5, 6\}$

$$\begin{aligned}\beta_{l,\ell}(\Theta_{j,k}) &= \big\langle b_l(\Theta_{j,k}), e^{\ell-3}\big\rangle \\ &= \delta_{l,j}\begin{cases}\langle e^k \times x_{l,1}, e^{\ell-3}\rangle = \langle e^{\ell-3} \times e^k, x_{l,1}\rangle, & \text{if } k \in \{1,2,3\},\\ \langle e^{k-3}, e^{\ell-3}\rangle = \delta_{\ell,k}, & \text{if } k \in \{4,5,6\},\end{cases}\end{aligned}$$

i.e.,

$$\begin{aligned}\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{1,k} \\ + \delta_{2,k} + \delta_{3,k})(x_{l,1})_{\widehat{\ell-3,k}}, \qquad k \in \{1,\dots,6\},\end{aligned}$$

where we define

$$\begin{aligned}(x_{l,1})_{\widehat{\ell-3,k}} &:= \langle e^{\ell-3} \times e^k, x_{l,1}\rangle = \sum_{i=1}^{3}\langle e^{\ell-3} \times e^k, e^i\rangle(x_{l,1})_i \\ &= \begin{cases}(x_{l,1})_i, & \exists i \in \{1,2,3\} : (\ell-3, k, i) \text{ even permutation of } (1,2,3),\\ -(x_{l,1})_i, & \exists i \in \{1,2,3\} : (\ell-3, k, i) \text{ odd permutation of } (1,2,3),\\ 0, & \forall i \in \{1,2,3\} : (\ell-3, k, i) \text{ no permutation of } (1,2,3).\end{cases}\end{aligned}$$

In particular, $(x_{l,1})_{\widehat{\ell-3,k}} = 0$ if $\ell - 3 = k$ or $\ell \in \{1, 2, 3\}$ or $k \in \{4, 5, 6\}$. Therefore, we have for $l, j \in \{1, \dots, p\}$ and $k, \ell \in \{1, \dots, 6\}$

$$\begin{aligned}\beta_{l,\ell}(\Theta_{j,k}) &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}} \\ &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{\ell,4} + \delta_{\ell,5} + \delta_{\ell,6})(\delta_{1,k} \\ &\quad + \delta_{2,k} + \delta_{3,k})(1 - \delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}}.\end{aligned} \tag{12.30}$$

Let Assumption 10.2 be satisfied. For the elasticity complex, similar to (3.3), (3.5), see also (12.2), (12.11), (12.21), we have the orthogonal decompositions

$$\begin{aligned}L^{2,3\times3}_{\mathbb{S}}(\Omega) &= \operatorname{ran}(\operatorname{symGrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega),\\ \ker(\operatorname{CurlCurl}^{\top}_{\mathbb{S}}, \Omega) &= \operatorname{ran}(\operatorname{symGrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega).\end{aligned} \tag{12.31}$$

Remark 12.36 [39, Lemma 3.2] yields $\operatorname{dom}(\operatorname{symGrad}, \Omega) = H^{1,3}(\Omega)$. Thus, as before, a combination of Rellich's selection theorem and a contradiction argument implies the Poincaré type estimate

$$\exists\, c > 0 \quad \forall\, \phi \in H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \qquad |\phi|_{L^{2,3}(\Omega)} \le c|\operatorname{symGrad}\phi|_{L^{2,3\times3}(\Omega)}.$$

Thus, the range in (12.31) is closed.

Let $\pi : L^{2,3\times3}_{\mathbb{S}}(\Omega) \to \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega)$ denote the orthogonal projector along $\operatorname{ran}(\operatorname{symGrad}, \Omega)$ according to (12.31). We infer $\pi\big(\ker(\operatorname{CurlCurl}^{\top}_{\mathbb{S}}, \Omega)\big) = \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)$. Thus, using Lemma 12.35, for all $j \in \{1, \dots, p\}$ and $k \in \{1, \dots, 6\}$ we find $\psi_{j,k} \in H^{1,3}(\Omega)$ such that

$$\begin{aligned}\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \ni \pi\Theta_{j,k} &= \Theta_{j,k} - \operatorname{symGrad}\psi_{j,k},\\ (\Theta_{j,k} - \operatorname{symGrad}\psi_{j,k})\big|_{\Omega_F} &= \operatorname{symGrad}(\theta_{j,k} - \psi_{j,k}).\end{aligned}$$

Next, Lemma 12.2 implies $\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \subseteq C^{\infty,3\times 3}(\Omega)$. Thus, $\pi\Theta_{j,k}, \Theta_{j,k} \in C^{\infty,3\times 3}(\Omega)$ implies $\operatorname{symGrad}\psi_{j,k} \in C^{\infty,3\times 3}(\Omega)$ and hence $\psi_{j,k} \in C^{\infty,3}(\Omega)$. Therefore, all path integrals over the closed curves ζ_l are well-defined. Furthermore, we observe by Lemma 12.31

$$\begin{aligned} a_l(\operatorname{symGrad}\psi_{j,k}) &= \int_{\zeta_l} (\operatorname{Curl}\operatorname{symGrad}\psi_{j,k})^\top \,\mathrm{d}\,\lambda \\ &= \frac{1}{2}\big(\operatorname{curl}\psi_{j,k}(x_{l,1}) - \operatorname{curl}\psi_{j,k}(x_{l,1})\big) = 0, \end{aligned}$$

and

$$\begin{aligned} b_l(\operatorname{symGrad}\psi_{j,k}) &= \int_{\zeta_l} \operatorname{symGrad}\psi_{j,k}\,\mathrm{d}\,\lambda \\ &\quad + \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad}\psi_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1}-y) \\ &= \int_{\zeta_{x_{l,1},x_{l,1}}} \Big(\operatorname{symGrad}\psi_{j,k}(y) \\ &\quad + \int_{\zeta_{x_{l,1},y}} \operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad}\psi_{j,k})^\top\,\mathrm{d}\,\lambda\big)\Big)\,\mathrm{d}\,\lambda_y \\ &= \psi_{j,k}(x_{l,1}) - \psi_{j,k}(x_{l,1}) - \frac{1}{2}\operatorname{curl}\psi_{j,k}(x_{l,1}) \times (x_{l,1}-x_{l,1}) = 0. \end{aligned}$$

Hence, $\beta_{l,\ell}(\operatorname{symGrad}\psi_{j,k}) = 0$ and by (12.30)

$$\begin{aligned} \beta_{l,\ell}(\pi\Theta_{j,k}) &= \beta_{l,\ell}(\Theta_{j,k}) - \beta_{l,\ell}(\operatorname{symGrad}\psi_{j,k}) = \beta_{l,\ell}(\Theta_{j,k}) \\ &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}} \\ &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{\ell,4}+\delta_{\ell,5} \\ &\quad + \delta_{\ell,6})(\delta_{1,k}+\delta_{2,k}+\delta_{3,k})(1-\delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}} \end{aligned} \tag{12.32}$$

for all $l, j \in \{1,\dots,p\}$ and all $\ell, k \in \{1,\dots,6\}$. We shall show that

$$\mathcal{B}^{\mathsf{ela}}_N := \big\{\pi\Theta_{j,k} : j \in \{1,\dots,p\},\ k \in \{1,\dots,6\}\big\} \subseteq \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \tag{12.33}$$

defines a basis of $\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)$.

The tensor fields $\Theta_{j,k}$ being constructed explicitly, we provide a way of finding $\psi_{j,k}$ by means of solutions of PDEs.

Remark 12.37 (Characterisation by PDEs) Let $j \in \{1,\dots,p\}$ and $k \in \{1,\dots,6\}$. Then $\psi_{j,k} \in H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$ can be found by solving the following elasticity PDE formulated in the standard variational formulation

$$\forall\,\phi \in H^{1,3}(\Omega) \qquad \langle \operatorname{symGrad}\psi_{j,k}, \operatorname{symGrad}\phi\rangle_{L^{2,3\times 3}(\Omega)} = \langle \Theta_{j,k}, \operatorname{symGrad}\phi\rangle_{L^{2,3\times 3}(\Omega)},$$

i.e., $\psi_{j,k} = \Delta^{-1}_{\mathbb{S},N}\big(\operatorname{Div}_{\mathbb{S}}\Theta_{j,k}, \Theta_{j,k}\nu|_\Gamma\big)$, where $\Delta_{\mathbb{S},N} \subseteq \operatorname{Div}_{\mathbb{S}}\operatorname{symGrad}$ is the 'symmetric' Laplacian with inhomogeneous Neumann boundary conditions restricted to a subset of $H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$. Therefore,

$$\pi\Theta_{j,k} = \Theta_{j,k} - \operatorname{symGrad}\psi_{j,k} = \Theta_{j,k} - \operatorname{symGrad}\Delta^{-1}_{\mathbb{S},N}\big(\operatorname{Div}_{\mathbb{S}}\Theta_{j,k}, \Theta_{j,k}\nu|_\Gamma\big).$$

In classical terms, $\psi_{j,k}$ solves the Neumann elasticity problem

$$\begin{aligned}
-\Delta_{\mathbb{S}}\psi_{j,k} &= -\operatorname{Div}_{\mathbb{S}}\Theta_{j,k} && \text{in } \Omega,\\
(\operatorname{Grad}\psi_{j,k})\nu &= \Theta_{j,k}\nu && \text{on } \Gamma,\\
\int_{\Omega_l}(\psi_{j,k})_\ell &= 0 && \text{for } l\in\{1,\dots,n\},\quad \ell\in\{1,2,3\},\\
\int_{\Omega_l}\big(x\times\psi_{j,k}(x)\big)_\ell\,\mathrm{d}\lambda_x &= 0 && \text{for } l\in\{1,\dots,n\},\quad \ell\in\{1,2,3\},
\end{aligned}\tag{12.34}$$

which is uniquely solvable.

Lemma 12.38 *Let Assumptions 10.2 and 10.3 be satisfied. Then* $\mathcal{H}^{ela}_{N,\mathbb{S}}(\Omega)=\operatorname{lin}\mathcal{B}^{ela}_N$.

Proof Let $H\in\mathcal{H}^{\mathrm{ela}}_{N,\mathbb{S}}(\Omega)=\ker(\mathring{\operatorname{Div}}_{\mathbb{S}},\Omega)\cap\ker(\operatorname{CurlCurl}^\top_{\mathbb{S}},\Omega)\subseteq C^{\infty,3\times3}_{\mathbb{S}}(\Omega)$, see Lemma 12.2. With the above introduced functionals a_l and b_l, $l\in\{1,\dots,p\}$, we recall

$$\mathbb{R}^3\ni a_l(H)=\int_{\zeta_l}(\operatorname{Curl}H)^\top\,\mathrm{d}\lambda,$$
$$\mathbb{R}^3\ni b_l(H)=\int_{\zeta_l}H\,\mathrm{d}\lambda+\int_{\zeta_l}\operatorname{spn}\big((\operatorname{Curl}H)^\top(y)\,\mathrm{d}\lambda_y\big)(x_{l,1}-y),$$

and define for $l\in\{1,\dots,p\}$ the numbers

$$\begin{aligned}
\gamma_{l,\ell}:=\gamma_{l,\ell}(H)&:=\big\langle a_l(H),e^\ell\big\rangle=\beta_{l,\ell}(H), && \text{for } \ell\in\{1,2,3\},\\
\gamma_{l,\ell}:=\gamma_{l,\ell}(H)&:=\Big\langle b_l(H)-\sum_{k=1}^3\beta_{l,k}(H)e^k\times x_{l,1},e^{\ell-3}\Big\rangle\\
&=\beta_{l,\ell}(H)-\sum_{k=1}^3\beta_{l,k}(H)(x_{l,1})_{\widehat{\ell-3,k}}, && \text{for } \ell\in\{4,5,6\},
\end{aligned}$$

where we recall $(x_{l,1})_{\widehat{\ell-3,k}}=(\delta_{\ell,4}+\delta_{\ell,5}+\delta_{\ell,6})(\delta_{1,k}+\delta_{2,k}+\delta_{3,k})(1-\delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}}$ by definition, see (12.30). We shall show that

$$\mathcal{H}^{\mathrm{ela}}_{N,\mathbb{S}}(\Omega)\ni\widehat{H}:=H-\sum_{j=1}^p\sum_{k=1}^6\gamma_{j,k}\pi\Theta_{j,k}=0\quad\text{in }\Omega.$$

Similar to the proof of Lemma 12.6 (or 12.18, 12.28) the aim is to prove the existence of $v\in H^{1,3}(\Omega)$ such that $\operatorname{symGrad}v=\widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}=\langle\operatorname{symGrad}v,\widehat{H}\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}=0.$$

For the construction of v, we apply Corollary 12.33 and Remark 12.34 to $S=\widehat{H}$. In order to show condition Corollary 12.33 (a), (i), we observe for $l\in\{1,\dots,p\}$ and $\ell\in\{1,2,3\}$ by (12.32)

$$\begin{aligned}
\Big(\int_{\zeta_l}(\operatorname{Curl}\widehat{H})^\top\,\mathrm{d}\lambda\Big)_\ell=\big(a_l(\widehat{H})\big)_\ell=\beta_{l,\ell}(\widehat{H})&=\beta_{l,\ell}(H)-\sum_{j=1}^p\sum_{k=1}^6\gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k})\\
&=\gamma_{l,\ell}-\sum_{j=1}^p\sum_{k=1}^6\gamma_{j,k}\delta_{l,j}\delta_{\ell,k}=0.
\end{aligned}$$

Note that $\operatorname{Curl}(\operatorname{Curl}\widehat{H})^\top = \operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top \widehat{H} = 0$, i.e., $(\operatorname{Curl}\widehat{H})^\top \in \ker(\operatorname{Curl}) \cap C^{\infty,3\times 3}(\Omega)$. Thus, by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_\zeta (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\lambda = 0. \tag{12.35}$$

Hence, by Corollary 12.33 (a), (ii), there exists $w \in C^{\infty,3}(\Omega)$ such that

$$\operatorname{Grad} w = (\operatorname{Curl}\widehat{H})^\top.$$

We define $T := \widehat{H} + \operatorname{spn} w$. Let $l \in \{1, \dots, p\}$. Then $\zeta_{x_{l,0},x_{l,1}} \subseteq \zeta_l \subseteq \Omega_0$ for some $\Omega_0 \in \mathsf{cc}(\Omega)$. With $c := w(x_{l,1}) \in \mathbb{R}^3$ we compute for all $x \in \zeta_l$

$$\begin{aligned} w(x) = w(x) - w(x_{l,1}) + c &= \int_{\zeta_{x_{l,1},x}} \operatorname{Grad} w \,\mathrm{d}\lambda + c \\ &= \int_{\zeta_{x_{l,1},x}} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\lambda + c, \end{aligned}$$

and

$$\int_{\zeta_l} (\operatorname{spn} c)\,\mathrm{d}\lambda = (\operatorname{spn} c)\int_{\zeta_l} \operatorname{Id}\mathrm{d}\lambda = (\operatorname{spn} c)\int_{\zeta_l} \operatorname{Grad} x \,\mathrm{d}\lambda_x = 0.$$

Again, we consider the curve ζ_l as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. By Lemma 12.31 and by the definition of b_l we have for $l \in \{1, \dots, p\}$

$$\begin{aligned} \int_{\zeta_l} T \,\mathrm{d}\lambda &= \int_{\zeta_l} \widehat{H}\,\mathrm{d}\lambda + \int_{\zeta_l} (\operatorname{spn} w)\,\mathrm{d}\lambda \\ &= \int_{\zeta_l} \widehat{H}\,\mathrm{d}\lambda + \int_{\zeta_{x_{l,1},x_{l,1}}} \operatorname{spn}\Big(\int_{\zeta_{x_{l,1},y}} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\lambda\Big)\,\mathrm{d}\lambda_y \\ &= \int_{\zeta_l} \widehat{H}\,\mathrm{d}\lambda + \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl}\widehat{H})^\top(y)\,\mathrm{d}\lambda_y\big)(x_{l,1} - y) = b_l(\widehat{H}). \end{aligned}$$

Hence, for $\ell \in \{4, 5, 6\}$ we get by (12.32)

$$\begin{aligned} \Big(\int_{\zeta_l} T\,\mathrm{d}\lambda\Big)_{\ell-3} &= \Big\langle \int_{\zeta_l} T\,\mathrm{d}\lambda, e^{\ell-3}\Big\rangle \\ &= \langle b_l(\widehat{H}), e^{\ell-3}\rangle = \beta_{l,\ell}(\widehat{H}) \\ &= \beta_{l,\ell}(H) - \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) \\ &= \beta_{l,\ell}(H) - \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}(\delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}}) \\ &= \beta_{l,\ell}(H) - \gamma_{l,\ell} - \sum_{k=1}^{3} \gamma_{l,k}(x_{l,1})_{\widehat{\ell-3,k}} \\ &= \beta_{l,\ell}(H) - \gamma_{l,\ell} - \sum_{k=1}^{3} \beta_{l,k}(H)(x_{l,1})_{\widehat{\ell-3,k}} = 0. \end{aligned}$$

Therefore, $\int_{\zeta_l} T \,\mathrm{d}\lambda = 0$ for all $l \in \{1, \ldots, p\}$. Note that $T \in \ker(\operatorname{Curl}) \cap C^{\infty,3\times 3}(\Omega)$ by Remark 12.34 as $S = \widehat{H} \in C^{\infty,3\times 3}_{\mathbb{S}}(\Omega)$ and $T = S + \operatorname{spn} w$ with $\operatorname{Grad} w = (\operatorname{Curl} \widehat{H})^\top$. Thus, by Assumption 10.3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} T \,\mathrm{d}\lambda = 0. \tag{12.36}$$

Hence, by the symmetry of $\widehat{H}$ and Corollary 12.33 (b), (c), there exists $v \in C^{\infty,3}(\Omega)$ such that $\operatorname{Grad} v = T$ as well as $\operatorname{symGrad} v = \widehat{H}$. Similar to the end of the proof of Lemma 12.6, elliptic regularity and, e.g., [22, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)] show that $v \in C^{\infty,3}(\Omega)$ with $\operatorname{symGrad} v \in L^{2,3\times 3}_{\mathbb{S}}(\Omega)$ implies $v \in H^{1,3}(\Omega)$, completing the proof. (Let us note that $v \in H^{1,3}(\Omega)$ implies also $T \in L^{2,3\times 3}(\Omega)$ and hence $w \in L^{2,3}(\Omega)$.) □

Lemma 12.39 *Let Assumptions 10.2 and 10.3 be satisfied. Then $\mathcal{B}^{ela}_N$ is linearly independent.*

Proof Let $\gamma_{j,k} \in \mathbb{R}$, $j \in \{1, \ldots, p\}$, $k \in \{1, \ldots, 6\}$, be such that $\sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\pi\Theta_{j,k} = 0$. Then (12.32) implies for $l \in \{1, \ldots, p\}$

$$0 = \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell}, \qquad \ell \in \{1, 2, 3\},$$

$$0 = \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell} + \sum_{k=1}^{3} \gamma_{l,k}(x_{l,1})_{\widehat{\ell-3,k}} = \gamma_{l,\ell}, \qquad \ell \in \{4, 5, 6\},$$

finishing the proof. □

Theorem 12.40 *Let Assumptions 10.2 and 10.3 be satisfied. Then* $\dim \mathcal{H}^{ela}_{N,\mathbb{S}}(\Omega) = 6p$ *and a basis of $\mathcal{H}^{ela}_{N,\mathbb{S}}(\Omega)$ is given by (12.33).*

Proof Use Lemmas 12.38 and 12.39. □

13 Conclusion

The index theorems presented rest on the abstract construction principle provided in [11] and the results on the newly found biharmonic complex from [37,38] and the elasticity complex from [39–41]. With this insight it is possible to construct basis fields for the generalised harmonic Dirichlet and Neumann tensor fields. The number of basis fields of the considered cohomology groups provides additional topological invariants. The construction of the generalised Dirichlet fields being somewhat similar to the de Rham complex, the same for the generalised Neumann fields requires some more machinery particularly relying on the introduction of Poincaré maps defining the functionals.

Furthermore, we have outlined numerical strategies to compute the generalised Neumann and Dirichlet fields in practice. In passing we have also provided a set of Friedrichs–Poincaré type estimates and included variable coefficients relevant for numerical studies.

All these constructions heavily rely on the choice of boundary conditions and we emphasise that the considered mixed order operators *cannot* be viewed as leading order plus relatively compact perturbation, when it comes to computation of the Fredholm index. In particular, techniques from pseudo-differential calculus successfully applied to obtain index formulas for operators defined on non-compact manifolds or compact manifolds without boundary, see e.g. [19,20], are likely to be very difficult to be applicable in the present situation. It would be interesting to see, whether the operators considered above defined on an unbounded domain enjoy similar index formulas (maybe a comparable Witten index of some sort) even though the operator itself might not be of Fredholm type anymore.

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