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Seshadri constants on principally polarized abelian surfaces with real multiplication

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Abstract
Seshadri constants on abelian surfaces are fully understood in the case of Picard number one. Little is known so far for simple abelian surfaces of higher Picard number. In this paper we investigate principally polarized abelian surfaces with real multiplication. They are of Picard number two and might be considered the next natural case to be studied. The challenge is to not only determine the Seshadri constants of individual line bundles, but to understand the whole *Seshadri function* on these surfaces. Our results show on the one hand that this function is surprisingly complex: on surfaces with real multiplication in $\mathbb{Z}[\sqrt{e}]$ it consists of linear segments that are never adjacent to each other—it behaves like the Cantor function. On the other hand, we prove that the Seshadri function is invariant under an infinite group of automorphisms, which shows that it does have interesting regular behavior globally.

Keywords Abelian surface · Seshadri constant · Real multiplication · Cantor function

Mathematics Subject Classification 14C20 · 14K12 · 26A30

Introduction

The purpose of this paper is to contribute to the study of Seshadri constants on abelian surfaces. Recall that for an ample line bundle L on a smooth projective variety X, the *Seshadri constant* of L at a point $x \in X$ is by definition the real number

$$\varepsilon(L, x) = \inf \left\{ \frac{L \cdot C}{\operatorname{mult}_x(C)} \;\middle|\; C \text{ irreducible curve through } x \right\}.$$

On abelian varieties, where this invariant is independent of the chosen point x, we write simply $\varepsilon(L)$. Seshadri constants are highly interesting invariants for numerous reasons: They are related to minimal period lengths [1,14], to syzygies [13,16], and they govern quite

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generally the geometry of linear series in many respects [11,12] (we refer to [15, Chapt. 5] and [6] for more background on Seshadri constants).

On abelian surfaces, Seshadri constants are fully understood in the case of Picard number $\rho = 1$ [5]. For $\rho > 1$, only self-products of elliptic curves have been studied, while the important case of simple abelian surfaces is completely unexplored so far. In contrast to the case of $\rho = 1$, the challenge on these surface is not only to determine the Seshadri constant of one ample line bundle, but to understand the behavior of the *Seshadri function*,

$$\varepsilon : \mathrm{Amp}(X) \to \mathbb{R}, \quad L \mapsto \varepsilon(L),$$

which associates to each ample line bundle its Seshadri constant. To our knowledge, there are—also beyond abelian surfaces—hardly any cases where this function is known explicitly, the exception being certain self-products $E \times E$ of elliptic curves [4]. In general, the Seshadri function of an abelian variety is known to be concave and continuous [4, Prop. 3.1], but at present it is unclear what kind of behavior to expect beyond these basic properties.

We attack this problem on abelian surfaces of Picard number $\rho = 2$, which seems to be the natural next case to investigate. As the Seshadri function is homogeneous, it is completely determined by its values on a cross-section of $\mathrm{Amp}(X)$. So, when $\mathrm{Amp}(X)$ is two-dimensional, we may consider it as a function $\varepsilon : I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$. We always take this point of view when we speak of the Seshadri function.

For clarity of exposition let us introduce a piece of terminology:

Definition Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *broken linear*, if it is continuous and there is a non empty and nowhere dense subset $M \subset I$ such that the following holds:

(i) Around every point of $I \backslash M$ there is an open interval, contained in $I \backslash M$, on which f is linear.
(ii) If I_1 and I_2 are maximal open subintervals of I on which f is linear, then I_1 and I_2 are contained in $I \backslash M$, and I_1 and I_2 are not adjacent to each other (i.e., an endpoint of I_1 is never an endpoint of I_2).

Note that these conditions imply that M is a perfect set (i.e., that every point of M is an accumulation point of M) and, thus, M is uncountable. More concretely, condition (ii) implies that whenever a linear piece of f ends (i.e., one of the maximal subintervals mentioned in the definition), then no other linear piece begins at that point, but instead there is a sequence of linear pieces converging to that point. And the same applies to the converging pieces: each of them is again approached by a sequence of pieces. The Cantor function (see e.g. [10]) is an example of a broken linear function (in which case the Cantor set is the perfect set M).

Our first result shows that on abelian surfaces with real multiplication, the Seshadri function is of the same baffling complexity as the Cantor function:

Theorem A *Let X be a principally abelian surface, whose endomorphism ring is isomorphic to $\mathbb{Z}[\sqrt{e}]$ for some non-square integer $e > 0$. Then the Seshadri function of X is broken linear.*

This result is in stark contrast to what had been observed so far: When E is a general elliptic curve, then the restriction of the Seshadri function on $E \times E$ to any rational line is a piecewise linear function, in the usual sense that each piece is adjacent to another piece [4]. The situation in Theorem A is at the other extreme: At no point are two pieces connected to each other.

The endomorphism ring of an abelian surface with real multiplication is an order of a quadratic number field $\mathbb{Q}(\sqrt{d})$. As the integer e appearing in Theorem A is not required to be square-free, the only orders not covered there are those of the form $\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{e}]$, where $e \equiv 1 \pmod 4$. Surfaces with these endomorphism rings add another level of complexity: We show that on certain surfaces of this type, every line bundle has only one submaximal curve (which then computes its Seshadri constant), while there also exist surfaces of this type carrying line bundles with two submaximal curves (see Propositions 5.2 and 5.4). Interestingly, the conclusion of Theorem A extends to the former surfaces (see Theorem 3.7), whereas on the latter surfaces there exist boundary points of linear segments which are accumulation points, as well as boundary points where linear segments meet (see Remark 3.8).

The discussion so far has shown how complex and subtle the Seshadri function on surfaces with real multiplication is. Our next result states that globally it has more structure than one might expect at this point:

Theorem B *There exists a decomposition of the ample cone into infinitely many subcones $\mathcal{C}_k$, $k \in \mathbb{Z}$, such that the group G of isometries of $\mathrm{NS}(X)$ that leave the Seshadri function on $\mathrm{Amp}(X)$ invariant acts transitively on the set of subcones. In particular, the values of the Seshadri function on any subcone of the subcones $\mathcal{C}_k$ completely determine the Seshadri function on the entire ample cone.*

There are only few known cases where one has effective computational access to the Seshadri constants of all line bundles on the surface (the self-product $E \times E$ of a general elliptic curve being an exception again). Our methods provide such computational access for the surfaces studied here.

Theorem C *There is an algorithm that computes the Seshadri constant of every given ample line bundle on principally polarized abelian surfaces with real multiplication.*

The algorithm enables us to efficiently compute Seshadri functions and thus to provide a graphical representation for any given endomorphism ring (see examples in Sect. 4). Also, the analysis of the method underlying the proof of Theorem C allows us to answer the question as to which data in fact determine the Seshadri function. A priori, the function could depend on the individual surface (or rather, on its isomorphism class). However, the numerical data entering the computation ultimately stems from the endomorphism ring, and this implies:

Corollary D *Let X and Y be principally polarized abelian surfaces with real multiplication, such that $\mathrm{End}(X) \simeq \mathrm{End}(Y)$. Then, in suitable linear coordinates on $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$, their Seshadri functions coincide.*

Note that the assumption that $\mathrm{End}(X)$ and $\mathrm{End}(Y)$ be isomorphic is strictly weaker than requiring that X and Y be isomorphic. In fact, the corollary shows that only countably many Seshadri functions occur, while the surfaces vary in two-dimensional families.

Concerning the organization of this paper, we start in Sect. 1 by establishing crucial properties of Pell divisors and submaximal curves. We study in Sect. 2 the intervals on which curves can be submaximal. Sect. 3 is devoted to the proofs of Theorems A and C, as well as Corollary D. In Sect. 4 we study the decomposition of the ample cone and prove Theorem B. Finally, in Sect. 5 we investigate in terms of $\mathrm{End}(X)$ the question on which surfaces there are line bundles with two submaximal curves.

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Throughout we work over the field of complex numbers.

1 Pell divisors and submaximal curves on abelian surfaces

As in the introduction, we refer to [15, Chapt. 5] and [6] for background on Seshadri constants. Let us just fix a few matters of terminology here. When we speak of the *general upper bound*, then we mean the bound $\varepsilon(L,x) \le \sqrt{L^2}$, which is valid for every ample line bundle L on a smooth projective surface S and for every point $x \in S$. An effective divisor D on S is called *submaximal* (for L at x), if $L \cdot D/\text{mult}_x D < \sqrt{L^2}$. If an irreducible curve $C \subset S$ satisfies the equation $L \cdot C/\text{mult}_x C = \varepsilon(L,x)$, then we say that C *computes* $\varepsilon(L,x)$. An irreducible curve which computes $\varepsilon(L,x)$ for some ample line bundle L on S will be called a *Seshadri curve* on S.

It was shown in [5] that on an abelian surface of Picard number one, with ample generator L of the Néron–Severi group, there is for suitable $k \ge 1$ a divisor $D \in |2kL|$ that computes the Seshadri constant of L. The number k and the multiplicity of D at 0 are governed by a Pell equation. We will see that a suitable notion of *Pell divisors* (in the sense of the subsequent definition) also play a crucial role in the present investigation. The results in this section work on all abelian surfaces and do not require that the surface has real multiplication.

Definition 1.1 Let A be an abelian surface, and let L be an ample primitive symmetric line bundle such that $\sqrt{L^2} \notin \mathbb{Z}$. Consider the Pell equation

$$\ell^2 - L^2 \cdot k^2 = 1$$

and let (ℓ, k) be its primitive solution. A divisor $D \in |2kL|^+$ with $\text{mult}_0 D \ge 2\ell$ is called a *Pell divisor* for L.

Here $|2kL|^+$ denotes the linear subsystem of even divisors in $|2kL|$, i.e, those defined by even theta functions (see [7, Sect. 4.7]). It will be convenient to extend the notion of Pell divisors to non-primitive bundles, and even to $\mathbb{Q}$-divisors:

Definition 1.2 Let A be an abelian surface, and let M be any ample $\mathbb{Q}$-line bundle on A such that $\sqrt{M^2} \notin \mathbb{Q}$. Write $M = qL$ with a primitive ample line bundle L and $q \in \mathbb{Q}$. A *Pell divisor* for M is then by definition a Pell divisor for L.

It was shown in [2, Theorem A.1] that Pell divisors exist for every ample line bundle L with $\sqrt{L^2} \notin \mathbb{Z}$. Their crucial feature is that they are submaximal for L. By contrast, the existence of submaximal divisors is not guaranteed when $\sqrt{L^2} \in \mathbb{Z}$. However, a dimension count shows that for such bundles there exist divisors $D \in |2L|^+$ satisfying in any event the weak inequality $L \cdot D/\text{mult}_0 D \le \sqrt{L^2}$.

We will see that on abelian surfaces with real multiplication it is almost never true that $\varepsilon(L)$ is computed by a Pell divisor of L. However, it will turn out that $\varepsilon(L)$ is always computed by a Pell divisor of *some* ample bundle on L, whenever $\sqrt{L^2}$ is irrational. It is for this reason that Pell divisors are crucial players in the present investigation.

The following statement will prove to be a valuable tool, as it provides strong restrictions on submaximal curves. Also, it exhibits a situation where Pell divisors are unique.

Proposition 1.3 *Let A be an abelian surface, and let $C \subset A$ be an irreducible curve that is submaximal for some ample line bundle. Then, putting $m = \text{mult}_0 C$, one has*

$$C^2 - m^2 = -1 \quad \text{or} \quad C^2 - m^2 = -4.$$

Furthermore, suppose that C is not an elliptic curve, and write $\mathcal{O}_A(C) = pM$ with a primitive ample bundle M and an integer $p > 0$. Then $\sqrt{M^2}$ is irrational, and letting (ℓ_0, k_0) be the primitive solution of the Pell equation $\ell^2 - M^2 k^2 = 1$, we have:

(i) *If* $C^2 - m^2 = -1$*, then the divisor* $2C$ *is the only Pell divisor for* M *and* $(\ell_0, k_0) = (m, p)$.
(ii) *If* $C^2 - m^2 = -4$*, then the curve* C *is the only Pell divisor for* M *and* $(2\ell_0, 2k_0) = (m, p)$. *In this case, the origin is the only halfperiod that lies on* C.

Proof The first half of the following argument is implicit in the proof of [3, Thm. 2]. To provide easier access, we briefly make it explicit here. As the claim on $C^2 - m^2$ is certainly true for elliptic curves, we may assume that C is non-elliptic, and hence that $\mathcal{O}_A(C)$ is ample. The assumption that C is submaximal for some ample line bundle L then implies that C is submaximal also for $\mathcal{O}_A(C)$, and in fact computes $\varepsilon(\mathcal{O}_A(C))$ (see [4, Prop. 1.2]). Therefore C must be symmetric and therefore descends to a (-2)-curve on the smooth Kummer surface of A (cf. proof of [5, Thm. 6.1]). The multiplicities $m_i = \mathrm{mult}_{e_i}(C)$ at the sixteen halfperiods e_i of A therefore satisfy the equation

$$C^2 - \sum_{i=1}^{16} m_i^2 = -4. \tag{1}$$

Putting $m = m_1$, one shows as in the proof of [3, Thm. 1.2] that only the two cases

$$C^2 - m^2 = -1 \quad \text{or} \quad C^2 - m^2 = -4, \tag{2}$$

are possible. This proves the first statement in the proposition.

Write now $\mathcal{O}_A(C) = pM$ with a primitive ample bundle M and $p > 0$. It follows from Eq. (2) that C^2 cannot be a perfect square, and hence that $\sqrt{M^2}$ is irrational. In the first case of (2), the pair (m, p) satisfies the Pell equation $m^2 - M^2 p^2 = 1$. The minimality of the solution (ℓ_0, k_0) implies then that $m \geq \ell_0$ and $p \geq k_0$. On the other hand, as C computes $\varepsilon(M)$, we have for every Pell divisor $P \in |2k_0 M|$ of L,

$$\frac{M \cdot C}{m} \leq \frac{M \cdot P}{\mathrm{mult}_0 P} \leq \frac{M \cdot P}{2\ell_0}$$

This implies $\frac{p}{m} \leq \frac{k_0}{\ell_0}$. Using the fact that both pairs (m, p) and (ℓ_0, k_0) solve the Pell equation, we find $m \leq \ell_0$, and hence $(m, p) = (\ell_0, k_0)$. So we have $P = 2C$ in this case.

In the second case of (2), the number m is clearly even. But also p is even in this case, because all multiplicities m_i are even (since we have $(m_1, \ldots, m_{16}) = (m, 0, \ldots, 0)$). Therefore $\mathcal{O}(C)$ is totally symmetric, and it can therefore be written as an even multiple of another bundle (see [7, Sect. 2, Cor. 4]). The upshot of this argument is that the pair $(\frac{m}{2}, \frac{p}{2})$ satisfies the Pell equation $(\frac{m}{2})^2 - M^2(\frac{p}{2})^2 = 1$. The minimality assumption implies then that $\frac{m}{2} \geq \ell_0$ and $\frac{p}{2} \geq k_0$. But C must be a component of any Pell divisor $P \in |2k_0 M|$ by [5, Lemma 6.2], and so $p = 2k_0$ and $m = 2\ell_0$. So we have $P = C$ in this case. □

Further, we show how two curves can intersect if they are submaximal for the same bundle:

Proposition 1.4 *Let* A *be an abelian surface, and let* C_1 *and* C_2 *be two irreducible curves on* A *that are submaximal for the same ample line bundle* L *on* A*, i.e.,*

$$\frac{L \cdot C_i}{\mathrm{mult}_0(C_i)} < \sqrt{L^2}$$

for $i = 1, 2$*. Then, putting* $m_i = \mathrm{mult}_0(C_i)$*, we have*

$$C_1 \cdot C_2 = m_1 m_2,$$

i.e., the curves C_1 *and* C_2 *meet only at the origin, and their tangent cones have no common components there.*

Proof Consider the blow-up $f : Y = \mathrm{Bl}_0(A) \to A$, let E be its exceptional divisor, and let $C_i' \subset Y$ be the proper transform of C_i. For rational numbers $t < \sqrt{L^2}$, the $\mathbb{Q}$-divisor

$$B := f^*L - tE$$

is big, because $B^2 = L^2 - t^2 > 0$ and $B \cdot f^*L = L^2 > 0$. If we take t strictly between $\max\left\{\frac{L\cdot C_1}{m_1}, \frac{L\cdot C_2}{m_2}\right\}$ and $\sqrt{L^2}$, then we moreover have

$$B \cdot C_i' = (f^*L - tE) \cdot (f^*C_i - m_i E) = L \cdot C_i - tm_i < 0.$$

As a consequence, both C_1' and C_2' must be contained in the negative part of the Zariski decomposition of B, and hence their intersection matrix is negative definite. This implies that $C_1'^2 C_2'^2 > (C_1' C_2')^2$, i.e.,

$$(C_1^2 - m_1^2)(C_2^2 - m_2^2) > (C_1 \cdot C_2 - m_1 m_2)^2. \tag{3}$$

We know from Proposition 1.3 that $C_i^2 - m_i^2 \in \{-1, -4\}$. Let us first consider the case $C_1^2 - m_1^2 = C_2^2 - m_2^2 = -1$. Then inequality (3) directly implies $C_1 \cdot C_2 - m_1 m_2 = 0$. Suppose next that $C_1^2 - m_1^2 = -4$ and $C_2^2 - m_2^2 = -1$. In that case inequality (3) tells us that

$$C_1 \cdot C_2 - m_1 m_2 < 2,$$

since we have in any event $C_1 \cdot C_2 - m_1 m_2 \geq 0$ because of the intersection inequality. Using now Proposition 1.3, we see that m_1 is an even number and that $C_1 \equiv 2B_1$ for some line bundle B_1 on A. So we obtain $B_1 \cdot C_2 - \frac{m_1}{2} m_2 < 1$ and hence $B_1 \cdot C_2 = \frac{m_1}{2} m_2$, which implies $C_1 \cdot C_2 = m_1 m_2 = 0$, as claimed. Finally, if both $C_1^2 - m_1^2$ and $C_2^2 - m_2^2$ equal -4, then we get

$$C_1 \cdot C_2 - m_1 m_2 < 4$$

from inequality (3), and we have $C_1 \equiv 2B_1$, $C_2 \equiv 2B_2$. As both m_1 and m_2 are even, this yields $B_1 \cdot B_2 - \frac{m_1}{2}\frac{m_2}{2} = 0$, and this implies the assertion. □

2 Submaximal curves on intervals

Abelian surfaces with real multiplication. Let X be a simple abelian surface with real multiplication, i.e., such that $\mathrm{End}_{\mathbb{Q}}(X) = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d \geq 2$. The endomorphism ring is an order in $\mathrm{End}_{\mathbb{Q}}(X)$ and hence of the form $\mathrm{End}(X) = \mathbb{Z} + f\omega\mathbb{Z}$, where $f \geq 1$ is an integer and

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4 \\ \frac{1}{2}(1 + \sqrt{d}) & \text{if } d \equiv 1 \pmod 4). \end{cases}$$

For our purposes an alternative distinction of the possible cases will be more convenient:

- *Case 1:* $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$, with a non-square integer $e > 0$.
- *Case 2:* $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ with a non-square integer $e > 0$ such that $e \equiv 1 \pmod 4$.

If X carries a principal polarization L_0, then we have an isomorphism $\varphi : \mathrm{NS}(X) \to \mathrm{End}^{\mathrm{sym}}(X) = \mathrm{End}(X)$. It provides us with a lattice basis of $\mathrm{NS}(X)$, given by $L_0 = \varphi^{-1}(1)$

and $L_\infty := \varphi^{-1}(\sqrt{e})$ (resp. $\varphi^{-1}(\frac{1}{2}+\frac{1}{2}\sqrt{e})$). The intersection matrix of this basis is

$$\begin{pmatrix} 2 & 0 \\ 0 & -2e \end{pmatrix} \text{ in Case 1, and } \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-e}{2} \end{pmatrix} \text{ in Case 2.} \tag{4}$$

This follows by considering the characteristic polynomials of $\sqrt{e}$ and $\frac{1}{2}+\frac{1}{2}\sqrt{e}$ in $\mathbb{Q}(\sqrt{e})$ (which coincides with the analytic characteristic polynomial of the endomorphism) and applying [7, Prop. 5.2.3].

Using the Nakai–Moishezon criterion (in the version of [7, Cor. 4.3.3]), and the fact that X does not contain any elliptic curves, we find:

Lemma 2.1 *Let L be a line bundle on X with numerical class given by $L = aL_0 + bL_\infty$ for $a, b \in \mathbb{Z}$. If* $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$*, then L is ample if and only if*

$$a > 0 \quad \textit{and} \quad a^2 - eb^2 > 0,$$

and if $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{e}]$*, then L is ample if and only if*

$$a > 0 \quad \textit{and} \quad a^2 + ab + \tfrac{1-e}{4} b^2 > 0.$$

In either case, L is ample if and only if $|L| \neq \emptyset$.

From now on we will assume that X is a principally polarized abelian surface with real multiplication. We are interested in its *Seshadri function*

$$\varepsilon : \mathrm{Nef}(X) \to \mathbb{R}, \quad L \mapsto \varepsilon(L) = \varepsilon(L, 0).$$

Thanks to homogeneity, it is enough to consider this function on a compact cross-section of the nef cone. Any non-trivial nef class $L \in \mathrm{NS}_\mathbb{R}(X)$ is a positive multiple of a class of the form $L_t := L_0 + tL_\infty$ with suitable $t \in \mathbb{R}$. Applying Lemma 2.1, we see that if $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$, then the line bundle L_t is nef if and only if $|t| \le \frac{1}{\sqrt{e}}$, and if $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{e}]$, then L_t is nef if and only if $-\frac{2}{\sqrt{e}+1} \le t \le \frac{2}{\sqrt{e}-1}$. Ampleness holds when the inequalities are strict. We denote by $\mathcal{N}(X) = [-\frac{1}{\sqrt{e}}, \frac{1}{\sqrt{e}}]$ and $\mathcal{N}(X) = [-\frac{2}{\sqrt{e}+1}, \frac{2}{\sqrt{e}-1}]$, respectively, the interval where L_t is nef. This interval $\mathcal{N}(X)$ is a model for the cross-section of the nef cone, and therefore we will also write $L_t \in \mathcal{N}(X)$ instead of $t \in \mathcal{N}(X)$. So for every nef $\mathbb{R}$-line bundle, the ray $\mathbb{R}_{>0}L$ has a unique representative in $\mathcal{N}(X)$, and we may consider the Seshadri function as

$$\varepsilon : \mathcal{N}(X) \to \mathbb{R}, \quad t \mapsto \varepsilon(L_t).$$

Any effective divisor D defines a linear function

$$\ell_D : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \frac{D \cdot L_t}{\mathrm{mult}_0 D}$$

which computes the Seshadri quotient of the divisor D for any line bundle L_t. We denote the open subset containing all ample line bundles L_t, whose Seshadri quotient with D is submaximal, by I_D, i.e.,

$$I_D := \left\{ t \in \mathcal{N}(X) \,\middle|\, \ell_D(t) < \sqrt{L_t^2} \right\},$$

and we call I_D the *submaximality interval* of D.

It is a result of Szemberg [17, Prop. 1.8] that on any smooth projective surface S an ample line bundle can have at most $\rho(S)$ (two, in our case) submaximal curves at any given point.

Using the restrictions derived from Propositions 1.3 and 1.4 we show that in many cases only *one* curve can exist:

Theorem 2.2 *Let L be any ample line bundle on X with $\varepsilon(L) < \sqrt{L^2}$. Suppose that either*

- $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$ *for a non-square integer* $e > 0$*, or*
- $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ *for a non-square integer* $e > 0$*, such that* $e \equiv 1$ *modulo* 4 *and* e *has a prime factor* p *with* $p \equiv 5$ *or* 7 *modulo* 8*,*

holds. Then there exists exactly one irreducible curve C that is submaximal for L.

Proof We will show that the restrictions given in Propositions 1.3 and 1.4 cannot hold for two submaximal curves. In fact, we will show that the following equations can never be satisfied by *any* two ample line bundles L_1 and L_2 and two positive integers m_1 and m_2:

(i) $L_1^2 = m_1^2 - 1$ for $m_1 > 1$,
(ii) $L_2^2 = m_2^2 - 1$ for $m_2 > 1$,
(iii) $L_1 \cdot L_2 = m_1 m_2$.

Note that this also includes the case $C^2 = m^2 - 4$ from Proposition 1.3, because in this case $\mathcal{O}_X(C)$ is an even multiple of another line bundle and dividing the equation by 4 leads to an equation of the form (i).

First we treat the more immediate case: $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$, where e is a non-square positive integer. Assume that there exist ample line bundles L_1 and L_2 satisfying (i) and (ii), with their numerical classes given by $L_i \equiv a_i L_0 + b_i L_\infty$ for $i = 1, 2$. Then m_1 and m_2 must be odd, since L_i^2 is even. But the intersection number for any two line bundles on X is even,

$$(a_1 L_0 + b_1 L_\infty) \cdot (a_2 L_0 + b_2 L_\infty) = 2a_1 a_2 - 2e b_1 b_2,$$

and hence it can never equal $m_1 m_2$.

Next we treat the more subtle case: $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$, where e is a non-square positive integer with $e \equiv 1$ modulo 4, which has a prime factor p with $p \equiv 5$ or 7 modulo 8. The crucial idea in this case is to consider the three equations modulo p. Assume that L_1 and L_2 are two line bundles satisfying (i)–(iii), with their numerical classes given by $L_i \equiv a_i L_0 + b_i L_\infty$ for $i = 1, 2$. If we consider the equations

(i) $2L_1^2 = 4a_1^2 + 4a_1 b_1 + (1-e)b_1^2 = 2m_1^2 - 2$,
(ii) $2L_2^2 = 4a_2^2 + 4a_2 b_2 + (1-e)b_2^2 = 2m_2^2 - 2$,
(iii) $2L_1 \cdot L_2 = 4a_1 a_2 + 2a_1 b_2 + 2a_2 b_1 + (1-e)b_1 b_2 = 2m_1 m_2$,

modulo p and replace $2a_i + b_i$ by c_i for $i = 1, 2$, then the equations can be expressed by bilinear forms over the finite field $\mathbb{F}_p$. For (i) and (ii) we obtain

$$\text{(I)} \quad \begin{pmatrix} c_1 \\ m_1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ m_1 \end{pmatrix} = c_1^2 - 2m_1^2 = -2,$$

$$\text{(II)} \quad \begin{pmatrix} c_2 \\ m_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ m_2 \end{pmatrix} = c_2^2 - 2m_2^2 = -2.$$

It follows that $(c_i, m_i) \neq (0, 0) \in \mathbb{F}_p^2$. For equation (iii) we find that

$$\text{(III)} \quad \begin{pmatrix} c_1 \\ m_1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ m_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ -2m_1 \end{pmatrix}^T \cdot \begin{pmatrix} c_2 \\ m_2 \end{pmatrix} = 0$$

and, therefore, we obtain

$$\begin{pmatrix} c_2 \\ m_2 \end{pmatrix} \in \ker \begin{pmatrix} c_1 \\ -2m_1 \end{pmatrix}^T = \left\{ \lambda \begin{pmatrix} -2m_1 \\ c_1 \end{pmatrix} \middle| \lambda \in \mathbb{F}_p \right\},$$

i.e., $c_2 = -2m_1\lambda$ and $m_2 = c_1\lambda$ for some $\lambda \in \mathbb{F}_p$. Using (I) and (II) we obtain

$$-2 = \begin{pmatrix} c_2 \\ m_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ m_2 \end{pmatrix} = \lambda^2(4m_1^2 - 2c_1^2) = 4\lambda^2.$$

This implies that -2 is a quadratic residue modulo p. But as $p \equiv 5$ or 7 modulo 8, this is impossible, and thus we arrive at a contradiction. □

As a consequence of Theorem 2.2 we observe:

Corollary 2.3 *Let* $\operatorname{End}(X)$ *be as in Theorem 2.2. Then for every ample* $\mathbb{R}$*-line bundle* L_λ *with* $\varepsilon(L_\lambda) < \sqrt{L_\lambda^2}$ *the Seshadri function is given by a linear function in a neighborhood of* L_λ*.*

Proof Let L_λ be an ample $\mathbb{R}$-line bundle with $\varepsilon(L_\lambda) < \sqrt{L_\lambda^2}$ and let C be any Seshadri curve of L_λ. By the previous Theorem 2.2 the curve C is the only submaximal curve for every $\mathbb{Q}$-line bundle in I_C. Assume now that there exists an ample $\mathbb{R}$-line bundle $L_t \in I_C$ with two submaximal curves. By continuity both curves remain submaximal in a neighborhood of L_t and, thus, there also exist $\mathbb{Q}$-line bundles which also have two submaximal curves. This, however, is impossible by Theorem 2.2. □

We will see that the assumption $\varepsilon(L_\lambda) < \sqrt{L_\lambda^2}$ is essential for the validity of the statement in the corollary, and in fact we will show that the local behavior in the remaining case $\varepsilon(L_\lambda) = \sqrt{L_\lambda^2}$ is surprisingly intricate (see Corollary 3.6).

Computer-assisted calculations suggest that Theorem 2.2 is in fact an "if and only if" statement, which means that in the remaining cases there should always exist a line bundle with two submaximal curves. In Sect. 5 we will show how the existence of a line bundle with two submaximal curves can be verified using computer-assisted calculations. Furthermore, we will provide a sequence of numbers e_n with the property that there exists a line bundle with two submaximal curves on any abelian surface with $\operatorname{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e_n}]$.

Before we continue studying the local behavior of the Seshadri function in the case where line bundles can have two submaximal curves, we prove a useful relation between submaximality intervals and reducibility of effective divisors:

Lemma 2.4 *Let D be an effective divisor on X which is submaximal for some ample line bundle. If there exists another effective divisor D' whose submaximality interval $I_{D'}$ satisfies $I_D \subsetneq I_{D'}$, then D is reducible.*

Proof Let $I_D = (a, b)$ and $I_{D'} = (c, d)$ be the submaximality intervals of D and D', respectively. Denoting by F the general upper bound function $t \mapsto \sqrt{L_t^2}$, the linear function ℓ_D is given by the straight line joining the points $(a, F(a))$ and $(b, F(b))$ and, respectively, $\ell_{D'}$ by joining the points $(c, F(c))$ and $(d, F(d))$. Since F is strictly concave, the linear function $\ell_{D'}$ is strictly smaller than ℓ_D in $I_{D'}$. Therefore D can never compute the Seshadri constant for any line bundle. However, if D were irreducible, then D would compute its own Seshadri constant (see [4, Prop. 1.2]), which is a contradiction. □

By [17, Prop. 1.8], the number of curves that can be submaximal for an individual ample line bundle L_t is bounded. We will now show that the number remains bounded even when all line bundles in an open neighborhood of L_t are considered, provided that $\varepsilon(L_t) < \sqrt{L_t^2}$. This is a consequence of the following lemma.

Lemma 2.5 *Let D be an effective divisor on X which is submaximal for an ample line bundle L_t. Then there exists at most four irreducible curves which are submaximal for some line bundles in I_D.*

Moreover, if D is irreducible, then there exists at most three irreducible curves which are submaximal for some line bundles in I_D.

Proof Assume, there exists five pairwise distinct irreducible curves $C_1, \ldots, C_5$ which are submaximal for some line bundles in $I_D = (a, b)$. Let $I_{C_i} = (a_i, b_i)$ be the submaximality interval of C_i and let $L_{t_i} \in \mathcal{N}(X)$ be the unique representative of $\mathcal{O}_X(C_i)$. We will show that the submaximality interval of C_3 is contained in I_D, which by Lemma 2.4 would imply that C_3 is reducible.

Since C_i is submaximal for some ample line bundle, C_i is submaximal for $\mathcal{O}_X(C_i)$ by [4, Prop. 1.2]. Therefore, C_i is submaximal for L_{t_i} and, thus, $t_i \in I_{C_i}$. Moreover, since C_i is the only submaximal curve for $\mathcal{O}_X(C_i)$, we have $t_i \notin I_{C_j}$ for $i \neq j$. By assuming $t_1 < t_2 < t_3 < t_4 < t_5$ we deduce for $i = 2, 3, 4$ that

$$(a_i, b_i) \subset (t_{i-1}, t_{i+1}) \quad \text{and} \quad t_i \in (b_{i-1}, a_{i+1})\,. \tag{*}$$

The submaximality intervals (a_1, b_1) and (a_5, b_5) have to intersect with (a, b), because by assumption C_1 and C_5 are submaximal for some line bundles in I_D and, thus, we have $a < b_1$ and $a_5 < b$. Furthermore, $(*)$ implies that $t_2, t_3, t_4 \in (b_1, a_5)$ and, as a consequence, the interval (t_2, t_4) is contained in (b_1, a_5) and, therefore, in I_D. Since (a_3, b_3) is contained in (t_2, t_4), it is also contained in I_D. This, however, implies that C_3 is reducible by Lemma 2.4, which is a contradiction.

For the second statement, we assume there exists three irreducible curves. Using the same notation and arguments as above, it follows that $t_2 \in (b_1, a_3)$, $a < b_1$, and $a_3 < b$. Hence, we have $t_2 \in (b_1, a_3) \subset I_D$. But this means that L_{t_2} has C_2 and D as submaximal curves, which is a contradiction. □

Hence, we conclude for the local structure of the Seshadri function:

Corollary 2.6 *For every ample $\mathbb{R}$-line bundle L_t with $\varepsilon(L_t) < \sqrt{L_t^2}$ the Seshadri function is locally a piecewise linear function, i.e., it is locally the minimum of at most two linear functions.*

As before, the assumption $\varepsilon(L_t) < \sqrt{L_t^2}$ is essential for this statement to be true (see Remark 3.8).

Clearly, if a line bundle L has two submaximal curves, then there exists a neighborhood of L such that every line bundle has two submaximal curves, since any submaximal curve will remain submaximal in a neighborhood of L. On the other hand, we show that every submaximal curve gives rise to an open interval, in which it is the only submaximal curve:

Proposition 2.7 *Let $C \equiv qL_0 + pL_\infty$ be an irreducible curve that is submaximal for some ample line bundle L on X. Then there exists a neighborhood U of $L_{\frac{p}{q}}$ in $\mathcal{N}(X)$ such that C is the only submaximal curve for all line bundles in U. In particular, the Seshadri function coincides with ℓ_C in U.*

Proof Since C is submaximal for some ample line bundle L, we know that C is also submaximal for $\mathcal{O}_X(C)$ by [4, Prop. 1.2], and in fact C is the only submaximal curve for $\mathcal{O}_X(C)$, since every $\mathcal{O}_X(C)$-submaximal curve has to be a component of C by [5, Lemma 5.2]. Thus, C is the only submaximal curve for $L_{\frac{p}{q}}$. Applying Lemma 2.5, there exist at most two other curves, which are submaximal for some line bundle $L' \in I_C$. Thus, the only possibility in which no such neighborhood of $L_{\frac{p}{q}}$ exists, is the case where one of the other curves C' satisfies $\mathcal{O}_X(C) \cdot C'/\mathrm{mult}_0(C') = \sqrt{C^2}$. This, however, implies that C' is a component of C by [5, Lemma 5.2]. □

3 Seshadri function on abelian surfaces with real multiplication

In this section we will develop a method to algorithmically compute the Seshadri constant for any ample $\mathbb{Q}$-line bundle on X, proving Theorem C stated in the introduction. Furthermore, we will see that the local structure of the Seshadri function has unexpected behavior at L_λ if $\varepsilon(L_\lambda) = \sqrt{L_\lambda^2}$. Our strategy is to make use of Pell divisors in such a way that it is not necessary to explicitly know their multiplicity, but to use their *expected multiplicity* given by the Pell solution.

Definition 3.1 Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ and let $q \in \mathbb{N}$ be the unique integer such that qL_λ is a primitive $\mathbb{Z}$-line bundle, i.e., q is the denominator of a coprime representation of $\lambda = \frac{p}{q}$. Denote by (l, k) the primitive solution of the Pell equation $x^2 - (qL_\lambda)^2 y^2 = 1$. We call

$$\pi_\lambda : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \frac{kqL_\lambda \cdot L_t}{l}$$

the *Pell bound* at L_λ, and

$$J_\lambda = \{t \in \mathcal{N}(X) \,\Big|\, \pi_\lambda(t) < \sqrt{L_t^2}\}$$

the *submaximality interval* of π_λ.

So if $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ and P is a Pell divisor of L_λ, then we have the following chain of inequalities:

$$\varepsilon(L_\lambda) \le \frac{L_\lambda \cdot P}{\mathrm{mult}_0 P} \le \pi_\lambda(\lambda) < \sqrt{L_\lambda^2}.$$

Moreover, π_λ is an upper bound for the Seshadri function in the submaximality interval J_λ:

$$\varepsilon(L_t) \le \frac{L_t \cdot P}{\mathrm{mult}_0 P} \le \pi_\lambda(t) < \sqrt{L_\lambda^2} \qquad \text{for all } t \in J_\lambda.$$

We will now establish two important connections between submaximal curves and Pell bounds. First we prove that every submaximal curve has a unique representative in the set of Pell bounds. Secondly, we will exhibit a relation between the submaximality interval of a Seshadri curve C of L_λ and the submaximality interval of the Pell bound π_λ.

Proposition 3.2 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ and let $C \equiv qL_0 + pL_\infty$ be an irreducible curve that is submaximal for some ample line bundle L on X. Then the following are equivalent:*

(i) *Either C or $2C$ is the unique Pell divisor of L_λ.*
(ii) *The linear functions ℓ_C and π_λ coincide.*
(iii) *We have $\lambda = \frac{p}{q}$.*

Proof The equivalence of (i) and (iii) is an immediate consequence of Proposition 1.3. Furthermore, the implication (i) $\Rightarrow$ (ii) also follows from Proposition 1.3, since it shows that the multiplicity of C coincides with the expected multiplicity given by the Pell solution, and therefore the linear functions ℓ_C and $\pi_{\frac{p}{q}}$ coincide.

For the implication (ii) $\Rightarrow$ (iii) we have to show that $\ell_C = \pi_\lambda$ implies $\lambda = \frac{p}{q}$. By Proposition 2.7 the linear function ℓ_C coincides with the Seshadri function in an open neighborhood U of $L_{\frac{p}{q}}$. For any Pell divisor P of L_λ we have

$$\varepsilon(L_t) = \ell_C(t) \le \ell_P(t) \le \pi_\lambda(t) \quad \text{for all} \quad t \in U,$$

and hence the linear function ℓ_P coincides with ℓ_C, since by assumption $\pi_\lambda = \ell_C$.

We claim that for every component C' of P the linear functions $\ell_{C'}$ and ℓ_C also coincide. For this, assume that there exists a $t_0 \in U$ such that $\ell_C(t_0) < \ell_{C'}(t_0)$. Then, upon writing $P = C' + R$, we have

$$\frac{C \cdot L_{t_0}}{\mathrm{mult}_0 C} = \ell_C(t_0) = \ell_P(t_0) = \frac{(C' + R) \cdot L_{t_0}}{\mathrm{mult}_0 C' + \mathrm{mult}_0 R}.$$

This, however, implies that

$$\frac{C \cdot L_{t_0}}{\mathrm{mult}_0 C} > \frac{R \cdot L_{t_0}}{\mathrm{mult}_0 R},$$

which is impossible, since C computes the Seshadri constant $\varepsilon(L_{t_0})$.

So we have shown that C and any component C' of the Pell divisor P define the same linear function. This means, in particular, that any component C' of P is also submaximal for the line bundle $\mathcal{O}_X(C)$. But $\mathcal{O}_X(C)$ has only C as a submaximal curve by [5, Lem. 5.2], and, therefore, $P = kC$ for $k \in \mathbb{N}$. This implies that $L_{\frac{p}{q}}$ and L_λ are rational multiples of each other. But in $\mathcal{N}(X)$ this is only possible if $\lambda = \frac{p}{q}$. ◻

Proposition 3.3 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ and let $J_\lambda = (t_1, t_2)$ be the submaximality interval of the Pell bound π_λ. Then every Seshadri curve C of L_λ is submaximal on (t_1, λ) or on (λ, t_2).*

Proof Assume that C is not submaximal on (λ, t_2), i.e., $\ell_C(t_2) > \sqrt{L_{t_2}^2} = \pi_\lambda(t_2)$. Furthermore, since C is a Seshadri curve of L_λ, we have $\ell_C(\lambda) \le \pi_\lambda(\lambda)$. Therefore the slopes m_λ of π_λ and m_C of ℓ_C satisfy $m_\lambda < m_C$. But this implies that $\ell_C(t_1) < \pi_\lambda(t_1)$ and therefore C is submaximal on (t_1, λ). ◻

We will need the submaximality intervals of Pell bounds in the following explicit form:

Lemma 3.4 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$, and let l, k and q be as in Definition 3.1. If $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$, then the submaximality interval J_λ of π_λ is given by*

$$J_\lambda = \left(\frac{2ek^2q^2\lambda - l\sqrt{e}}{e(2k^2q^2+1)}, \frac{2ek^2q^2\lambda + l\sqrt{e}}{e(2k^2q^2+1)} \right),$$

and, if $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$*, then the submaximality interval* J_λ *of* π_λ *is given by*

$$J_\lambda = \left(\frac{2 + 2ek^2q^2\lambda - 2l\sqrt{e}}{(e-1) + 2eq^2k^2}, \frac{2 + 2ek^2q^2\lambda + 2l\sqrt{e}}{(e-1) + 2eq^2k^2} \right).$$

Proof The interval limits of the submaximality interval $J_\lambda = (t_1, t_2)$ are the solutions t of the equation

$$\sqrt{L_t^2} = \pi_\lambda(t).$$

In the case $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$, the solutions are

$$t_{1,2} = \frac{2ek^2q^2\lambda \mp l\sqrt{e}\sqrt{l^2 - (qL_\lambda)^2k^2}}{e(2ek^2q^2\lambda^2 + l^2)}.$$

Upon applying the Pell equation $l^2 - (qL_\lambda)^2k^2 = 1$, these solutions can be expressed by

$$t_{1,2} = \frac{2ek^2q^2\lambda \mp l\sqrt{e}}{e(2k^2q^2 + 1)}.$$

The case $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ is computed analogously. □

As the linear function ℓ_C of a submaximal curve C coincides with a Pell bound, the interval borders for Seshadri curves have the same structure. This reveals an interesting behavior of submaximality intervals of Seshadri curves:

Proposition 3.5 *Let* C_1 *and* C_2 *be two submaximal curves on* X*. Then the submaximality intervals* I_{C_1} *and* I_{C_2} *are never adjacent to each other, i.e., if* $I_{C_1} = (t_1, t_2)$ *and* $I_{C_2} = (s_1, s_2)$*, then* $t_1 \neq s_2$ *and* $t_2 \neq s_1$*.*

Proof Using the computation of the interval limits of Lemma 3.4, it follows that the left-hand side of the interval is always of the form $a - b\sqrt{e}$ for some $a \in \mathbb{Q}$ and $b \in \mathbb{Q}^+$, whereas the right-hand side is of the form $a' + b'\sqrt{e}$ for some $a' \in \mathbb{Q}$ and $b' \in \mathbb{Q}^+$. Since 1 and $\sqrt{e}$ form a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}(\sqrt{e})$, they can never coincide. □

Corollary 3.6 *Let* L_λ *be any ample* $\mathbb{R}$*-line bundle such that* $\varepsilon(L_\lambda) = \sqrt{L_\lambda^2}$*. For every neighborhood* U *of* λ*, the Seshadri function is the pointwise infimum of infinitely many linear functions* π_μ*, but it is not a piecewise linear function on* U*.*

Proof In any neighborhood U of λ, the rational numbers $\mu \in U$ with $\varepsilon(L_\mu) < \sqrt{L_\mu^2}$ are dense in U. Thus, by continuity of the Seshadri function we can express the Seshadri constant for any value $t \in U$ as an infimum of linear functions π_μ. The only possibility for the Seshadri function to be a piecewise linear function in a neighborhood of λ is, if the Seshadri function is computed near λ by two linear functions ℓ_1 and ℓ_2 with $\ell_1(\lambda) = \ell_2(\lambda) = \sqrt{L_\lambda^2}$. This, however, is impossible by Proposition 3.5. □

By combining Corollaries 2.3 and 3.6 we deduce the following more general version of Theorem A stated in the introduction.

Theorem 3.7 *Let* $\mathrm{End}(X)$ *be as in Theorem 2.2. Then the Seshadri function is broken linear.*

Proof It follows from Corollary 2.3 that for every point $t \in \mathcal{N}(X)$ with $\varepsilon(L_t) < \sqrt{L_t^2}$ the Seshadri function is a linear function in a neighborhood of t. By Proposition 3.5 the maximal intervals, on which the Seshadri function is linear are never adjacent to each other. Lastly, we have to argue that the set $M(X) = \left\{ t \in \mathcal{N}(X) \;\middle|\; \varepsilon(L_t) = \sqrt{L_t^2} \right\}$ is nowhere dense and non empty. For this we consider ample line bundles of the form $L = qL_0 + 4qL_\infty$ for odd $q \in \mathbb{N}$ and $p \in \mathbb{Z}$. In this cases L^2 can never be a square number as $L^2 \equiv 2 \pmod 4$. This yields a dense subset of lines bundles $L_{4q/p}$ in $\mathcal{N}(X)$ with $\varepsilon(L_{4p/q}) < \sqrt{L_{4p/q}^2}$. As the Seshadri function is continuous, we get for each line bundle $L_{4q/p}$ an open neighborhood on which the Seshadri function is submaximal. Thus, $M(X)$ is a nowhere dense subset of $\mathcal{N}(X)$. Explicit computations show that the Seshadri curve $C \in |4L_0|$ of L_0 with $\mathrm{mult}_0 C = 6$ is not submaximal on $\mathcal{N}(X)$ and, therefore, the interval borders of the submaximality interval I_C are contained in $M(X)$. □

Remark 3.8 Suppose that on X there is a line bundle L_λ with two submaximal curves. Then there exists a neighborhood of L_λ in which every line bundle has two submaximal curves, and thus there exist linear segments of the Seshadri function that are adjacent to each other. On the other hand, we have seen in Proposition 2.7 that there are also neighborhoods, in which only one submaximal curve exists. Furthermore, using Proposition 1.4 one can show that every line bundle in the submaximality interval I_0 of L_0 has only one submaximal curve C, which is the unique Pell divisor of L_0. Consequently, the limit points of this submaximal interval are accumulation points of (piecewise) linear segments. So in this case, as in the situation of Theorem 3.7, the Seshadri function does not consist of only finitely many linear pieces.

We return to the submaximality interval $J_\lambda = (t_1, t_2)$ of a Pell bound π_λ by providing an upper bound for its length:

Lemma 3.9 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$, and let l, k and q be as in Definition 3.1. Then the interval length of $J_\lambda = (t_1, t_2)$ is bounded by*

$$t_2 - t_1 < \frac{\sqrt{11}}{q\sqrt{e}}.$$

Proof Using the fact that the Pell equation is equivalent to

$$\frac{l}{kq} = \sqrt{L_\lambda^2 + \frac{1}{k^2q^2}},$$

the interval length can be determined via Lemma 3.4: In the case $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$ we get

$$t_2 - t_1 = \frac{2\frac{l}{kq}}{\sqrt{e}(2kq + \frac{1}{kq})} = \frac{2\sqrt{L_\lambda^2 + \frac{1}{k^2q^2}}}{\sqrt{e}(2kq + \frac{1}{kq})} = \frac{2\sqrt{2 - \frac{2ep^2k^2-1}{k^2q^2}}}{\sqrt{e}(2kq + \frac{1}{kq})} < \frac{\sqrt{2}}{q\sqrt{e}},$$

and in the case $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ we obtain

$$t_2 - t_1 = \frac{4\frac{l}{kq}\sqrt{e}}{\frac{e-1}{kq} + 2ekq} = \frac{4\sqrt{e}\sqrt{2 + 2\lambda - \frac{e-1}{2}\lambda^2 + \frac{1}{k^2q^2}}}{\frac{e-1}{kq} + 2ekq} < \frac{2\sqrt{2 + \frac{2}{e-1} + \frac{1}{k^2q^2}}}{q\sqrt{e}} < \frac{\sqrt{11}}{q\sqrt{e}}.$$

In the third step we replaced the term $2\lambda - \frac{e-1}{2}\lambda^2$ with its maximum value $\frac{2}{e-1}$, and we use $e \geq 5$ in the last step. Also, we made use of the inequality $\frac{1}{k^2q^2} \leq \frac{1}{4}$, which can be verified by explicitly considering all possible Pell solutions for $q = 1$. □

Remark 3.10 In the last step of the proof we could have used the more direct estimate $\frac{1}{k^2q^2} \leq 1$, which implies

$$\frac{2\sqrt{2 + \frac{2}{e-1} + \frac{1}{k^2q^2}}}{q\sqrt{e}} < \frac{\sqrt{14}}{q\sqrt{e}}$$

However, it turns out that this upper bound is not sufficient for our purposes (in particular, for the proof of Proposition 5.4).

The previous Lemma yields the following:

Corollary 3.11 *For any given interval $I \subset \mathcal{N}(X)$ there exist only finitely many Pell bounds π_λ that are submaximal on I.*

Proof Let s be the length of I and let π_λ be a Pell bound that is submaximal on I. Then s is at most the length of J_λ, so that it follows from Lemma 3.9 that the denominator of $\lambda = \frac{p}{q}$ satisfies

$$s \leq \frac{\sqrt{11}}{q\sqrt{e}}.$$

Thus, λ has to be contained in the finite set

$$\left\{ \frac{a}{b} \in \mathcal{N}(X) \;\middle|\; 1 \leq b \leq \frac{\sqrt{11}}{s\sqrt{e}},\ \gcd(a, b) = 1 \right\}.$$

□

By combining Proposition 3.3 and Corollary 3.11 we will obtain a purely numerical method to compute the Seshadri constant and determine the Seshadri curves of L_λ:

Proposition 3.12 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$, and $J_\lambda = (t_1, t_2)$ be the submaximality interval of π_λ. Let $s(\lambda) = \min\{\lambda - t_1, t_2 - \lambda\}$, and consider the finite set $A_\lambda := \left\{ \frac{a}{b} \in \mathcal{N}(X) \;\middle|\; 1 \leq b \leq \frac{\sqrt{11}}{s(\lambda)\sqrt{e}},\ \gcd(a, b) = 1 \right\}$. Then the Seshadri constant of L_λ is given by*

$$\varepsilon(L_\lambda) = \min\{\pi_\mu(\lambda) \mid \mu \in A_\lambda\}.$$

Moreover, every Seshadri curve C of L_λ is represented by a unique Pell bound π_τ with $\tau \in A_\lambda$ and $\varepsilon(L_\lambda) = \pi_\tau(\lambda)$.

Proof By Prop. 3.3 any Seshadri curve C of L_λ is submaximal either on (t_1, λ) or on (λ, t_2), and therefore it is submaximal on an interval of length $s = \min\{\lambda - t_1, t_2 - \lambda\}$. By Proposition 3.2 the linear function ℓ_C coincides with a unique Pell bound π_τ, and therefore their submaximality intervals coincide. Thus, τ is an element of the finite set A_λ by Corollary 3.11. □

Furthermore, we can identify those Pell bounds which uniquely represent submaximal curves:

Proposition 3.13 *Let L_λ be an ample $\mathbb{Q}$-line bundle with $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ and let A_λ be as in Proposition 3.12. Then the following conditions are equivalent:*

(i) *The Pell bound π_λ coincides with ℓ_C for some submaximal irreducible curve C.*
(ii) *Every Pell bound π_μ with $\mu \in A_\lambda \setminus \{\lambda\}$ satisfies $\pi_\lambda(\lambda) < \pi_\mu(\lambda)$.*

Proof Assume that $\pi_\lambda = \ell_C$, and let π_μ be any Pell bound with $\mu \neq \lambda$. Proposition 3.2 shows that the unique representative of $\mathcal{O}_X(C)$ in $\mathcal{N}(X)$ is L_λ. As C computes the Seshadri constant of L_λ, we have $\pi_\lambda(\lambda) \le \pi_\mu(\lambda)$. Thus, we have to show that equality does not occur for $\lambda \neq \mu$.

Assume that $\pi_\lambda(\lambda) = \pi_\mu(\lambda)$ holds. We will show that this implies $\lambda = \mu$. By Proposition 2.7 the submaximal curve C computes the Seshadri constant in an open neighborhood U at L_λ, hence

$$\pi_\lambda(t) \le \pi_\mu(t) \quad \text{for} \quad t \in U.$$

This implies that the linear functions π_λ and π_μ coincide, since otherwise we would have $\pi_\lambda(t) < \pi_\mu(t)$ for either $t < \lambda$ or $t > \lambda$, which is impossible because π_λ computes the Seshadri function locally. But by Proposition 3.2 the linear function ℓ_C only coincides with the Pell bound π_λ and, thus, we have $\lambda = \mu$.

For the other implication, we argue as in the proof of Proposition 3.12: For every Seshadri curve C of L_λ there is a unique π_τ with $\ell_C = \pi_\tau$ and $\tau \in A_\lambda$. Since C computes the Seshadri constant of L_λ, the Pell bound π_τ computes the Seshadri constant in λ. In particular, we have $\pi_\tau(\lambda) \le \pi_\lambda(\lambda)$. Since by assumption $\pi_\lambda(\lambda) < \pi_\mu(\lambda)$ for $\mu \neq \lambda$, we conclude that $\tau = \lambda$, and therefore $\pi_\lambda = \ell_C$. □

So far, the assumption $\sqrt{L_\lambda^2} \notin \mathbb{Q}$ was crucial for our arguments since they depended on the existence of Pell divisors. We will now show that the Seshadri constant can in fact be effectively computed for *any* ample $\mathbb{Q}$-line bundle. This will complete the proof of Theorem C stated in the introduction.

Theorem 3.14 *There is an algorithm that computes the Seshadri constant of every given ample line bundle on principally polarized abelian surfaces with real multiplication.*

Proof If L_λ is a $\mathbb{Q}$-line bundle such that $\sqrt{L_\lambda^2} \notin \mathbb{Q}$, then the assertion follows from the fact, that the set A_λ from Proposition 3.12 is finite. Suppose then that $\sqrt{L_\lambda^2} \in \mathbb{Q}$. We will construct a theoretical interval around L_λ, on which every Seshadri curve of L_λ must be submaximal, if $\varepsilon(L_\lambda) < \sqrt{L_\lambda^2}$. By Corollary 3.11, there are only finitely many Pell bounds on this interval, and $\varepsilon(L_\lambda)$ is the minimum of those.

Assume that the Seshadri constant satisfies $\varepsilon(L_\lambda) < \sqrt{L_\lambda^2}$. Let $\lambda = \frac{p}{q}$ be a coprime representation. Then, $L := qL_\lambda$ is a primitive $\mathbb{Z}$-line bundle. Denote by C any Seshadri curve of L_λ. As explained in Sect. 1, there exists an effective Divisor $D \in |2L|^+$ such that D satisfies $D \cdot L/\text{mult}_0(D) \le \sqrt{L^2}$. As C is the Seshadri curve of L, C is a component of D by [5, Lemma 6.2]. It follows that the intersection-number $C \cdot L$ is bounded by $D \cdot L = 2L^2$. As a consequence, the Seshadri constant can only take certain rational values:

$$\varepsilon(L) \in \left\{1 \le \frac{a}{b} < \sqrt{L^2} \;\middle|\; 1 \le a \le 2L^2, b \in \mathbb{N}\right\}.$$

Therefore, we find that the Seshadri constant is at most

$$\varepsilon(L) \leq \frac{2L^2 - 1}{2\sqrt{L^2}}.$$

For the construction of the interval, we will give a lower and an upper bound for the slope of the linear function ℓ_C. To this end, we will chose any two rational numbers $\mu_i \in \mathcal{N}(X)$ with $\mu_1 < \lambda < \mu_2$ and $\sqrt{L^2_{\mu_i}} \notin \mathbb{Q}$ for $i = 1, 2$. Next, we compute a Seshadri curve C_i of L_{μ_i} using Proposition 3.13. We denote by m_i the slope of the linear function ℓ_{C_i}. As the Seshadri function is a concave function, the slope m of the linear function ℓ_C is bounded, $m_2 \leq m \leq m_1$. Let r_i be the linear function passing through the point $(\lambda, (2L^2 - 1)/(2\sqrt{L^2}))$ with slope m_i. Then the function $u(t) = \max\{r_1(t), r_2(t)\}$ is an upper bound for ℓ_C, since we have

$$\ell_C(t) \leq r_1(t) \quad \text{for } \lambda \leq t \quad \text{and} \quad \ell_C(s) \leq r_2(s) \quad \text{for} \quad s \leq \lambda.$$

Denote by I the submaximality interval of u. It follows that C has to be submaximal on I, and so we have constructed a computable interval on which C is submaximal. Now, by following the same argument from Proposition 3.12 we can compute the Seshadri constant of L_λ by taking the minimum of Pell bounds in λ, which are submaximal on I. Clearly, if none of these Pell bounds are submaximal in λ, then the Seshadri constant satisfies $\varepsilon(L_\lambda) = \sqrt{L^2_\lambda}$. □

Remark 3.15 In the proof of Theorem 3.14 we have shown how to algorithmically distinguish the cases $\varepsilon(L^2_\lambda) < \sqrt{L_\lambda}$ and $\varepsilon(L_\lambda) = \sqrt{L^2_\lambda}$. Both cases do, in fact, occur for line bundles L with $\sqrt{L^2} \in \mathbb{Q}$: Consider a principally polarized abelian surface X with $\mathrm{End}(X) = \mathbb{Z}[\sqrt{2}]$. Then the line bundle $L = 2L_0 + L_\infty$ satisfies $\varepsilon(L) = \sqrt{L^2} = 2$, whereas the line bundle $L' = 58L_0 + L_\infty$ satisfies $\varepsilon(L') < \sqrt{L'^2}$, since the Seshadri curve C of L_0 is also submaximal for L'.

It is an important consequence of Theorem 3.14 that the Seshadri function depends only on the endomorphism ring of X, but not on the isomorphism class of the surface:

Theorem 3.16 *Let X and Y be (not necessarily isomorphic) principally polarized abelian surfaces with real multiplication with* $\mathrm{End}(X) \cong \mathrm{End}(Y)$. *Then their Seshadri functions coincide in the following sense: Choosing suitable bases of the Néron-Severi groups* $\mathrm{NS}(X)$ *and* $\mathrm{NS}(Y)$ *yields an isomorphism* $\mathrm{Nef}(X) \simeq \mathrm{Nef}(Y)$, *under which we have* $\varepsilon_X = \varepsilon_Y$.

This implies Corollary D stated in the introduction.

Proof The proof of Theorem 3.14 shows that the numerical data that enters the computation of the Seshadri functions stems from the endomorphism ring. Therefore, an isometry of $\mathrm{NS}(X)$ that leaves the ample cone invariant also leaves the Seshadri function invariant. □

4 Fundamental cone and sample plots for Seshadri functions

We will now determine the subgroup $G \subset \mathrm{Aut}(\mathrm{NS}(X))$ of isometries with respect to the intersection product that leave the Seshadri function on $\mathrm{Amp}(X)$ invariant. This group gives rise to a decomposition of the ample cone into subcones on which G acts transitively.

With respect to the basis (L_0, L_∞), an automorphism $\varphi \in \mathrm{Aut}(\mathrm{NS}(X))$ is given by a matrix

$$M_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

By Theorem 3.16 the Seshadri function remains invariant under the automorphism M_φ if it is an isometry of $\mathrm{NS}(X)$ and additionally leaves the ample cone invariant. These conditions can be expressed by:

(i) $L_0^2 = (\alpha L_0 + \gamma L_\infty)^2$,
(ii) $L_\infty^2 = (\beta L_0 + \delta L_\infty)^2$,
(iii) $L_0 \cdot L_\infty = (\alpha L_0 + \gamma L_\infty) \cdot (\beta L_0 + \delta L_\infty)$,
(iv) $\alpha > 0$.

The conditions (i)–(iii) are equivalent to φ being an isometry, whereas condition (vi) ensures that the ample cone is left invariant.

In the case of $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$ we find by solving (i)–(iv) that M_φ is of the form

$$\begin{pmatrix} \alpha & e\beta \\ \beta & \alpha \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & -e\beta \\ \beta & -\alpha \end{pmatrix}, \quad \text{with } \alpha > 0 \text{ and } \alpha^2 - e\beta^2 = 1.$$

Since any other Pell solution of $x^2 - ey^2 = 1$ is generated by the minimal solution (α_0, β_0), the group G is generated by

$$\varphi_0 := \begin{pmatrix} \alpha_0 & e\beta_0 \\ \beta_0 & \alpha_0 \end{pmatrix} \text{ and } \tau := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A line bundle $L = aL_0 + bL_\infty$ is a principal polarization if and only if (a, b) is a solution of Pell's equations $x^2 - ey^2 = 1$ with $a > 0$. Therefore, we can express every principal polarization by $L_k := x_k L_0 + y_k L_\infty$, where (x_k, y_k) satisfies

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \alpha_0 & e\beta_0 \\ \beta_0 & \alpha_0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \varphi_0^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad k \in \mathbb{Z}.$$

So we have $\varphi_0(L_k) = L_{k+1}$. Next, we consider for $k \in \mathbb{Z}$ the subcone $\mathcal{D}_k \subset \mathrm{Amp}(X)$ generated by L_k and L_{k+1}. We have $\varphi_0^k(\mathcal{D}_0) = \mathcal{D}_k$. Additionally, by also considering the automorphism τ we can further divide the subcone $\mathcal{D}_0$ into two subcones $\mathcal{D}_{0,1}$ and $\mathcal{D}_{0,2}$ as follows: The automorphism $\varphi_0 \circ \tau$ is of order two and maps the cone $\mathcal{D}_0$ onto itself. The line bundle $L' := e\beta_0 L_0 + (\alpha_0 - 1)L_\infty$, which satisfies $\varphi_0 \circ \tau(L') = L'$, divides the subcone $\mathcal{D}_0$ into two subcones $\mathcal{D}_{0,1}$ and $\mathcal{D}_{0,2}$, where $\mathcal{D}_{0,1}$ is generated by L_0 and L',

$$\mathcal{D}_{0,1} = \left\{ \lambda_1 L_0 + \lambda_2 L' \,\middle|\, \lambda_1, \lambda_2 \geq 0 \right\},$$

and $\mathcal{D}_{0,2}$ is generated by L' and L_1. The subcones $\mathcal{D}_{0,1}$ and $\mathcal{D}_{0,2}$ satisfy $\varphi_0 \circ \tau(\mathcal{D}_{0,1}) = \mathcal{D}_{0,2}$ and, again by construction of τ, the Seshadri constants remain invariant. We call $\mathcal{D}_{0,1}$ the *fundamental cone* of $\mathrm{Amp}(X)$. This cone corresponds to the interval $[0, \frac{\alpha_0 - 1}{e\beta_0}]$ in $\mathcal{N}(X)$. The decomposition of the subcone $\mathcal{D}_0$ into $\mathcal{D}_{0,1}$ and $\mathcal{D}_{0,2}$ extends via φ_0^k to every subcone $\mathcal{D}_k$. After renumbering we obtain a decomposition of $\mathrm{Amp}(X)$ into subcones $\mathcal{C}_k$ with $\mathcal{C}_0 = \mathcal{D}_{0,1}$.

We now deal with the case of $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$. In this case we find that M_φ is given by

$$\begin{pmatrix} \alpha & \frac{e-1}{4}\beta \\ \beta & \alpha + \beta \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & \alpha - \frac{e-1}{4}\beta \\ \beta & -\alpha \end{pmatrix}, \quad \text{with } \alpha > 0 \text{ and } \alpha^2 + \alpha\beta - \frac{e-1}{4}\beta^2 = 1.$$

Note that we have the following bijection

$$\left\{(x, y) \in \mathbb{Z}^2 \middle| x^2 - ey^2 = 4\right\} \xrightarrow{\sim} \left\{(x, y) \in \mathbb{Z}^2 \middle| x^2 + xy - \tfrac{e-1}{4} y^2 = 1\right\}$$
$$(x, y) \mapsto (\tfrac{x-y}{2}, y).$$

By [8, Prop. 6.3.16] the set of solutions for the Pell-type equation $x^2 - ey^2 = 4$ can be expressed through a minimal solution (x_0, y_0) (we may assume $x_0 > y_0 > 0$) as follows:

$$\left\{ \pm \frac{1}{2^k} \begin{pmatrix} x_0 & ey_0 \\ y_0 & x_0 \end{pmatrix}^k \begin{pmatrix} 2 \\ 0 \end{pmatrix} \middle| \, k \in \mathbb{Z} \right\}.$$

With some calculation, the set of solutions of $\alpha^2 + \alpha\beta - \frac{e-1}{4}\beta^2 = 1$ can be determined as

$$\left\{ \pm \begin{pmatrix} \alpha_0 & \frac{e-1}{4}\beta_0 \\ \beta_0 & \alpha_0 + \beta_0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \, k \in \mathbb{Z} \right\},$$

where $(\alpha_0, \beta_0) := (\frac{x_0 - y_0}{2}, y_0)$ and, hence, the group G is generated by

$$\psi_0 := \begin{pmatrix} \alpha_0 & \frac{e-1}{4}\beta_0 \\ \beta_0 & \alpha_0 + \beta_0 \end{pmatrix} \text{ and } \sigma := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Using the exact same argument as before, we get a decomposition of the ample cone: We can express every principal polarization by $L_k := x_k L_0 + y_k L_\infty$ with $(x_k, y_k) = \psi_0^k(1, 0)$. The subcones $\mathcal{D}_k$ generated by L_k and L_{k+1} satisfy $\psi_0^k(\mathcal{D}_0) = \mathcal{D}_k$. Furthermore, $\psi_0 \circ \sigma$ divides the subcone $\mathcal{D}_0$ into two subcones $\mathcal{D}_{0,1}$ and $\mathcal{D}_{0,2}$, which are generated by L_0 and $L' := (\alpha_0 + 1)L_0 + \beta_0 L_\infty$ and, respectively, L' and L_1. In this case, the fundamental cone $\mathcal{D}_{0,1}$ corresponds to the interval $[0, \frac{\beta_0}{\alpha_0+1}]$ in $\mathcal{N}(X)$.

The considerations above prove Theorem B stated in the introduction.

We now provide some sample plots in order to illustrate the behavior of Seshadri functions. Concretely, we compute for fixed e all Seshadri curves $C = qL_0 + pL_\infty$ with $q \leq 3.000$ that are contained in $\mathcal{C}_0$. From this set of curves we derive further Seshadri curves by applying the automorphisms in G. In the pictures, the dotted lines indicate the fundamental interval from which the complete Seshadri function can be computed by Theorem B.

The values of e in Figs. 1, 2, 3 and 4 are chosen in such a way that they illustrate different kinds of behavior: In the case of $\mathbb{Z}[\sqrt{2}]$ there exist $\mathbb{Q}$-line bundles L_λ with $\varepsilon(L_\lambda) = \sqrt{L_\lambda^2} \in \mathbb{Q}$ whereas in the case of $\mathbb{Z}[\sqrt{5}]$ no such bundles exist. These line bundles generate "gaps" in the graph, because they do not give rise to a linear segment. In fact, at each of these gaps there are infinitely many linear segments which converge from both sides. In the plots for the case $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ for $e = 5$ and 33 the ample cone is not symmetric at 0 in the case of $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$.

The Seshadri function for $e = 5$ consists only of linear segments which by Theorem 2.2 are never adjacent to each other. In the case of $e = 33$ there exist line bundles with two submaximal curves, e.g., at $t = 0.37$. In fact, calculations show that there are chains of linear segments which overlap. It should also be noted that the size of the fundamental interval depends heavily on the minimal solution of $x^2 - ey^2 = 1$ or, respectively, $x^2 + xy - \frac{e-1}{4}x^2 = 1$: In the first three cases the minimal solutions are small, which leads to a small fundamental interval. However, experience with further examples has shown that the limit of the fundamental interval can be arbitrarily close to the interval limit of $\mathcal{N}(X)$.

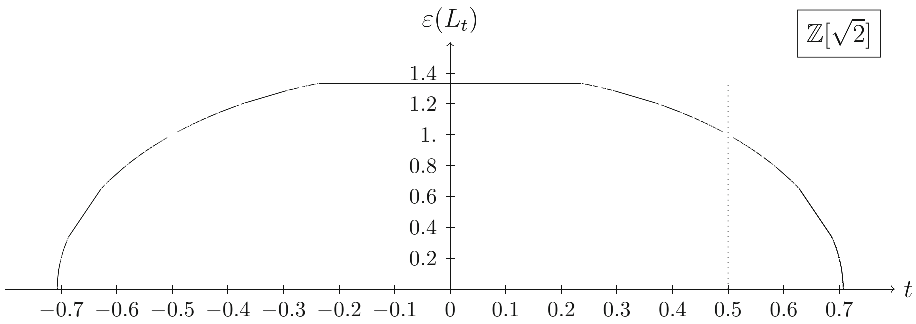

Fig. 1 The Seshadri function of an abelian surface with real multiplication in $\mathbb{Z}[\sqrt{2}]$

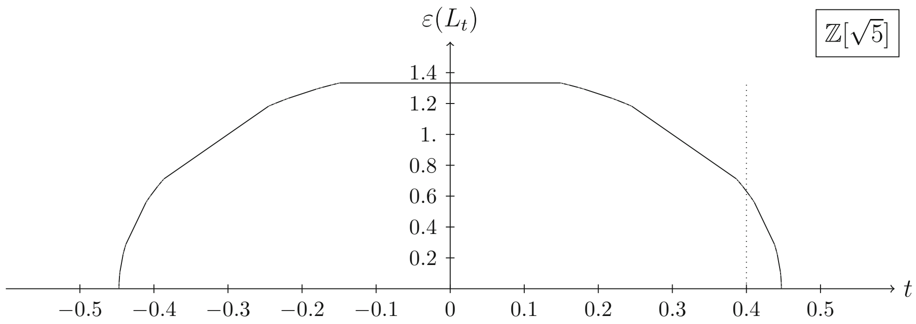

Fig. 2 The Seshadri function of an abelian surface with real multiplication in $\mathbb{Z}[\sqrt{5}]$

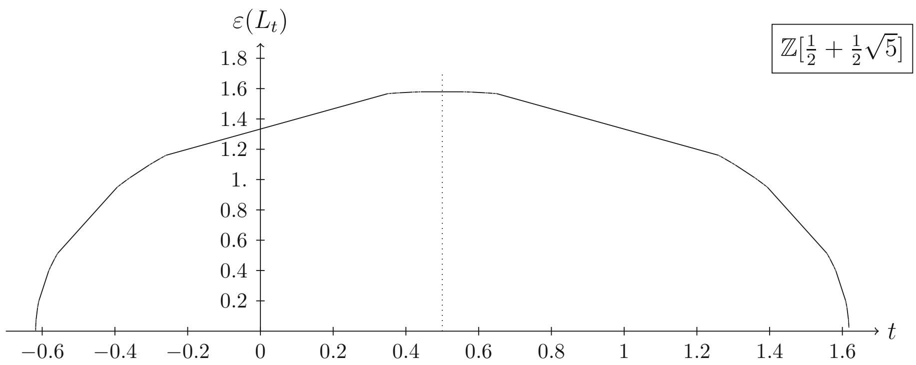

Fig. 3 The Seshadri function of an abelian surface with real multiplication in $\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{5}]$

5 Distinguishing the cases of one and two submaximal curves

In this section we derive a method that allows one to distinguish whether all line bundles on X have at most one submaximal curve or if there exists a line bundle which has 2 submaximal curves. By Theorem 2.2 we already know that there are infinitely many cases for $\mathrm{End}(X) =$

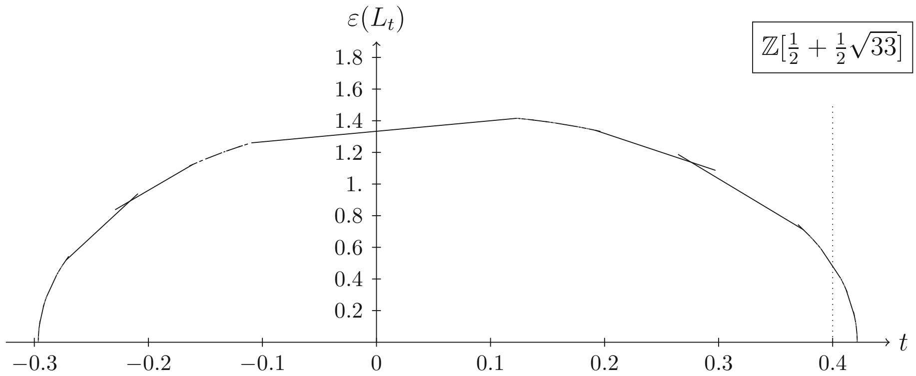

Fig. 4 The Seshadri function of an abelian surface with real multiplication in $\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{33}]$

$\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{e}]$, where every line bundle has at most one submaximal curve. We will show that the case with two submaximal curves also appears infinitely many times.

Proposition 5.1 *There exists a line bundle on X that has two submaximal curves if and only if there exist two Pell bounds π_λ and π_μ such that the following two conditions are met:*

(i) *Their submaximality intervals J_λ and J_μ intersect and one is not contained in the other.*
(ii) *There does not exist a Pell bound π_τ such that the submaximality interval J_τ contains $J_\lambda \cup J_\mu$.*

Proof Assume that there exists a line bundle L on X with two submaximal curves C_1 and C_2. The linear functions ℓ_{C_1} and ℓ_{C_2} are Pell bounds by Proposition 3.2, and their submaximality intervals I_{C_1} and I_{C_2} must intersect, because C_1 and C_2 are both L-submaximal. By Lemma 2.4 one submaximality interval can not be contained in the other. Assume that there exists a Pell bound π_τ such that $I_{C_1} \cap I_{C_2} \subset J_\tau$. Then any Pell divisor P of L_τ is submaximal on $I_{C_1} \cap I_{C_2}$, and therefore C_1 and C_2 are reducible by Lemma 2.4, a contradiction.

Suppose now that there exist two Pell bounds π_λ and π_μ such that (i) and (ii) holds. The Pell bounds yields an upper bound for the Seshadri function in $J_\lambda \cup J_\mu$: We have

$$\varepsilon(t) \le \min\{\pi_\lambda(t), \pi_\mu(t)\} < \sqrt{L_t^2} \qquad \text{for } t \in J_\lambda \cup J_\mu.$$

Let C_1 be a Seshadri curve for a line bundle L_{t_1} with $t_1 \in J_\lambda \cup J_\mu$. Due to (ii) the submaximality interval I_{C_1} of C_1 cannot cover the complete interval $J_\lambda \cup J_\mu$. Therefore, by continuity there exists a $t_2 \in (J_\lambda \cup J_\mu) \cap I_{C_1}$ such that

$$\varepsilon(L_{t_2}) \le \min\{\pi_\lambda(t_2), \pi_\mu(t_2)\} < \frac{C_1 \cdot L_{t_2}}{\mathrm{mult}_0 C_1} < \sqrt{L_{t_2}^2},$$

i.e., C_1 is submaximal for L_{t_2} but does not compute its Seshadri constant. But the Seshadri constant of L_{t_2} is computed by a curve, and thus there exists for L_{t_2} another submaximal curve C_2 that computes the Seshadri constant. It follows that L_{t_2} has two submaximal curves. □

The criterion in Proposition 5.1 provides us with a numerical method to search for line bundles with two submaximal curves: First, we search for Pell bounds whose submaximality intervals intersect. After that, one checks by using Proposition 3.12 whether there exists another Pell bound which contains both intervals. Using computer-assisted computation, this yields the following:

Proposition 5.2 *Suppose that* $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ *for a non-square integer e with* $0 < e \le 25.000$*, such that we have* $e \equiv 1$ *modulo* 4 *and e does not have a prime factor p with* $p \equiv 5$ *or* 7 *modulo* 8*. Then there exists a line bundle on X with two submaximal curves.*

Theorem 2.2 and the previous proposition suggest the following conjecture:

Conjecture 1 *Let L be any ample* $\mathbb{Q}$*-line bundle on X. Then there exists at most one irreducible curve C that is submaximal for L if and only if* $\mathrm{End}(X)$ *satisfies either*

- $\mathrm{End}(X) = \mathbb{Z}[\sqrt{e}]$ *for a non-square integer* $e > 0$*, or*
- $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e}]$ *for a non-square integer* $e > 0$*, such that* $e \equiv 1$ *modulo* 4 *and e has a prime factor p with* $p \equiv 5$ *or* 7 *modulo* 8*.*

Remark 5.3 One can show by applying well-known results on quadratic residues and binary quadratic forms (see e.g. [8, Prop. 2.2.4] and [9, Lemma 2.5]) that the following conditions are equivalent for a non-square integer e with $e \equiv 1$ modulo 4:

(i) e does not have any prime factor p with $p \equiv 5$ or 7 modulo 8.
(ii) -2 is a quadratic residue modulo e.
(iii) $e = A^2 + 8B^2$ for some $A, B \in \mathbb{N}$ with $\gcd(A, B) = 1$.

Finally, we will show that the case with two submaximal curves occurs infinitely often.

Proposition 5.4 *Let* $e_n := 1 + 8n^2$*. If* e_n *is not a perfect square, then every principally polarized abelian surface with* $\mathrm{End}(X) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{e_n}]$ *has a line bundle with two submaximal curves.*

Proof Consider the ample line bundles $L = 2nL_0 + L_\infty$ and $L' = (2n-1)L_0 + L_\infty$. The Pell solution of $x^2 - L^2y^2 = 1$ is given by $(2n+1, 1)$, and the Pell solution of $x^2 - L'^2y^2 = 1$ is $(2n-1, 1)$. Hence, the submaximality intervals of the corresponding Pell bounds $\pi_{\frac{1}{2n}}$ and $\pi_{\frac{1}{2n-1}}$ are given by

$$J_{\frac{1}{2n}} = \left(\frac{16n^3 + (2n+1)(1 - \sqrt{8n^2+1})}{8n^2(4n^2+1)}, \frac{16n^3 + (2n+1)(1 + \sqrt{8n^2+1})}{8n^2(4n^2+1)} \right)$$

and, respectively,

$$J_{\frac{1}{2n-1}} = \left(\frac{2n + (2n-1)(8n^2 - \sqrt{8n^2+1})}{32n^4 - 32n^3 + 16n^2 - 4n + 1}, \frac{2n + (2n-1)(8n^2 + \sqrt{8n^2+1})}{32n^4 - 32n^3 + 16n^2 - 4n + 1} \right).$$

Explicit computations show that both Pell bounds $\pi_{\frac{1}{2n}}$ and $\pi_{\frac{1}{2n-1}}$ are submaximal at $\frac{2}{4n-1}$, and therefore their submaximality intervals intersect.

So the first condition of Proposition 5.1 is satisfied. In order to conclude that a line bundle with two submaximal curves exists, it remains to show that there does not exist another Pell bound π_λ whose submaximality interval covers the interval $I := J_{\frac{1}{2n}} \cup J_{\frac{1}{2n-1}}$. For this, we will derive an upper bound and a lower bound for the denominator q of $\lambda = \frac{p}{q}$ which must be satisfied if the Pell bound π_λ covers I. As we will see, the upper and lower bound contradict each other and thus there cannot exist such a Pell bound.

Upper bound for q: The Pell bound π_λ has to cover both submaximality intervals, i.e., the interval

$$I = \left(\frac{16n^3 + (2n+1)(1-\sqrt{8n^2+1})}{8n^2(4n^2+1)}, \frac{2n + (2n-1)(8n^2+\sqrt{8n^2+1})}{32n^4 - 32n^3 + 16n^2 - 4n + 1} \right).$$

One can show that the length of this interval is at least $(\sqrt{2}+1)/(4n^2)$, and using Lemma 3.9 we derive the upper bound

$$q \leq \frac{4\sqrt{11}n^2}{(\sqrt{2}+1)\sqrt{8n^2+1}} \leq \frac{35}{18}n.$$

Lower bound for q: First, we observe that the unique Pell bound π_0 is not submaximal for L and L' and, thus, we may assume that $\lambda \neq 0$, i.e. $p \neq 0$. We obtain a preliminary lower bound for q by taking into account that the line bundle L_λ has to be ample, i.e.,

$$L_\lambda^2 = 2 + \frac{2p}{q} - \frac{4n^2p^2}{q^2} > 0,$$

and, thus,

$$q \geq \frac{p}{2}(\sqrt{8n^2+1} - 1) \geq \sqrt{2}\,(n-1).$$

Unfortunately, this lower bound yields no contradiction with our upper bound. However, it provides us with a method to refine the lower bound. Using the computation from Lemma 3.9, we find a maximal possible length for the submaximality interval $J_\lambda = (t_1, t_2)$ provided that $\sqrt{2}\,(n-1) \leq q \leq \frac{35}{18}n$:

$$t_2 - t_1 < \frac{2\sqrt{2 + \frac{2}{e_n - 1} + \frac{1}{k^2q^2}}}{q\sqrt{e_n}} \leq \frac{2\sqrt{2 + \frac{1}{4n^2} + \frac{1}{2\,(n-1)^2}}}{\sqrt{2}(n-1)\sqrt{1+8n^2}} < \frac{\sqrt{2 + \frac{1}{4n^2} + \frac{1}{2\,(n-1)^2}}}{2n(n-1)}.$$

This in turn, gives us an upper bound for λ, since the submaximality interval of π_λ can cover at most $t_1 - t_2$:

$$\lambda \leq \frac{16n^3 + 2n + 1 - (2n+1)\sqrt{8n^2+1}}{8n^2(4n^2+1)} + \frac{\sqrt{2 + \frac{1}{4n^2} + \frac{1}{2\,(n-1)^2}}}{2n(n-1)}.$$

It follows that $\lambda \leq \frac{1}{2n-3}$ and, therefore, the denominator q of λ must be at least $2n-3$.

This shows that for $n \geq 55$ there cannot exist a Pell bound whose submaximality interval covers $J_{\frac{1}{2n}}$ and $J_{\frac{1}{2n-1}}$. Thus, the assertion follows for $n \geq 55$ from Proposition 5.1. The explicit computations from Proposition 5.2 cover the remaining cases for $n \leq 54$. □

Remark 5.5 The case where e_n is a square number, i.e., $e_n = 1 + 8n^2 = r^2$ for an integer $r \in \mathbb{N}$, is equivalent to the case where (r, n) is a solution for the Pell equation $x^2 - 8y^2 = 1$, and hence there are infinitely many n such that e_n is not a square number.

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