Non-negatively curved GKM orbifolds

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Abstract

In this paper we study non-negatively curved and rationally elliptic GKM_4 manifolds and orbifolds. We show that their rational cohomology rings are isomorphic to the rational cohomology of certain model orbifolds. These models are quotients of isometric actions of finite groups on non-negatively curved torus orbifolds. Moreover, we give a simplified proof of a characterisation of products of simplices among orbit spaces of locally standard torus manifolds. This characterisation was originally proved in Wiemeler (J Lond Math Soc 91(3): 667–692, 2015) and was used there to obtain a classification of non-negatively curved torus manifolds.

Keywords Non-negative curvature · Rationally elliptic orbifolds · GKM orbifolds

1 Introduction

The classification of Riemannian manifolds with positive or non-negative sectional curvature is one of the most prominent open problems in differential geometry. Many authors have investigated it under the additional assumption of an isometric Lie group action, e.g. [1, 8, 9, 11–13, 16, 19, 29, 32, 33]. In this paper we continue our investigation [17] of isometric torus actions of GKM type on Riemannian manifolds with sectional curvature bounded from below.

The GKM condition—introduced in [14]—requires that the one-skeleton M_1 of the action, i.e. the union of all orbits of dimension less than or equal to one, is of a particularly simple type. Namely, it is required that M_1 is a union of two-dimensional spheres, such that the T -action restricts to a cohomogeneity one action on each two-sphere. The orbit space $\Gamma = M_1/T$ is then an n -valent graph, where $2n = \dim M$. The isotropy representations at the fixed points induce a labeling of the edges of the graph, as explained in Sect. 2.

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From this labelled graph one can compute the equivariant and non-equivariant rational cohomology rings of a GKM manifold or orbifold. This is made explicit in the GKM Theorem [14], see Theorem 2.8 below.

Similarly to the GKM condition, we say that an action is GKM_k if for all $0 \leq k' < k$ the union of the orbits of dimension at most k' is a union of $2k'$ -dimensional invariant submanifolds. Their GKM graphs are then the k' -dimensional faces of the GKM graph of M .

In [17] we showed that a positively curved Riemannian manifold admitting an isometric GKM_3 torus action has the same real cohomology ring as a compact rank one symmetric space. The assumption of positive curvature forces, by the classification of 4-dimensional positively curved T^2 -manifolds [16], the two-dimensional faces of the GKM graph to be just biangles or triangles – this condition turned out to be a severe enough restriction to classify all occurring graphs.

Considering the same setting for non-negatively instead of positively curved manifolds, we observe that now also quadrangles appear as two-dimensional faces [19, 29], which increases the possibilities for the GKM graphs greatly. Still, we are able to show the following theorem on the structure of the GKM graph (without the labelling):

Theorem 1.1 *Let O be a orientable GKM_4 orbifold with an invariant metric of non-negative curvature. Then the GKM graph of O is finitely covered by the vertex-edge graph of a finite product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$.*

In the above theorem and later on, Δ^n denotes an n -dimensional simplex and Σ^m the orbit space of the linear effective action of the m -dimensional torus on S^{2m} .

The stronger GKM_4 condition implies that the GKM graph of O has three-dimensional faces. The restriction on the two-dimensional faces of the graph imply that the combinatorial types of the three-dimensional faces is also very restricted. They are all combinatorially equivalent to one of the following: I^3 , Σ^3 , Δ^3 , $\Sigma^2 \times I$, $\Delta^2 \times I$.

Using these restrictions, we show that the combinatorial type of a neighborhood of a vertex in the GKM graph Γ of a non-negatively curved GKM_4 manifold is the same as that of a neighborhood of a vertex in the vertex edge graph $\tilde{\Gamma}$ of a finite product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$. Extending this local result to all of $\tilde{\Gamma}$ then yields the covering described in Theorem 1.1.

Since the number of the vertices in the GKM graph is equal to the total Betti number of the orbifold we get the following gap phenomenon:

Corollary 1.2 *Let O of dimension $2n$ be as in the previous theorem, then the total Betti number $b(O) = \sum_i b_i(O)$ is either smaller than or equal to $2^{n-2} \cdot 3$ or equal to 2^n . The latter case appears if and only if the GKM graph of O is combinatorially equivalent to the vertex-edge graph of I^n .*

Note that the upper bound on the total Betti number is sharp. Therefore we are verifying a conjecture of Gromov [15] in the special case of non-negatively curved GKM_4 -manifolds. He conjectured that for general non-negatively curved manifolds of dimension d the total Betti number is bounded from above by 2^d .

By the GKM Theorem, the GKM graph determines the rational cohomology of a GKM orbifold. Therefore, if we can show that all GKM graphs appearing in the above theorem can be realised as GKM graphs of certain model GKM orbifolds, any non-negatively curved GKM orbifold will have rational cohomology isomorphic to the rational cohomology of one of the model orbifolds.

To construct the models, we have to show that GKM_4 graphs with underlying graph equal to the vertex-edge graph of $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$ extend — in the sense of Kuroki [23] — to

GKM_n graphs, i.e. to GKM graphs of torus orbifolds over $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$. This reasoning then leads to

Theorem 1.3 *If O is a GKM₄ orbifold such that the GKM graph of O is the vertex-edge graph of a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$, then the rational cohomology of O is isomorphic to the rational cohomology of a non-negatively curved torus orbifold.*

To get the models in the general case, we show that the deck transformation group G of the covering $\tilde{\Gamma} \rightarrow \Gamma$ from Theorem 1.1 acts on the model torus orbifold $\tilde{O}$ associated to the extended GKM graph $\tilde{\Gamma}$ as in Theorem 1.3. The quotient $\tilde{O}/G$ is a GKM₄ orbifold realising the GKM graph Γ . Hence we get

Theorem 1.4 *Let O be a non-negatively curved GKM₄ orbifold. Then there is a non-negatively curved torus orbifold $\tilde{O}$ and an isometric action of a finite group G on $\tilde{O}$ such that*

$$H^*(O; \mathbb{Q}) \cong H^*(\tilde{O}/G; \mathbb{Q}).$$

Moreover, if O is a manifold then $\tilde{O}$ is a simply connected manifold.

In the literature it is often assumed that a GKM manifold has an invariant almost complex structure. This assumption results in the fact that in this case the weights of the GKM graph have preferred signs. We consider this special case in Sect. 7. We show that in the situation of Theorem 1.1 this implies that the covering of Γ is trivial, and that the covering graph is the vertex edge graph of $\prod_i \Delta^{n_i}$. Moreover, the torus manifold corresponding to the extended GKM_n graph will also admit an invariant almost complex structure. Torus manifolds over $\prod_i \Delta^{n_i}$ admitting an invariant almost complex structure were classified in [6]. They are all diffeomorphic to so-called generalised Bott manifolds. These manifolds X are total spaces of iterated $\mathbb{C}P^{n_i}$ -bundles

$$X = X_k \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = \{pt\},$$

where each X_i is the total space of the projectivisation of a Whitney sum of $n_i + 1$ complex line bundles over X_{i-1} . Their cohomology rings can be easily determined from the Chern classes of the involved line bundles. Indeed, if $P(E)$ is the projectivisation of a complex vector bundle E of dimension n over a base space B , then we have

$$H^*(P(E); \mathbb{Z}) \cong H^*(B)[x]/(f(x)),$$

where x has degree two and $f(x) = \sum_{i=0}^n c_i(E)x^{n+1-i}$. Here $c_i(E)$ denotes the i -th Chern class of E . By iterating this formula one gets the cohomology rings of all generalised Bott manifolds. By the above discussion we get

Theorem 1.5 *Let M be a non-negatively curved GKM₄ manifold which admits an invariant almost complex structure. Then the rational cohomology ring of M is isomorphic to the rational cohomology ring of a generalised Bott manifold.*

In [32] a classification of non-negatively curved simply connected torus manifolds was given. A crucial step in the proof was to show that the orbit space of such a torus manifold is combinatorially equivalent to a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$. The proof of this intermediate result was very long and highly technical. With the methods of the paper at hand we can give a short conceptual proof of this result.

The Bott conjecture asks if any simply connected non-negatively curved manifold is rationally elliptic. Therefore it is natural to consider the question if the above theorems also hold for rationally elliptic GKM₄ manifolds or orbifolds.

By [11], the two-dimensional faces of the corresponding GKM graphs have at most four vertices. Moreover, since our arguments are purely graph-theoretic we conclude that all the above theorems also hold for rationally elliptic GKM_4 orbifolds instead of non-negatively curved ones.

The remaining sections of this paper are structured as follows. In Sect. 2 we gather background material about GKM manifolds and orbifolds as well as on torus manifolds and orbifolds. Then in Sect. 3 we prove Theorem 1.1.

In Sect. 4 we show that a GKM_4 graph with underlying graph the vertex edge graph of a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$ always extends to a GKM_n graph with $n = \sum_i n_i + \sum_i m_i$. This is then used in Sect. 5 to prove Theorems 1.3 and 1.4.

In Sect. 6 we give an example of a non-negatively curved GKM_4 manifold which does not have the same rational cohomology as a torus manifold. This shows that the group G from Theorem 1.4 cannot be omitted.

In Sect. 7 we consider GKM manifolds with invariant almost complex structure and prove Theorem 1.5. Moreover, in the last Sect. 8 we give a short proof of the “big lemma” which is used in the classification of non-negatively curved torus manifolds.

Throughout, cohomology will be taken with rational coefficients.

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2 Preliminaries

2.1 GKM manifolds

We begin with a review of GKM theory for manifolds; below we will describe the changes that are necessary to treat orbifolds as well.

Consider an effective action of a compact torus T on a smooth, compact, orientable manifold M of dimension $2n$ with finite fixed point set, such that the one-skeleton

$$M_1 = \{p \in M \mid \dim Tp \leq 1\}$$

of the action is a union of invariant 2-spheres. Given that the fixed point set is finite, the second condition is equivalent to the condition that for every fixed point, the weights of the isotropy representation are pairwise linearly independent. To such an action one associates its *GKM graph*: its vertices are given by the fixed points of the action; to any invariant 2-sphere – which contains exactly two fixed points – one associates an edge connecting the corresponding vertices. Finally, any edge is labeled with the weight of the isotropy representation in any of the two fixed points which corresponds to the two-dimensional submodule given by the tangent space of this sphere. These labels are linear forms on the Lie algebra $\mathfrak{t}$ of T and are well-defined up to sign.

We need to abstract from group actions and consider the occurring graphs detached from any geometric situation, as in [18] or [3].

For a graph Γ we denote its set of vertices by $V(\Gamma)$ and its set of edges by $E(\Gamma)$. We consider only graphs with finite vertex and edge set, but we allow multiple edges between vertices. Edges are oriented; for $e \in E(\Gamma)$ we denote by $i(e)$ its initial vertex and by $t(e)$ its terminal vertex. The edge e , with opposite orientation, is denoted $\bar{e}$. For a vertex $v \in V(\Gamma)$ we denote the set of edges starting at v by E_v .

Definition 2.1 Let Γ be a graph. Then a *connection* on Γ is a collection of bijective maps $\nabla_e : E_{i(e)} \rightarrow E_{t(e)}$, for $e \in E(\Gamma)$, such that

1. $\nabla_e(e) = \bar{e}$ and
2. $\nabla_{\bar{e}} = \nabla_e^{-1}$, for all $e \in E(\Gamma)$.

Definition 2.2 Let $k \geq 2$. A GKM_k graph (GKM graph for $k = 2$) (Γ, α) consists of an n -valent connected graph Γ and a map $\alpha : E(\Gamma) \rightarrow H^2(BT^m)/\{\pm 1\}$ such that

1. If $e_1, \dots, e_k$ are edges of Γ which meet in a vertex v of Γ then the $\hat{\alpha}(e_i)$, $i = 1, \dots, k$, are linearly independent. Here $\hat{\alpha}(e_i) \in H^2(BT^m)$ denotes a lift of $\alpha(e_i)$. Note that this property is independent of the choice of $\hat{\alpha}(e_i)$.
2. There exists a connection ∇ on Γ such that for any two edges e_1, e_2 which meet in a vertex v there are $p, q \in \mathbb{Q}$ such that

$$\hat{\alpha}(\nabla_{e_1}(e_2)) = p\hat{\alpha}(e_2) + q\hat{\alpha}(e_1). \quad (2.1)$$

Note here that p, q are determined up to sign by α . Moreover, if we fix a sign for $\hat{\alpha}(\nabla_{e_1}(e_2))$, $\hat{\alpha}(e_2)$, and $\hat{\alpha}(e_1)$, then p, q are uniquely determined.

3. For each edge e we have $\alpha(e) = \alpha(\bar{e})$.

Definition 2.2 is slightly more general than usual, as p and q are allowed to be rational numbers. The reason will become clear below, when we consider orbifolds. The construction of the graph described above always results in GKM graphs:

Proposition 2.3 For any action of a compact torus T on a smooth, compact, orientable manifold M of dimension $2n$ with finite fixed point set, and whose one-skeleton M_1 is the union of invariant 2-spheres, the graph associated to the action in the way prescribed above admits a connection for which Eq. (2.1) holds, with $p = \pm 1$ and q an integer. In particular the graph is a GKM graph.

Moreover, if the weights at every vertex are 3-independent, i.e. if any three weights at every vertex are linearly independent, then the connection is unique.

Proof Let N be one of the invariant 2-spheres and T_N the principal isotropy group of the T -action on N . Moreover, let x_1, x_2 be the two T -fixed points in N . Then we have two T -representations $T_{x_1}M \otimes \mathbb{C}$ and $T_{x_2}M \otimes \mathbb{C}$ on the complexified tangent spaces at these fixed points. Let

$$T_{x_i}M \otimes \mathbb{C} = \bigoplus_j W_{ij} \quad W_{ij} = \{v \in T_{x_i}M \otimes \mathbb{C}; \quad tv = \alpha_{ij}(t)v \text{ for all } t \in T\},$$

be the decomposition of $T_{x_i}M \otimes \mathbb{C}$ into weight spaces. Here the α_{ij} are some homomorphisms $T \rightarrow S^1$. By the GKM condition each W_{ij} has complex dimension one. Moreover, the derivatives of the α_{ij} agree –up to sign– with the weights of the edges starting in x_i .

Since the T_N -action on N is trivial, it follows from the proof of Proposition 2.2 of [30] that $TM \otimes \mathbb{C}|_N$ splits T_N -equivariantly as

$$TM \otimes \mathbb{C}|_N \cong \bigoplus_{i=1}^k V_i \otimes E_i,$$

where the V_i are complex irreducible T_N -representations and the E_i are complex vector bundles with trivial T_N -action. Therefore the isomorphism type of the T_N -representation

on $T_x M \otimes \mathbb{C}$ is independent of $x \in N$. In particular, there is an isomorphism of T_N -representations

$$T_{x_1} M \otimes \mathbb{C} \cong T_{x_2} M \otimes \mathbb{C}.$$

So the homomorphisms α_{1j} and α_{2j} agree after restriction to T_N (and reordering).

Because T_N has codimension one in T there is a homomorphism $\alpha : T \rightarrow S^1$ —unique up to complex conjugation on $S^1 \subset \mathbb{C}$ —such that $\ker \alpha = T_N$. By the definition of T_N , this α —or its conjugate $\bar{\alpha}$ —induces the T -action on $T_{x_i} N \cong \mathbb{C}$, $i = 1, 2$. Moreover, every homomorphism $T \rightarrow S^1$ which is trivial on T_N factors through α .

We now apply this to $\alpha_{1j} \cdot \bar{\alpha}_{2j}$, where $\cdot$ and $\bar{}$ denote multiplication and complex conjugation in S^1 , respectively. So we get some factorisation

$$\alpha_{1j} \cdot \bar{\alpha}_{2j} = \alpha^{k_j} \quad (2.2)$$

Forming the derivative of this expression leads to equation (2.1) with $p = \pm 1$ and $q = \pm k_j \in \mathbb{Z}$. Moreover, in case the weights at x_2 are 3-independent, for given α_{1j} there is at most one α_{2j} for which equation (2.2) holds. So our claim follows. $\square$

Remark 2.4 An alternative proof of this proposition was given in [18, p. 6]; there, the fact that $q \in \mathbb{Z}$ followed from the Atiyah–Bott–Berline–Vergne localization theorem.

Remark 2.5 For GKM_3 -graphs the connection ∇ in Condition 2 of 2.2 is unique.

Remark 2.6 A T -invariant almost complex structure on a manifold M allows to speak about weights of the isotropy representation that are well-defined elements of $\mathfrak{t}^*$, not only up to sign. On the level of graphs we say that a GKM graph admits an invariant almost complex structure if there is a lift $\hat{\alpha} : E(\Gamma) \rightarrow H^2(BT^m)$ of α such that (2.1) holds with $p = 1$ and q an integer and $\hat{\alpha}(e) = -\hat{\alpha}(\bar{e})$, for all edges e .

The relevance of this type of actions is grounded in the fact that for manifolds M with vanishing odd-dimensional (rational) cohomology, the (equivariant) cohomology is determined by the associated GKM graph. We define

Definition 2.7 We say that an action of a compact torus T on a smooth, compact, orientable manifold M is of type GKM_k (simply GKM for $k = 2$) if $H^{\text{odd}}(M) = 0$, the fixed point set is finite, and for every fixed point any k weights of the isotropy representation are linearly independent.

Theorem 2.8 [14, Theorem 7.2] For a T -action of GKM type with fixed points $p_1, \dots, p_r$, the inclusion $M^T \rightarrow M$ induces an injection

$$H_T^*(M) \rightarrow H_T^*(M^T) = \bigoplus_{i=1}^r H^*(BT)$$

whose image is given by the set of tuples $(f_i) \in \bigoplus_{i=1}^r H^*(BT)$ such that $f_i - f_j = \alpha\beta$ with some $\beta \in H^*(BT)$ whenever p_i and p_j are joined by an edge with label α .

Obviously, the image in particular depends only on the labelled graph Γ , so it is sensible to use the notation $H_\Gamma^*(\Gamma)$ for the $H^*(BT)$ -subalgebra of $\bigoplus_{v \in V(\Gamma)} H^*(BT)$ defined in the theorem above.

It is well-known that the vanishing of the odd rational cohomology implies that the canonical map $H_T^*(M) \rightarrow H^*(M)$ is surjective. In particular, the rational cohomology ring of M is determined by the GKM graph of the action.

2.2 GKM orbifolds

The fact that GKM-theory works equally well for torus actions on orbifolds was already remarked in [18, Sect. 1.2]. In this paper we consider general (orientable) orbifolds O , which are given by orbifold atlases on a topological space, consisting of good local charts $(\tilde{U}_p, \Gamma_p)$ for any point $p \in O$. For the precise definition, and all basics on orbifolds and Lie group actions we use Sect. 2 of [11] as a general reference. Other sources of background material on orbifolds are [20–22, 25, 34]. But note that the notation differs slightly from source to source.

We consider an action of a torus on a compact, orientable orbifold O , in the sense of [11, Definition 2.10]. In [11] it is shown that orbits, as well as components of fixed point sets O^H , where $H \subset T$ is a connected Lie subgroup, are strong suborbifolds of O . Moreover, for any $p \in O$, with good local chart $\pi : (\tilde{U}_p, \Gamma_p) \rightarrow U_p$ there is an extension $\tilde{T}_p$ of T_p by Γ_p , acting on $\tilde{U}_p$. The $\tilde{T}_p$ -action fixes the single point $\tilde{p}$ in the preimage $\pi^{-1}(p)$. We thus obtain a well-defined isotropy representation of $\tilde{T}_p$ on $T_p \tilde{U}_p$. Its restriction to the identity component $\tilde{T}_p^o$ of $\tilde{T}_p$ has well-defined weights α , which we consider, via the projection $\tilde{T}_p^o \rightarrow T_p$, as elements in $\mathfrak{t}_p^*/\{\pm 1\}$.

With this definition of weights, Definition 2.7 applies to torus actions on orbifolds equally well, and we can speak about torus actions on orbifolds of type GKM_k . This leads to the following definition.

Definition 2.9 An action of a compact torus T on a smooth, compact, orientable orbifold O is of type GKM_k (simply GKM for $k = 2$) if $H^{\text{odd}}(O) = 0$, the fixed point set is finite, and for every fixed point any k weights of the isotropy representation are linearly independent.

To any such action we can associate an n -valent graph Γ as in the case of manifolds, because any non-trivial torus action on a two-dimensional compact, orientable orbifold with a fixed point has exactly two fixed points, see [11, Lemma 3.9]. For the labelling, we rescale the weights as follows: for a weight α at a fixed point p , the intersection of $\mathbb{R}\alpha$ with the integer lattice in $\mathfrak{t}^*$ is isomorphic to $\mathbb{Z}$. We let α' be a generator of this group, and k be the number of components of the principal isotropy group of the T -action on the 2-sphere to which the weight space of α is tangent. We define $\beta = k\alpha'$.

We remark that the factor k is irrelevant for what follows: we include it in order for the GKM graph to encode the full isotropy groups. Considering β as a homomorphism $T \rightarrow S^1$, its kernel is precisely the principal isotropy group of the corresponding 2-sphere.

In order to construct a connection on Γ , we now restrict to actions of type GKM_3 . Let v be a vertex of the graph, corresponding to the fixed point $p \in O$, and e an edge with $i(e) = p$, with label $\alpha \in \mathfrak{t}^*/\{\pm 1\}$. Let q be the fixed point corresponding to $t(e)$. For any other edge e' at v , with weight β , we consider the connected subgroup $H \subset T$ with Lie algebra $\ker \alpha \cap \ker \beta$. By the GKM_3 -condition and the slice theorem for actions on orbifolds [11, Theorem 2.18] the connected component of O^H is a four-dimensional strong suborbifold of O . It contains q , and there exists an edge f with $i(f) = t(e)$, with weight γ , such that $\ker \gamma \cap \ker \alpha = \mathfrak{h}$. We define a connection on Γ by $\nabla_e e' := f$. By construction, γ is a (rational) linear combination of α and β , so that Equation (2.1) holds.

Also Theorem 2.8 holds true for GKM actions on orbifolds. This was (for torus orbifolds) already observed in [11, Theorem 4.2].

2.3 Torus manifolds and orbifolds

Here we gather the facts we need to know about torus manifolds and torus orbifolds. General references for the constructions used here are [4, 5, 7, 11, 27].

We start with a general construction of such manifolds and orbifolds. An n -dimensional manifold with corners P is called nice if at each vertex of P there meet exactly n facets of P , that is, exactly n codimension-one faces. The faces of P ordered by inclusion form a poset $\mathcal{P}(P)$, the so-called face poset of P . We also denote by $\mathcal{F} = \mathcal{F}(P)$ the set of facets of P and let $m = |\mathcal{F}|$.

Now let P be a nice manifold with corners with only contractible faces, and assume that there is a map $\lambda : \mathcal{F} \rightarrow \mathbb{Z}^n$ such that for every vertex v of P ,

$$\lambda(F_1), \dots, \lambda(F_n)$$

are linearly independent, where the F_i are the facets of P which meet in v .

Then we can construct a torus orbifold, i.e. an orientable $2n$ -dimensional orbifold O with an action of the n -dimensional torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$ with $O^{T^n} \neq \emptyset$, such that the orbit space of the T^n action on O is homeomorphic to P . The orbifold O is defined as

$$O = (P \times \mathbb{R}^n / \mathbb{Z}^n) / \sim,$$

where $(x, v) \sim (x', v')$ if and only if $x = x'$ and $v - v'$ is contained in the $\mathbb{R}$ -span of all the $\lambda(F)$ with $x \in F$. Here T^n acts on the second factor of O . Note that replacing the $\lambda(F)$ by nonzero multiples does not change the associated orbifold O .

If for every vertex v the $\lambda(F_i)$ of the facets F_i which meet at v form a basis of $\mathbb{Z}^n$, then O is a manifold and the T^n -action is locally standard, i.e., locally modelled on effective n -dimensional complex representations of T^n .

The preimages of the facets of P under the orbit map are invariant suborbifolds of codimension two in O . Therefore their equivariant Poincaré duals $v_1, \dots, v_m \in H_T^2(O; \mathbb{Q})$ are defined. Moreover, these Poincaré duals form a basis of $H_T^2(O; \mathbb{Q})$ because the faces of P are contractible (see [27] and [28]).

The contractibility of the faces of P also implies that the rational cohomology of a torus orbifold O as above is concentrated in even degrees.

Equivalently to giving the labels λ of the facets, one can also define O by a labeling $\hat{\alpha}$ of the edges of O in such a way that for edges $e_1, \dots, e_n$ meeting a vertex, $\hat{\alpha}(e_1), \dots, \hat{\alpha}(e_n)$ is the basis of $\mathbb{Q}^{n*}$ dual to $\lambda(F_1), \dots, \lambda(F_n) \in \mathbb{Z}^n \otimes \mathbb{Q}$. In this way the vertex-edge graph of P becomes a so-called torus graph (see [26]), i.e. a GKM graph of a torus manifold or orbifold.

Similar to the definition of the torus orbifold O associated to the pair (P, λ) one can associate a moment angle manifold of dimension $n + m$ to P . This goes as follows.

Denote the facets of P by $F_1, \dots, F_m$ and for each $i = 1, \dots, m$ let S_i^1 be a copy of the circle group. Then define

$$Z_P = (P \times (\prod_{i=1}^m S_i^1)) / \sim,$$

where $(x, t) \sim (x', t')$ if and only if $x = x'$ and

$$t' t^{-1} \in \prod_{i: x \in F_i} S_i^1 \subset \prod_{i=1}^m S_i^1.$$

There is an action of $T^m = \prod_{i=1}^m S_i^1$ on Z_P induced by multiplication on the second factor. Moreover the torus orbifold O from above is the quotient of the action of the kernel of

a homomorphism $\varphi : T^m \rightarrow T^n = \mathbb{R}^n / \mathbb{Z}^n$. Here φ is defined by the condition that its restriction to S_i^1 induces an isomorphism $S_i^1 \rightarrow \mathbb{R}\lambda(F_i) / (\mathbb{Z}^n \cap \mathbb{R}\lambda(F_i))$.

Note that if P is a product of simplices Δ^{n_i} and quotients $\Sigma^{m_i} = S^{2m_i} / T^{m_i}$, then Z_P is a product of spheres. We can equip this product with the product metric of the round spheres. If we do so, T^m is identified with a maximal torus of the isometry group of Z_P .

Example 2.10 At the end of this section we give examples of torus orbifolds. Let $P = \Delta^2$ be a triangle. Then we have $n = 2$ and $m = 3$. Denote by F_1, F_2, F_3 the facets of P . Moreover, let

$$\lambda(F_1) = (1, 0), \quad \lambda(F_2) = (0, 1), \quad \lambda(F_3) = (\alpha, \beta)$$

with $\alpha, \beta \in \mathbb{Z} - \{0\}$. Then by the above discussion the pair (P, λ) defines a torus orbifold O . Note that O is a torus manifold if and only if $|\alpha| = |\beta| = 1$.

The moment angle manifold Z_P associated to P is $S^5 \subset \mathbb{C}^3$ with a linear $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$ -action. The map $Z_P \rightarrow O$ constructed above is the orbit map for the action of the subtorus of T^3 whose Lie-algebra is generated by $(-\alpha, -\beta, 1)$. Therefore O is a so-called weighted projective space.

3 Coverings of GKM graphs

In this section we construct a covering of a GKM_k graph, $k \geq 4$, with small three-dimensional faces by the vertex edge graph of a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$. We start with the definition of what we mean by the faces of a graph.

Definition 3.1 Let Γ be a graph with a connection ∇ . Then an l -dimensional face of Γ is a connected l -valent ∇ -invariant subgraph of Γ .

Lemma 3.2 Let Γ be a GKM_k -graph, where $k \geq 3$. Then for any vertex v of Γ and any edges $e_1, \dots, e_l$ that meet at v , where $1 \leq l \leq k - 1$, there exists a unique l -dimensional face of Γ that contains $e_1, \dots, e_l$.

Proof First recall that by Remark 2.5 there is a unique connection on Γ which is compatible with the weights. Therefore it makes sense to speak about the faces of a GKM_k -graph, $k \geq 3$.

Let V be the (l -dimensional) span of the $\alpha(e_i)$ in $H^2(BT^m)$. Consider the subgraph of Γ that consists of those edges whose labeling is contained in V , and let $\tilde{\Gamma}$ be its connected component of v . We claim that this subgraph is ∇ -invariant and l -valent.

As Γ is GKM_k , with $k > l$, the only edges at v contained in $\tilde{\Gamma}$ are $e_1, \dots, e_l$. Moreover, whenever w is an l -valent vertex of $\tilde{\Gamma}$ and $e \in E(\Gamma)$ with $i(e) = w$, then also $t(e)$ is l -valent. In fact, (2.1) shows that for any edge e' at w in $E(\tilde{\Gamma})$, also $\nabla_e e'$ is an edge of $E(\tilde{\Gamma})$, and the GKM_k property of Γ shows that there is no further edge at $t(e)$ contained in $\tilde{\Gamma}$. $\square$

A simply connected space X is called rationally elliptic if it has finite dimensional rational homotopy and finite dimensional rational cohomology, i.e.,

$$\sum_{i \geq 2} \dim \pi_i(X) \otimes \mathbb{Q} < \infty \quad \text{and} \quad \sum_{i \geq 2} \dim H^i(X) \otimes \mathbb{Q} < \infty. \quad (3.1)$$

In this case the minimal Sullivan model of X is elliptic. Moreover, for simply connected spaces the other implication also holds. However, there are non-simply connected spaces whose minimal Sullivan model is elliptic which do not satisfy the conditions (3.1).

Lemma 3.3 *Let O be a GKM_3 manifold or orbifold which admits an invariant metric of non-negative curvature or whose minimal Sullivan model is elliptic. Then each two-dimensional face of the GKM graph of O has at most four vertices.*

Proof The proof is essentially the same as the proof of Lemma 4.2 in [32]. In the non-negatively curved case it was first discussed in [12]. It has been translated to the orbifold setting in [11]. Here we repeat it for the sake of completeness.

The two-dimensional faces of the GKM graph of O are GKM graphs of four-dimensional invariant totally geodesic suborbifolds of O . These submanifolds are fixed point components of codimension-two subtori of T . They are non-negatively curved if O is non-negatively curved. Let O' be one of these suborbifolds. Then O'/T is a two-dimensional non-negatively curved Alexandrov space with totally geodesic boundary, such that the points in $(O')^T$ correspond to corners of the orbit space, i.e. points on the boundary whose space of directions has diameter $\pi/2$. Let $a_0, \dots, a_{k-1}$ be these corners such that for $i \in \mathbb{Z}/k\mathbb{Z}$, a_i is connected to a_{i+1} by a totally geodesic arc which is contained in the boundary. For $i \neq 0$ choose geodesics γ_i from a_0 to a_i , such that γ_1 and γ_{k-1} are part of the boundary.

Then by Toponogov's Theorem the sum of angles in each of the triangles spanned by γ_i , γ_{i+1} and the part of the boundary between a_i and a_{i+1} is at least π . Summing over all these triangles we get the inequality

$$\pi(k-2) \leq \frac{\pi}{2}k.$$

Hence the claim follows in this case.

Now assume that the minimal Sullivan model of O is elliptic. Then, by [2], the minimal model $(\Lambda V, d)$ of O' is also elliptic. Since $\chi(O'^T) = \chi(O')$, the number of vertices in the GKM graph of O' is equal to the Euler characteristic of O' . Moreover, since $H^*(O)$ is evenly graded, $H^{\text{odd}}(O') = 0$ by localization in equivariant cohomology. In particular, $H^1(O') = 0$.

Hence, by [10, Theorem 32.6 (ii)] and [10, Proposition 12.2], we have

$$4 \geq 2 \dim V^2 = 2b_2(O').$$

Therefore we have $\chi(O') \leq 4$ and there are at most four vertices in the GKM graph of O' . $\square$

Lemma 3.4 *Let O be a orientable GKM_4 orbifold such that all two-dimensional faces of the GKM graph of O have at most four vertices. Then each three-dimensional face of the GKM graph of M has one of the following combinatorial types: Δ^3 , Σ^3 , $\Delta^2 \times I$, $\Sigma^2 \times I$, I^3 . Moreover, the face structure of the three-dimensional faces induced by the connection is the natural one.*

Proof First note that the three-dimensional faces of the GKM graph of O are GKM graphs of six-dimensional torus orbifolds $N_1, \dots, N_k$.

Note that since O is orientable, all N_i (and therefore all the orbit spaces N_i/T) are orientable orbifolds without (and with, respectively) boundary. Moreover, because the cohomology of O (and therefore that of the N_i) is concentrated in even degrees, the orbit space N_i/T and all its faces are acyclic over the rationals. Because two-dimensional orbifolds are homeomorphic to surfaces it follows from the classification of surfaces that the facets, i.e. 2-dimensional faces, of N_i/T are homeomorphic to discs D^2 . Moreover, by Lemma 3.2, any two edges of the GKM graph of N_i meeting in a vertex span a unique 2-dimensional face of N_i/T .

There are the following cases:

Fig. 1 These graphs do not occur

1. There is a facet F_1 of N_i/T , which has two vertices. And, there is another facet F_2 of N_i/T , which intersects with F_1 in an edge and has
 - (a) two vertices, or
 - (b) three vertices, or
 - (c) four vertices.
2. There is a facet F_1 of N_i/T , which has three vertices. And, there is another facet F_2 of N_i/T , which intersects with F_1 in an edge and has
 - (a) three vertices, or
 - (b) four vertices.
3. All facets of N_i/T have four vertices.

In case 1.(a), it is clear that $P = N_i/T$ is combinatorially equivalent to Σ^3 . Moreover, in case 1.(c), using the 3-valence of the graph of N_i one easily sees that P is of type $\Sigma^2 \times I$.

Next assume that F_2 is of type Δ^2 , that is, we are in case 1.(b). Let v_0 be the vertex of F_2 which does not belong to F_1 and v_1 and v_2 be the other vertices. Then one sees by considering the faces which meet at v_1 that the two edges v_0v_1 and v_0v_2 belong to two different faces which both contain all three vertices, as in the first graph in Fig. 1. But any two edges at v_0 must span a unique face, a contradiction.

Alternatively, the following argument also leads to a contradiction. By the 3-valence of the graph there is a third edge e starting from v_0 . Since it is contained in a two-dimensional face of N_i/T , the end point of e must be v_1 or v_2 (or must be connected via one edge to one of v_1, v_2). But this cannot happen because of the 3-valence of the graph (at v_1 and v_2).

Hence the case 1.(b) does not occur.

We now assume that F_1 is of type Δ^2 . That is, we are in one of the cases 2.(a) or 2.(b).

If all other facets of P which have an edge with F_1 in common are also of type Δ^2 , then P must be of type Δ^3 . If all faces which have an edge with F_1 in common are of type I^2 , then P is of type $\Delta^2 \times I$.

Therefore we have to exclude the case that there is a face F_2 of type Δ^2 and a face F_3 of type I^2 such that $F_1 \cap F_2 \cap F_3$ is a vertex, as in the second graph in Fig. 1.

In this case the third face F_4 which has an edge with F_1 in common must be of type I^2 . Using the 3-valence of the graph one now gets a contradiction in a similar way as in case 1.b).

Fig. 2 Small three-dimensional faces

In the remaining case 3) let F_1 be one of the two-dimensional faces of N_i/T . Then by Lemma 3.2 there are four not necessarily pairwise distinct two-dimensional faces $F_2, \dots, F_5$ of N_i/T which have non-trivial intersection with F_1 . If all the intersections $F_i \cap F_j$, $i = 1, \dots, 5$, are connected or empty, then $F_2, \dots, F_5$ are pairwise distinct and it is clear that P is of type I^3 .

Therefore assume that $F_1 \cap F_2$ has two components. These two components must then both be edges of P .

Let $v_1, \dots, v_4$ be the vertices of F_1 such that v_i and v_{i+1} are connected by an edge $v_i v_{i+1}$ for all $i \in \mathbb{Z}/4\mathbb{Z}$.

Assume that $v_1 v_2$ and $v_3 v_4$ are contained in the intersection of F_1 and F_2 . If v_1 and v_4 were connected by an edge in F_2 , then there would be a facet of type Σ^2 in P , contradicting our assumption. Therefore v_1 and v_3 are connected by an edge in F_2 (and similarly v_2 and v_4). So $F_1 \cup F_2$ is homeomorphic to a Moebius strip, contradicting our orientability assumption on O . $\square$

Definition 3.5 Let Γ be a connected graph with a connection ∇ . We say that Γ is a *graph with small three-dimensional faces* if the following conditions hold true:

1. For any $x \in V(\Gamma)$ and distinct edges e_1, e_2, e_3 meeting at x , there exists a unique 3-dimensional face of Γ containing e_1, e_2 and e_3 .
2. The conclusion of Lemma 3.4 holds true, i.e., any three-dimensional face of Γ has the combinatorial type of Δ^3 , Σ^3 , $\Delta^2 \times I$, $\Sigma^2 \times I$ or I^3 (see Fig. 2) and these faces have the natural face structure.

Lemmas 3.2 and 3.4 thus say that the GKM graph of a connected nonnegatively curved GKM₄ orbifold is a graph with small three-dimensional faces.

Definition 3.6 Let $x \in V(\Gamma)$, where Γ is a graph with small three-dimensional faces. We call a subgraph $\Gamma' \subset \Gamma$ a *local factor* at x if Γ' contains x , is ∇ -invariant, has the combinatorial type of Δ^k or Σ^k , ∇ induces the natural face structure on Γ' and Γ' is maximal with these properties.

Lemma 3.7 Let Γ be a graph with small three-dimensional faces. For two edges e, e' emanating from x we have:

1. e and e' belong to the same local factor of type Δ^k if and only if e and e' span a triangle.
2. e and e' belong to the same local factor of type Σ^k if and only if e and e' span a biangle.
3. e and e' do not belong to the same local factor if and only if e and e' span a square.

Proof By definition, all two dimensional faces of Γ have at most four vertices. Therefore the lemma follows from the definition of the local factors, because two-dimensional faces of Δ^k and Σ^k are triangles and biangles, respectively. $\square$

Lemma 3.8 *Let Γ be a graph with small three-dimensional faces. Then the local factors at $x \in V(\Gamma)$ are partitioning $E_x = G_1 \sqcup \dots \sqcup G_{n_x}$ in such a way that each G_i contains the edges which span a local factor at x .*

Proof We have to show that belonging to the same local factor is an equivalence relation on E_x . We only have to show transitivity.

Let $e, e', e'' \in E_x$ be such that e and e' belong to one local factor σ_i and e' and e'' belong to another local factor. Then, by Lemma 3.7, e and e' and e' and e'' span a triangle or a biangle, respectively.

Consider the three-dimensional face F of Γ spanned by e, e', e'' . It has one of the combinatorial types described in Definition 3.5. Since none of the faces spanned by e, e' and e', e'' are squares, it follows that F has the combinatorial type of Δ^3 or Σ^3 . Hence, we have shown transitivity and the claim follows. $\square$

Lemma 3.9 *Let Γ be a graph with small three-dimensional faces and let e be an oriented edge of Γ . Then the connection $\nabla_e : E_{i(e)} \rightarrow E_{t(e)}$ preserves the partitions $E_{i(e)} = G_1 \sqcup \dots \sqcup G_{n_{i(e)}}$ and $E_{t(e)} = G'_1 \sqcup \dots \sqcup G'_{n_{t(e)}}$. Moreover, the combinatorial types of the local factors spanned by G_i and $\nabla_e(G_i)$ are the same.*

Proof Let e', e'' be other edges of Γ emanating from $i(e)$. By Lemma 3.7, we have to show that the two-dimensional faces spanned by any choice of two edges e_1, e_2 from e, e', e'' , respectively, has the same combinatorial types as the two-dimensional face spanned by $\nabla_e(e_1), \nabla_e(e_2)$.

To do so, we consider the three-dimensional face of Γ spanned by e, e', e'' . It contains the two-dimensional faces spanned by e, e' and e, e'' , respectively. Moreover, these two-dimensional faces are also spanned by $e, \nabla_e(e')$ and $e, \nabla_e(e'')$, respectively.

The three-dimensional face also contains the face spanned by e' and e'' . By an inspection of the list of three-dimensional faces in Definition 3.5, it follows that the two-dimensional faces spanned by e', e'' and $\nabla_e(e'), \nabla_e(e'')$, respectively, have the same combinatorial types. Hence the lemma is proved. $\square$

Lemma 3.10 *Let Γ be a graph with small three-dimensional faces. Let e, e' and f be edges in Γ with the same initial point.*

1. If e and e' span a biangle, then $\nabla_e f = \nabla_{e'} f$.
2. If e and e' span a square, then $\nabla_{e'_1} \nabla_e f = \nabla_{e_1} \nabla_{e'} f$, where e_1, e'_1 are the edges opposite to e and e' , respectively, in the square spanned by e and e' .

Proof To see the first claim, one has to show that if e, e' span a biangle Σ^2 and f is an edge with the same initial point as e and e' then

$$\nabla_e f = \nabla_{e'} f.$$

To see this, first assume that e and f span a biangle. Then by Definition 3.5, e, e', f span a face of Γ which is combinatorially equivalent to Σ^3 . Hence, $\nabla_e f = \tilde{f} = \nabla_{e'} f$ follows.

Next assume that e and f span a square. Then, by Definition 3.5, e , e' , and f span a face with the combinatorial type of $\Sigma^2 \times I$. Therefore $\nabla_e f = g = \nabla_{e'} f$, where g is the third edge at $t(e) = t(e')$. Hence the claim follows in this case.

By Definition 3.5, the case that e and f span a triangle does not occur. So the claim follows.

The second claim follows in a similar way, again by considering the three-dimensional faces of Γ . $\square$

The following theorem states that any graph with small three-dimensional faces is covered by a product of Δ^k s and Σ^k s. Here by a covering of a graph by another graph we mean the following: We consider graphs as one-dimensional CW-complexes, and coverings should be cellular. Note that the graphs we consider are n -valent, for some $n \geq 1$; for $n \neq 2$ a covering of n -valent graphs is automatically cellular, because in this case the neighbourhoods of points in the interior of a one-cell and the neighbourhoods of the vertices are not homeomorphic.

Theorem 3.11 *Let Γ be a graph with small three-dimensional faces, $x \in V(\Gamma)$, and $\sigma_1, \dots, \sigma_k$ the local factors at x . Consider the product graph $\tilde{\Gamma} := \prod_{i=1}^k \sigma_i$, equipped with its natural connection $\tilde{\nabla}$. Let $x_0 \in \tilde{\Gamma}$ be a base point, and $f : E(\tilde{\Gamma})_{x_0} \rightarrow E(\Gamma)_x$ a bijection sending the edges of a local factor to the edges of a local factor. Then there exists a unique covering $\pi : \tilde{\Gamma} \rightarrow \Gamma$ extending f that is compatible with the connections, i.e., which satisfies $\nabla_{\pi(e)} \circ \pi = \pi \circ \tilde{\nabla}_e$ for all edges $e \in E(\tilde{\Gamma})$.*

Proof The compatibility condition $\nabla_{\pi(e)} \circ \pi = \pi \circ \tilde{\nabla}_e$ shows that if e, e' are edges meeting at some vertex v , and $\pi(e)$ and $\pi(e')$ are given, then $\pi(\nabla_e e')$ is uniquely determined. This implies the uniqueness of π .

We have to show the existence of π . Note that for any path γ in $\tilde{\Gamma}$, say from x_0 to y , the connection $\tilde{\nabla}$ on $\tilde{\Gamma}$ induces a bijection

$$\tilde{\nabla}_\gamma : E(\tilde{\Gamma})_{x_0} \rightarrow E(\tilde{\Gamma})_y \quad \tilde{\nabla}_\gamma = \tilde{\nabla}_{e_m} \circ \dots \circ \tilde{\nabla}_{e_1},$$

where $\gamma = e_1 * \dots * e_m$. Similarly, for paths in Γ from x to y there is a bijection

$$\nabla_\gamma : E(\Gamma)_x \rightarrow E(\Gamma)_y$$

induced by ∇ .

For $n \geq 0$, we let $\tilde{\Gamma}_n$ be the subgraph of $\tilde{\Gamma}$ whose vertices are those that have distance at most n to x_0 , and whose edges are all the edges of $\tilde{\Gamma}$ connecting two such vertices. We prove by induction that we can construct a map of graphs $\pi : \tilde{\Gamma}_n \rightarrow \Gamma$ extending f that satisfies

$$\nabla_{\pi(\gamma)} \circ \pi(e) = \pi \circ \tilde{\nabla}_\gamma e$$

for all shortest paths γ in $\tilde{\Gamma}_n$ starting at x_0 , and all edges e of $\tilde{\Gamma}_n$ with $i(e) = i(\gamma)$. For $n = 0$ there is nothing to do. For $n = 1$, the existence of π is guaranteed by the fact that f sends local factors to local factors.

We assume that $\pi : \tilde{\Gamma}_{n-1} \rightarrow \Gamma$ is already constructed, and wish to construct $\pi : \tilde{\Gamma}_n \rightarrow \Gamma$. Let e be an edge of $\tilde{\Gamma}_n$ which is not an edge of $\tilde{\Gamma}_{n-1}$, but whose initial vertex $i(e)$ is a vertex of $\tilde{\Gamma}_{n-1}$, and choose a shortest path γ from x_0 to $i(e)$. Note that γ is a path in $\tilde{\Gamma}_{n-1}$. We want to define $\pi(e) = \nabla_{\pi(\gamma)} \pi(\tilde{\nabla}_\gamma^{-1} e)$; in order to do so we have to show that this definition is independent of the choice of γ .

If x', y' are vertices of $\tilde{\Gamma}$, then the shortest paths between x' and y' are of the following form:

$$e_{i_1} * \dots * e_{i_l},$$

where

- e_{ij} is an edge tangent to σ_{ij}
- the i_j are unique up to ordering and $i_j \neq i_{j'}$ if $j \neq j'$
- If $\sigma_{i_j} = \Delta^k$, then e_{ij} is unique
- If $\sigma_{i_j} = \Sigma^k$, $k \geq 2$, then by Lemma 3.10, $\nabla_{e_{ij}}$ does not depend on the choice of edge in σ_{ij} .

By Lemma 3.10, we have

$$\tilde{\nabla}_{e_{ij}} \circ \tilde{\nabla}_{e_{ij+1}} = \tilde{\nabla}_{e'_{ij+1}} \circ \tilde{\nabla}_{e'_{ij}},$$

and

$$\nabla_{\pi(e_{ij})} \circ \nabla_{\pi(e_{ij+1})} = \nabla_{\pi(e'_{ij+1})} \circ \nabla_{\pi(e'_{ij})},$$

where e'_{ij} and e'_{ij+1} are the opposite edges to e_{ij} and e_{ij+1} , respectively, in the square spanned by e_{ij} and e_{ij+1} . Hence $\tilde{\nabla}_{\gamma}$ and $\nabla_{\pi(\gamma)}$ do not depend on the chosen shortest path from x_0 to $i(e)$.

Next, we show that for two such edges e, e' with initial point in $\tilde{\Gamma}_{n-1}$ and same end point $t(e) = t(e')$, which is not a vertex of $\tilde{\Gamma}_{n-1}$, we have $\pi(t(e)) = \pi(t(e'))$. To do so let γ and γ' be two minimising curves from x_0 to $i(e)$ and $i(e')$, respectively. Then

$$\gamma * e \qquad \qquad \qquad \gamma' * e'$$

are minimising curves from x_0 to $t(e)$. Therefore they coincide up to ordering of the edges (replacing edges by parallel edges) and possible disambiguity with multiple edges.

At first assume that $i(e) = i(e')$; then e and e' span a triangle. Moreover, by the induction hypothesis, we can assume that $\gamma = \gamma'$. Hence, it follows from Lemma 3.9 that $\pi(e)$ and $\pi(e')$ have the same end points.

Next assume that $i(e) \neq i(e')$. Then we may assume

$$\gamma = \gamma_1 * \tilde{e}' \qquad \qquad \qquad \gamma' = \gamma_1 * \tilde{e},$$

where γ_1 is some minimising path and $\tilde{e}', \tilde{e}$ are parallel to e' and e respectively. Hence $\tilde{e}'$ and $\tilde{e}$ span a square at the end point of γ_1 . Hence, it follows from Lemma 3.9 that $\pi(\tilde{e}')$ and $\pi(\tilde{e})$ also span a square in Γ . Therefore it follows that $\pi(e)$ and $\pi(e')$ have the same end points.

Finally we have to consider edges e of $\tilde{\Gamma}_n$ such that both $i(e)$ and $t(e)$ are not vertices in $\tilde{\Gamma}_{n-1}$. In this case there are shortest paths from x_0 to $i(e)$, as well as to $t(e)$, of the following form:

$$\gamma_1 * e_1 \qquad \qquad \qquad \gamma_1 * f_1,$$

where the edges in the two paths satisfy the same relations as above and e_1 and f_1 are tangent to the same factor. This is because if two vertices of $\tilde{\Gamma}$ are connected by an edge, then they only differ in one coordinate. We have to show that the two possible definitions $\pi(e) = \nabla_{\pi(\gamma_1 * e_1)} \pi(\tilde{\nabla}_{\gamma_1 * e_1}^{-1} e)$ and $\pi(\tilde{e}) = \nabla_{\pi(\gamma_1 * f_1)} \pi(\tilde{\nabla}_{\gamma_1 * f_1}^{-1} \tilde{e})$ for the image of e are compatible.

Since the length of the two paths are the same and they are minimising, it follows that e_1 and f_1 span a triangle, with e as third edge.

Then we have, using the induction hypothesis,

$$\begin{aligned} \nabla_{\pi(\gamma_1 * e_1)} \pi(\tilde{\nabla}_{\gamma_1 * e_1}^{-1} e) &= \nabla_{\pi(e_1)} \nabla_{\pi(\gamma_1)} \pi(\tilde{\nabla}_{\gamma_1}^{-1} \tilde{\nabla}_{e_1}^{-1}(e)) \\ &= \nabla_{\pi(e_1)} \nabla_{\pi(\gamma_1)} \pi(\tilde{\nabla}_{\gamma_1}^{-1} f_1) = \nabla_{\pi(e_1)} \pi(f_1), \end{aligned}$$

and analogously,

$$\nabla_{\pi(\gamma_1 * f_1)} \pi(\tilde{\nabla}_{\gamma_1 * f_1}^{-1} \bar{e}) = \nabla_{\pi(f_1)} \pi(e_1).$$

Thus we are done if we can show that $\pi(e_1)$ and $\pi(f_1)$ span a triangle in Γ . But this follows from Lemma 3.9.

We have thus shown that $\pi : \tilde{\Gamma} \rightarrow \Gamma$ is a well-defined map of graphs. Next, we confirm that π is compatible with the connections $\tilde{\nabla}$ and ∇ .

By construction π has the following properties:

- π maps two-dimensional faces of $\tilde{\Gamma}$ of a given type $(I^2, \Sigma^2, \Delta^2)$ to a face of Γ of the same type.
- If e is an edge of $\tilde{\Gamma}$ which is part of a shortest path to the base point, then we have

$$\nabla_{\pi(e)} \circ \pi = \pi \circ \tilde{\nabla}_e,$$

i.e. π is compatible with $\tilde{\nabla}_e$ and $\nabla_{\pi(e)}$.

Hence, it only remains to be shown that π is also compatible with the connection of those edges which are not part of a minimising path to the base point. These edges e are tangent to factors Δ^n , $n \geq 2$ and are opposite to the closest point $y_0 \in \Delta^n$ to x_0 . Then e together with two edges f_1, f_2 which connect y_0 with the initial and end point of e form a triangle Δ_0^2 . Note that f_1 and f_2 are part of minimising paths to x_0 . Therefore for an edge e' starting at the same point as e and not tangent to Δ_0^2 we have:

$$\pi(\tilde{\nabla}_e e') = \pi(\tilde{\nabla}_{f_2} \circ \tilde{\nabla}_{\tilde{f}_1} e') = \nabla_{\pi(f_2)} \circ \nabla_{\pi(\tilde{f}_1)} \circ \pi(e') = \nabla_{\pi(e)} \circ \pi(e'),$$

because $\pi(e), \pi(f_1), \pi(f_2)$ form a triangle in Γ .

Moreover, by the same reason, we have

$$\pi(\tilde{\nabla}_e \tilde{f}_1) = \pi(\tilde{f}_1) = \nabla_{\pi(e)} \pi(\tilde{f}_2).$$

Hence π is compatible with the connections. Finally it follows that π is a covering. $\square$

This theorem directly implies Theorem 1.1:

Proof of Theorem 1.1 The GKM graph of a GKM₄ orbifold with an invariant metric of non-negative curvature is a graph with small three-dimensional faces. By Theorem 3.11, any such graph is finitely covered by the vertex-edge graph of a finite product of simplices. $\square$

Definition 3.12 Let Γ be a graph with small three-dimensional faces, $x \in V(\Gamma)$, and $\pi : \tilde{\Gamma} = \prod_{i=1}^k \sigma_i \rightarrow \Gamma$ a covering as in Theorem 3.11. Then a deck transformation of π is an automorphism $\psi : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$ such that $\pi \circ \psi = \pi$ and $\nabla_{\psi(e)} \circ \psi = \psi \circ \tilde{\nabla}_e$ for all edges $e \in E(\tilde{\Gamma})$.

Clearly, the deck transformations of π form a group.

Proposition 3.13 *The covering π is Galois, i.e., the deck transformation group of π acts simply transitively on the fibers of π .*

Proof Let $x, y \in V(\tilde{\Gamma})$ such that $\pi(x) = \pi(y) =: z$. The covering π induces a bijection

$$f : E(\tilde{\Gamma})_x \xrightarrow{\pi} E(\Gamma)_z \xrightarrow{\pi^{-1}} E(\tilde{\Gamma})_y.$$

We first claim that f respects the combinatorial types of the local factors at x and y . In fact, this property is clear for $\pi : E(\tilde{\Gamma})_x \rightarrow E(\Gamma)_z$. At the vertex y , the covering π necessarily

maps a local factor attached to y to a local factor of the same type. As the combinatorial structure of the local factors at y is, by Lemma 3.9, the same as that of the local factors at x , and hence also the same as that of the local factors at z , it follows that π has to respect the combinatorial structures of the local factors at y as well.

Thus Theorem 3.11, applied to $\Gamma := \tilde{\Gamma}$, implies that f extends uniquely to an automorphism $\psi : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$ respecting the natural connection of $\tilde{\Gamma}$. By the uniqueness statement of Theorem 3.11, the maps π and $\pi \circ \psi$ are identical, hence ψ is a deck transformation.

This shows that the deck transformation group acts transitively on the fibers of π . The uniqueness statement of Theorem 3.11 implies that the action of the deck transformation group on the fibers is also free. $\square$

Now let Γ be the GKM graph of a nonnegatively curved GKM_4 manifold, and $\pi : \tilde{\Gamma} \rightarrow \Gamma$ as above. Using π , we pull back the labeling of Γ ; in this way $\tilde{\Gamma}$ becomes a GKM_4 graph. By construction, the deck transformation group of π leaves invariant the labeling of $\tilde{\Gamma}$.

4 Extending GKM graphs

We say that a GKM graph is effective if at each vertex $v \in \Gamma$, we have that $\alpha(e_1), \dots, \alpha(e_k)$ generate $\mathfrak{t}^*$, where $e_1, \dots, e_k$ are the edges emanating from v .

Note that every GKM graph can be made effective by replacing the torus T by $T / \bigcap_{i=1}^k \exp(\ker \alpha(e_i))$.

Definition 4.1 Let $m \geq k$. We consider two effective GKM_3 graphs with the same underlying graph with connection, (Γ, α, ∇) and (Γ, β, ∇) , for an k - respectively m -dimensional torus T^k respectively T^m . If there exists a linear map $\phi : (\mathfrak{t}^m)^* \rightarrow (\mathfrak{t}^k)^*$, such that $\alpha = \phi \circ \beta$, then we say that (Γ, β, ∇) is an extension of (Γ, α, ∇) .

Recall that β is only well-defined up to signs, i.e., is a map $\beta : E(\Gamma) \rightarrow (\mathfrak{t}^m)^* / \{\pm 1\}$. The composition $\phi \circ \beta : E(\Gamma) \rightarrow (\mathfrak{t}^k)^* / \{\pm 1\}$ is thus also well-defined only up to signs.

Because a linear map sends linearly dependent vectors to linearly dependent vectors, any extension of a GKM_k graph is again GKM_k .

Let (Γ, β) be an extension of an effective GKM graph (Γ, α) in the sense of Definition 4.1. If e_1, e_2 are two edges that meet in a vertex v , then by (2.1)

$$\begin{aligned}\hat{\beta}(\nabla_{e_1}(e_2)) &= q'\hat{\beta}(e_1) + p'\hat{\beta}(e_2), \\ \hat{\alpha}(\nabla_{e_1}(e_2)) &= q\hat{\alpha}(e_1) + p\hat{\alpha}(e_2),\end{aligned}$$

for some $p, p', q, q' \in \mathbb{Q}$. If we have chosen the signs of $\hat{\beta}$ in such a way that $\phi(\hat{\beta}(e_i)) = \hat{\alpha}(e_i)$ and $\phi(\hat{\beta}(\nabla_{e_1}e_2)) = \hat{\alpha}(\nabla_{e_1}e_2)$, then we see by applying ϕ to the first equation and by using the 2-independence of the weights that $q = q'$ and $p = p'$.

From this argument we get a necessary and sufficient condition for extending an n -valent GKM_3 graph (Γ, α) to a GKM_n graph. This goes as follows.

At first choose a basis $b_1, \dots, b_n$ of $(\mathfrak{t}^n)^*$ and a base point v_0 of Γ . Let $e_1, \dots, e_n$ be the edges with initial point v_0 . We want to define $\beta(e_i) = \pm b_i$. Hence ϕ must be defined by the equations $\phi(b_i) = \hat{\alpha}(e_i)$, for some choices of signs $\hat{\alpha}(e_i)$. As the weights on the edges emanating from v_0 are defined, we next want to define the weights for the edges emanating from a vertex connected to v_0 by one edge, say e_1 .

As in the above computation for these edges $\nabla_{e_1}(e_j)$, the numbers $p, q \in \mathbb{Q}$ such that

$$\hat{\alpha}(\nabla_{e_1}(e_j)) = q\hat{\alpha}(e_1) + p\hat{\alpha}(e_j)$$

(for an arbitrary choice of sign of $\hat{\alpha}(\nabla_{e_1}(e_j))$) also have to satisfy

$$\hat{\beta}(\nabla_{e_1}(e_j)) = q\hat{\beta}(e_1) + p\hat{\beta}(e_j),$$

where the sign of $\hat{\beta}(\nabla_{e_1}(e_j))$ is chosen such that $\phi(\hat{\beta}(\nabla_{e_1}(e_j))) = \hat{\alpha}(\nabla_{e_1}(e_j))$. Hence the weights $\beta(\nabla_{e_i}e_j)$ are uniquely determined by the weights of the edges at v_0 .

Next one can consider edges emanating from vertices which are two edges away from v_0 . By the above argument these are uniquely determined by the weights of edges at vertices which are one edge away from v_0 . Iterating this argument, one sees that if one fixes the weights at the vertex v_0 , there is at most one way to define the weights of the other edges.

One can define these edges consistently if and only if weights at a vertex v_1 are independent of the paths from v_0 to v_1 which is used to transport the weights from v_0 to v_1 . That is, we can define the extension if and only if transporting the weights $\beta(e_i)$ at v_0 around a loop based at v_0 in a way prescribed by the weights α leads to the same weights $\beta(e_i)$ at v_0 . Formally, we fix an arbitrary lift $\hat{\alpha}$ of α , and consider arbitrary paths γ based at v_0 of the form

$$f_1 * \cdots * f_l,$$

as well as another edge e at v_0 . We define, inductively on $j = 1, \dots, l$, the weight of the edge $\nabla_{f_1 * \cdots * f_j} e$ which is obtained by transporting $\hat{\beta}(e)$ along the path $f_1 * \cdots * f_j$ in a way prescribed by the weights α : for $j = 1$, we put

$$A_{f_1}^\gamma(e) := p\hat{\beta}(e) + q\hat{\beta}(f_1),$$

where p and q are determined by the equation

$$\hat{\alpha}(\nabla_{f_1} e) = p\hat{\alpha}(e) + q\hat{\alpha}(f_1).$$

If $A_{f_1 * \cdots * f_{j-1}}^\gamma(e)$ is already defined for all edges e at v_0 , then we set

$$A_{f_1 * \cdots * f_j}^\gamma(e) := pA_{f_1 * \cdots * f_{j-1}}^\gamma(e) + qA_{f_1 * \cdots * f_{j-1}}^\gamma(\nabla_{f_1 * \cdots * f_{j-1}}^{-1} f_j), \quad (4.1)$$

where p and q are given by

$$\hat{\alpha}(\nabla_{f_1 * \cdots * f_j} e) = p\hat{\alpha}(\nabla_{f_1 * \cdots * f_{j-1}} e) + q\hat{\alpha}(f_j).$$

Then, the prescribed $\beta(e_i)$ can be completed to a well-defined extension of (Γ, α) to a GKM_n -graph if and only if

$$A_\gamma^\gamma(e) = \pm \hat{\beta}(\nabla_\gamma e) \quad (4.2)$$

for all closed loops γ based at v_0 . (Note that $\nabla_\gamma e$ is an edge at v_0 .)

This reasoning leads to the following lemma:

Lemma 4.2 *Let (Γ, α) be an n -valent GKM_4 graph which is combinatorially equivalent to a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$. Then (Γ, α) extends to an GKM_n graph. This extension is uniquely determined by the weights at a single vertex in Γ .*

Proof Consider first the situation of a loop γ at v_0 containing two successive edges that are tangent to different factors of the product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$, which hence span a square. Denoting by γ' the loop obtained from γ by replacing the two edges by its opposite edges, we note that as in the proof of Theorem 3.11 the connection actions of γ and γ' coincide. We wish to show that the relations (4.2) are equivalent for γ and for γ' . Concretely, we write

$$\gamma = \gamma_1 * f_1 * f_2 * \gamma_2,$$

where f_1 and f_2 are two edges that span a square, with opposite edges f'_1 and f'_2 , and put

$$\gamma' = \gamma_1 * f'_2 * f'_1 * \gamma_2.$$

Then, for any edge e at v_0 ,

$$\begin{aligned} A_{\gamma_1 * f_1 * f_2}^{\gamma}(e) &= p_1 A_{\gamma_1 * f_1}^{\gamma}(e) + q_1 A_{\gamma_1 * f_1}^{\gamma}(\nabla_{\gamma_1 * f_1}^{-1} f_2) \\ &= p_1(p_2 A_{\gamma_1}^{\gamma}(e) + q_2 A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f_1)) + q_1 A_{\gamma_1 * f_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f'_2) \\ &= p_1(p_2 A_{\gamma_1}^{\gamma}(e) + q_2 A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f_1)) + q_1(p_3 A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f'_2) + q_3 A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f_1)) \end{aligned}$$

for some rational numbers p_i and q_i . Thus, $A_{\gamma_1 * f_1 * f_2}^{\gamma}(e)$ is contained in the three-dimensional span V of $A_{\gamma_1}^{\gamma}(e) = A_{\gamma_1}^{\gamma'}(e)$, $A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f_1) = A_{\gamma_1}^{\gamma'}(\nabla_{\gamma_1}^{-1} f_1)$ and $A_{\gamma_1}^{\gamma}(\nabla_{\gamma_1}^{-1} f'_2) = A_{\gamma_1}^{\gamma'}(\nabla_{\gamma_1}^{-1} f'_2)$. The same computation shows that $A_{\gamma_1 * f'_2 * f'_1}^{\gamma'}$ is contained in the same space V . Now as by construction the restriction map ϕ sends the above basis of V to $\hat{\alpha}(\nabla_{\gamma_1} e)$, $\hat{\alpha}(f_1)$ and $\hat{\alpha}(f'_2)$, it is injective on V . As $\phi(A_{\gamma_1 * f_1 * f_2}^{\gamma}(e)) = \pm \hat{\alpha}(\nabla_{\gamma_1 * f_1 * f_2} e) = \phi(A_{\gamma_1 * f'_1 * f'_2}^{\gamma'}(e))$, we conclude that $A_{\gamma_1 * f_1 * f_2}^{\gamma}(e) = \pm A_{\gamma_1 * f'_1 * f'_2}^{\gamma'}(e)$ for all edges e at v_0 .

Then, by successively arguing along the edges of γ_2 using Equation (4.1), we conclude that $A_{\gamma}^{\gamma}(e) = A_{\gamma'}^{\gamma'}(e)$ for all edges e at v_0 , which shows the claim.

Hence, it remains to show Equation (4.2) for loops γ of the form $\gamma_1 * \dots * \gamma_k$, where γ_i is a closed path in the i -th factor Γ_i of Γ . However for this it suffices to prove the equation for loops γ in one of the factors Γ_i .

Consider the case that Γ_i is a simplex Δ^{n_i} . Note that if a loop γ is of the form $\gamma_1 * f_1 * \bar{f}_1 * \gamma_2$, then Equation (4.2) holds for γ if and only if it holds for the loop $\gamma_1 * \gamma_2$. As any loop in a simplex can, up to insertion of paths of the form $f * \bar{f}$, be written as a composition of boundaries of triangles, it suffices to show (4.2) for $\gamma = f_1 * f_2 * f_3$ the boundary of a triangle. For such a loop one computes, analogously to the computation above, that $A_{\gamma}^{\gamma}(e)$, for any edge e at v_0 , is contained in the linear span V of $\hat{\beta}(e)$, $\hat{\beta}(f_1)$ and $\hat{\beta}(f_3)$, and is sent to $\pm \hat{\alpha}(\nabla_{\gamma} e)$ by ϕ . On the other hand, $\hat{\beta}(\nabla_{\gamma} e)$ restricts to $\hat{\alpha}(\nabla_{\gamma} e)$, and is also contained in V (this is because if e is any edge different from f_1 or f_3 , then $\nabla_{\gamma} e = e$; if $e = f_1$, then $\nabla_{\gamma} e = \bar{f}_3$, and if $e = \bar{f}_3$, then $\nabla_{\gamma} e = f_1$). As ϕ is injective on V , we conclude that $A_{\gamma}^{\gamma}(e) = \pm \hat{\beta}(\nabla_{\gamma} e)$.

A similar argument, only easier, goes through for the factors of type Σ^{m_i} . This concludes the proof. $\square$

5 A model

In this section we prove the following theorem which implies Theorem 1.4:

Theorem 5.1 *Let O be a GKM_4 orbifold which is rationally elliptic or non-negatively curved. Then there is a torus orbifold $\tilde{O}$ and an action of a finite group G on $\tilde{O}$ which normalises the torus action on $\tilde{O}$ such that*

$$H^*(O; \mathbb{Q}) \cong H^*(\tilde{O}/G; \mathbb{Q}).$$

Moreover, $\tilde{O}$ admits a metric of non-negative curvature, which is invariant both under the torus action and the G -action.

If O is a manifold, then also $\tilde{O}$ is a manifold.

For the proof of the theorem we need several lemmas. Throughout this section we use the following notation. We denote the (k -dimensional) torus acting on O by T^k , the GKM graph of O by (Γ, α) and the covering of Γ provided by Theorem 3.11 by $\pi : \tilde{\Gamma} \rightarrow \Gamma$. We let G be the deck transformation group of this covering. Denoting the lifted labeling on $\tilde{\Gamma}$ by $\tilde{\alpha}$, we can extend $(\tilde{\Gamma}, \tilde{\alpha})$ by Lemma 4.2 to a GKM $_n$ graph, for an n -dimensional torus T^n . We denote its weights by β , and we have a linear map $\phi : (\mathfrak{t}^n)^* \rightarrow (\mathfrak{t}^k)^*$ such that $\alpha = \phi \circ \beta$.

Moreover, by P we denote the product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$ of which $\tilde{\Gamma}$ is the vertex-edge-graph.

Lemma 5.2 *In the above situation G acts by face-preserving homeomorphisms on P .*

Proof We will show that G acts by automorphisms on the face poset of P . The action on P can then be constructed inductively because all the faces of P are homeomorphic to cones over their respective boundaries.

The vertex-edge graphs of the k -dimensional faces of P are the ∇ -invariant k -valent subgraphs of $\tilde{\Gamma}$. Since the G -action on $\tilde{\Gamma}$ is compatible with ∇ , it leaves the set of these subgraphs invariant. Hence, we have a G -action on the face poset of P and the claim follows. $\square$

Let $I^* \subset \mathfrak{t}^{n*}$ be the integer lattice spanned by all the $\beta(e)$ where e runs through the edges of $\tilde{\Gamma}$. Moreover denote by $I \subset \mathfrak{t}^n$ the dual lattice and by $\check{T}^n$ the quotient torus $\mathfrak{t}^n/I$. Note that T^n finitely covers $\check{T}^n$.

Note also that if O is a manifold, then I^* is spanned by the weights of the edges meeting in one vertex, by the fact that in this case we can choose $p = \pm 1$ and q to be an integer in Equation (2.1).

Lemma 5.3 *In the above situation G acts by automorphisms on the torus $\check{T}^n$ such that for all $g \in G$, $e \in E(\tilde{\Gamma})$:*

$$\beta(ge) = g\beta(e)$$

Proof Let

$$H = \{(g, f) \in G \times \text{Aut}(\check{T}^n); \beta(ge) = f \circ \beta(e) \text{ for all } e \in E(\tilde{\Gamma}) \text{ and } \phi \circ f = \phi\}.$$

Recall that β takes values in $\mathfrak{t}^{n*}/\{\pm 1\}$. We claim that the projection on the first factor of H is an isomorphism. We first show surjectivity. Let $g \in G$.

Let v_0 be a vertex of $\tilde{\Gamma}$ and $e_1, \dots, e_n$ be the edges of $\tilde{\Gamma}$ meeting at v_0 . Then, after choosing signs, the $\hat{\beta}(e_1), \dots, \hat{\beta}(e_n)$ form a basis of $\mathfrak{t}^*$. Moreover, we can choose signs for the $\hat{\beta}(ge_1), \dots, \hat{\beta}(ge_n)$ in such a way that

$$\phi(\hat{\beta}(e_i)) = \hat{\alpha}(\pi(e_i)) = \phi(\hat{\beta}(ge_i))$$

holds for all i . Therefore, for each $g \in G$, the automorphism A_g of $\mathfrak{t}^{n*}$ defined by

$$A_g \hat{\beta}(e_i) = \hat{\beta}(ge_i) \tag{5.1}$$

satisfies $\phi \circ A_g = \phi$. Now by

$$e \longmapsto \beta(e) \quad \text{and} \quad e \longmapsto A_g^{-1} \beta(ge)$$

there are two GKM $_n$ extensions of $\tilde{\alpha}$ defined which agree on edges at the vertex v_0 . Therefore by Lemma 4.2 they agree everywhere. Hence, A_g maps the lattice I^* isomorphically to itself and therefore its dual A_g^* defines an element of $\text{Aut}(\check{T}^n)$. Thus surjectivity is proven.

To prove injectivity, let $(\text{Id}, f) \in \ker(H \rightarrow G)$. Then for $i = 1, \dots, n$ we have

$$f \circ \beta(e_i) = \beta(e_i).$$

Since $\phi \circ \beta(e_i) \neq 0$, it follows from the requirement that $\phi \circ f = \phi$ that f must be the identity and the claim follows. $\square$

By the above lemma we have a G -action by automorphism on $\check{T}^n$. For later reference we note here also the following

Lemma 5.4 *In the above situation the map $\phi^* : \mathfrak{t}^k \rightarrow \mathfrak{t}^n$ descends to a homomorphism $\phi^* : T^k \rightarrow \check{T}^n$. The above G -action restricts to the trivial action on $\check{T}^k = \phi^*(T^k) \subset \check{T}^n$, where the torus T^k is associated to the covered GKM graph Γ .*

Proof As $\phi \circ \beta = \alpha$, the map ϕ sends I^* onto the lattice J^* in $\mathfrak{t}^{k*}$ spanned by all the $\alpha(e)$. Thus, ϕ^* sends the dual lattice J injectively to I . As $T^k = \mathfrak{t}^k/J$, it follows that ϕ^* induces a well-defined map on T^k .

In the notation of the proof of Lemma 5.3 we have $\phi^* = A_g \circ \phi^*$ for all $g \in G$. $\square$

By dualising the weights β we get a labeling λ of the facets of P by one-dimensional subgroups of T^n which is also compatible with the G -actions on P and $\check{T}^n$ by the above lemmas.

Therefore there is a continuous G -action on the torus orbifold

$$\tilde{O} = (P \times \check{T}^n)/\sim$$

where $(x_1, t_1) \sim (x_2, t_2)$ if and only if $x_1 = x_2$ and $t_1 t_2^{-1} \in \langle \lambda(F); x_1 \in F \rangle$, which normalises the $\check{T}^n$ -action.

Note that in the case that O is a manifold, $\tilde{O}$ is also a manifold. This is because in this case the weights at every vertex $v \in P$ form a basis of I^* . Hence in this case $\check{T}^n = \prod_{v \in F} \lambda(F)$ for all vertices $v \in P$. This implies that $\tilde{O}$ is locally modelled on effective complex n -dimensional $\check{T}^n$ -representations. So it is a manifold.

Moreover, since G acts trivially on T^k , we have

$$\begin{aligned} H^*(O) &= H_{T^k}^*(O)/(H^2(BT^k)) \\ &= H_{T^k}^*(\Gamma)/(H^2(BT^k)) && \text{by the GKM Theorem} \\ &= H_{\check{T}^k}^*(\Gamma)/(H^2(B\check{T}^k)) \\ &= H_{\check{T}^k}^*(\tilde{\Gamma})^G/(H^2(B\check{T}^k)) && \text{because } \Gamma = \tilde{\Gamma}/G \\ &= H_{\check{T}^k}^*(\tilde{O})^G/(H^2(B\check{T}^k)) && \text{by the GKM Theorem again} \\ &= (H_{\check{T}^k}^*(\tilde{O})/(H^2(B\check{T}^k)))^G \\ &= H^*(\tilde{O})^G = H^*(\tilde{O}/G). \end{aligned}$$

Hence we have proven Theorem 5.1 except for the claim about the invariant non-negatively curved metric on $\tilde{O}$.

This will follow from the next lemma because the orbit space of an isometric group action on a non-negatively curved manifold equipped with the quotient metric is a non-negatively curved Alexandrov space.

Lemma 5.5 *Let $Z_P = \prod_i S^{2n_i+1} \times \prod_i S^{2m_i}$ be the moment angle complex associated to $P = \prod_i \Delta^{n_i} \times \prod_i \Sigma^{m_i}$ equipped with the natural product metric. Then the G -action on $\tilde{O}$ lifts to an isometric action on Z_P .*

Proof Let $F_1, \dots, F_m$ be the facets of P . Then the moment angle complex associated to P is given by

$$Z_P = P \times T^m / \sim,$$

where $T^m = \prod_{i=1}^m S_i^1$ and $(x_1, t_1) \sim (x_2, t_2)$ if and only if $x_1 = x_2$ and

$$t_1 t_2^{-1} \in \prod_{i; x_1 \in F_i} S_i^1.$$

We have a map $\varphi : T^m \rightarrow \check{T}^n$ such that the restriction of φ to S_i^1 is an isomorphism $S_i^1 \rightarrow \lambda(F_i)$. With this map we can realise $\tilde{O}$ as a quotient of an almost free action of an abelian Lie group on Z_P .

We can lift the G -action on $\tilde{O}$ to a G -action on Z_P by requiring that for $g \in G$, $g(S_i^1) = S_j^1$ with $gF_i = F_j$ and that the following diagram commutes:

$$\begin{array}{ccc} S_i^1 & \xrightarrow{g} & S_j^1 \\ \varphi \downarrow & & \downarrow \varphi \\ \lambda(F_i) & \xrightarrow{g} & \lambda(F_j) \end{array}$$

Now note that in the case that P is a product of Δ^{n_i} and Σ^{m_i} , Z_P is equivariantly diffeomorphic to a product of round spheres of radius one. Moreover, note that the combinatorial type of the G -action on Z_P , that is, the action of G on $\mathcal{P}(P) \times T^m$ can be realised by elements of the Weyl-group of $(\prod_i O(2n_i + 2) \times \prod_i O(2m_i + 1)) \rtimes H$, where H is the group of permutations of the factors of the same dimension in the product of spheres. Here $\mathcal{P}(P)$ denotes the face poset of P as defined in Sect. 2.3. Therefore the claim follows. $\square$

Proof of Theorem 1.3 In the special case that the graph Γ is the vertex-edge graph of a product $\prod_i \Delta^{n_i} \times \prod_j \Sigma^{m_j}$, the covering of graphs $\tilde{\Gamma} \rightarrow \Gamma$ is trivial. Hence the deck transformation group G is the trivial group. From the proof of Theorem 5.1 it then follows that the group G in that theorem is trivial. Therefore Theorem 1.3 follows from Theorem 5.1. Note that for this theorem no assumptions on rational ellipticity or non-negative curvature are needed, as they are only used to obtain information on the structure of the GKM graph; here we instead assume the graph to be of the simplest possible type. $\square$

6 Upper bound for the dimension of the acting torus

In this section we give an upper bound for the dimension of a torus which can act on a non-negatively curved GKM₄ orbifold O . Moreover, we give an example of a non-negatively curved GKM₄-manifold whose rational cohomology ring is not isomorphic to that of a torus manifold. This shows that in Theorem 1.4 the group G is not always trivial.

We denote by T the torus acting on O . Otherwise we use the same notation as in the previous section.

Theorem 6.1 *Let O be a GKM₄-orbifold and T be the torus acting on O .*

Then the dimension of a maximal torus T' to which the T -action on O can be extended is bounded from above by

$$a - b_2(O),$$

where a is the number of orbits of the G -action on the set of facets of P .

Proof T' is a subtorus of $(T^n)^G$, where G acts on T^n as in Lemma 5.3. We have a short exact sequence of G -representations

$$0 \rightarrow H^2(BT^n) \rightarrow H_{T^n}^2(\tilde{O}) \rightarrow H^2(\tilde{O}) \rightarrow 0.$$

$H_{T^n}^2(\tilde{O})$ has a basis $v_1, \dots, v_n$ consisting out of the Poincaré duals of the facial suborbifolds of $\tilde{O}$. Since the G -action sends facial suborbifolds to facial submanifolds, the $\pm v_i$ are permuted by the G -action. Hence we have $\dim H_{T^n}^2(\tilde{O})^G \leq a$. Since $b_2(O) = \dim H^2(\tilde{O})^G$, it follows that

$$\dim T' \leq \dim H^2(BT^n)^G = \dim H_{T^n}^2(\tilde{O})^G - \dim H^2(\tilde{O})^G \leq a - b_2(O).$$

□

Next we want to give an example of a GKM_{n-1} -manifold of dimension $2n$ where the action does not extend to an effective action of an n -dimensional torus.

Example 6.2 Let P be $I^n = [-1, +1]^n$ with facets $F_{1,\pm 1}, \dots, F_{n,\pm 1}$ such that $F_{i,+1}$ and $F_{i,-1}$ belong to the same factor.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and set $\lambda(F_{n,\pm 1}) = \pm e_n$, $\lambda(F_{i,\pm 1}) = e_i \pm e_n$ for $i = 1, \dots, n-1$. Then the pair (P, λ) defines a quasitoric manifold M over P by $M = (P \times T)/\sim$, where $T = \mathbb{R}^n/\mathbb{Z}^n$ and $(x_1, t_1) \sim (x_2, t_2)$ if and only if $x_1 = x_2$ and

$$t_1 - t_2 \in \left(\bigoplus_{x_1 \in F_{ij}} \mathbb{R}\lambda(F_{ij}) \right) / \left(\bigoplus_{x_1 \in F_{ij}} \mathbb{Z}\lambda(F_{ij}) \right).$$

The T -action on the second factor of $P \times T$ induces the T -action on M . The weights of the T -representation at the fixed point $F_{1,\epsilon_1} \cap \dots \cap F_{n,\epsilon_n}$ are given by $e_1^*, \dots, e_{n-1}^*, \epsilon_n e_n^* - \epsilon_n \sum_{i \neq n} \epsilon_i e_i^*$.

The map

$$P \times T \rightarrow P \times T \quad \left(x, \sum_i \alpha_i e_i \right) \mapsto \left(-x, \sum_{i \neq n} \alpha_i e_i - \alpha_n e_n + \frac{1}{2} e_1 \right)$$

induces a free involution τ on M which is T -equivariant with respect to the involution

$$\phi : T \rightarrow T \quad \sum_i \alpha_i e_i \mapsto \sum_{i \neq n} \alpha_i e_i - \alpha_n e_n.$$

Note, that τ is orientation preserving if and only if n is odd.

Hence, τ commutes with the action of the torus $T' = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ and $N = M/\tau$ becomes a GKM_{n-1} -manifold with the action of T' . Lemma 6.3 below shows that T' is a maximal torus acting on N . Moreover, Corollary 6.4 shows that N does not have the rational cohomology of a torus manifold.

Lemma 6.3 *If, in the situation of Example 6.2, $n \geq 3$ then we have $b_2(N) = 1$ and there is an element $v \in H^2(N)$ with $v^2 \neq 0$. In particular, for all non-zero $v' \in H^2(N)$ we have $v'^2 \neq 0$.*

Proof First note that $H^2(N; \mathbb{Q}) \cong H^2(M; \mathbb{Q})^{\mathbb{Z}/2}$. Hence it suffices to describe the $\mathbb{Z}/2$ -action on $H^2(M)$.

We have a short exact sequence of $\mathbb{Z}/2$ -representations

$$0 \rightarrow H^2(BT) \rightarrow H_T^2(M) \rightarrow H^2(M) \rightarrow 0.$$

Hence it suffices to describe the $\mathbb{Z}/2$ -actions on $H^2(BT)$ and $H_T^2(M)$. The first action is induced by ϕ . Hence we have $\dim H^2(BT)^{\mathbb{Z}/2} = n - 1$. The action on $H_T^2(M)$ can be described as follows. We have an isomorphism

$$H_T^2(M) \cong \bigoplus_{i=1}^n (\mathbb{Q}v_{i,1} \oplus \mathbb{Q}v_{i,-1}),$$

Here, after possible changes of the orientations of the facial submanifolds, the $\mathbb{Z}_2$ action is given by $v_{i,\epsilon} \mapsto v_{i,-\epsilon}$ for $i = 1, \dots, n$ because τ interchanges opposite facial submanifolds of M and the $v_{i,\epsilon}$ are the equivariant Poincaré duals of these manifolds.

Therefore we have $\dim H_T^2(M)^{\mathbb{Z}/2} = n$ and the first claim follows.

To see the second claim note that $v_{n,-1} + v_{n,+1} \in H_T^2(M)^{\mathbb{Z}/2\mathbb{Z}}$ maps to a non-zero element $v \in H^2(N) \cong H^2(M)^{\mathbb{Z}/2\mathbb{Z}}$. Moreover, by [7, Theorem 4.14], there are the following relations in $H^*(M)$:

$$\begin{aligned} [v_{i,1}] &= -[v_{i,-1}] && \text{for } i = 1, \dots, n-1 \\ [v_{n,1}] &= [v_{n,-1}] - \sum_{i=1}^{n-1} ([v_{i,1}] - [v_{i,-1}]) \\ [v_{i,1}][v_{i,-1}] &= 0 && \text{for } i = 1, \dots, n \\ \prod_{i=1}^n [v_{i,\pm 1}] &\neq 0. \end{aligned}$$

Therefore we see

$$\begin{aligned} v^2[v_{n,1}]^{n-2} &= [v_{n,1}]^n = [v_{n,1}](-2)^{n-1} \left(\sum_{i=1}^{n-1} [v_{i,1}] \right)^{n-1} \\ &= (-2)^{n-1}(n-1)! \prod_{i=1}^n [v_{i,1}] \neq 0 \in H^{2n}(M). \end{aligned}$$

Hence we have $v^2 \neq 0$. □

Corollary 6.4 Assume $n \geq 5$. Then the manifold N from Example 6.2 does not have the rational cohomology of a torus manifold.

Proof Assume that there is a torus manifold O with the same rational cohomology as N . Then because the total Betti number 2^{n-1} of N is equal to the number of vertices in the GKM graph Γ of O , the covering graph $\tilde{\Gamma}$ must be the vertex-edge graph of I^n or $I^{n-2} \times \Sigma^2$.

In the second case the number of vertices in Γ and $\tilde{\Gamma}$ are equal. Therefore the covering must be trivial so that we have $\tilde{\Gamma} = \Gamma$. Therefore $H_T^2(O; \mathbb{Q})$ has dimension $2n - 2$. From the exactness of

$$0 \rightarrow H^2(BT^n) \rightarrow H_T^2(O) \rightarrow H^2(O) \rightarrow 0,$$

we now get a contradiction to Lemma 6.3.

Therefore assume that we are in the first case. Then the order of the deck transformation group G of the covering $\tilde{\Gamma} \rightarrow \Gamma$ is two. Let $g \in G$ be the non-trivial element. Let $\tilde{O}$ be the torus manifold constructed from O in Theorem 5.1. Denote by $v_{i,\pm 1}$, $i = 1, \dots, n$, the equivariant Poincaré duals of the facial submanifolds $\tilde{O}_{1,\pm 1}, \dots, \tilde{O}_{n,\pm 1}$ of $\tilde{O}$. Then as in the proof of Theorem 6.1, we see that for each $i \in \{1, \dots, n\}$ there is an $j_i \in \{1, \dots, n\}$ and $\epsilon \in \pm 1$ such that $g^* v_{i,\pm 1} = \pm v_{j_i, \pm \epsilon_i}$. Since g has order two, we can orient the facial submanifolds in such a way that the minus sign only appears when $j_i = i$ and $\epsilon_i = 1$.

However, because the G -action commutes with the torus action, g preserves the orientation of the normal bundle of $\tilde{O}_{i,\pm 1}$. Hence we also have a $+$ -sign in this case. Moreover, by the same reason, the labeling of the facets of I^n is G -invariant. Because the labels of intersecting facets are linearly independent, it follows that $j_i = i$ for all i .

Form the exactness of

$$0 \rightarrow H^*(BT^n) \rightarrow H_T^*(\tilde{O})^G \rightarrow H^2(O) \rightarrow 0$$

and Lemma 6.3 it follows, that the G -action on the set of facets of I^n has two fixed points and $n - 1$ non-trivial orbits.

Assume that the $v_{i,\pm 1}$, $i = 1, \dots, n - 1$ are not fixed by g . Let $\lambda_{i,\pm 1} \in \mathbb{Z}^n$, $i = 1, \dots, n$, be primitive generators of the Lie algebra of the circle subgroups of $T^n = \mathbb{R}^n / \mathbb{Z}^n$ that fix $\tilde{O}_{i,\pm 1}$. Then for $i = 1, \dots, n - 1$ we have $\lambda_{i,1} = \lambda_{i,-1}$. Moreover, $\lambda_{1,1}, \dots, \lambda_{n,1}$ is a basis of $\mathbb{Z}^n$ and

$$\lambda_{n,-1} = \pm \lambda_{n,1} + \sum_{i=1}^{n-1} \mu_i \lambda_{i,1},$$

with some integers μ_i . Therefore, by [7, Theorem 4.14], we have the following relations in $H^*(\tilde{O})$:

$$\begin{aligned} [v_{i,1}] &= -[v_{i,-1}] - \mu_i [v_{n,-1}] && \text{for } i = 1, \dots, n - 1 \\ [v_{n,1}] &= \pm [v_{n,-1}] \\ [v_{i,1}][v_{i,-1}] &= 0 && \text{for } i = 1, \dots, n. \end{aligned}$$

In particular, we see that $[v_{n,1}] \neq 0 \in H^2(\tilde{O})^{\mathbb{Z}/2\mathbb{Z}}$ and $[v_{n,1}]^2 = 0$. Therefore we have a contradiction to Lemma 6.3. $\square$

7 GKM manifolds with invariant almost complex structures

In this section we let M^{2n} be a GKM_4 manifold which admits an invariant almost complex structure whose GKM graph Γ has small three-dimensional faces. Denote by $\tilde{\Gamma}$ the covering graph of Γ constructed in Theorem 3.11 and G the deck transformation group. We let $\tilde{\alpha}$ be the weights of the edges of $\tilde{\Gamma}$. Moreover, β are the weights of the extension to an GKM_n graph of $(\tilde{\Gamma}, \tilde{\alpha})$.

We will show in this section that under the above conditions M has the same rational cohomology as a generalised Bott manifold. To do so we first show that $(\tilde{\Gamma}, \beta)$ is the GKM graph of a generalised Bott manifold. Then in a second step we show that the covering $\tilde{\Gamma} \rightarrow \Gamma$ is trivial. Together these two steps imply the above claim, which is stated in Corollary 7.6.

The combinatorial consequence of the existence of an almost complex structure is that the weights of the oriented edges of the GKM graph have preferred signs. By going through the

arguments one then also sees that the weights of the extensions constructed in Sect. 2 have preferred signs.

Denote by $\hat{\alpha}$ and $\hat{\beta}$ the weights $\tilde{\alpha}$ and β with the preferred sign, respectively.

We will show that in this case the GKM graph of M is a product of simplices. In particular, by the following theorem, M has the same rational cohomology as a generalised Bott manifold.

Theorem 7.1 *With the notation as above we have that $(\tilde{\Gamma}, \beta)$ is the GKM graph of a generalised Bott manifold.*

Proof By the argument in [17, Lemma 5.6] there are no local factors in the GKM graph of M with the combinatorial type of Σ^k . Therefore $(\tilde{\Gamma}, \beta)$ is the GKM graph of a quasitoric manifold over a product of simplices $P = \prod_i \Delta^{n_i}$. We embed each factor Δ^{n_i} of P as a convex polytope in $\mathbb{R}^{n_i}$. From these embeddings we get an embedding of P in $\mathbb{R}^n = \prod_i \mathbb{R}^{n_i}$.

By Theorem 6.4 of [6], a quasitoric manifold over a product of simplices is equivariantly homeomorphic to a generalised Bott manifold if and only if it admits an invariant almost complex structure.

By [24] (see also Theorem 7.3.24 in [5]), a quasitoric manifold admits an invariant almost complex structure if and only if the following product is independent of the vertex v we are looking at:

$$\det \sigma_v \cdot \det A_v. \quad (7.1)$$

Here we use the following notation:

- σ_v is the matrix with columns the vectors in direction of the edges emanating from v . That means σ_v is the matrix with the column vectors $v_i - v$ where $v_1, \dots, v_n$ are the vertices of P connected to v by an edge of $\tilde{\Gamma}$.
- A_v has as columns the labels $\hat{\beta}(e)$ in the same order as in the first matrix.

Note here that our matrix A_v is the inverse of the matrix appearing in [24] with columns the labels $\lambda(F)$ of the facets F containing v .

Therefore it suffices to show that the sign of the product (7.1) is independent of the vertex v of P . To do so let v_1 and v_0 be vertices connected by an edge. Then $v_1 - v_0$ is tangent to one of the direct Euclidean factors of the product $\mathbb{R}^n = \prod_i \mathbb{R}^{n_i}$, say the i_0 -th. Note that if we go from v_1 to v_0 the following happens:

- The column $v_1 - v_0$ in the first matrix changes sign.
- To other columns of this matrix which correspond to vertices v_2 of P , such that $v_2 - v_0$ is tangent to the i_0 -th factor of the above product, $v_0 - v_1$ is added.
- The columns of this matrix which correspond to vertices which differ in another coordinate are unchanged.
- The column $\hat{\beta}(v_1 - v_0)$ in the second matrix changes sign.
- To the other columns of this matrix a multiple of this weight is added.

Hence, in total the sign of the product does not change. Since any GKM graph is connected the claim follows. $\square$

Our next goal is to show that the covering $\tilde{\Gamma} \rightarrow \Gamma$ is trivial. To do so we have to analyse the action of G on $\tilde{\Gamma}$. By Lemma 5.2, G acts by automorphisms of the face poset of $\tilde{\Gamma}$.

Moreover, by Lemma 5.3, G acts on $(t^n)^*$ in a way that is compatible with the weights β . In the situation that we have an T -invariant complex structure we can strengthen the result of that lemma. This is the content of the next lemma.

Lemma 7.2 *There is an action $\varphi : G \rightarrow \text{Aut}((\mathfrak{t}^n)^*)$ such that*

$$\varphi(g)\hat{\beta}(e) = \hat{\beta}(ge)$$

for all $g \in G$, $e \in E(\tilde{\Gamma})$.

Proof By Lemma 5.3, we only have to show that the G -action constructed there is compatible with the signs of the weights.

Note that by construction the G -action is compatible with the signs of the weights at the base point v_0 . Moreover, it is compatible with the connection on $\tilde{\Gamma}$. Since the connection is compatible with the signs of the weights, it follows that the G -action is compatible with these signs. $\square$

With this lemma we can show that there is a factor of $\tilde{\Gamma}$ such that not all facets belonging to this factor are in the same G -orbit.

Lemma 7.3 *Let $(\tilde{\Gamma}, \hat{\beta})$ be the GKM graph of a generalised Bott manifold M . Assume that the group G acts on this graph. Then by dualising the action we get an action on the characteristic pair $(\mathcal{P}, \lambda)$ associated to M . Here $\mathcal{P}$ is the face poset of the orbit space of the T -action on M and λ denotes its labeling by Lie algebras of isotropy groups.*

Assume that for each factor of M/T all facets of this factor belong to the same G -orbit. Then the G -action on $\mathfrak{t}$ does not have any non-zero fixed points.

Proof Since M is a generalised Bott manifold, we can order the factors $\Delta_1, \dots, \Delta_n$ of M/T and filter $V_1 \subset V_2 \subset \dots \subset V_n = \mathfrak{t}$ the Lie algebra of T in such a way that

- $\dim V_i = \dim \prod_{k \leq i} \Delta_k$
- $\lambda(F_i) \in V_i$ for each facet of M/T belonging to Δ_i .

The above filtration of $\mathfrak{t}$ is related to the fact that the reduced characteristic matrix of M can be assumed to be an upper triangular vector matrix (see Proposition 6.2 and Remark 6.3 of [6]). Here the characteristic matrix of M is the matrix which has as its columns the vectors $\lambda(F)$, $F \in \mathcal{F}(M/T)$. Moreover the reduced characteristic matrix has precisely one column $\lambda(F_i)$ for each factor Δ_i of M/T such that F_i is a facet of Δ_i (see [6] for a more precise description).

After one more reordering of the facets we can assume that there is an $i_0 \geq 1$ such that

$$\dim \langle \lambda(F_k); F_k \text{ is a facet of } \Delta_i \rangle = \dim \Delta_i$$

for all $i \leq i_0$ and

$$\dim \langle \lambda(F_k); F_k \text{ is a facet of } \Delta_i \rangle = \dim \Delta_i + 1$$

for $i > i_0$. So the factors Δ_i , $i \leq i_0$ are precisely those factors for which the corresponding column in the reduced characteristic matrix does not have non-zero off-diagonal entries.

Then for each $i \leq i_0$,

$$\sum_{F_{ki}} \lambda(F_{ki}) = 0,$$

where the sum is over those facets F_{ki} which belong to Δ_i . Note that this sum is non-zero for $i > i_0$.

Since V_{i_0} is generated by the $\lambda(F_{ki})$ with $i \leq i_0$ it follows that V_{i_0} is invariant under the G -action and $V_{i_0}^G = 0$. Hence the claim follows by considering the G -representation $\mathfrak{t}/V_{i_0}$ and induction over the dimension of $\mathfrak{t}$. $\square$

Corollary 7.4 *Let M be a GKM_4 manifold with an invariant metric of non-negative sectional curvature and an invariant almost complex structure with GKM graph Γ .*

Then there is a factor of $\tilde{\Gamma}$ such that not all its facets belong to the same G -orbit.

Proof This follows from Lemmas 5.4 and 7.3. $\square$

Now we can prove the following

Theorem 7.5 *The covering $\tilde{\Gamma} \rightarrow \Gamma$ is trivial.*

Proof Let Δ be a factor of $\tilde{\Gamma}$ such that not all facets belonging to this factor are in the same G -orbit. Let H be one of the G -orbits of the facets of Δ . Then $B_1 = \bigcap_{F \in H} F$ is a G -invariant non-empty face of $\tilde{\Gamma}$. Moreover, the preimage in the generalised Bott manifold of this face is a generalised Bott manifold. Hence, by the above discussion there is a factor of B_1 such that not all facets of this factor belong to the same G -orbit.

Hence by induction we can construct a non-empty face fixed by G . This is a contradiction because G acts freely on the set of vertices of $\tilde{\Gamma}$. $\square$

As a consequence of the discussion in this section we get:

Corollary 7.6 *Let M be a non-negatively curved or rationally elliptic GKM_4 manifold which admits an invariant almost complex structure. Then the rational cohomology ring of M is isomorphic to the rational cohomology ring of a generalised Bott manifold.*

8 Torus manifolds revisited

The proof of the classification of simply connected non-negatively curved torus manifolds given in [32] proceeds as follows. First one uses results of Spindeler [31] to show that a simply connected non-negatively curved torus manifold M is locally standard and that all faces of M/T are diffeomorphic to standard discs after smoothing the corners. Moreover, one knows that the two-dimensional faces have at most four vertices. Using a combinatorial argument (Proposition 4.5 in the cited paper and Theorem 8.1 below) shows that the orbit space of M is combinatorially equivalent to a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{n_i}$. From this one can then deduce that M/T is diffeomorphic to that product and it follows that M is equivariantly diffeomorphic to a quotient of a product of spheres by a free linear torus action.

The proof of Proposition 4.5 given in [32] is highly technical, very long and not really enlightening. Therefore we want to give a short proof of that proposition based on the results of our paper in this section.

We start by stating the proposition as the following theorem.

Theorem 8.1 *Let Q be the orbit space of a locally standard torus manifold M^{2n} , such that all faces of Q are acyclic over the integers. Assume that each two-dimensional face of Q has at most four vertices. Then Q is combinatorially equivalent to a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{n_i}$.*

Proof For $n \leq 3$ this follows as in the proof of Lemma 3.4. Therefore assume $n \geq 4$. Then M is a GKM_4 -manifold. Let Γ be the vertex-edge graph of Q . It is the same as the GKM graph of M . Therefore we have a normal covering $\tilde{\Gamma} \rightarrow \Gamma$ by the vertex edge graph $\tilde{\Gamma}$ of a product $\prod_i \Delta^{n_i} \times \prod_i \Sigma^{n_i}$. Let G be the deck transformation group of this covering. Since the covering is compatible with the connections on Γ and $\tilde{\Gamma}$ and the connections determine the face structure, it suffices to show that the covering is trivial.

We do this by induction on n starting with $n = 4$. If $n = 4$ then because the covering respects the local face structure, its restriction to any 3-dimensional face of $\tilde{\Gamma}$ is a trivial covering. Moreover, if $n > 4$ the same holds for $(n - 1)$ -dimensional faces of $\tilde{\Gamma}$ by the induction hypothesis.

Therefore in both cases the action of G on the set of facets of $\tilde{\Gamma}$ is free. Moreover, because the labeling λ of the facets with isotropy groups is invariant under the G -action, we must have

$$F \cap gF = \emptyset$$

for all facets F and all non-trivial $g \in G$.

There is only one possibility how this can happen: $\tilde{\Gamma}$ is the vertex-edge graph of $[-1, 1]^n$ and $G = \mathbb{Z}/2$ acts by multiplication with -1 on each factor. So the intersection of any two facets of Q has precisely two components and all facets are combinatorially equivalent to cubes $[-1, 1]^{n-1}$. But by Lemma 4.4 of [32] such an orbit space Q with only acyclic faces does not exist. $\square$

Remark 8.2 Using the same proof and Proposition 5.1 (f) of [11] one can show that the same conclusion as in the above theorem holds, if Q is the orbit space of a not necessary locally standard rationally elliptic torus orbifold.

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