The Chow ring of hyperkähler varieties of $K3^{[2]}$ -type via Lefschetz actions

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Abstract

We propose an explicit conjectural lift of the Neron–Severi Lie algebra of a hyperkähler variety X of $K3^{[2]}$ -type to the Chow ring of correspondences $\mathrm{CH}^*(X \times X)$ in terms of a canonical lift of the Beauville–Bogomolov class obtained by Markman. We give evidence for this conjecture in the case of the Hilbert scheme of two points of a $K3$ surface and in the case of the Fano variety of lines of a very general cubic fourfold. Moreover, we show that the Fourier decomposition of the Chow ring of X of Shen and Vial agrees with the eigenspace decomposition of a canonical lift of the cohomological grading operator.

Keywords Chow ring · Hyperkähler manifold · Hilbert scheme of points · Fano variety of lines

Mathematics Subject Classification 14C15 · 14C25 · 14J35

1 Introduction and results

The main purpose of this note is to hint at a connection between two lines of reasoning which are both relevant for the study of the Chow ring of hyperkähler varieties of $K3^{[2]}$ -type. One focuses on Hilbert schemes of points of $K3$ surfaces [14, 15] using Nakajima operators, and the other investigates an analog of the Fourier transform and injectivity results for the cycle class map on subrings of universal classes [4, 16]. The present text suggests a connection in two directions. The first is provided by Conjecture 1.2, asking for an extension of the main theorem of [15] to the $K3^{[2]}$ -case. The second comes in the form of Theorem 1.7, stating that the Fourier decomposition of the Chow ring from [16, Theorem 2] agrees with the eigenspace decomposition of a canonical lift of the cohomological grading operator h .

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1.1 Conventions

We denote the Chow ring of a smooth projective variety X over $\mathbb{C}$ by $\mathrm{CH}^*(X)$, and it will always be with coefficients in $\mathbb{Q}$. Similarly, we often abbreviate the cohomology ring $H^*(X, \mathbb{Q})$ by $H^*(X)$. For a cycle class $Z \in \mathrm{CH}^*(X)$ or a cohomology class $\beta \in H^*(X)$ we denote the pullbacks to $X \times X$ via the two projections by Z_1, Z_2 and β_1, β_2 , and similar conventions will be followed throughout.

1.2 Lefschetz actions on the Chow ring

Let X be a hyperkähler variety of $K3^{[2]}$ -type and $(-, -)$ the Beauville–Bogomolov bilinear form on $H^2(X)$. Let $L \in \mathrm{CH}^2(X \times X)$ be Markman's canonical lift of the associated cohomology class and $l := \Delta^*(L) \in \mathrm{CH}^2(X)$, where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding, see Sect. 2 for details. We denote the second Betti number of X by $r = 23$.

Proposition 1.1 *For every divisor class $a \in \mathrm{CH}^1(X)$ with $(a, a) \neq 0$ the cycle classes*

$$F_a := \frac{4}{(r+2)(a, a)}(l_1 a_1 + l_2 a_2) + \frac{2}{(a, a)}L(a_1 + a_2) \in \mathrm{CH}^3(X \times X),$$

$$H := \frac{4}{r(r+2)}(l_2^2 - l_1^2) + \frac{2}{r+2}L(l_2 - l_1) \in \mathrm{CH}^4(X \times X)$$

are lifts of the Lefschetz dual operator f_a and the grading operator h , respectively.

Conjecture 1.2 *Let $\mathfrak{g}_{\mathrm{NS}}(X)$ be the Neron–Severi Lie algebra of X and cl the cycle class map. The linear map φ in the commutative diagram*

$$\begin{array}{ccc} \mathfrak{g}_{\mathrm{NS}}(X) & \xrightarrow{\varphi} & \mathrm{CH}^*(X \times X) \\ & \searrow & \downarrow \mathrm{cl} \\ & & \mathrm{End}_{\mathbb{Q}}(H^*(X, \mathbb{Q})), \end{array}$$

given by $\varphi(e_a) = \Delta_(a)$, $\varphi(f_a) = F_a$ and $\varphi(h) = H$ is a well-defined Lie algebra homomorphism.*

In the case $X = S^{[2]}$ for a projective $K3$ surface S we use the explicit description from Theorem 2.4 of Markman's lift L in order to prove in Sect. 3.3 that our formulas for the lifts F_a and H agree with the canonical lifts provided in [15] in terms of Nakajima operators. This is the content of Proposition 3.21. Hence, the main theorem of *loc. cit.* shows that Conjecture 1.2 is true if $X = S^{[2]}$. In the case of the Fano variety of lines we use a result of Fu, Laterveer, Vial and Shen [4] and the explicit description of Markman's lift from Theorem 2.7 to obtain the following partial confirmation of Conjecture 1.2.

Theorem 1.3 *Let $X = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y and $g \in \mathrm{CH}^1(X)$ the Plücker polarization class. Let $L \in \mathrm{CH}^2(X \times X)$ be Markman's lift. Then there is a Lie algebra homomorphism*

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{Q}) &\rightarrow \mathrm{CH}^*(X \times X), \\ e &\mapsto \Delta_*(g), \\ f &\mapsto F_g, \\ h &\mapsto H, \end{aligned}$$

lifting the $\mathfrak{sl}_2(\mathbb{Q})$ -action on cohomology given by the Lefschetz triple (e_g, f_g, h) .

Whenever X has Picard rank 1, i.e., $\mathrm{CH}^1(X) = \langle g \rangle$, which is true for very general cubic fourfolds Y , then Conjecture 1.2 reduces to Theorem 1.3. We also propose two new relations in Conjecture 3.13 which would yield a generalization of Theorem 1.3 to the case of divisor classes different from g , see Proposition 3.15. More evidence for Conjecture 1.2 is provided by Remark 3.12, where we note that $[F_a, F_b] = 0$ in full generality.

The obstacle to proving the conjecture in the case of Picard rank > 1 is that we neither have at our disposal an analog of the machinery of Nakajima operators nor sufficiently strong injectivity results for the cycle class map yet, involving non-tautological divisor classes and hence extending results such as Theorem 2.9, which is the main geometrical input for Theorem 1.3.

1.3 Eigenspace decomposition of H

A natural question concerning the operator H lifting the cohomological grading operator h is that of diagonalizability. This has a positive answer if Markman's lift L satisfies a subset of the core relations necessary to establish a Fourier decomposition of the Chow ring in the sense of [16, Theorem 2], which we recall explicitly below. Moreover, the eigenspace decomposition of H could be expected to be multiplicative with respect to the intersection product because it can be viewed as an analog of the Beauville decomposition in the abelian variety case, as discussed in the introduction of [14]. As it turns out, the Fourier decomposition and the eigenspace decomposition of H actually agree.

In [16], see especially Theorem 1 and the paragraphs after Theorem 2, the following four relations involving Markman's lift L are proven for the Hilbert scheme of two points of a $K3$ surface and the Fano variety of lines of a smooth cubic fourfold. The first is referred to as the *quadratic equation for L* :

$$L^2 = 2\Delta - \frac{2}{r+2}L(l_1 + l_2) - \frac{1}{r(r+2)}(2l_1^2 - r l_1 l_2 + 2l_2^2). \quad (1.1)$$

The remaining three relations involve arbitrary $\sigma \in \mathrm{CH}^4(X)$ and $\tau \in \mathrm{CH}^2(X)$:

$$L_*(l^2) = 0, \quad (1.2)$$

$$L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma), \quad (1.3)$$

$$(L^2)_*(l \cdot (L^2)_*(\tau)) = 0. \quad (1.4)$$

To our knowledge, these relations are not currently known for arbitrary hyperkähler varieties of $K3^{[2]}$ -type.

Theorem 1.4 *Let X be a hyperkähler variety of $K3^{[2]}$ -type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying the relations (1.1)–(1.3). Let $\Lambda_\lambda^i \subseteq \mathrm{CH}^i(X)$ be the eigenspace for the eigenvalue λ of H_* and denote $\mathrm{CH}^i(X)_s := \Lambda_{2i-4-s}^i$. The operator $H_* \in \mathrm{End}_{\mathbb{Q}}(\mathrm{CH}^*(X))$ is diagonalizable with eigenspace decomposition*

$$\begin{aligned} \mathrm{CH}^0(X) &= \mathrm{CH}^0(X)_0, \\ \mathrm{CH}^1(X) &= \mathrm{CH}^1(X)_0, \\ \mathrm{CH}^2(X) &= \mathrm{CH}^2(X)_0 \oplus \mathrm{CH}^2(X)_2, \\ \mathrm{CH}^3(X) &= \mathrm{CH}^3(X)_0 \oplus \mathrm{CH}^3(X)_2, \\ \mathrm{CH}^4(X) &= \mathrm{CH}^4(X)_0 \oplus \mathrm{CH}^4(X)_2 \oplus \mathrm{CH}^4(X)_4. \end{aligned} \quad (1.5)$$

All direct summands outside the leftmost column belong to the homologically trivial cycle classes $\mathrm{CH}^*(X)_{\mathrm{hom}}$. We have

$$\mathrm{CH}^3(X)_2 = \Lambda_0^3 = \mathrm{CH}^3(X)_{\mathrm{hom}}, \quad \mathrm{CH}^4(X)_0 = \Lambda_4^4 = \langle l^2 \rangle, \quad \mathrm{CH}^4(X)_2 = \Lambda_2^4 = l \cdot L_*(\mathrm{CH}^4(X)),$$

so that the cycle class map is injective on $\mathrm{CH}^i(X)_0$ except maybe for $i = 2$. Moreover, all elements of $\mathrm{CH}^4(X)_0 \oplus \mathrm{CH}^4(X)_2$ are multiples of l , and multiplication by l gives an injective map $\mathrm{CH}^1(X) \rightarrow \mathrm{CH}^3(X)_0$. Furthermore, $L_*(\mathrm{CH}^4(X)_4) = 0$.

The number s should be seen as a sort of defect, e.g., the terms with $s = 0$ give the expected eigenvalue $2i - 4$ of H_* . The eigenspace decomposition only shows terms with $s \geq 0$. In fact, there is a conjecture by Beauville in the abelian variety case which predicts this behavior, see [1]. It is also expected that the cycle class map is injective on $\mathrm{CH}^*(X)_0$, and Theorem 1.4 confirms this in all codimensions except 2.

Conjecture 1.5 *Let X be as in Theorem 1.4 such that L additionally satisfies (1.4). For all occurring $s, t \in \mathbb{Z}$ we conjecture that the intersection product gives a well-defined map*

$$\mathrm{CH}^i(X)_s \times \mathrm{CH}^j(X)_t \longrightarrow \mathrm{CH}^{i+j}(X)_{s+t}.$$

This can be rewritten in several ways, see [14, Eq. (6) and (44)]. Evidence for the conjecture is provided by Corollary 1.8 below.

1.4 The Fourier decomposition

We now compare the eigenspace decomposition of H_* in Theorem 1.4 to the Fourier decomposition of [16, Theorem 2] which needs the additional relation (1.4). The Fourier transform is given by the correspondence

$$e^L = (X \times X) + L + \frac{1}{2}L^2 + \frac{1}{6}L^3 + \frac{1}{24}L^4 \in \mathrm{CH}^*(X \times X).$$

The Fourier decomposition groups are defined by

$$e^L \mathrm{CH}^i(X)_s := \{Z \in \mathrm{CH}^i(X) : (e^L)_*(Z) \in \mathrm{CH}^{4-i+s}(X)\}.$$

They do not depend on the precise values of the coefficients of the powers L^k as long as they are non-zero.

Theorem 1.6 ([16, Theorem 2]) *Let X be a hyperkähler variety of $K3^{[2]}$ -type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying all the relations (1.1)–(1.4). Then the decomposition (1.5) equally holds with $\mathrm{CH}^i(X)_s$ replaced by $e^L \mathrm{CH}^i(X)_s$.*

Theorem 1.7 *Let X be a hyperkähler variety of $K3^{[2]}$ -type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying all the relations (1.1)–(1.4). Then the eigenspace decomposition (1.5) of H_* agrees with the Fourier decomposition of Theorem 1.6.*

The only place where the relation (1.4) is needed is the existence of the Fourier decomposition of $\mathrm{CH}^2(X)$, see the proof of [16, Theorem 2.4].

Corollary 1.8 *Assumptions as in Theorem 1.7. In the eigenspace decomposition of Theorem 1.4 all occurring direct summands are non-trivial. Moreover, Conjecture 1.5 is true if $X = S^{[2]}$ is Hilbert scheme of two points of a projective $K3$ surface S and if $X = F(Y)$ is the Fano variety of lines of a very general cubic fourfold Y with L being Markman's lift in both cases.*

Proof For the non-vanishing of every occurring direct summand, see [16, p. 7]. For the multiplicativity in the $S^{[2]}$ case we even have two independent possibilities. One is [16, Theorem 3] and the other is [14, Theorem 1.4] together with Proposition 3.21, showing that the two canonical lifts of the cohomological grading operator h agree. The case of a Fano variety of lines $F(Y)$ of a very general cubic fourfold Y is again [16, Theorem 3]. $\square$

Remark 1.9 It becomes clear from the proof of [16, Theorem 3] in the case of a Fano variety $X = F(Y)$ that the assumption on Y to be very general is only needed for the inclusions

$$\mathrm{CH}^1(X)_0 \cdot \mathrm{CH}^2(X)_0 \subseteq \mathrm{CH}^3(X)_0,$$

$$\mathrm{CH}^2(X)_0 \cdot \mathrm{CH}^2(X)_0 \subseteq \mathrm{CH}^4(X)_0,$$

and the first one has been dealt with for an arbitrary smooth cubic fourfold Y in [4, Proposition A.7]. The second inclusion, however, still remains open if Y is not very general. In the very general case both inclusions are even equalities.

2 Preliminaries

2.1 The Neron–Severi Lie algebra

Let X be a smooth projective variety over $\mathbb{C}$ of complex dimension n . An element $a \in H^2(X, \mathbb{Q})$ is called *Lefschetz* if the conclusion of the hard Lefschetz theorem holds, i.e., for the operator e_a of cup product by a the k -fold iteration $e_a^k : H^{n-k}(X) \rightarrow H^{n+k}(X)$ is an isomorphism for all $1 \leq k \leq n$. The operator e_a is called the *Lefschetz operator*. If a is Lefschetz, there exists a unique operator f_a , the *Lefschetz dual*, such that the commutator $[e_a, f_a] = h$ is the *grading operator*, given by multiplication with the integer $k - n$ in degree k . The triple (e_a, f_a, h) satisfies the $\mathfrak{sl}_2$ -commutation relations and is called a *Lefschetz triple*. [8] and [17] introduced the *total Lie algebra* $\mathfrak{g}(X) \subseteq \mathrm{End}_{\mathbb{Q}}(H^*(X))$ of X which is generated by all Lefschetz triples (e_a, f_a, h) . The *Neron–Severi Lie algebra* of X is the Lie subalgebra $\mathfrak{g}_{\mathrm{NS}}(X) \subseteq \mathfrak{g}(X)$ generated by only those Lefschetz triples where a is algebraic, i.e., $a \in H^{1,1}(X, \mathbb{Q})$. In view of the Grothendieck standard conjecture of Lefschetz type, a natural question to ask, then, is whether the Neron–Severi Lie algebra action on the cohomology ring can be lifted to an action on the Chow ring or, slightly stronger, whether there is a Lie algebra homomorphism $\mathfrak{g}_{\mathrm{NS}}(X) \rightarrow \mathrm{CH}^*(X \times X)$ to the ring of correspondences $\mathrm{CH}^*(X \times X)$ such that

$$\begin{array}{ccc} \mathfrak{g}_{\mathrm{NS}}(X) & \longrightarrow & \mathrm{CH}^*(X \times X) \\ & \searrow & \downarrow \mathrm{cl} \\ & & \mathrm{End}_{\mathbb{Q}}(H^*(X, \mathbb{Q})) \end{array}$$

commutes, where cl denotes the cycle class map. By the main theorem of [15], this is the case if X is the Hilbert scheme of points of a projective $K3$ surface. We investigate this question for the Fano variety of lines of a smooth cubic fourfold, obtaining Theorem 1.3 as a partial analog. We view the Fano variety of lines as a test case for Conjecture 1.2.

2.2 Markman’s lift of the Beauville–Bogomolov class

Let X be a hyperkähler variety and $(-, -) : H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \rightarrow \mathbb{Q}$ the non-degenerate symmetric bilinear form associated to the Beauville–Bogomolov quadratic form on X [6].

Via the Künneth isomorphism we obtain from this the Beauville–Bogomolov cohomology class $\mathfrak{B} \in H^4(X \times X, \mathbb{Q})$. Extending the coefficients to $\mathbb{C}$ and choosing an orthonormal basis (e_i) of $H^2(X, \mathbb{C})$ with respect to $(-, -)$, we can write $\mathfrak{B} = \sum_{i=1}^r e_i \otimes e_i$, where $r = b_2(X)$ is the second Betti number. We denote by $\mathfrak{b} := \Delta^*(\mathfrak{B}) \in H^4(X, \mathbb{Q})$ the pullback along the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Over $\mathbb{C}$ we can write $\mathfrak{b} = \sum_{i=1}^r e_i^2$.

Theorem 2.1 ([10]) *Let X be a hyperkähler variety deformation equivalent to the Hilbert scheme of $n \geq 2$ points of a K3 surface. Then there exists a lift $L \in \text{CH}^2(X \times X)$ of the Beauville–Bogomolov class $\mathfrak{B}$.*

A summary of Markman's construction of L in [10] is given in [16, Section 1.9], particularly Theorem 9.15 where Shen and Vial use the κ -class $\kappa_2(M) \in \text{CH}^2(X \times X)$ of Markman's sheaf M , a twisted sheaf in the sense of [10, Definition 2.1], in order to define L . The twisted sheaf M is constructed as in the theorem below, following closely [16, Theorem 9.12].

Theorem 2.2 ([10]) *Let X be a hyperkähler manifold of K3^[n]-type. There exists a K3 surface S and a suitable Mukai vector v with primitive v_1 and with a v -generic ample divisor H such that there is a proper flat family $\pi : \mathcal{X} \rightarrow C$ of compact hyperkähler manifolds having the following properties:*

- (1) *The curve C is connected but possibly reducible of arithmetic genus 0.*
- (2) *There exist $t_1, t_2 \in C$ such that $\mathcal{X}_{t_1} = X$ and $\mathcal{X}_{t_2} = M_H(v)$, the latter denoting the moduli space of stable sheaves on S with Mukai vector v .*
- (3) *There is a twisted universal sheaf $\mathcal{E}$ on $M_H(v) \times S$.*
- (4) *There is a torsion-free reflexive coherent twisted sheaf $\mathcal{G}$ on $\mathcal{X} \times_C \mathcal{X}$, flat over C , satisfying $\mathcal{G}_{t_2} \cong \mathcal{E} \otimes_{\mathcal{O}_{P_{13}}}^L (\mathcal{E}_{12}, \mathcal{E}_{32})$ on $M_H(v) \times M_H(v)$.*

Then set Markman's sheaf to be the twisted sheaf $M := \mathcal{G}_{t_1}$ on $X \times X$. A priori, M might depend on the chosen deformation, but as it turns out, $\kappa_2(M)$ does not. As in [11] we denote by $\mathfrak{M}_\Lambda$ the moduli space of isomorphism classes of marked hyperkähler manifolds.

Proposition 2.3 ([11]) *The κ -class $\kappa_2(M)$ is independent of the chosen deformation and starting point $M_H(v) \in \mathfrak{M}_\Lambda$.*

In the paragraph preceding Definition 6.16 of [10] the notion of a parametrized twistor path $\gamma : C \rightarrow \mathfrak{M}_\Lambda$ is defined. We stress that Markman requires γ to map the irreducible components of the possibly reducible curve C isomorphically onto twistor lines in the moduli space $\mathfrak{M}_\Lambda$. Now, by [11, Theorem 1.11], the Azumaya algebra $\mathcal{E}nd(M)$ (see [11, Definition 1.1]) associated with M is independent of the twistor path and also of the starting point $M_H(v) \in \mathfrak{M}_\Lambda$ up to possibly dualizing M . Hence, $\kappa_2(M)$ is independent of both as it is a rational multiple of the second Chern class of $\mathcal{E}nd(M)$, see [10, Lemma 2.4]. Shen and Vial now define first

$$l' := \frac{1}{\deg(c_{2n}(T_X))} (p_2)_* \left(\kappa_2(M) \cdot (p_1)^*(c_{2n}(T_X)) \right) + c_2(T_X) \in \text{CH}^2(X),$$

where $p_i : X \times X \rightarrow X$ are the projections and $c_i(T_X)$ are the Chern classes of the tangent bundle of X . This is indeed a lift of $\mathfrak{b}$ by [16, Lemma 9.14]. They then set

$$L := -\kappa_2(M) + \frac{1}{2}(p_1)^*(l' - c_2(T_X)) + \frac{1}{2}(p_2)^*(l' - c_2(T_X)) \in \text{CH}^2(X \times X), \quad (2.1)$$

which is indeed a lift of $\mathfrak{B}$ by the same lemma. By Proposition 2.3, L is, in fact, a canonical lift of its cohomology class, and we are justified in calling L Markman's lift of $\mathfrak{B}$ without

ambiguity. The equation $\Delta^*(L) = l'$ holds if and only if $\Delta^*(\kappa_2(M)) = -c_2(T_X)$ in $\text{CH}^2(X)$. This is the case for the Hilbert scheme of two points of a K3 surface by [16, Eq. (103)] and also for the Fano variety of lines of a smooth cubic fourfold by an argument similar to the one in Remark 2.8 below. We will henceforth only use $l := \Delta^*(L)$.

2.3 Markman's lift in terms of tautological classes for the Hilbert scheme

In the case of the Hilbert scheme $S^{[2]}$ of two points of a projective K3 surface S we have an explicit description of Markman's lift of $\mathfrak{B}$, see [16] for details. Denote $X = S^{[2]}$ and let $\mathcal{X} \subseteq X \times S$ be the universal family. Its set of closed points consists of the pairs (η, x) where $x \in \text{supp}(\eta)$. This is a codimension 2 closed subscheme of the product, and we denote by

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \swarrow p & \downarrow & \searrow q & \\ X & \xleftarrow{\pi} & X \times S & \xrightarrow{\rho} & S \end{array}$$

the projections. Let $c \in \text{CH}_0(S)$ be the canonical 0-cycle, represented by any point on a rational curve in S [3, Theorem 1]. We let

$$S_c := p_* q^*(c) \in \text{CH}^2(X).$$

Moreover, denote by $\Delta_{\text{Hilb}} \in \text{CH}^1(X)$ the divisor class of X parametrizing the non-reduced length 2 subschemes of S and set

$$\delta := \frac{1}{2} \Delta_{\text{Hilb}} \in \text{CH}^1(X).$$

This agrees with the convention of [16] which differs in the sign from [15]. Finally, let $I \subseteq X \times X$ be the subset of pairs of length 2 subschemes which share a common support point. This is closed and irreducible (see the proof of [16, Lemma 11.2]), and endowed with the reduced induced subscheme structure it gives a closed subvariety of codimension 2, called the *incidence subscheme*. Its cycle class in $\text{CH}^2(X \times X)$ is also denoted I and called the *incidence correspondence*. By *loc. cit.* we have $I = {}^t \mathcal{X} \circ \mathcal{X}$.

Theorem 2.4 ([16, p. 67]) *Let $X = S^{[2]}$ for a projective K3 surface S . An explicit lift of the Beauville–Bogomolov class $\mathfrak{B}$ is given by*

$$L = I - 2(S_c)_1 - 2(S_c)_2 - \frac{1}{2} \delta_1 \delta_2. \quad (2.2)$$

Moreover, by [16, Proposition 16.1], L agrees with Markman's lift. For its pullback along the diagonal embedding $\Delta : X \hookrightarrow X \times X$ we have

$$l = \Delta^*(L) = 20S_c - \frac{5}{2} \delta^2. \quad (2.3)$$

In fact, $l = \frac{5}{6} c_2(T_X)$ where $c_2(T_X)$ is the second Chern class of the tangent bundle.

2.4 Markman's lift in terms of tautological classes for the Fano variety

For the Fano variety of lines $X = F(Y) \subseteq \text{Gr}(2, 6)$ of a smooth cubic fourfold Y we define several tautological cycle classes as follows.

Definition 2.5 Let $\mathcal{E}$ be the tautological bundle on $\mathrm{Gr}(2, 6)$. Let $g \in \mathrm{CH}^1(X)$ be the first Chern class $c_1(\mathcal{E}|_{F(Y)}^\vee)$. Then g is called the *Plücker polarization class*. Moreover, we denote by $c \in \mathrm{CH}^2(X)$ the second Chern class $c_2(\mathcal{E}|_{F(Y)}^\vee)$.

Definition 2.6 The *incidence subscheme* is the closed subset $I \subseteq X \times X$ with the reduced subscheme structure, given by the set of pairs of intersecting lines inside Y . Its cycle class in $\mathrm{CH}^2(X \times X)$, also denoted I , is called the *incidence correspondence*. The *tautological subring* $R^*(X \times X) \subseteq \mathrm{CH}^*(X \times X)$ is the $\mathbb{Q}$ -subalgebra generated by $I, \Delta, c_1, c_2, g_1, g_2$ where $\Delta \subseteq X \times X$ denotes the diagonal and g_i, c_i for $i = 1, 2$ are the pullbacks via the two projections $X \times X \rightarrow X$.

Theorem 2.7 ([16, p. 81]) *Let Y be a smooth cubic fourfold and $X = F(Y)$ its Fano variety of lines. An explicit lift L of the Beauville–Bogomolov class $\mathfrak{B}$ is given by*

$$L = \frac{1}{3}(g_1^2 + \frac{3}{2}g_1g_2 + g_2^2 - c_1 - c_2) - I. \quad (2.4)$$

Moreover, we have $l = \Delta^*(L) = \frac{5}{6}c_2(T_X)$, where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding and $c_2(T_X)$ is the second Chern class of the tangent bundle, satisfying $c_2(T_X) = 5g^2 - 8c$. Hence, the tautological subring $R^*(X \times X)$ contains L, l_1, l_2 . By the quadratic equation (1.1), $R^*(X \times X)$ can in fact be generated by L, l_1, l_2, g_1, g_2 .

Remark 2.8 Let M be Markman’s sheaf on $F(Y) \times F(Y)$ as in Theorem 2.2. By [9, Corollary 1.6] and [11], the Azumaya algebra $\mathcal{E}nd(M)$ is universally defined over the moduli space of Fano varieties. Hence, by [4, Theorem 1.10], the explicit lift of $\mathfrak{B}$ from (2.4), which clearly is universally defined as well, agrees with Markman’s lift (2.1).

The following result is crucial for the proof of Theorem 1.3.

Theorem 2.9 ([4, Proposition 6.4]) *Let $X = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y . Then the restriction of the cycle class map to*

$$\mathrm{cl} : R^\bullet(X \times X) \rightarrow H^{2\bullet}(X \times X)$$

is injective.

The four relations (1.1)–(1.4) in the Fano case all follow more directly from cohomology, Theorem 2.9 and the fact that the tautological subring $R^*(X \times X)$ is closed under the composition of correspondences. The latter is a consequence of [4, Proposition 6.3]. Indeed, relations (1.2)–(1.4) can be stated in an obvious way (and slightly more generally) as equations of correspondences in $\mathrm{CH}^*(X \times X)$.

3 Proofs of the main results

3.1 Hyperkähler fourfolds

Let X be a hyperkähler variety of complex dimension 4 without odd cohomology over $\mathbb{Q}$ or, equivalently, $H^3(X) = 0$. Then the cup product is commutative and we always write $\alpha\beta$ instead of $\alpha \cup \beta$. By (e_i) we denote an arbitrary orthonormal basis of $H^2(X, \mathbb{C})$ with respect to the Beauville–Bogomolov form, and we denote the second Betti number by $r := b_2(X)$. We normalize the Fujiki constant c_X such that $\int_X \beta^4 = 3c_X(\beta, \beta)^2$ for all $\beta \in H^2(X)$. For hyperkähler varieties of $K3^{[2]}$ -type we then have $c_X = 1$. Finally, we write $\mathbf{1} \in H^8(X)$ for the generator of $H^8(X)$ with integral 1.

Proposition 3.1 Let $a \in H^2(X, \mathbb{Q})$ and define $\tilde{f}_a : H^\bullet(X) \rightarrow H^{\bullet-2}(X)$ by

$$\tilde{f}_a(\beta) := \begin{cases} 0 & \beta \in H^0(X), \\ 4(a, \beta)[X] & \beta \in H^2(X), \\ \frac{2}{c_X} \mathfrak{B}_*(a\beta) & \beta \in H^4(X), \\ \frac{2}{c_X} a \mathfrak{B}_*(\beta) & \beta \in H^6(X), \\ \frac{4}{(r+2)c_X} (\int_X \beta) \mathfrak{b} a & \beta \in H^8(X). \end{cases}$$

Then $\tilde{f}_a$ satisfies $[e_a, \tilde{f}_a] = (a, a)h$ and $[h, \tilde{f}_a] = -2\tilde{f}_a$. Moreover, $[h, e_a] = 2e_a$.

If $(a, a) \neq 0$, we set $f_a := \frac{1}{(a, a)} \tilde{f}_a$ to obtain the usual $\mathfrak{sl}_2(\mathbb{Q})$ -commutation relations. By Lemma 3.5 below, $(a, a) \neq 0$ is equivalent to a being Lefschetz in the sense of Sect. 2.1.

Remark 3.2 If $(a, a) \neq 0$, then $\tilde{f}_a$ is uniquely determined by the commutation relations while this fails if $(a, a) = 0$ in which case the zero map also satisfies them. It should be emphasized also that $\tilde{f}_a$ is linear in $a \in H^2(X)$ because for one, f_a is not, and neither is it clear from the abstract description of f_a that it would suffice to multiply it by some quadratic form (a, a) in order to make it linear in a .

Before proving Proposition 3.1, we need some lemmas. They are simple computations in cohomology using the definition of the Beauville–Bogomolov form, so we omit the proofs.

Lemma 3.3 For arbitrary $\gamma, \gamma' \in H^2(X)$ we have $\int_X \mathfrak{b} \gamma \gamma' = c_X(r+2)(\gamma, \gamma')$.

In the same vein and using that $(-, -)$ is non-degenerate, one proves:

Lemma 3.4 The linear map $H^2(X) \rightarrow H^6(X)$ given by the cup product with $\frac{1}{(r+2)c_X} \mathfrak{b}$ is an isomorphism with inverse $\mathfrak{B}_*$.

Lemma 3.5 Let $a \in H^2(X, \mathbb{Q})$. Then the following are equivalent:

- (1) $(a, a) \neq 0$,
- (2) $e_a^2 : H^2(X) \rightarrow H^6(X)$ is an isomorphism,
- (3) $e_a^4 : H^0(X) \rightarrow H^8(X)$ is an isomorphism.

In particular, a is Lefschetz if and only if $(a, a) \neq 0$.

Proof The equation $\int_X a^4 = 3c_X(a, a)^2$ proves that $(a, a) \neq 0$ if and only if e_a^4 is an isomorphism. Next, for $(a, a) \neq 0$ let $\beta \in H^2(X)$ be in the kernel of e_a^2 , i.e., $a^2\beta = 0$. It suffices to show $\beta = 0$ by Poincaré duality. Now, if $a^2\beta = 0$, then $a^2\beta\gamma = 0$ for every $\gamma \in H^2(X)$. Setting $\gamma = a$ gives $0 = \int_X a^3\beta = 3c_X(a, a)(a, \beta)$, hence $(a, \beta) = 0$. Then for arbitrary γ we get $0 = \int_X a^2\beta\gamma = c_X(a, a)(\beta, \gamma) + 2c_X(a, \beta)(a, \gamma) = c_X(a, a)(\beta, \gamma)$, so $\beta = 0$ as $(-, -)$ is non-degenerate. Conversely, let e_a^2 be an isomorphism. Then certainly $a \neq 0$, and so $0 \neq e_a^2(a) = a^3$. By Poincaré duality, there is some $\beta \in H^2(X)$ with $a^3\beta \neq 0$, so $0 \neq \int_X a^3\beta = 3c_X(a, a)(a, \beta)$, in particular $(a, a) \neq 0$. $\square$

Proof of Proposition 3.1 We show $[e_a, \tilde{f}_a] = (a, a)h$ case by case. For $\beta \in H^0(X)$ assume by linearity $\beta = [X]$, in which case indeed

$$[e_a, \tilde{f}_a]([X]) = 0 - \tilde{f}_a(a) = -4(a, a)[X] = (a, a)h([X]).$$

For $\beta \in H^2(X)$, we first compute

$$\mathfrak{B}_*(a^2\beta) = (p_2)_* \left(\sum_{i=1}^r (a^2\beta e_i) \otimes e_i \right)$$

$$\begin{aligned}
&= \sum_{i=1}^r \left(\int_X a^2 \beta e_i \right) e_i \\
&= c_X \sum_{i=1}^r ((a, a)(\beta, e_i) + 2(a, \beta)(a, e_i)) e_i \\
&= c_X(a, a)\beta + 2c_X(a, \beta)a.
\end{aligned}$$

Hence,

$$[e_a, \tilde{f}_a](\beta) = a\tilde{f}_a(\beta) - \tilde{f}_a(a\beta) = 4(a, \beta)a - \frac{2}{c_X}\mathfrak{B}_*(a^2\beta) = -2(a, a)\beta = (a, a)h(\beta),$$

as desired. For $\beta \in H^4(X)$, we have $[e_a, \tilde{f}_a](\beta) = \frac{2}{c_X}a\mathfrak{B}_*(a\beta) - \frac{2}{c_X}a\mathfrak{B}_*(a\beta) = 0$. In the case $\beta \in H^6(X)$, by Lemma 3.4, we can write $\beta = \frac{1}{r+2}\mathfrak{b}\hat{\beta}$ for a unique $\hat{\beta} \in H^2(X)$. As $h(\beta) = 2\beta$ we need to show $[e_a, \tilde{f}_a](\beta) - 2(a, a)\beta = 0$, which, by Poincaré duality, is equivalent to $\int_X ([e_a, \tilde{f}_a](\beta) - 2(a, a)\beta) \gamma = 0$ for all $\gamma \in H^2(X)$. Indeed,

$$\begin{aligned}
\int_X ([e_a, \tilde{f}_a](\beta) - 2(a, a)\beta) \gamma &= \int_X (2a^2\hat{\beta}\gamma) \\
&\quad - \int_X \left(\frac{4}{(r+2)^2} \left(\int_X \mathfrak{b}\hat{\beta}a \right) \mathfrak{b}a\gamma \right) - \int_X \left(\frac{2(a, a)}{r+2} \mathfrak{b}\hat{\beta}\gamma \right) \\
&= c_X (2(a, a)(\hat{\beta}, \gamma) + 4(a, \hat{\beta})(a, \gamma)) \\
&\quad - 4c_X(a, \hat{\beta})(a, \gamma) - 2c_X(a, a)(\hat{\beta}, \gamma) \\
&= 0.
\end{aligned}$$

Finally, let $\beta \in H^8(X)$. Here,

$$[e_a, \tilde{f}_a](\beta) = a\tilde{f}_a(\beta) - 0 = \frac{4}{(r+2)c_X} \left(\int_X \beta \right) \mathfrak{b}a^2 = 4(a, a)\beta = (a, a)h(\beta),$$

using Lemma 3.3 and the fact that $H^8(X)$ is of dimension 1. The relations $[h, \tilde{f}_a] = -2\tilde{f}_a$ and $[h, e_a] = 2e_a$ follow directly from the fact that $\tilde{f}_a$ decreases and e_a increases the degree by 2.

Corollary 3.6 *Let X be a hyperkähler variety of complex dimension 4 with $H^3(X, \mathbb{Q}) = 0$. Let $\mathfrak{B}$ admit a lift $L \in \text{CH}^2(X \times X)$, and as before we set $l := \Delta^*(L)$. Then the cycle class*

$$\widetilde{F}_a := \frac{4}{(r+2)c_X}(l_1a_1 + l_2a_2) + \frac{2}{c_X}L(a_1 + a_2) \in \text{CH}^3(X \times X)$$

is a lift of $\tilde{f}_a$.

Definition 3.7 Under the assumptions of Corollary 3.6 we set $F_a := \frac{1}{(a, a)}\widetilde{F}_a$ if $(a, a) \neq 0$. Moreover, we denote the commutator by

$$\widetilde{H}_a := [\Delta_*(a), \widetilde{F}_a] = (a_2 - a_1)\widetilde{F}_a$$

and $H_a := \frac{1}{(a, a)}\widetilde{H}_a$ for $(a, a) \neq 0$.

Automatically, $\widetilde{H}_a$ is a lift of $(a, a)h$ where h is the cohomological grading operator.

Proposition 3.8 For all $a \in H^2(X, \mathbb{Q})$ the following relation holds in $H^8(X \times X, \mathbb{Q})$:

$$(a, a)\mathfrak{B}b_1 = (r+2)\mathfrak{B}a_1^2 - 2b_1a_1a_2.$$

Proof First observe that it suffices to show

$$((r+2)\mathfrak{B}a_1^2 - 2b_1a_1a_2 - (a, a)\mathfrak{B}b_1)\gamma_1 = 0$$

for all $\gamma \in H^2(X)$ by Poincaré duality. A direct computation then shows indeed

$$\begin{aligned} \text{LHS} &= (r+2)\mathfrak{B}a_1^2\gamma_1 - 2b_1a_1a_2\gamma_1 - (a, a)\mathfrak{B}b_1\gamma_1 \\ &= (r+2) \sum_{i=1}^r (e_i a^2 \gamma) \otimes e_i - 2 \sum_{i=1}^r (e_i^2 a \gamma) \otimes a - (a, a) \sum_{i,j=1}^r (e_i e_j^2 \gamma) \otimes e_i \\ &= (r+2)c_X \cdot \mathbf{1} \otimes \left(\sum_{i=1}^r e_i (2(a, \gamma)(a, e_i) + (a, a)(\gamma, e_i)) \right) \\ &\quad - 2c_X \cdot \mathbf{1} \otimes a \cdot \sum_{i=1}^r ((a, \gamma) + 2(a, e_i)(\gamma, e_i)) \\ &\quad - c_X(a, a) \cdot \mathbf{1} \otimes \left(\sum_{i,j=1}^r e_i ((\gamma, e_i) + 2\delta_{ij}(\gamma, e_j)) \right) \\ &= 2(r+2)c_X(a, \gamma) \cdot \mathbf{1} \otimes a + (r+2)c_X(a, a) \cdot \mathbf{1} \otimes \gamma \\ &\quad - 2(r+2)c_X(a, \gamma) \cdot \mathbf{1} \otimes a \\ &\quad - (r+2)c_X(a, a) \cdot \mathbf{1} \otimes \gamma \\ &= 0. \end{aligned}$$

□

An easy consequence of Proposition 3.8 is the next relation.

Proposition 3.9 For all $a \in H^2(X, \mathbb{Q})$ we have $r\mathfrak{B}b_1a_1 = b_1^2a_2$ in $H^{10}(X \times X, \mathbb{Q})$.

In the same vein, one shows:

Proposition 3.10 For all $a \in H^2(X, \mathbb{Q})$ the following relation holds in $H^{10}(X \times X, \mathbb{Q})$:

$$(r+2)\mathfrak{B}^2a_1 = 2\mathfrak{B}b_1a_2 + b_1b_2a_1.$$

Proposition 3.11 Assumptions as in Corollary 3.6. Then

$$H := \frac{4}{r(r+2)}(l_2^2 - l_1^2) + \frac{2}{r+2}(l_2 - l_1)L \in \text{CH}^4(X \times X)$$

is a lift of the cohomological grading operator h .

Proof Transposing the relation from Proposition 3.8 and subtracting the two equations yields

$$(a, a)\mathfrak{B}(b_2 - b_1) = (r+2)\mathfrak{B}(a_2^2 - a_1^2) - 2a_1a_2(b_2 - b_1).$$

Now,

$$\begin{aligned}
\frac{r+2}{2}[\widetilde{H}_a] &= \frac{r+2}{2}(a_2 - a_1)[\widetilde{F}_a] \\
&= (r+2)\mathfrak{B}(a_2^2 - a_1^2) - 2a_1a_2(b_2 - b_1) + 2(b_2a_2^2 - b_1a_1^2) \\
&= (a, a)\mathfrak{B}(b_2 - b_1) + 2(b_2a_2^2 - b_1a_1^2) \\
&= (a, a)\mathfrak{B}(b_2 - b_1) + \frac{2}{r}(a, a)(b_2^2 - b_1^2) \\
&= \frac{r+2}{2}(a, a)[H].
\end{aligned}$$

We used the above equation in line three and Lemma 3.3 in line four. Hence, $\widetilde{H}_a$ and $(a, a)H$ agree in cohomology, and therefore H is a lift of h as long as there exists some a with $(a, a) \neq 0$ which is always the case for non-zero symmetric bilinear forms over a field of characteristic $\neq 2$. $\square$

The following proof is inspired by the proof of [16, Theorem 2.2].

Proof of Theorem 1.4 In codimensions 0 and 1 the cycle class map injects into cohomology. Also, $L_*(\mathrm{CH}^i(X)) \subseteq \mathrm{CH}^{i-2}(X)$ and $(l_1^2)_*$ acts only on $\mathrm{CH}^0(X)$ while $(l_2^2)_*$ acts only on $\mathrm{CH}^4(X)$. Now, for $Z \in \mathrm{CH}^2(X)$ we have $L_*(Z) = 0$ by cohomology. Hence, $H_*(Z) = \frac{-2}{r+2}L_*(lZ)$. By the hypotheses, $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \mathrm{CH}^4(X)$, so

$$\begin{aligned}
((H_* + 2\mathrm{id}) \circ H_*)(Z) &= (H_* + 2\mathrm{id})\left(\frac{-2}{r+2}L_*(lZ)\right) \\
&= \frac{4}{(r+2)^2}L_*(l \cdot L_*(lZ)) - \frac{4}{r+2}L_*(lZ) = 0,
\end{aligned}$$

giving the decomposition of $\mathrm{CH}^2(X)$. Next, let $Z \in \mathrm{CH}^3(X)$. Then $H_*(Z) = \frac{2}{r+2}lL_*(Z)$, and $L_*(Z)$ is a divisor class. Therefore and by Lemma 3.4, $L_*(Z) = 0$ if and only if $Z \in \mathrm{CH}^3(X)_{\mathrm{hom}}$ is homologically trivial. Again by Lemma 3.4, in cohomology the cup product by $\frac{1}{r+2}[l]$ is the inverse isomorphism of $[L]_* : H^6(X) \rightarrow H^2(X)$. Hence, if $H_*(Z) = 0$, then in cohomology

$$0 = [H_*(Z)] = \frac{2}{r+2}[l] \cup [L_*(Z)] = 2[Z],$$

hence $Z \in \mathrm{CH}^3(X)_{\mathrm{hom}}$. We have shown $\Lambda_0^3 = \mathrm{CH}^3(X)_{\mathrm{hom}}$. Next, by the quadratic equation for L , we have $((L^2)_* - 2\mathrm{id})(Z) = \frac{-2}{r+2}lL_*(Z) = -H_*(Z)$, or equivalently,

$$(H_* - 2\mathrm{id})(Z) = -(L^2)_*(Z).$$

But here, $(L^2)_*(Z) \in \mathrm{CH}^3(X)_{\mathrm{hom}} = \Lambda_0^3$ for degree reasons. Therefore, $H_* \circ (H_* - 2\mathrm{id}) = 0$ on $\mathrm{CH}^3(X)$, giving the desired decomposition. In order to see that $l \cdot D$ is an element of Λ_2^3 for every divisor class D , just note $H_*(l \cdot D) = \frac{2}{r+2}l \cdot L_*(lD)$, and $L_*(lD)$ is a divisor which in cohomology agrees with $(r+2)D$. At last, let $Z \in \mathrm{CH}^4(X)$. Then

$$H_*(Z) = \frac{4}{r(r+2)}\left(\int_X [Z]\right)l^2 + \frac{2}{r+2}lL_*(Z).$$

Now, $\int_X [H_*(Z)] = 4 \int_X [Z]$, implying

$$H_*(H_*(Z)) = \frac{16}{r(r+2)}\left(\int_X [Z]\right)l^2 + \frac{4}{r+2}lL_*(Z),$$

in particular $(H_*^2 - 4H_*)(Z) = \frac{-4}{r+2}lL_*(Z)$. Applying H_* once again yields

$$(H_* \circ (H_*^2 - 4H_*))(Z) = \frac{-8}{r+2}lL_*(Z) = 2(H_*^2 - 4H_*)(Z),$$

using that $\int_X [lL_*(Z)] = 0$. This is because $L_*(Z) \in \text{CH}^2(X)_{\text{hom}}$ for degree reasons. We have seen, then, that $H_* \circ (H_*^2 - 4H_*) = 2(H_*^2 - 4H_*)$, or equivalently,

$$H_* \circ (H_* - 2\text{id}) \circ (H_* - 4\text{id}) = 0,$$

giving the desired decomposition for $\text{CH}^4(X)$. Finally, let $Z \in \Lambda_4^4$. We want to show that Z is a multiple of l^2 . Indeed,

$$4Z = H_*(Z) = l^2 \cdot \frac{4}{r(r+2)} \int_X [Z] + \frac{2}{r+2} l \cdot L_*(Z).$$

Applying L_* to the equation and using again the hypothesis $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \text{CH}^4(X)$, we get

$$4L_*(Z) = L_*(l^2) \cdot \frac{4}{r(r+2)} \int_X [Z] + \frac{2}{r+2} L_*(l \cdot L_*(Z)) = 2L_*(Z),$$

hence $L_*(Z) = 0$. But then, the previous equation implies that Z is a multiple of l^2 . A similar argument shows $L_*(\Lambda_0^4) = 0$. The equation $\Lambda_2^4 = l \cdot L_*(\text{CH}^4(X))$ is immediate from the explicit formula for H after observing $\Lambda_2^4 \subseteq \text{CH}^4(X)_{\text{hom}}$.

Proof of Theorem 1.7 In codimensions 0 and 1 there is nothing to show. For codimension 2 it suffices to show only the two inclusions

$$e^L \text{CH}^2(X)_0 \subseteq \text{CH}^2(X)_0 \quad \text{and} \quad e^L \text{CH}^2(X)_2 \subseteq \text{CH}^2(X)_2,$$

because in both cases their sum is $\text{CH}^2(X)$, and the sums are direct. Let $Z \in e^L \text{CH}^2(X)_0$, i.e., $(e^L)_*(Z) \in \text{CH}^2(X)$ which is easily seen to be equivalent to $(L^3)_*(Z) = 0$. Now, applying the quadratic equation for L twice, it can be checked that

$$L^3 = \frac{2r}{r+2} \Delta_*(l) + \frac{r+10}{(r+2)^2} Ll_1l_2 + \text{cst}_1 \cdot L(l_1^2 + l_2^2) + \text{cst}_2 \cdot (l_1^2l_2 + l_1l_2^2),$$

where we used $L \cdot \Delta = L \cdot \Delta_*(X) = \Delta_*(\Delta^*(L)) = \Delta_*(l)$. Applying this to Z gives

$$0 = (L^3)_*(Z) = \frac{2r}{r+2} l \cdot Z + \frac{r+10}{(r+2)^2} l \cdot L_*(l \cdot Z) + \text{cst} \cdot l^2.$$

Applying L_* then yields $L_*(l \cdot Z) = 0$ by the relations $L_*(l^2) = 0$ and $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \text{CH}^4(X)$. But now, the explicit formula for H yields $H_*(Z) = \frac{2}{r+2} l \cdot L_*(Z) - \frac{2}{r+2} L_*(l \cdot Z) = 0$. For the second inclusion let $Z \in e^L \text{CH}^2(X)_2$ which is now equivalent to $(L^2)_*(Z) = 0$. Using the quadratic equation for L , we get

$$0 = (L^2)_*(Z) = 2Z - \frac{2}{r+2} L_*(l \cdot Z) + \frac{1}{r+2} \left(\int_X [l \cdot Z] \right) l.$$

Now, by [16, Proposition 4.1], $l \cdot Z \in e^L \text{CH}^4(X)_2$, and the latter by [16, Theorem 4] agrees with $l \cdot L_*(\text{CH}^4(X)) \subseteq \text{CH}^4(X)_{\text{hom}}$, hence $\int_X [l \cdot Z] = 0$. We thus obtain $L_*(l \cdot Z) = (r+2)Z$, and this is precisely equivalent to $H_*(Z) = 2Z$. In codimension 3, let first $Z \in e^L \text{CH}^3(X)_2$. This is equivalent to $L_*(Z) = 0$. But now, $H_*(Z) = \frac{2}{r+2} l \cdot L_*(Z)$, so $e^L \text{CH}^3(X)_2 =$

$\mathrm{CH}^3(X)_2$, as desired. Next, $Z \in {}^{e^L}\mathrm{CH}^3(X)_0$ is equivalent to $(L^2)_*(Z) = 0$, i.e., by the quadratic equation for L ,

$$0 = (L^2)_*(Z) = 2Z - \frac{2}{r+2}l \cdot L_*(Z),$$

hence $H_*(Z) = \frac{2}{r+2}l \cdot L_*(Z) = 2Z$, concluding the codimension 3 case. In codimension 4, by [16, Theorem 4], we already know the equality of the direct summands

$$\begin{aligned} {}^{e^L}\mathrm{CH}^4(X)_0 &= \langle l^2 \rangle = \Lambda_4^4 = \mathrm{CH}^4(X)_0, \\ {}^{e^L}\mathrm{CH}^4(X)_2 &= l \cdot L_*(\mathrm{CH}^4(X)) = \Lambda_2^4 = \mathrm{CH}^4(X)_2. \end{aligned}$$

Finally, let $Z \in {}^{e^L}\mathrm{CH}^4(X)_4$ which is equivalent to the vanishing of both $L_*(Z)$ and $\int_X [Z]$. Thus,

$$H_*(Z) = \frac{4}{r(r+2)} \left(\int_X [Z] \right) l^2 + \frac{2}{r+2}l \cdot L_*(Z) = 0,$$

i.e., $Z \in \Lambda_0^4 = \mathrm{CH}^4(X)_4$.

Remark 3.12 Let X be a hyperkähler variety of complex dimension 4 with $H^3(X) = 0$ and fix a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$. Then for all divisor classes $a, b \in \mathrm{CH}^1(X)$ we have $[\widetilde{F}_a, \widetilde{F}_b] = 0$. We omit the proof as it is a straightforward formal computation, using that all terms symmetric in a and b cancel out and that $(p_{13})_*(L_{12} \cdot L_{23} \cdot a_2)$ is a divisor on $X \times X$ vanishing in cohomology.

Conjecture 3.13 Let X be any hyperkähler variety of $K3^{[2]}$ -type and let L be Markman's lift. We conjecture the following relations in the Chow ring for all divisor classes $a \in \mathrm{CH}^1(X)$ with $(a, a) \neq 0$:

$$(a, a)Ll_1 = (r+2)La_1^2 - 2l_1a_1a_2, \quad (3.1)$$

$$rLl_1a_1 = l_1^2a_2, \quad (3.2)$$

$$(r+2)L^2a_1 = 2Ll_1a_2 + l_1l_2a_1. \quad (3.3)$$

Remark 3.14 In the case of the Fano variety of lines, the second relation (3.2) follows from (3.1) by [18, Theorem 1.4], showing that $(r+2)a^3 = 3(a, a)la$ and from pulling back (3.1) along the diagonal, obtaining $(a, a)l^2 = rla^2$. Moreover, (3.1) and its transpose would imply the important equality $H = H_a$ by the proof of Proposition 3.11 carried out in the Chow ring, replacing $\mathfrak{B}$ by L and $\mathfrak{b}$ by $l = \Delta^*(L)$. This equality is necessary for establishing Conjecture 1.2. Note also that we indirectly show $H = H_a$ in the Hilbert scheme setting in Sect. 3.3, see Remark 3.22. In the Hilbert scheme setting, the above relations (3.1)–(3.3) can be checked by computations similar to those in Sect. 3.4, and we expect them to hold true.

If Markman's lift satisfies relations (1.1) and (1.2) in general, then under Conjecture 3.13 we have the following generalization of Theorem 1.3.

Proposition 3.15 Let X be a hyperkähler variety of $K3^{[2]}$ -type endowed with a symmetric lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying the relations (1.1) and (1.2). Let $a \in \mathrm{CH}^1(X)$ be Lefschetz and such that the relations of Conjecture 3.13 hold with respect to L . Then the Lie algebra homomorphism

$$\mathfrak{sl}_2(\mathbb{Q}) \rightarrow \mathrm{CH}^*(X \times X),$$

$$e \mapsto \Delta_*(a),$$

$$f \mapsto F_a,$$

$$h \mapsto H,$$

is well-defined.

Proof We need to show the three commutation relations

$$[H, \Delta_*(a)] = 2\Delta_*(a), \quad [H, F_a] = -2F_a, \quad \text{and} \quad [\Delta_*(a), F_a] = H.$$

For the last one of these we already indicated this in Remark 3.14. For the first one, observe that the left hand side equals

$$(a_1 - a_2)H = \frac{4}{r(r+2)}(l_1^2 a_2 + l_2^2 a_1) + \frac{2}{r+2}L(l_1 a_2 + l_2 a_1) - \frac{2}{r+2}L(l_1 a_1 + l_2 a_2).$$

For the right hand side use the quadratic equation (1.1) and the second and third relation of the conjecture. Then

$$2\Delta_*(a) = \frac{2}{r+2}L(l_1 a_2 + l_2 a_1) + \frac{2}{r(r+2)}(l_1^2 a_1 + l_2^2 a_2).$$

Hence, the difference equals $\frac{2}{r(r+2)}(l_1^2 a_2 + l_2^2 a_1) - \frac{2}{r+2}L(l_1 a_1 + l_2 a_2) = 0$ by another application of the second relation of the conjecture and its transpose. For the second commutation relation we show more generally $[H, \widetilde{F}_a] = -2\widetilde{F}_a$. Here, consider first the composition $H \circ \widetilde{F}_a$. The only summand of the latter which is not easily computed using equation (1.2) and the second relation of Conjecture 3.13 is

$$-\frac{4}{r+2}(p_{13})_*\left((p_{12})^*(Ll_2 a_1) \cdot (p_{23})^*(L)\right),$$

where $p_{ij} : X \times X \times X \rightarrow X \times X$ denote the projections to the factors according to the indices. Here, we use the transpose of the third relation of the conjecture in order to write $Ll_2 a_1 = \frac{r+2}{2}L^2 a_2 - \frac{1}{2}l_1 l_2 a_2$. From this we deduce that the above summand equals

$$2l_1 a_2 - 2(p_{13})_*\left((p_{13})^*(L^2) \cdot (p_{23})^*(La_1)\right).$$

Next, using the quadratic equation (1.1) we obtain that the latter agrees with $-4La_1$. We obtain $H \circ \widetilde{F}_a = -\frac{16}{r+2}l_1 a_1 + \frac{8}{r+2}l_2 a_2 - 4La_1$. Now, ${}^t H = -H$ and ${}^t \widetilde{F}_a = \widetilde{F}_a$, so

$$[H, \widetilde{F}_a] = -4L(a_1 + a_2) - \frac{8}{r+2}(l_1 a_1 + l_2 a_2) = -2\widetilde{F}_a,$$

as desired. $\square$

3.2 Fano variety of lines of a smooth cubic fourfold

Proof of Theorem 1.3 By Corollary 3.6, F_g lifts f_g , and by Proposition 3.11, H lifts h . Moreover, $\Delta_*(g)$ lifts e_g . By the explicit formulas for $\Delta_*(g) = \Delta g_1$, F_g and H we know that all of them lie in the tautological subring $R^*(F \times F)$ of Theorem 2.9. As a consequence of [4, Proposition 6.3], so do their compositions. Hence all the commutators lie in $R^*(F \times F)$, and as the commutation relations are true in cohomology, they are true in the Chow ring as well.

Proposition 3.16 *Conjecture 3.13 is true for the Fano variety of lines $X = F(Y)$ of a smooth cubic fourfold Y if a is a multiple of the Plücker polarization class g . In particular, $H = H_g$.*

Proof By the foregoing Propositions 3.8–3.10 the relations are true in cohomology for all a . If a is a multiple of g , then by Theorem 2.9 they hold in the Chow ring as well because all occurring terms then lie in the tautological subring $R^*(X \times X)$. Moreover, $(g, g) \neq 0$ by [2]. $\square$

3.3 Hilbert scheme of two points of a K3 surface

Let S be a smooth projective surface. Recall the Nakajima operators on the Chow ring of Hilbert schemes of points of S ,

$$q_i : \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n]}) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n+i]} \times S).$$

We refer to [5, 13] and to [15, Sect. 2.3] for a brief overview. Recall that q_i for $i \geq 1$ is given as a correspondence by the reduced closed subscheme

$$Z_{n,n+i} := \{(\xi, x, \eta) \in S^{[n]} \times S \times S^{[n+i]} : \mathrm{supp}(\mathcal{I}_\xi / \mathcal{I}_\eta) = \{x\}\}$$

of the product $S^{[n]} \times S \times S^{[n+i]}$, where $\mathcal{I}_\xi$ is the ideal sheaf corresponding to the length n closed subscheme ξ of S . Recall also that q_{-i} is just the transpose of $(-1)^i q_i$. Let $\mathcal{Z}_n \subseteq S^{[n]} \times S$ be the universal family. We write $\mathcal{Z} := \mathcal{Z}_2$.

Lemma 3.17 *Let $n \geq 1$ and $p_{S \times S^{[n]}} : S^{[n-1]} \times S \times S^{[n]} \rightarrow S \times S^{[n]}$ be the projection. Then in $\mathrm{CH}^*(S \times S^{[n]})$ we have*

$$(p_{S \times S^{[n]}})_* (Z_{n-1,n}) = {}^t \mathcal{Z}_n,$$

where ${}^t \mathcal{Z}_n$ is the transpose of $\mathcal{Z}_n$.

Proof First recall that $Z_{n-1,n}$ is a closed subvariety of the product. This can be seen using the projection map $Z_{n-1,n} \rightarrow S^{[n-1,n]}$ onto the nested Hilbert scheme [7, Sect. 1.2] and the fact that $S^{[n-1,n]}$ is irreducible by [7, Theorem 1.9]. Indeed, the projection is proper, hence closed, and it actually is a bijection on closed points. It follows that $Z_{n-1,n}$ is irreducible, hence a closed subvariety as it is endowed with the reduced subscheme structure.

Now, to prove the claim, note that the equation is clearly true set-theoretically. As $Z_{n-1,n}$ is a closed subvariety of the product, it suffices to show that the degree of $p_{S \times S^{[n]}}|_{Z_{n-1,n}}$ is 1. Considering a general fiber suffices, and if $(x, \eta) \in {}^t \mathcal{Z}_n$ with η supported at n distinct points $x, x_2, \dots, x_n$ then the only preimage in $Z_{n-1,n}$ is $([x_2, \dots, x_n], x, \eta)$. $\square$

Corollary 3.18 *As correspondences in $\mathrm{CH}^*(S^{[2]} \times S^{[2]})$ we have*

$${}^t \mathcal{Z} \circ \mathcal{Z} = -q_1(S) \circ q_{-1}(S)$$

in terms of Nakajima operators.

Proof This follows from the fact that the Hilbert–Chow morphism $S^{[1]} \rightarrow S$, sending a length 1 subscheme to its support point, is an isomorphism, and from the definition of $q_{\pm 1}(S)$ as correspondences in $\mathrm{CH}^*(S^{[2]} \times S^{[1]})$. Indeed,

$$\begin{aligned} q_1(S) &= (p_{S^{[1]} \times S^{[2]}})_* (Z_{1,2}), \\ q_{-1}(S) &= (-1) \cdot (p_{S^{[1]} \times S^{[2]}})_* (Z_{1,2}). \end{aligned}$$

The operator $q_{-1}(S)$ is then a correspondence from $S^{[2]}$ to $S^{[1]}$ while $q_1(S)$ is a correspondence in the opposite direction. Now, using $S^{[1]} = S$ by the Hilbert–Chow morphism, Lemma 3.17 for $n = 2$ gives the desired result. $\square$

3.4 Computations

Let S be a projective $K3$ surface and $\Delta_S \subseteq S \times S$ the diagonal, $\Delta_{123} \subseteq S \times S \times S$ the small diagonal. In contrast, the diagonal in $S^{[2]} \times S^{[2]}$ will still be denoted Δ . By a result of Beauville and Voisin [3] we have the following relations in the Chow ring of S , $S \times S$ and $S \times S \times S$ for every divisor class a :

$$c_2(T_S) = 24c, \quad (3.4)$$

$$\Delta_S c_1 = \Delta_S c_2 = c_1 c_2, \quad (3.5)$$

$$\Delta_S^2 = 24c_1 c_2, \quad (3.6)$$

$$\Delta_S a_1 = a_1 c_2 + c_1 a_2, \quad (3.7)$$

$$\Delta_{123} = \Delta_{123} c_3 + \Delta_{132} c_2 + \Delta_{231} c_1 - c_1 c_2 - c_1 c_3 - c_2 c_3. \quad (3.8)$$

Here, (3.6) is a consequence of equation (3.4) and the self-intersection formula.

We show next that the canonical lifts F_a and H of the Lefschetz dual f_a and the grading operator h , respectively, agree with Oberdieck's canonical lifts in [15]. Recall the notation set up in Subsection 2.3.

Proposition 3.19 *Let S be a projective $K3$ surface and $X = S^{[2]}$. We have the following expressions for correspondences in terms of Nakajima operators:*

$$\begin{aligned} X \times X &= \frac{1}{4} q_1 q_1 q_{-1} q_{-1} (S^4), \\ \text{mult}_{S_c} &= -q_1 q_{-1} (c_1 c_2) - q_2 q_{-2} (c_1 c_2), \\ \text{mult}_{\delta^2} &= 12 q_2 q_{-2} (c_1 c_2) - \frac{1}{2} q_1 q_1 q_{-1} q_{-1} (\Delta_{1234}), \\ \text{mult}_l &= -20 q_1 q_{-1} (c_1 c_2) - 50 q_2 q_{-2} (c_1 c_2) + \frac{5}{4} q_1 q_1 q_{-1} q_{-1} (\Delta_{1234}). \end{aligned}$$

Of course, as correspondences we interpret mult_Z to be $\Delta_*(Z)$.

Proof The first of these equations follows immediately from Corollary A.1 and $X = \frac{1}{2} q_1(S) q_1(S) \cdot S^{[0]}$. The last equation for mult_l follows from the two preceding ones together with (2.3). For the second equation we observe $S_c = p_* q^*(c) = \pi_*(\text{ch}_2(\mathcal{O}_{\mathcal{X}}) \rho^*(c))$, as $\mathcal{X} = \text{ch}_2(\mathcal{O}_{\mathcal{X}})$ by [16, Eq. (101) on p. 75]. Then the first formula of [12, Theorem 1.6] shows the claim. Finally, for mult_{δ^2} we use the operator e_δ of multiplication by δ and its expression in Nakajima operators from [12] in order to compute

$$\begin{aligned} \text{mult}_{\delta^2} &= e_\delta \circ e_\delta = \frac{1}{4} \left(q_2 q_{-1} q_{-1} (\Delta_{123}) + q_1 q_1 q_{-2} (\Delta_{123}) \right)^{\circ 2} \\ &= \frac{1}{4} \left(q_2 q_{-1} q_{-1} q_1 q_1 q_{-2} (\Delta_{123} \Delta_{456}) + q_1 q_1 q_{-2} q_2 q_{-1} q_{-1} (\Delta_{123} \Delta_{456}) \right). \end{aligned}$$

Using the commutation relations for Nakajima operators, the second summand inside the brackets equals $-2 q_1 q_1 q_{-1} q_{-1} (\Delta_{1234})$. For the first summand, we obtain

$$q_2 q_{-1} q_{-1} q_1 q_1 q_{-2} (\Delta_{123} \Delta_{456}) = -2 q_2 q_{-1} q_1 q_{-2} (\Delta_{1234})$$

$$\begin{aligned}
&= 2q_2q_{-2}\left((\rho_{14})_*(\Delta_{23} \cdot \Delta_{1234})\right) \\
&= 48q_2q_{-2}(c_1c_2),
\end{aligned}$$

where in the last step we used $\Delta_{1234} = \Delta_{12}\Delta_{23}\Delta_{34}$ and equations (3.5) and (3.6). $\square$

Proposition 3.20 *Let S be a projective $K3$ surface and $X = S^{[2]}$. We can express Markman's lift L in terms of Nakajima operators as*

$$L = -q_1q_{-1}(S \times S) - \frac{1}{8}q_2q_{-2}(S \times S) - q_1q_1q_{-1}q_{-1}(c_1 + c_4).$$

Proof We express each summand of (2.2) in terms of Nakajima operators. For I we have $I = {}^t\mathcal{L} \circ \mathcal{L} = -q_1q_{-1}(S \times S)$ by [16, Lemma 11.2] and Corollary 3.18. Next,

$$\begin{aligned}
\delta = e_\delta(X) &= \frac{1}{2}\left(q_2q_{-1}q_{-1}(\Delta_{123}) + q_1q_1q_{-2}(\Delta_{123})\right) \circ \frac{1}{2}q_1q_1(S \times S) \cdot S^{[0]} \\
&= \frac{1}{2}q_2(S) \cdot S^{[0]}.
\end{aligned}$$

Together with Corollary A.1 this gives

$$\begin{aligned}
-\frac{1}{2}\delta_1\delta_2 &= -\frac{1}{8}\left(q_2(S) \cdot S^{[0]}\right) \boxtimes \left(q_2(S) \cdot S^{[0]}\right) \\
&= -\frac{1}{8}q_2(S)q'_2(S) \cdot \Delta_{S^{[0]}} \\
&= -\frac{1}{8}q_2q_{-2}(S \times S).
\end{aligned}$$

Similarly, using Proposition 3.19, we get $S_c = \text{mult}_{S_c}(X) = q_1q_1(c_1) \cdot S^{[0]}$. Applying again Corollary A.1 yields

$$-2(S_c)_1 = -q_1q_1q_{-1}q_{-1}(c_4) \quad \text{and} \quad -2(S_c)_2 = -q_1q_1q_{-1}q_{-1}(c_1),$$

and putting everything together gives the claimed formula for L . $\square$

On $S^{[2]}$ recall Oberdieck's canonical lifts [15, Eqs. (5) and (6)]

$$-2 \sum_{n \geq 1} \frac{1}{n^2} q_n q_{-n} (a_1 + a_2) = -2q_1q_{-1}(a_1 + a_2) - \frac{1}{2}q_2q_{-2}(a_1 + a_2)$$

of $\tilde{f}_a$ if a is in the surface part $\text{CH}^1(S) \subseteq \text{CH}^1(S^{[2]}) = \text{CH}^1(S) \oplus \mathbb{Q}\delta$, and

$$\begin{aligned}
&-\frac{1}{3} \sum_{i+j+k=0} : q_i q_j q_k \left(\frac{1}{k^2} \Delta_{12} + \frac{1}{j^2} \Delta_{13} + \frac{1}{i^2} \Delta_{23} + \frac{2}{j \cdot k} c_1 + \frac{2}{i \cdot k} c_2 + \frac{2}{i \cdot j} c_3 \right) : \\
&= 2q_2q_{-1}q_{-1} \left(c_3 - c_1 - \Delta_{12} - \frac{1}{8} \Delta_{23} \right) + 2q_1q_1q_{-2} \left(c_1 - c_3 - \Delta_{23} - \frac{1}{8} \Delta_{12} \right),
\end{aligned}$$

of $\tilde{f}_\delta$. Recall also the lift

$$2 \sum_{n \geq 1} \frac{1}{n} q_n q_{-n} (c_2 - c_1) = 2q_1q_{-1}(c_2 - c_1) + q_2q_{-2}(c_2 - c_1).$$

of h .

Proposition 3.21 *Let $X = S^{[2]}$ and $a \in \mathrm{CH}^1(X)$. Let $\widetilde{F}_a$ be as in Corollary 3.6 and H as in Proposition 3.11 with respect to the canonical lift L from Theorem 2.4. Then both cycles agree with Oberdieck's canonical lifts.*

Proof First, let a be in the surface part $\mathrm{CH}^1(S) \subseteq \mathrm{CH}^1(S^{[2]})$. We begin by expressing $l_1 a_1$ in terms of Nakajima operators, $l_2 a_2$ being its transpose. We use

$$l_1 a_1 = (X \times X) \circ \mathrm{mult}_l \circ \mathrm{mult}_a$$

as correspondences and the formulas from Proposition 3.19 as well as the formula for $e_a = \mathrm{mult}_a$ from [15, Eq. (4)]. We have

$$\begin{aligned} l_1 a_1 &= q_1 q_1 q_{-1} q_{-1} (S^4) \circ \\ &\quad \circ \left(5q_1 q_{-1} (c_1 c_2) - \frac{5}{16} q_1 q_1 q_{-1} q_{-1} (\Delta_{1234}) \right) \circ q_1 q_{-1} (\Delta_*(a)) \\ &= q_1 q_1 q_{-1} q_{-1} (S^4) \circ \left(5q_1 q_1 q_{-1} q_{-1} (\Delta_{14} a_1 c_2 c_3) + \frac{5}{8} q_1 q_1 q_{-1} q_{-1} (\Delta_{1234} a_1) \right). \end{aligned} \quad (3.9)$$

For the first summand in (3.9), we get

$$5q_1 q_1 q_{-1} q_1 q_{-1} q_1 q_{-1} q_{-1} (\Delta_{48} a_8 c_6 c_7) - 5q_1 q_1 q_{-1} q_1 q_{-1} q_{-1} (c_4 c_5 a_6) = 10q_1 q_1 q_{-1} q_{-1} (c_3 a_4).$$

The second summand in (3.9) equals

$$\frac{5}{8} q_1 q_1 q_{-1} q_1 q_{-1} q_1 q_{-1} q_{-1} (\Delta_{4678} a_8) - \frac{5}{8} q_1 q_1 q_{-1} q_1 q_{-1} q_{-1} (\Delta_{456} a_6) = \frac{5}{2} q_1 q_1 q_{-1} q_{-1} (c_3 a_4),$$

where we used (3.7). Hence, $l_1 a_1 = \frac{25}{2} q_1 q_1 q_{-1} q_{-1} (c_3 a_4)$, and therefore

$$\frac{4}{25} (l_1 a_1 + l_2 a_2) = 2q_1 q_1 q_{-1} q_{-1} (a_1 c_2 + c_3 a_4).$$

We now calculate $La_1 = (-L) \circ (-e_a)$ employing the formula for L from Proposition 3.20. We have

$$\begin{aligned} La_1 &= \left(q_1 q_{-1} (S \times S) + \frac{1}{8} q_2 q_{-2} (S \times S) + q_1 q_1 q_{-1} q_{-1} (c_1 + c_4) \right) \\ &\quad \circ \left(q_1 q_{-1} (\Delta_*(a)) + q_2 q_{-2} (\Delta_*(a)) \right) \\ &= q_1 q_{-1} q_1 q_{-1} (\Delta_{34} a_4) + \frac{1}{8} q_2 q_{-2} q_2 q_{-2} (\Delta_{34} a_4) + q_1 q_1 q_{-1} q_{-1} q_1 q_{-1} ((c_1 + c_4) \Delta_{56} a_6) \\ &= -q_1 q_{-1} (a_2) - \frac{1}{4} q_2 q_{-2} (a_2) + q_1 q_1 q_{-1} q_{-1} (a_1 c_4 - c_1 a_4 - c_3 a_4). \end{aligned}$$

After adding La_2 , clearly $q_1 q_1 q_{-1} q_{-1} (a_1 c_4 - c_1 a_4)$ cancels out. We obtain

$$2L(a_1 + a_2) = -2q_1 q_{-1} (a_1 + a_2) - \frac{1}{2} q_2 q_{-2} (a_1 + a_2) - 2q_1 q_1 q_{-1} q_{-1} (a_1 c_2 + c_3 a_4),$$

and putting both pieces together yields the desired formula. Let now $a = \delta$. Using the formula for e_δ from [15, Eq. (4)], we can analogously compute

$$l_1 \delta_1 = (F \times F) \circ \mathrm{mult}_l \circ e_\delta = -25q_1 q_1 q_{-2} (c_3),$$

so that

$$\frac{4}{25}(l_1\delta_1 + l_2\delta_2) = -4q_1q_1q_{-2}(c_3) - 4q_2q_{-1}q_{-1}(c_1). \quad (3.10)$$

Using the formula for L , we now consider $2L\delta_1 = 2(-L) \circ (-e_\delta)$ and get

$$2L\delta_1 = -2q_1q_1q_{-2}(\Delta_{23}) - \frac{1}{4}q_2q_{-1}q_{-1}(\Delta_{23}) + 2q_1q_1q_{-2}(c_1 + c_3).$$

Therefore,

$$\begin{aligned} 2L(\delta_1 + \delta_2) &= q_1q_1q_{-2} \left(2(c_1 + c_3) - 2\Delta_{23} - \frac{1}{4}\Delta_{12} \right) \\ &\quad + q_2q_{-1}q_{-1} \left(2(c_1 + c_3) - 2\Delta_{12} - \frac{1}{4}\Delta_{23} \right). \end{aligned} \quad (3.11)$$

Putting equations (3.10) and (3.11) together yields the claimed formula for $\widetilde{F}_\delta$. The proof for H is very similar and uses the decomposition of the small diagonal (3.8) as well as the equation $q_1q_{-1}(\Delta_S) = -\text{id}_{S^{[1]}}$ as correspondences on $S^{[1]} \times S^{[1]}$. $\square$

Remark 3.22 Proposition 3.21 together with the main theorem of [15] shows the equation $(a, a)H = \widetilde{H}_a$ (see Remark 3.14) for all divisor classes a in the Hilbert scheme case and so yields further evidence that this might be true for hyperkähler varieties X of $K3^{[2]}$ -type in general. It also gives another proof of Conjecture 1.5 for Hilbert schemes of two points of $K3$ surfaces by the more general [14, Theorem 1.4].

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A An auxiliary lemma

Lemma A.1 *Let X , Y and Z be smooth projective varieties and $\Gamma \in \mathrm{CH}^*(X \times Y)$, $\tilde{\Gamma} \in \mathrm{CH}^*(X \times Z)$ correspondences. Then as correspondences in $\mathrm{CH}^*(Y \times Z)$ we have*

$$(\Gamma \times \tilde{\Gamma})_*(\Delta_X) = \tilde{\Gamma} \circ {}^t\Gamma.$$

Proof Writing out the definitions of both sides, we see that we only need to show

$$p_{X_1Y}^*(\Gamma) \cdot p_{X_2Z}^*(\tilde{\Gamma}) \cdot p_{X_1X_2}^*(\Delta_X) = (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)_* (p_{YX}^*({}^t\Gamma) \cdot p_{XZ}^*(\tilde{\Gamma})). \quad (\text{A.1})$$

In here, p_{YX} and p_{XZ} are the projections from the triple product $Y \times X \times Z$ to $Y \times X$ and $X \times Z$, respectively. The projections p_{X_1Y} , p_{X_2Z} and $p_{X_1X_2}$ are the projections from the product $X \times Y \times X \times Z$ to the factors indicated by the indices. Moreover, Δ_X denotes both the diagonal embedding $X \hookrightarrow X \times X$ and the cycle class of its image. We can now rewrite the argument of the right hand side of (A.1) as

$$\begin{aligned} p_{YX}^*({}^t\Gamma) \cdot p_{XZ}^*(\tilde{\Gamma}) &= (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* (p_{YX_1}^*({}^t\Gamma)) \cdot (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* (p_{X_2Z}^*(\tilde{\Gamma})) \\ &= (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* (p_{YX_1}^*({}^t\Gamma) \cdot p_{X_2Z}^*(\tilde{\Gamma})). \end{aligned}$$

Hence, by the projection formula, the entire right hand side of (A.1) equals

$$\begin{aligned} (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)_* (p_{YX}^*({}^t\Gamma) \cdot p_{XZ}^*(\tilde{\Gamma})) &= p_{YX_1}^*({}^t\Gamma) \cdot p_{X_2Z}^*(\tilde{\Gamma}) \cdot p_{X_1X_2}^*(\Delta_X) \\ &= p_{X_1Y}^*(\Gamma) \cdot p_{X_2Z}^*(\tilde{\Gamma}) \cdot p_{X_1X_2}^*(\Delta_X). \end{aligned}$$

□

This simple result has an interesting consequence in the case $X = \mathrm{Spec}(\mathbb{C})$ where Δ_X is an isomorphism and is probably known to experts. It was pointed out to me by my advisor Georg Oberdieck and only uses Lemma A.1 and the fact that the Nakajima correspondences q_i and q_{-i} are the transpose of each other up to sign.

Corollary A.1 *Let S be a smooth projective surface and recall the definition of the Nakajima operators on Hilbert schemes of points of surfaces. Denote by q_i and q'_j the operators*

$$q_i, q'_j : \bigoplus_{m,n} \mathrm{CH}^*(S^{[m]} \times S^{[n]}) \rightarrow \bigoplus_{m,n} \mathrm{CH}^*(S^{[m]} \times S^{[n]} \times S)$$

acting as the Nakajima operator q_i on the first factor and as the identity on the second factor respectively as q_j on the second factor and as the identity on the first factor. Obviously, q_i commutes with q'_j . Let $\Gamma \in \mathrm{CH}^(S^{k+l})$. Then we have the equation of correspondences*

$$q_{i_1} \cdots q_{i_k} q'_{j_1} \cdots q'_{j_l}(\Gamma) \cdot S^{[0]} = (-1)^{i_1 + \cdots + i_k} q_{j_1} \cdots q_{j_l} q_{-i_k} \cdots q_{-i_1}(\tau_*(\Gamma)),$$

where $\tau : S^{k+l} \rightarrow S^{k+l}$ permutes the factors according to the permutation of the indices.

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