Equivariant splitting of the Hodge–de Rham exact sequence

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Abstract

Let X be an algebraic curve with a faithful action of a finite group G over a field k . We show that if the Hodge–de Rham short exact sequence of X splits G -equivariantly then the action of G on X is weakly ramified. In particular, this generalizes the result of Köck and Tait for hyperelliptic curves. We discuss also converse statements and tie this problem to lifting coverings of curves to the ring of Witt vectors of length 2.

Keywords De Rham cohomology · Equivariant sheaf cohomology

Mathematics Subject Classification Primary 14F40; Secondary 14G17 · 14H37

1 Introduction

Let X be a smooth proper algebraic variety over a field k . Recall that its de Rham cohomology may be computed in terms of Hodge cohomology via the spectral sequence

$$E_1^{ij} = H^j(X, \Omega_{X/k}^i) \Rightarrow H_{dR}^{i+j}(X/k). \quad (1.1)$$

Suppose that the spectral sequence (1.1) degenerates at the first page. This is automatic if $\text{char } k = 0$. For a field of positive characteristic, this happens for instance if X is a smooth projective curve or an abelian variety, or (by a celebrated result of Deligne and Illusie from [4]) if $\dim X > \text{char } k$ and X lifts to $W_2(k)$, the ring of Witt vectors of length 2. Under this assumption we obtain the following exact sequence:

$$0 \rightarrow H^0(X, \Omega_{X/k}) \rightarrow H_{dR}^1(X/k) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0. \quad (1.2)$$

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If X is equipped with an action of a finite group G , the terms of the sequence (1.2) become $k[G]$ -modules. In case when $\text{char } k \nmid \#G$, Maschke theorem allows one to conclude that the sequence (1.2) splits equivariantly. However, this might not be true in case when $\text{char } k = p > 0$ and $p \mid \#G$, as shown in [10]. The goal of this article is to show that for curves the sequence (1.2) *usually* does not split equivariantly.

Let X be a curve over an algebraically closed field of characteristic $p > 0$ with an action of a finite group G . For $P \in X$, denote by $G_{P,n}$ the n -th ramification group of G at P . Let also:

$$n_P := \max\{n : G_{P,n} \neq 0\}. \quad (1.3)$$

Following [9], we say that the action of G is **weakly ramified** if $n_P \leq 1$ for every $P \in X$.

Main Theorem *Suppose that X is a smooth projective curve over an algebraically closed field k of a finite characteristic $p > 2$ with a faithful action of a finite group G . If*

$$H_{dR}^1(X/k) \cong H^0(X, \Omega_{X/k}) \oplus H^1(X, \mathcal{O}_X) \quad (1.4)$$

as $k[G]$ -modules then the action of G on X is weakly ramified.

The example below is a direct generalization of results proven in [10].

Example 1.1 Suppose that k is an algebraically closed field of characteristic p . Let X/k be the smooth projective curve with the affine part given by the equation:

$$y^m = f(z^p - z),$$

where f is a separable polynomial and $p \nmid m$. Denote by $\mathcal{P}$ the set of points of X at infinity. One checks that $\#\mathcal{P} = \delta := \text{GCD}(m, \deg f)$ (cf. [22, Section 1]). The group $G = \mathbb{Z}/p$ acts on X via the automorphism $\varphi(z, y) = (z + 1, y)$. In this case

$$n_P = \begin{cases} m/\delta, & \text{if } P \in \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases} \quad (1.5)$$

(cf. Example 4.3). Thus if the exact sequence (1.2) splits G -equivariantly, then by Main Theorem either $p = 2$, or $m \mid \deg f$.

The main idea of the proof of Main Theorem is to compare $H_{dR}^1(X/k)^G$ and $H_{dR}^1(Y/k)$, where $Y := X/G$. The discrepancy between those groups is measured by the sheafified version of group cohomology, introduced by Grothendieck in [6]. This allows us to compute the 'defect'

$$\begin{aligned} \delta(X, G) &:= \dim_k H^0(X, \Omega_{X/k})^G + \dim_k H^1(X, \mathcal{O}_X)^G \\ &\quad - \dim_k H_{dR}^1(X/k)^G \end{aligned}$$

in terms of some local terms connected to Galois cohomology (cf. Proposition 3.1). We compute these local terms in case of Artin-Schreier coverings, which leads to the following theorem.

Theorem 1.2 *Suppose that X is a smooth projective curve over an algebraically closed field k of characteristic $p > 0$ with a faithful action of the group $G = \mathbb{Z}/p$. Then:*

$$\delta(X, G) = \sum_{P \in X} \left(n_P - 1 - 2 \cdot \left\lfloor \frac{n_P}{p} \right\rfloor \right).$$

Theorem 1.2 shows that if the group action of $G = \mathbb{Z}/p$ on a curve is not weakly ramified and $p > 2$ then $\delta(X, G) > 0$. This immediately implies Main Theorem for $G \cong \mathbb{Z}/p$. The general case may be easily derived from this special one.

The natural question arises: *to which extent is the converse of Main Theorem true?* We give some partial answers. In particular, we prove the following theorem.

Theorem 1.3 *Suppose that the action of G on a smooth projective curve X over an algebraically closed field k is weakly ramified. Assume also that there exists $Q_0 \in Y := X/G$ such that $p \nmid \# \pi^{-1}(Q_0)$. Then the sequence*

$$0 \rightarrow H^0(X, \Omega_{X/k})^G \rightarrow H_{dR}^1(X/k)^G \rightarrow H^1(X, \mathcal{O}_X)^G \rightarrow 0$$

is exact also on the right.

To derive Theorem 1.3 we use the method of proof of Main Theorem and a result of Kock from [9]. We were also able to show the splitting of the Hodge–de Rham exact sequence of a curve with a weakly ramified group action under some additional assumptions.

Theorem 1.4 *Keep the above notation. If any of the following conditions is satisfied:*

- (1) *the action of G on X is weakly ramified, a p -Sylow subgroup of G is cyclic and there exists $Q_0 \in Y$ such that $p \nmid \# \pi^{-1}(Q_0)$,*
- (2) *the action of G on X lifts to $W_2(k)$,*
- (3) *X is ordinary.*

then there exists an isomorphism (1.4) of $\mathbb{F}_p[G]$ -modules.

Parts (1), (2), (3) of Theorem 1.4 are proven in Lemma 5.3, Theorem 5.4 and Corollary 5.9 respectively. In fact, we prove more precise statements, involving the conjugate spectral sequence. This allows to prove that the conditions (2) and (3) of Theorem 1.4 imply that the action of G on X is weakly ramified. In order to prove (1) we use a description of modular representations of cyclic groups. (2) and (3) are easy corollaries of the equivariant version of results of Deligne and Illusie from [4]. The connection of [4] with the splitting of the Hodge–de Rham exact sequence was observed by Piotr Achinger.

Notation. Throughout the paper we will use the following notation (unless stated otherwise):

- k is an algebraically closed field of a finite characteristic p .
- G is a finite group.
- X is a smooth projective curve equipped with a faithful action of G .
- $Y := X/G$ is the quotient curve, which is of genus g_Y .
- $\pi : X \rightarrow Y$ is the canonical projection.
- $R \in \text{Div}(X)$ is the ramification divisor of π .
- $R' := \left\lfloor \frac{\pi_* R}{\#G} \right\rfloor \in \text{Div}(Y)$, where for $\delta \in \text{Div}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, we denote by $\lfloor \delta \rfloor$ the integral part taken coefficient by coefficient.
- $k(X), k(Y)$ are the function fields of X and Y .
- $\text{ord}_Q(f)$ denotes the order of vanishing of a function f at a point Q .
- $\underline{A}_X$ denotes the constant sheaf on X associated to a ring A .

Fix now a (closed) point P in X . Denote:

- $G_{P,i}$ – the i th ramification group of π at P , i.e.

$$G_{P,i} := \{g \in G : g(f) \equiv f \pmod{\mathfrak{m}_{X,P}^{i+1}} \text{ for all } f \in \mathcal{O}_{X,P}\}.$$

Note in particular that (since k is algebraically closed) the inertia group $G_{P,0}$ coincides with the decomposition group at P , i.e. the stabilizer of P in G .

- d_P – the different exponent at P , i.e.

$$d_P := \sum_{i \geq 0} (\#G_{P,i} - 1).$$

Recall that $R = \sum_{P \in X} d_P \cdot (P)$ (cf. [18, IV §1, Proposition 4]).

- e_P – the ramification index of π at P , i.e. $e_P = \#G_{P,0}$.
- n_P is given by the formula (1.3).

Also, by abuse of notation, for $Q \in Y$ we write $e_Q := e_P$, $d_Q := d_P$, $n_Q := n_P$ for any $P \in \pi^{-1}(Q)$. Note that these quantities don't depend on the choice of P .

Let for any abelian category $\mathcal{A}$, $\mathcal{C}_+(\mathcal{A})$ denote the category of non-negative cochain complexes in $\mathcal{A}$. We denote by $h^i(\mathcal{F}^\bullet)$ the i th cohomology of a complex $\mathcal{F}^\bullet$.

Outline of the paper Section 2 presents some preliminaries on the group cohomology of sheaves. We focus on the sheaves coming from Galois coverings of a curve. We use this theory to express the 'defect' $\delta(X, G)$ as a sum of local terms coming from Galois cohomology of certain modules in Sect. 3. In Sect. 4 we compute these local terms for Artin-Schreier coverings, which allows us to prove of Main Theorem and Theorem 1.2. In the final section we discuss the converse statements to Main Theorem and its relation to the problem of lifting curves with a given group action.

2 Review of group cohomology

Recall that our goal is to compare $H_{dR}^1(X/k)^G$ and $H_{dR}^1(Y/k)$, where $Y := X/G$. To this end, we need to work in the G -equivariant setting.

2.1 Group cohomology of sheaves

Let A be any commutative ring and G a finite group. We define the i -th group cohomology, $H_A^i(G, -)$, as the i -th derived functor of the functor

$$(-)^G : A[G]\text{-mod} \rightarrow A\text{-mod}, \quad M \mapsto M^G := \{m \in M : g \cdot m = m\}.$$

One checks that if $A \rightarrow B$ is a morphism of rings and M is a $B[G]$ -module then $H_B^i(G, M)$ and $H_A^i(G, M)$ are isomorphic A -modules for all $i \geq 0$ (cf. [19, Lemma 0DVD]). In particular, $H_A^i(G, M)$ is isomorphic as a $\mathbb{Z}$ -module to the usual group cohomology ($H_{\mathbb{Z}}^i(G, M)$ in our notation). Thus without ambiguity we will denote it by $H^i(G, M)$. For a future use we note the following properties of group cohomology:

- If $M = \text{Ind}_H^G N$ is an induced module (which for finite groups is equivalent to being a coinduced module) then

$$H^i(G, M) \cong H^i(H, N), \quad (2.1)$$

(this property is known as **Shapiro's lemma**, cf. [18, Proposition VIII.2.1.]).

- If M is a $\mathbb{F}_p[G]$ -module and G has a normal p -Sylow subgroup P then:

$$H^i(G, M) \cong H^i(P, M)^{G/P} \quad (2.2)$$

(for a proof observe that $H^i(G/P, N)$ is killed by multiplication by p for any $\mathbb{F}_p[G/P]$ -module N and use [18, Theorem IX.2.4.] to obtain $H^i(G/P, N) = 0$ for $i \geq 1$. Then use Lyndon–Hochschild–Serre spectral sequence).

- Suppose that A is a local ring with the maximal ideal $\mathfrak{m}$. If M is a finitely generated A -module then

$$H^i(G, M) \cong H^i(G, \widehat{M}_{\mathfrak{m}}), \quad (2.3)$$

where $\widehat{M}_{\mathfrak{m}}$ denotes the completion of M with respect to $\mathfrak{m}$ (see e.g. proof of [1, Lemme 3.3.1] for a brief justification).

The above theory extends to sheaves, as explained e.g. in [1, 6]. We briefly recall this theory. Let $(Y, \mathcal{O})$ be a ringed space and let G be a finite group. By an $\mathcal{O}[G]$ -sheaf on $(Y, \mathcal{O})$ we understand a sheaf $\mathcal{F}$ equipped with an $\mathcal{O}$ -linear action of G on $\mathcal{F}(U)$ for every open subset $U \subset Y$, compatible with respect to the restrictions. The $\mathcal{O}[G]$ -sheaves form a category $\mathcal{O}[G]\text{-mod}$, which is abelian and has enough injectives. For any $\mathcal{O}[G]$ -sheaf $\mathcal{F}$ one may define a sheaf $\mathcal{F}^G$ by the formula

$$U \mapsto \mathcal{F}(U)^G := \{f \in \mathcal{F}(U) : \forall g \in G \quad g \cdot f = f\}.$$

We denote the i -th derived functor of

$$(-)^G : \mathcal{O}[G]\text{-mod} \rightarrow \mathcal{O}\text{-mod}$$

by $\mathcal{H}_{(Y, \mathcal{O})}^i(G, -)$. Similarly as in the case of modules, one may neglect the dependence on the sheaf $\mathcal{O}$ and write simply $\mathcal{H}^i(G, M)$. If $\mathcal{F} = \widetilde{M}$ is a quasicoherent $\mathcal{O}[G]$ -module coming from a $\mathcal{O}(Y)[G]$ -module M , one may compute the group cohomology of sheaves via the standard group cohomology:

$$\mathcal{H}^i(G, \mathcal{F}) \cong \widetilde{H^i(G, M)}.$$

In particular, group cohomology of a quasicoherent $\mathcal{O}[G]$ -sheaf is a quasicoherent $\mathcal{O}$ -module. Moreover for any $Q \in Y$:

$$\mathcal{H}^i(G, \mathcal{F})_Q \cong H^i(G, \mathcal{F}_Q). \quad (2.4)$$

The sheaf group cohomology may be also computed using Čech complex (cf. [1, section 3.1]). However, we will not use this fact in any way.

2.2 Galois coverings of curves

We now turn to the case of curves over a field k . Let X/k be a smooth projective curve with a faithful action of a finite group G , i.e. a homomorphism $G \rightarrow \text{Aut}_k(X)$. In this case one can define the quotient $Y := X/G$ of X by the G -action. It is a smooth projective curve. Its underlying space is the topological quotient X/G and the structure sheaf is given by $\pi_*^G(\mathcal{O}_X)$, where $\pi : X \rightarrow Y$ is the quotient morphism. We say that X is a **G -covering of Y** .

In this section we will investigate the G -sheaves on Y coming from its G -coverings. Suppose that $\pi : X \rightarrow Y$ is a G -covering of Y . Let $\mathcal{F}$ be a $\mathcal{O}_X$ -sheaf on X with a G -action lifting that on X . Then $\pi_*\mathcal{F}$ is an $\mathcal{O}_Y[G]$ -module. It is natural to try to relate the group cohomology of $\pi_*\mathcal{F}$ to the ramification of π . Suppose for a while that the action of G on X is free, i.e. that $\pi : X \rightarrow Y$ is unramified. In this case the functors

$$\begin{aligned} \mathcal{F} &\mapsto \pi_*^G(\mathcal{F}) \\ \pi^*(\mathcal{G}) &\leftarrow \mathcal{G} \end{aligned}$$

are exact and provide an equivalence between the category of coherent $\mathcal{O}_Y$ -modules and coherent $\mathcal{O}_X[G]$ -modules (cf. [13, Proposition II.7.2, p. 70]). In particular, $\mathcal{H}^i(G, \pi_*\mathcal{F}) = 0$

for all $i \geq 1$ and every coherent $\mathcal{O}_X[G]$ -module $\mathcal{F}$. The following Proposition treats the general case.

Proposition 2.1 *Keep the notation introduced in Sect. 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module, which is G -equivariant. Then for every $i \geq 1$*

$$\mathcal{H}^i(G, \pi_* \mathcal{F})$$

is a torsion sheaf, supported on the wild ramification locus of π .

To prove Proposition 2.1 we shall need the following lemma involving group cohomology of modules over Dedekind domains.

Lemma 2.2 *Let k be an algebraically closed field. Let B be a k -algebra, which is a Dedekind domain equipped with a k -linear action of the group G . Suppose that $A := B^G$ is a discrete valuation ring with a maximal ideal $\mathfrak{q}$. Let $G_{\mathfrak{p},i}$ denote the i -th higher ramification group of a prime ideal $\mathfrak{p} \in \text{Spec } B$ over $\mathfrak{q}$. Then for every B -module M we have an isomorphism of B -modules:*

$$H^i(G, M) \cong H^i(G_{\mathfrak{p},1}, M_{\mathfrak{p}})^{G_{\mathfrak{p},0}/G_{\mathfrak{p},1}}$$

(here $M_{\mathfrak{p}}$ denotes the localisation of M at $\mathfrak{p}$).

Proof One easily sees that we have an isomorphism of $B[G]$ -modules

$$\widehat{M}_{\mathfrak{q}} \cong \text{Ind}_{G_{\mathfrak{p},0}}^G \widehat{M}_{\mathfrak{p}}$$

(see [18, II §3, Proposition 4] for a proof for $M = B$. The general case follows by tensoring both sides by M). Thus by (2.3) and (2.1) $H^i(G, M) \cong H^i(G_{\mathfrak{p},0}, M_{\mathfrak{p}})$. Moreover, $G_{\mathfrak{p},1}$ is a normal p -Sylow subgroup of $G_{\mathfrak{p},0}$ (cf. [18, Corollary 4.2.3., p. 67]). Hence the proof follows by (2.2). $\square$

Proof of Proposition 2.1 Denote by ξ the generic point of Y . Recall that by the Normal Base Theorem (cf. [8, sec. 4.14]), $k(X) = \text{Ind}_G^k k(Y)$ is an induced G -module. Therefore $(\pi_* \mathcal{F})_{\xi}$ is also an induced G -module (since it is a $k(X)$ -vector space of finite dimension) and by (2.1):

$$\mathcal{H}^i(G, \pi_* \mathcal{F})_{\xi} = H^i(G, (\pi_* \mathcal{F})_{\xi}) = 0.$$

Thus, since the sheaf $\mathcal{H}^i(G, \pi_* \mathcal{F})$ is coherent, it must be a torsion sheaf. Note that if a point $Q \in Y$ is tamely ramified then $G_{P,1} = 0$ for any $P \in \pi^{-1}(Q)$ and thus $\mathcal{H}^i(G, \pi_* \mathcal{F})_Q = 0$ by Lemma 2.2. This concludes the proof. $\square$

We will recall now a standard formula describing G -invariants of a $\mathcal{O}_Y[G]$ -module coming from an invertible $\mathcal{O}_X$ -module. For a reference see e.g. the proof of [1, Proposition 5.3.2].

Lemma 2.3 *For any G -invariant divisor $D \in \text{Div}(X)$:*

$$\pi_*^G(\mathcal{O}_X(D)) = \mathcal{O}_Y\left(\left[\frac{\pi_* D}{\#G}\right]\right),$$

where for $\delta \in \text{Div}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, we denote by $[\delta]$ the integral part taken coefficient by coefficient.

Corollary 2.4 *Keep the notation of Sect. 1. Let:*

$$R' = \left[\frac{\pi_* R}{\#G}\right] \in \text{Div}(Y).$$

Then:

$$\pi_*^G \Omega_{X/k} = \Omega_{Y/k} \otimes \mathcal{O}_Y(R').$$

In particular:

$$\dim_k H^0(X, \Omega_{X/k}^1)^G = \begin{cases} g_Y, & \text{if } R' = 0, \\ g_Y - 1 + \deg R', & \text{otherwise.} \end{cases}$$

Proof The first claim follows by Lemma 2.3 by taking D to be the canonical divisor of X and using the Riemann-Hurwitz formula. To prove the second claim, observe that

$$H^0(X, \Omega_{X/k})^G = H^0(Y, \pi_*^G \Omega_{X/k}) = H^0(Y, \Omega_{Y/k} \otimes \mathcal{O}_Y(R'))$$

and apply the Riemann-Roch theorem (cf. [7, Theorem IV.1.3]). $\square$

We end this section with one more elementary observation.

Lemma 2.5 R' (given as above) vanishes if and only if the morphism $\pi : X \rightarrow Y$ is tamely ramified.

Proof Recall that $R = \sum_{P \in X} d_P \cdot (P)$. Hence

$$\begin{aligned} R' &= \sum_{Q \in Y} \left[\frac{d_Q \cdot \#\pi^{-1}(Q)}{\#G} \right] (Q) \\ &= \sum_{Q \in Y} \left[\frac{d_Q}{e_Q} \right] (Q). \end{aligned}$$

Note however that $d_Q \geq e_Q - 1$ with an equality if and only if π is tamely ramified at Q . This completes the proof. $\square$

3 Computing the defect

The goal of this section is to prove the following Proposition.

Proposition 3.1 We follow the notation introduced in Sect. 1. Suppose that there exists $Q_0 \in Y$ such that $p \nmid \#\pi^{-1}(Q_0)$.

Then:

$$\delta(X, G) = \sum_{Q \in Y} \dim_k \operatorname{im} \left(H^1(G, (\pi_* \mathcal{O}_X)_Q) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_Q) \right),$$

where

$$H^1(G, (\pi_* \mathcal{O}_X)_Q) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_Q)$$

is the map induced by the differential $\mathcal{O}_X \rightarrow \Omega_{X/k}$.

3.1 Proof: Preparation

Recall that the i -th hypercohomology $\mathbb{H}^i(Y, -)$ can be defined as the i -th derived functor of

$$\mathbb{H}^0(Y, -) : \mathcal{C}_+(k_Y\text{-mod}) \rightarrow k\text{-mod}, \quad \mathbb{H}^0(Y, \mathcal{F}^\bullet) := H^0(Y, h^0(\mathcal{F}^\bullet))$$

(cf. [24, ex. 5.7.4 (2)]). The hypercohomology may be computed in terms of the usual cohomology using the spectral sequences:

$$E_1^{ij} = H^j(Y, \mathcal{F}^i) \Rightarrow \mathbb{H}^{i+j}(Y, \mathcal{F}^\bullet), \quad (3.1)$$

$${}_{II}E_2^{ij} = H^i(Y, h^j(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{i+j}(Y, \mathcal{F}^\bullet). \quad (3.2)$$

The de Rham cohomology of X is defined as the hypercohomology of the de Rham complex $\Omega_{X/k}^\bullet$. Note that π is an affine morphism. Therefore π_* is an exact functor on the category of quasicoherent sheaves. Thus using the spectral sequence (3.1) we obtain:

$$H_{dR}^i(X/k) = \mathbb{H}^i(X, \Omega_{X/k}^\bullet) = \mathbb{H}^i(Y, \pi_* \Omega_{X/k}^\bullet).$$

We start with the following observation.

Lemma 3.2 *The spectral sequence*

$$E_1^{ij} = H^j(Y, \pi_*^G \Omega_{X/k}^i) \Rightarrow \mathbb{H}^{i+j}(Y, \pi_*^G \Omega_{X/k}^\bullet)$$

degenerates at the first page.

Proof We have a morphism of complexes $\Omega_{Y/k}^\bullet \rightarrow \pi_*^G \Omega_{X/k}^\bullet$, which is an isomorphism on the zeroth term. Thus for $j = 0, 1$ we obtain a commutative diagram:

$$\begin{array}{ccc} H^j(Y, \mathcal{O}_Y) & \xrightarrow{\cong} & H^j(Y, \pi_*^G \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^j(Y, \Omega_{Y/k}) & \longrightarrow & H^j(Y, \pi_*^G \Omega_{X/k}), \end{array}$$

where the upper arrow is an isomorphism. Note also that the left arrow in the above diagram is zero for $j = 0, 1$, since the Hodge–de Rham spectral sequence for Y degenerates on the first page. Therefore for $j = 0, 1$ the maps

$$H^j(Y, \pi_*^G \mathcal{O}_X) \rightarrow H^j(Y, \pi_*^G \Omega_{X/k})$$

are zero. This is the desired conclusion. $\square$

Corollary 3.3

$$\begin{aligned} \delta(X, G) = & \left(\dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}^\bullet) - \dim_k \mathbb{H}^1(Y, \pi_* \Omega_{X/k}^\bullet) \right)^G \\ & - \left(\dim_k H^1(Y, \pi_*^G \mathcal{O}_X) - \dim_k H^1(Y, \pi_* \mathcal{O}_X) \right)^G. \end{aligned}$$

Proof. By Lemma 3.2 we obtain an exact sequence:

$$0 \rightarrow H^0(Y, \pi_*^G \Omega_{X/k}) \rightarrow \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}^\bullet) \rightarrow H^1(Y, \pi_*^G \mathcal{O}_X) \rightarrow 0.$$

Hence:

$$\begin{aligned}
 \delta(X, G) &= \dim_k H^0(X, \Omega_{X/k})^G + \dim_k H^1(X, \mathcal{O}_X)^G - \dim_k H_{dR}^1(X/k)^G \\
 &= \left(\dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}^\bullet) - \dim_k H^1(Y, \pi_*^G \mathcal{O}_X) \right) \\
 &\quad + \dim_k H^1(X, \mathcal{O}_X)^G - \dim_k H_{dR}^1(X/k)^G \\
 &= (\dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}^\bullet) - \dim_k \mathbb{H}^1(Y, \pi_* \Omega_{X/k}^\bullet)^G) \\
 &\quad - (\dim_k H^1(Y, \pi_*^G \mathcal{O}_X) - \dim_k H^1(Y, \pi_* \mathcal{O}_X)^G).
 \end{aligned}$$

□

Corollary 3.3 implies that we need to compare the hypercohomology groups

$$\mathbb{H}^i(Y, (\mathcal{F}^\bullet)^G) \text{ and } \mathbb{H}^i(Y, \mathcal{F}^\bullet)^G.$$

for $\mathcal{F}^\bullet = \pi_* \mathcal{O}_X$ (treated as a complex concentrated in degree 0) and $\mathcal{F}^\bullet = \pi_* \Omega_{X/k}^\bullet$ (note that it is a complex of $k_Y[G]$ -modules rather than $\mathcal{O}_Y$ -modules, since the differentials in the de Rham complex are not $\mathcal{O}_Y$ -linear).

Consider the commutative diagram:

$$\begin{array}{ccc}
 \mathcal{C}_+(k_Y[G]\text{-mod}) & \xrightarrow{(-)^G} & \mathcal{C}_+(k_Y\text{-mod}) \\
 \downarrow \mathbb{H}^0(Y, -) & & \downarrow \mathbb{H}^0(Y, -) \\
 k[G]\text{-mod} & \xrightarrow{(-)^G} & k\text{-mod}
 \end{array} \quad (3.3)$$

Note that the categories in the diagram (3.3) are abelian and have enough injectives (cf. [17, Theorem 10.43. and the following Remark]). By applying the Grothendieck spectral sequence to compositions of the functors in the diagram (3.3), we obtain two spectral sequences:

$${}_I E_2^{ij} = \mathbb{H}^i(Y, \mathcal{H}^j(G, \mathcal{F}^\bullet)) \Rightarrow \mathbb{R}^{i+j} \Gamma^G(\mathcal{F}^\bullet) \quad (3.4)$$

$${}_I E_2^{ij} = H^i(G, \mathbb{H}^j(Y, \mathcal{F}^\bullet)) \Rightarrow \mathbb{R}^{i+j} \Gamma^G(\mathcal{F}^\bullet), \quad (3.5)$$

(note that here $\mathcal{H}^j(G, \mathcal{F}^\bullet)$ denotes a complex of k_Y -modules with l th term being $\mathcal{H}^j(G, \mathcal{F}^l)$). For motivation, suppose at first that the 'obstructions'

$$\mathcal{H}^i(G, \mathcal{F}^l) \quad \text{and} \quad H^i(G, \mathbb{H}^l(Y, \mathcal{F}^\bullet))$$

vanish for all $i \geq 1$ and $l \geq 0$ (this happens e.g. if $\text{char } k = 0$). Then the spectral sequences (3.4) and (3.5) lead us to the isomorphisms:

$$\mathbb{H}^i(Y, (\mathcal{F}^\bullet)^G) \cong \mathbb{R}^i \Gamma^G(\mathcal{F}^\bullet) \cong (\mathbb{H}^i(Y, \mathcal{F}^\bullet))^G.$$

In general case the relation between $\mathbb{H}^i(Y, (\mathcal{F}^\bullet)^G)$ and $\mathbb{H}^i(Y, \mathcal{F}^\bullet)^G$ is more complicated. However, in the case of the first hypercohomology group, one can extract some information from the low-degree exact sequences of (3.4) and (3.5):

$$\begin{aligned}
 0 &\rightarrow \mathbb{H}^1(Y, (\mathcal{F}^\bullet)^G) \rightarrow \mathbb{R}^1 \Gamma^G(\mathcal{F}^\bullet) \rightarrow \\
 &\rightarrow \mathbb{H}^0(Y, \mathcal{H}^1(G, \mathcal{F}^\bullet)) \rightarrow \mathbb{H}^2(Y, (\mathcal{F}^\bullet)^G) \rightarrow \\
 &\rightarrow \mathbb{R}^2 \Gamma^G(\mathcal{F}^\bullet)
 \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} 0 &\rightarrow H^1(G, \mathbb{H}^0(Y, \mathcal{F}^\bullet)) \rightarrow \mathbb{R}^1 \Gamma^G(\mathcal{F}^\bullet) \rightarrow \\ &\rightarrow \mathbb{H}^1(Y, \mathcal{F}^\bullet)^G \rightarrow H^2(G, \mathbb{H}^0(Y, \mathcal{F}^\bullet)) \rightarrow \\ &\rightarrow \mathbb{R}^2 \Gamma^G(\mathcal{F}^\bullet). \end{aligned} \quad (3.7)$$

This will be done in the Sect. 3.2.

3.2 Proof: Low-degree exact sequences

Note that if $p \nmid \#G$, then Proposition 3.1 is immediate. Thus, we may assume that π is wildly ramified and by Lemma 2.5 we have $R' \neq 0$. Then, as one easily sees by Lemma 3.2, Corollary 2.4 and the Riemann–Roch theorem (cf. [7, Theorem IV.1.3]):

$$\begin{aligned} \mathbb{H}^2(Y, \pi_*^G \Omega_{X/k}^\bullet) &= H^1(Y, \pi_*^G \Omega_{X/k}) \\ &= H^1(Y, \Omega_{Y/k} \otimes \mathcal{O}_Y(R')) \\ &= 0. \end{aligned}$$

By (3.6) we see that

$$\begin{aligned} \dim_k \mathbb{R}^1 \Gamma^G(\pi_* \Omega_{X/k}^\bullet) &= \dim_k \mathbb{H}^1(Y, (\pi_* \Omega_{X/k}^\bullet)^G) \\ &\quad + \dim_k \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \Omega_{X/k}^\bullet)). \end{aligned} \quad (3.8)$$

On the other hand, (3.7) yields:

$$\begin{aligned} \dim_k \mathbb{R}^1 \Gamma^G(\pi_* \Omega_{X/k}^\bullet) &= \dim_k H^1(G, \mathbb{H}^0(Y, \pi_* \Omega_{X/k}^\bullet)) \\ &\quad + \dim_k \mathbb{H}^1(Y, \pi_* \Omega_{X/k}^\bullet)^G - c_1, \end{aligned} \quad (3.9)$$

where

$$c_1 = \dim_k \ker \left(H^2(G, \mathbb{H}^0(Y, \pi_* \Omega_{X/k}^\bullet)) \rightarrow \mathbb{R}^2 \Gamma^G(\pi_* \Omega_{X/k}^\bullet) \right). \quad (3.10)$$

Thus by comparing (3.8) and (3.9):

$$\begin{aligned} \dim_k \mathbb{H}^1(Y, \pi_* \Omega_{X/k}^\bullet)^G &= \dim_k \mathbb{H}^1(Y, (\pi_* \Omega_{X/k}^\bullet)^G) \\ &\quad + \dim_k \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \Omega_{X/k}^\bullet)) \\ &\quad - \dim_k H^1(G, \mathbb{H}^0(Y, \pi_* \Omega_{X/k}^\bullet)) + c_1. \end{aligned} \quad (3.11)$$

By repeating the same argument for $\pi_* \mathcal{O}_X$, we obtain:

$$\begin{aligned} \dim_k H^1(Y, \pi_* \mathcal{O}_X)^G &= \dim_k H^1(Y, (\pi_* \mathcal{O}_X)^G) \\ &\quad + \dim_k H^0(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \\ &\quad - \dim_k H^1(G, H^0(Y, \pi_* \mathcal{O}_X)) + c_2, \end{aligned} \quad (3.12)$$

where:

$$c_2 = \dim_k \ker \left(H^2(G, H^0(Y, \pi_* \mathcal{O}_X)) \rightarrow R^2 \Gamma^G(\pi_* \mathcal{O}_X) \right). \quad (3.13)$$

By combining (3.11), (3.12) and Corollary 3.3 we obtain:

$$\begin{aligned} \delta(X, G) &= \dim_k \operatorname{im} \left(H^0(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \rightarrow H^0(Y, \mathcal{H}^1(G, \pi_* \Omega_{X/k})) \right) \\ &\quad + (c_2 - c_1). \end{aligned}$$

Note that since $\mathcal{H}^1(G, \pi_* \mathcal{O}_X)$, $\mathcal{H}^1(G, \pi_* \Omega_{X/k})$ are torsion sheaves, we can compute their sections by taking stalks and using (2.4):

$$\dim_k \operatorname{im} \left(H^0(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \rightarrow H^0(Y, \mathcal{H}^1(G, \pi_* \Omega_{X/k})) \right) = \sum_{Q \in Y} \dim_k \operatorname{im} \left(H^1(G, (\pi_* \mathcal{O}_X)_Q) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_Q) \right)$$

Thus we are left with showing that $c_1 = c_2$. This will be done in Sect. 3.3.

3.3 Proof: The end

Recall that in order to prove Proposition 3.1 we have to investigate the map

$$H^2(G, \mathbb{H}^0(Y, \mathcal{F}^\bullet)) \rightarrow \mathbb{R}^2 \Gamma^G(\mathcal{F}^\bullet) \quad (3.14)$$

arising from the exact sequence (3.7). Note that for any complex $\mathcal{F}^\bullet$ of $\mathcal{O}[G]$ -sheaves on Y there exists a natural map:

$$\mathbb{R}^2 \Gamma^G(\mathcal{F}^\bullet) \rightarrow_I E_\infty^{02} \hookrightarrow_I E_2^{02} = \mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{F}^\bullet)). \quad (3.15)$$

We will investigate this map for $\mathcal{F}^\bullet = \pi_* \mathcal{O}_X$ (treated as a complex concentrated in degree 0). Note that by Proposition 2.1, the support of the quasicoherent sheaf $\mathcal{H}^2(G, \pi_* \mathcal{O}_X)$ is finite. Therefore:

$$\mathcal{H}^2(G, \pi_* \mathcal{O}_X) \cong \bigoplus_{Q \in Y} i_{Q,*} \left(H^2(G, (\pi_* \mathcal{O}_X)_Q) \right),$$

where $i_Q : \operatorname{Spec}(\mathcal{O}_{Y,Q}) \rightarrow Y$ is the natural morphism.

Lemma 3.4 *Keep the above notation. The map (3.15) for $\mathcal{F}^\bullet = \pi_* \mathcal{O}_X$ is an isomorphism. Moreover, the map:*

$$H^2(G, k) \rightarrow H^0(Y, \mathcal{H}^2(G, \pi_* \mathcal{O}_X)) \cong \bigoplus_{Q \in Y} H^2(G, (\pi_* \mathcal{O}_X)_Q)$$

(composition of maps (3.14) and (3.15) for $\mathcal{F}^\bullet = \pi_* \mathcal{O}_X$) is induced by the natural maps $k \rightarrow (\pi_* \mathcal{O}_X)_Q$.

Proof. Observe that for $\mathcal{F}^\bullet = \pi_* \mathcal{O}_X$ one has ${}_I E_2^{ij} = 0$ for $i, j \geq 1$ and for $i \geq 2$. Therefore ${}_I E_\infty^{11} = {}_I E_2^{11} = 0$ and ${}_I E_\infty^{20} = {}_I E_2^{20} = 0$, which leads to the proof of the first claim.

The morphism of sheaves:

$$\pi_* \mathcal{O}_X \rightarrow \mathcal{G} := \bigoplus_{Q \in Y} i_{Q,*}((\pi_* \mathcal{O}_X)_Q)$$

induces by functoriality the commutative diagram:

$$\begin{array}{ccccc} H^2(G, H^0(Y, \pi_* \mathcal{O}_X)) & \longrightarrow & \mathbb{R}^2 \Gamma^G(\pi_* \mathcal{O}_X) & \xrightarrow{\sim} & H^0(Y, \mathcal{H}^2(G, \pi_* \mathcal{O}_X)) \\ \downarrow & & \downarrow & & \downarrow \sim \\ H^2(G, H^0(Y, \mathcal{G})) & \longrightarrow & \mathbb{R}^2 \Gamma^G(\mathcal{G}) & \longrightarrow & H^0(Y, \mathcal{H}^2(G, \mathcal{G})). \end{array} \quad (3.16)$$

Note that $H^i(Y, \mathcal{G}) = \bigoplus_{Q \in Y} H^i(Y, i_{Q,*}((\pi_* \mathcal{O}_X)_Q)) = 0$ for $i \geq 1$. Therefore for $\mathcal{F}^\bullet := \mathcal{G}$ we have ${}_I E_2^{ij} = 0$ for $i \geq 1$ and ${}_{II} E_2^{ij} = 0$ for $j \geq 1$. This implies that the two lower arrows in the diagram (3.16) are isomorphisms. Moreover, the composition of those arrows is given by the composition of the natural maps:

$$\begin{aligned} H^2(G, H^0(Y, \mathcal{G})) &= H^2\left(G, \bigoplus_{Q \in Y} (\pi_* \mathcal{O}_X)_Q\right) \cong \bigoplus_{Q \in Y} H^2(G, (\pi_* \mathcal{O}_X)_Q) \\ &\cong \mathbb{H}^0\left(Y, \bigoplus_{Q \in Y} i_{Q,*}(H^2(G, \mathcal{G}_Q))\right) \cong \mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{G})). \end{aligned}$$

Thus the second claim follows by the diagram (3.16). $\square$

We are now ready to finish the proof of Proposition 3.1. Recall that we are left with showing that $c_1 = c_2$ (where c_1 and c_2 are given by (3.10) and (3.13) respectively).

In fact, we will show that $c_1 = c_2 = 0$. We start by proving that the map:

$$\begin{aligned} H^2(G, k) &= H^2(G, \mathbb{H}^0(Y, \pi_* \mathcal{O}_X)) \rightarrow \mathbb{R}^2 \Gamma^G(\pi_* \mathcal{O}_X) \\ &\cong H^0(Y, \mathcal{H}^2(G, \pi_* \mathcal{O}_X)) \cong \bigoplus_{Q \in Y} H^2(G, (\pi_* \mathcal{O}_X)_Q) \end{aligned} \quad (3.17)$$

is injective. It suffices to show that the map:

$$H^2(G, k) \rightarrow H^2(G, (\pi_* \mathcal{O}_X)_{Q_0}) \quad (3.18)$$

is injective. Choose any $P_0 \in \pi^{-1}(Q_0)$. Observe that by Lemma 2.2 we have:

$$H^2(G, (\pi_* \mathcal{O}_X)_Q) \cong H^2(G_{P_0,1}, \mathcal{O}_{X,P_0})^{G_{P_0,0}/G_{P_0,1}}.$$

But $\mathcal{O}_{X,P_0} \cong k \oplus \mathfrak{m}_{X,P_0}$ as a $k[G_{P_0,1}]$ -module and therefore

$$H^2(G_{P_0,1}, \mathcal{O}_{X,P_0}) \cong H^2(G_{P_0,1}, k) \oplus H^2(G_{P_0,1}, \mathfrak{m}_{X,P_0}).$$

Note that G acts trivially on k . Hence the map (3.18) factors as

$$H^2(G, k) \rightarrow H^2(G_{P_0,1}, k) \hookrightarrow H^2(G_{P_0,1}, k) \oplus H^2(G_{P_0,1}, \mathfrak{m}_{X,P_0})^{G_{P_0,0}/G_{P_0,1}},$$

where the first map is the restriction map $\text{res}_{G_{P_0,1}}^G$. Now note that $p \nmid \#\pi^{-1}(Q_0) = [G : G_{P_0}]$ and thus $G_{P_0,1}$ is a p -Sylow subgroup of G by [18, Corollary 4.2.3., p. 67]. Thus by [18, Theorem IX.4, p. 140] $\text{res}_{G_{P_0,1}}^G$ is injective. This shows the injectivity of (3.17). Hence $c_2 = 0$. Consider now the diagram:

$$\begin{array}{ccc} H^2(G, \mathbb{H}^0(Y, \pi_* \mathcal{O}_X)) & \hookrightarrow & \mathbb{R}^2 \Gamma^G(\pi_* \mathcal{O}_X) \\ \cong \uparrow & & \uparrow \\ H^2(G, \mathbb{H}^0(Y, \pi_* \Omega_{X/k}^\bullet)) & \longrightarrow & \mathbb{R}^2 \Gamma^G(\pi_* \Omega_{X/k}^\bullet), \end{array}$$

which is obtained by functoriality for the map of complexes $\pi_* \Omega_{X/k}^\bullet \rightarrow \pi_* \mathcal{O}_X$ (where $\pi_* \mathcal{O}_X$ is treated as a complex concentrated in degree zero). Note that the left arrow is an isomorphism and the upper arrow is injective, as shown above. Therefore also the lower arrow must be injective, which proves that $c_1 = 0$. This concludes the proof of Proposition 3.1. Note that we obtain by (3.12) the following Corollary:

Corollary 3.5 *In the notation of Sect. 1 suppose that there exists $Q_0 \in Y$ such that $p \nmid \#\pi^{-1}(Q_0)$. Then:*

$$\dim_k H^1(X, \mathcal{O}_X)^G = g_Y + \sum_{Q \in Y} H^1(G, (\pi_* \mathcal{O}_X)_Q) - \dim_k H^1(G, k).$$

4 Computation of local terms

4.1 Proofs of main results

The main goal of this section is to compute the local terms occurring in Proposition 3.1. This is achieved in the following proposition.

Proposition 4.1 *Keep the notation introduced in Sect. 1 and suppose that $G \cong \mathbb{Z}/p$. Then for any $Q \in Y$ the dimension of*

$$\operatorname{im} \left(H^1(G, (\pi_* \mathcal{O}_X)_Q) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_Q) \right)$$

equals

$$n_Q - 1 - 2 \cdot \left\lfloor \frac{n_Q}{p} \right\rfloor.$$

Proposition 4.1 will be proven in the Sect. 4.2. We now show how the Proposition 4.1 implies the Theorems announced in the Introduction.

Proof of Theorem 1.2 If π is unramified, the action of G on X lifts to $W_2(k)$, cf. [21, Theorem 5.7.9]. Hence $\delta(X, G) = 0$ by Theorem 5.4. Suppose now that π is ramified. In this case Theorem 1.2 follows by combining Propositions 3.1 and 4.1. $\square$

Proof of Main Theorem We consider first the case $G = \mathbb{Z}/p$. An easy computation shows that for any $n \geq 1$, $p \geq 3$ one has:

$$n - 1 - 2 \cdot \left\lfloor \frac{n}{p} \right\rfloor \geq 0$$

with an equality only for $n = 1$ (here is where we use the assumption $p > 2$). Thus by Theorem 1.2, $\delta(X, G) = 0$ holds if and only if π is weakly ramified.

Suppose now that G is arbitrary and $G_{P,2} \neq 0$ for some $P \in X$. Note that $G_{P,2}$ is a p -group (cf. [18, Corollary 4.2.3., p. 67]) and thus contains a subgroup H of order p . Observe that $\pi : X \rightarrow X/H$ is an Artin-Schreier covering and it is non-weakly ramified, since $H_{P,2} = H \neq 0$. Therefore by the first paragraph of the proof, the sequence (1.2) does not split H -equivariantly and therefore it cannot split as a sequence of $k[G]$ -modules. $\square$

4.2 Galois cohomology of sheaves on Artin-Schreier coverings

We start by recalling the most important facts concerning Artin-Schreier coverings. For a reference see e.g. [16, sec. 2.2]. Let X be a smooth algebraic curve with a faithful action of

$G = \mathbb{Z}/p$ over an algebraically closed field k of characteristic p . By Artin–Schreier theory, the function field of X is given by the equation

$$z^p - z = f \quad (4.1)$$

for some $f \in k(Y)$, where $Y := X/G$. The action of $G = \langle \sigma \rangle \cong \mathbb{Z}/p$ is then given by $\sigma(z) := z + 1$. Let $\mathcal{P} \subset Y$ denote the set of points at which π is ramified. Note that $\mathcal{P}$ is contained in the set of poles of f and moreover for any $Q \in Y$:

$$\#\pi^{-1}(Q) = \begin{cases} p, & \text{for } Q \notin \mathcal{P}, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 4.2 *Keep the above setting. Fix a point $Q \in \mathcal{P}$ and let $\pi^{-1}(Q) = \{P\}$. Suppose that $p \nmid n := -\text{ord}_Q(f)$. Then for some $t \in \widehat{\mathcal{O}}_{X,P}$ and $x \in \widehat{\mathcal{O}}_{Y,Q}$:*

- $\widehat{\mathcal{O}}_{X,P} = k[[t]]$, $\widehat{\mathcal{O}}_{Y,Q} = k[[x]]$,
- $t^{-np} - t^{-n} = x^{-n}$,
- the action of $G \cong \mathbb{Z}/p$ on t is given by an automorphism:

$$\sigma(t) = \frac{t}{(1+t^n)^{1/n}} = t - \frac{1}{n}t^{n+1} + (\text{terms of order } \geq n+2). \quad (4.2)$$

In particular, n is equal to n_Q as defined by (1.5).

Proof Let x, t be arbitrary uniformizers at Q and P respectively. Then $\widehat{\mathcal{O}}_{Y,Q} = k[[x]]$ and $\widehat{\mathcal{O}}_{X,P} = k[[t]]$. Before the proof observe that an equation $u^m = h(x)$ has a solution $u \in k[[x]]$, whenever $p \nmid m$ and $m \mid \text{ord}(h)$ (this follows easily from Hensel’s lemma). We will denote any solution by $h(x)^{1/m}$. Note that:

$$f^{-1} = \frac{z^{-p}}{1 - z^{1-p}}.$$

By comparing the valuations we see that $\text{ord}_P(z) = -n$. Thus we may replace t by $z^{-1/n}$ to ensure that $z = t^{-n}$. Then:

$$\begin{aligned} \sigma(t)^n &= \sigma(t^n) = \sigma\left(\frac{1}{z}\right) \\ &= \frac{1}{z+1} = \frac{1}{t^{-n}+1} = \frac{t^n}{1+t^n} \end{aligned}$$

and thus we can assume without loss of generality (by replacing σ by its power if necessary) that $\sigma(t) = \frac{t}{(1+t^n)^{1/n}}$. Finally, we replace x by $f(x)^{-1/n}$ to ensure that $t^{-np} - t^{-n} = x^{-n}$. $\square$

Example 4.3 Let X be the curve considered in Example 1.1. Then X is a $\mathbb{Z}/p$ -covering of a curve Y with the affine equation:

$$y^m = f(x).$$

The function field of X is given by the equation $z^p - z = x$. As proven in [22] the function $x \in k(Y)$ has $\delta := \text{GCD}(m, \deg f)$ poles, each of them of order m/δ . This establishes the formula (1.5).

Remark 4.4 Suppose that $\pi : X \rightarrow Y$ is an Artin-Schreier covering. For every point $Q \in \mathcal{P}$ we can find functions $f_Q \in k(Y)$, $z_Q \in k(X)$ such that the function field of X is given by the equation $z_Q^p - z_Q = f_Q$ and either $f_Q \in \mathcal{O}_{Y,Q}$, or $p \nmid \text{ord}_Q(f_Q)$. Indeed, in order to obtain f_Q one can repeatedly subtract from f a function of the form $h^p - h$, where h is a power of a uniformizer at Q .

Example 4.5 It might not be possible to find a function f such the function field of X is given by (4.1) and for any pole Q of f one has $p \nmid \text{ord}_Q(f)$. Take for example an ordinary elliptic curve $X/\overline{\mathbb{F}}_p$. Let $\tau \in \text{Aut}(X)$ be a translation by a p -torsion point. Consider the action of $G = \langle \tau \rangle \cong \mathbb{Z}/p$ on X . This group action is free and hence $n_P = 0$ for all $P \in X$. Thus, if $k(X)$ would have an equation of the form $z^p - z = f$, where $p \nmid \text{ord}_Q(f)$ for all $Q \in \mathcal{P}$, then f would have no poles. This easily leads to a contradiction.

Keep the notation of Lemma 4.2. Fix an integer $a \in \mathbb{Z}$ and denote:

- $B := \widehat{\mathcal{O}}_{X,P} = k[[t]]$, $L := k((t))$, $I := t^a B$,
- $A := \widehat{\mathcal{O}}_{Y,Q} = k[[x]]$, $K := k((x))$.

In the below Lemma we will compute $H^1(G, I)$. The dimension of $H^1(G, I)$ is computed also in [1, Théorème 4.1.1] (see also [11, formula (18)]). However, we need an explicit description of a basis of $H^1(G, I)$.

Lemma 4.6 1. $H^1(G, I)$ may be identified with

$$M := \text{coker}(L^G \rightarrow (L/I)^G).$$

2. A basis of $H^1(G, I)$ is given by the images of the elements $(t^i)_{i \in J}$ in M , where

$$J := \{a - n \leq i \leq a - 1 : p \nmid i\}.$$

$$3. \dim_k H^1(G, I) = n - \left\lfloor \frac{a-1}{p} \right\rfloor + \left\lfloor \frac{a-1-n}{p} \right\rfloor.$$

4. The images of the elements:

$$t^i \quad \text{for } a - n \leq i \leq a - 1, \quad p \nmid i$$

are zero in M .

Proof. For any $h \in L$, we will denote its image in L/I by $[h]_{L/I}$. Analogously, if $h \in L^G$, we denote its image in M by $[h]_M$.

1. The proof follows by taking the long exact sequence of cohomology for the short exact sequence of $k[G]$ -modules:

$$0 \rightarrow I \rightarrow L \rightarrow L/I \rightarrow 0$$

and using the Normal Base Theorem (cf. [8, sec. 4.14]).

2. Note that for any $a - n \leq i \leq a - 1$, we have $[t^i]_{L/I} \in (L/I)^G$, since

$$\begin{aligned} \sigma([t^i]_{L/I}) &= [\sigma(t^i)]_{L/I} = \left[\left(t - \frac{1}{n} t^{n+1} + O(t^{2n}) \right)^i \right]_{L/I} \\ &= \left[t^i - \frac{i}{n} t^{i+n} + O(t^{i+2n}) \right]_{L/I} = [t^i]_{L/I}. \end{aligned}$$

We'll show now that the set $([t^i]_M)_{i \in J}$ spans M . Note that $L^G = K$. Therefore it suffices to show that for any $[h]_{L/I} \in (L/I)^G$, one has

$$h \in K + \bigoplus_{i \in J} k \cdot t^i. \quad (4.3)$$

Let $h = \sum_{i=N}^{a-1} a_i t^i \in L$, where $a_N \neq 0$. Observe that if $p|j$ and $a_j \neq 0$, then we may replace h by $h - c \cdot x^{j/p}$ for a suitable constant $c \in k$, since valuation of x in L equals p . Thus without loss of generality we may assume that $a_j = 0$ for $p|j$ and that $p \nmid N$. The equality $\sigma([h]_{L/I}) = [h]_{L/I}$ is equivalent to

$$\sum_{i=N}^{a-1} a_i \sigma(t)^i = \sum_{i=N}^{a-1} a_i t^i + \sum_{i=a}^{\infty} b_i t^i$$

for some $b_a, b_{a+1}, \dots \in k$. By using equality (4.2) this implies:

$$\sum_{i=N}^{a-1} a_i t^i \cdot \left(1 - \frac{i}{n} t^n + O(t^{2n})\right) = \sum_{i=N}^{a-1} a_i t^i + \sum_{i=a}^{\infty} b_i t^i.$$

By comparing coefficients of t^{N+n} , we see that either $N+n \geq a$, or

$$a_N \cdot \left(-\frac{N}{n}\right) + a_{N+n} = a_{N+n}.$$

The second possibility easily leads to a contradiction. This proves (4.3). We check now linear independence of the considered elements. Suppose that for some $a_i \in k$ not all equal to zero:

$$\sum_{i \in J} a_i [t^i]_M = 0$$

or equivalently,

$$\sum_{i \in J} a_i t^i = \sum_{j \geq N} b_j x^j + \sum_{j \geq a} c_j t^j \quad (4.4)$$

for some $b_j, c_j \in k$, $b_N \neq 0$. Consider the coefficient of t^{pN} in (4.4). Observe that $x = t^p + O(t^{p+1})$, since $\text{ord}_P(x) = p$. We see that either $pN \geq a$ (which is impossible, since $\sum_{i \in J} a_i t^i \notin I$) or $0 = b_N + 0$, which also leads to a contradiction. This ends the proof.

3. Follows immediately by (2).

4. Note that

$$\begin{aligned} x &= \frac{1}{(t^{-np} - t^{-n})^{1/n}} = \frac{t^p}{(1 - t^{n \cdot (p-1)})^{1/n}} \\ &= t^p \cdot (1 + O(t^{n \cdot (p-1)})) \end{aligned}$$

and thus for any $a - n \leq i \leq a - 1$, $p|i$:

$$x^{i/p} = t^i \cdot (1 + O(t^{n \cdot (p-1)})) = t^i + O(t^a),$$

and $[t^i]_{L/I} = [x^{i/p}]_{L/I}$, which shows that $[t^i]_M = 0$.

□

Proof of Proposition 4.1 Fix a point $Q \in \mathcal{P}$ and keep the above notation. Note that $(\pi_* \mathcal{O}_X)_Q \cong B$, $\pi_* \Omega_{X/k} = B dt$. Moreover, note that $\frac{dt}{t^{n+1}}$ is a G -invariant form, since from the equation $t^{-np} - t^{-n} = x^{-n}$ one obtains:

$$\frac{dt}{t^{n+1}} = -\frac{dx}{x^{n+1}}.$$

Thus we have the following isomorphism of $B[G]$ -modules:

$$\begin{aligned} B \, dt &\longrightarrow t^{n+1} B \\ h(t) \, dt = t^{n+1} h(t) \cdot \frac{dt}{t^{n+1}} &\longmapsto t^{n+1} h(t). \end{aligned}$$

(cf. [11, proof of Lemma 1.11.] for the “dual” version of this isomorphism). Lemma 4.6 implies that $H^1(G, B)$ and $H^1(G, B \, dt)$ may be identified with

$$M_1 := \operatorname{coker}(L^G \rightarrow (L/B)^G) \quad \text{and} \quad M_2 := \operatorname{coker}((L \, dt)^G \rightarrow (L \, dt/B \, dt)^G)$$

respectively. One easily checks that the morphism $d : H^1(G, B) \rightarrow H^1(G, B \, dt)$ corresponds to

$$d : M_1 \rightarrow M_2, \quad d([h(t)]_{M_1}) = [dh(t)]_{M_2} = [h'(t) \, dt]_{M_2}.$$

By using Lemma 4.6 (2), (4) for $a = 0$ and $a = n + 1$ we see that the basis of $\operatorname{im}(d : M_1 \rightarrow M_2)$ is

$$[dt^i]_{M_2} = [it^{i-1} \, dt]_{M_2} \quad \text{for } i = -n, -n + 1, \dots, -1, \quad p \nmid i, \quad i + n \not\equiv 0 \pmod{p}.$$

An elementary calculation allows one to compute the dimension of this space. $\square$

5 Converse results

This section will be devoted to proving various converse statements to Main Theorem.

5.1 The G -fixed subspaces

The methods used throughout the article seem to be insufficient to obtain a positive result regarding splitting of the exact sequence (1.2). However, we may say something about the G -fixed subspaces in the sequence (1.2).

Proof of Theorem 1.3 By Proposition 3.1 it is sufficient to show that the map

$$H^1(G, (\pi_* \mathcal{O}_X)_{\pi(P)}) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_{\pi(P)})$$

is zero for every $P \in X$. Just as in Sect. 3.3 we observe that

$$H^1(G, (\pi_* \mathcal{O}_X)_{\pi(P)}) \cong H^1(G_{P,0}, k) \oplus H^1(G_{P,0}, \mathfrak{m}_{X,P}).$$

However, the map $d : k \rightarrow \Omega_{X/k}$ is zero and thus the induced map

$$d : H^1(G_{P,0}, k) \rightarrow H^1(G, (\pi_* \Omega_{X/k})_{\pi(P)})$$

is also zero. Moreover, since π is weakly ramified, by a result of Kock (cf. [9, Theorem 1.1]), $H^1(G_{P,0}, \mathfrak{m}_{X,P}) = 0$. This ends the proof. $\square$

Note that if an action of a finite group G on X is weakly ramified then the action of any subgroup of G on X is also weakly ramified. Therefore the condition imposed by Theorem 1.3 on the Hodge–de Rham exact sequence of X seems to be strong from the group theoretical point of view. This raises the following question:

Question 5.1 Suppose that k is a field of characteristic $p > 0$ and G is a finite group. Let also

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (5.1)$$

be an exact sequence of $k[G]$ -modules of finite dimension over k . Assume that for every subgroup $H \leq G$ the sequence

$$0 \rightarrow A^H \rightarrow B^H \rightarrow C^H \rightarrow 0$$

is exact also on the right. Does it follow that the exact sequence (5.1) splits G -equivariantly?

The following lemma reduces the Question 5.1 to the case of p -groups.

Lemma 5.2 Let k be a field of characteristic $p > 0$ and let G be a finite group with a p -Sylow subgroup P . Suppose that

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (5.2)$$

is an exact sequence of $k[G]$ -modules. Then (5.2) splits as an exact sequence of $k[G]$ -modules if and only if it splits as an exact sequence of $k[P]$ -modules.

Proof The proof is adapted from the proof of Maschke theorem. Suppose that $s : C \rightarrow B$ is a $k[P]$ -equivariant section of the map $B \rightarrow C$. Let $P \backslash G = \{Pg_1, \dots, Pg_m\}$, where $p \nmid m = [G : P]$. Then, as one easily checks

$$\tilde{s} : C \rightarrow B, \quad \tilde{s}(x) := \frac{1}{m} \sum_{i=1}^m g_i^{-1} s(g_i x)$$

is a $k[G]$ -equivariant section of $B \rightarrow C$. □

Unfortunately we are able to answer Question 5.1 only for the class of groups that have 'tame' modular representation theory, i.e. groups with a cyclic p -Sylow subgroup.

Lemma 5.3 Suppose that k is a field of characteristic $p > 0$ and G is a finite group with a cyclic p -Sylow subgroup. Let

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (5.3)$$

be an exact sequence of $k[G]$ -modules. If the sequence

$$0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \rightarrow 0$$

is exact on the right then the exact sequence (5.3) splits G -equivariantly.

Proof Without loss of generality we can assume that $G = \mathbb{Z}/p^n$ is a cyclic p -group (by Lemma 5.2). Note that $k[\mathbb{Z}/p^n] \cong k[x]/(x-1)^{p^n}$. The classification theorem of finitely generated modules over the principal ideal domain $k[x]$ (cf. [5, Theorem 12.1.5]) implies that every finitely generated indecomposable $k[\mathbb{Z}/p^n]$ -module is of the form:

$$J_i = k[x]/(x-1)^i \quad \text{for some } i = 1, \dots, p^n.$$

Denote also $J_0 := 0$. Using Smith's Normal Form theorem (cf. [5, Theorem 12.1.4]) we obtain a commutative diagram:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_{i=1}^l J_{a_i} & \hookrightarrow & \bigoplus_{i=1}^m J_{b_i} \end{array}$$

where $l \leq m$, $a_i \leq b_i$ and $J_{a_i} \hookrightarrow J_{b_i}$ is the natural inclusion. Hence we are reduced to proving the claim for the exact sequence:

$$0 \rightarrow J_a \rightarrow J_b \rightarrow J_c \rightarrow 0,$$

where $a + b = c$, $0 \leq a, b, c \leq p^n$. However, the equality

$$\dim_k J_s^G = \begin{cases} 1, & \text{if } s \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

makes it obvious that $a = 0$ or $c = 0$. This finishes the proof. $\square$

5.2 Relation to the problem of lifting coverings

Let X be a smooth variety over an algebraically closed field k of characteristic p . Denote by X' the Frobenius twist of X and by $F : X \rightarrow X'$ – the absolute Frobenius morphism of X . Recall that in this case we have the following Cartier isomorphism (cf. [2]) of $\mathcal{O}_{X'}$ -modules:

$$\mathcal{C}^{-1} : \Omega_{X'}^i \rightarrow h^i(F_* \Omega_X^\bullet). \quad (5.4)$$

Therefore the spectral sequence (3.2) for the de Rham cohomology becomes:

$${}_II E^{ij} = H^i(X', \Omega_{X'}^j) \Rightarrow H_{dR}^{i+j}(X/k). \quad (5.5)$$

Let for any k -vector space V , V' denote the k -vector space with the same underlying abelian group as V and the scalar multiplication $(\lambda, v) \mapsto \lambda^p \cdot v$. Then one easily checks that:

$$H^1(X', \mathcal{O}_{X'}) \cong H^1(X, \mathcal{O}_X)', \quad H^0(X', \Omega_{X'/k}) \cong H^0(X, \Omega_{X/k})'$$

Therefore if the spectral sequence (5.5) degenerates on the second page, we obtain the following exact sequence:

$$0 \rightarrow H^1(X, \mathcal{O}_X)' \rightarrow H_{dR}^1(X/k) \rightarrow H^0(X, \Omega_{X/k})' \rightarrow 0. \quad (5.6)$$

Suppose now that X is equipped with a faithful action of a finite group G . We say that the pair (X, G) lifts to $W_2(k)$, if there exists a smooth scheme $\mathbf{X}$ over $W_2(k)$ and a homomorphism $G \rightarrow \text{Aut}_{W_2(k)}(\mathbf{X})$ such that

$$(\mathbf{X}, G \rightarrow \text{Aut}_{W_2(k)}(\mathbf{X})) \times_{W_2(k)} k = (X, G \rightarrow \text{Aut}_k(X)).$$

The following Theorem is a G -equivariant version of the main result of [4] and follows from the functoriality of the result of Deligne and Illusie.

Theorem 5.4 *Suppose that the pair (X, G) lifts to $W_2(k)$ and that $\dim X < p$. Then the exact sequence (5.6) of $k[G]$ -modules splits. In particular, there is an isomorphism (1.4) of $\mathbb{F}_p[G]$ -modules.*

Proof The article [4] provides for each lift $\tilde{X}/W_2(k)$ of X/k an isomorphism:

$$\varphi_{\tilde{X}} : \bigoplus_i \Omega_{\tilde{X}/k}^i[-i] \rightarrow F_* \Omega_{\tilde{X}/k}^\bullet \quad (5.7)$$

in $D(X')$, the derived category of coherent $\mathcal{O}_{X'}$ -modules. The isomorphism $\varphi_{\tilde{X}}$ is functorial with respect to open embeddings (in particular to isomorphisms). Recall that φ_X^0 is defined as the composition:

$$\mathcal{O}_{X'}[0] \xrightarrow{\mathcal{C}^{-1}} h^0(F_* \Omega_{X/k}^\bullet)[0] \hookrightarrow F_* \Omega_{X/k}^\bullet$$

(see [4, proof of Théorème 2.1, part (a)]). Thus, by applying the first cohomology to (5.7) we obtain an isomorphism:

$$\phi_{\tilde{X}} : H^0(X', \Omega_{X'/k}^1) \oplus H^1(X', \mathcal{O}_{X'}) \rightarrow H^1(X', F_* \Omega_{X/k}^\bullet) \cong H^1(X, \Omega_{X/k}^\bullet) \quad (5.8)$$

which yields a splitting of (5.6). Suppose now that (X, G) lifts to $W_2(k)$. Then (5.8) becomes an isomorphism of $k[G]$ -modules by functoriality of $\varphi_{\tilde{X}}$ and $\phi_{\tilde{X}}$. $\square$

Remark 5.5 If G is a cyclic p -group and V is a $k[G]$ -module with $\dim_k V < \infty$, one may easily prove that $V \cong V'$ as $k[G]$ -modules.

The following important question remains open.

Question 5.6 Suppose that the pair (X, G) lifts to $W_2(k)$. Does it follow that the exact sequence of $k[G]$ -modules (1.2) splits?

Proposition 5.7 Keep the notation introduced in Sect. 1. If the exact sequence (5.6) of $k[G]$ -modules splits, then the action of G on X is weakly ramified.

Proof Note that for a $k[G]$ -module V of finite k -dimension, $\dim V^G = \dim(V')^G$. Thus the proof follows by the method of proof of Main Theorem. $\square$

The following is an immediate consequence of Theorem 5.4 and Proposition 5.7.

Corollary 5.8 Suppose that $p > 2$, X is a smooth projective curve over k and the pair (X, G) lifts to $W_2(k)$. Then the action of G on X is weakly ramified.

Note that it was known previously that non-weakly ramified actions on curves do not lift to $W(k)$ (cf. [14, Corollary, Sec. 4]).

The problem of lifting Galois coverings of curves from characteristic p to $W_2(k)$ and $W(k)$ has been studied extensively in the literature, cf. e.g. [15] for the case $G = \mathbb{Z}/p$. In particular, it is possible to classify all weakly ramified group actions into liftable and non-liftable ones (cf. [1, Section 4.2] and [3, Section 4.1]). However, we weren't able to extract any information that would help us to understand the behaviour of the sequence (1.2) for curves with weakly ramified group action.

Corollary 5.9 Suppose that a finite group G acts on an ordinary curve X . Then the exact sequences (1.2) and (5.6) split G -equivariantly and

$$H^0(X, \Omega_{X/k})' \cong H^0(X, \Omega_{X/k}), \quad H^1(X, \mathcal{O}_X)' \cong H^1(X, \mathcal{O}_X).$$

as $k[G]$ -modules.

Proof Let A be the Jacobian variety of X . Observe that the Abel-Jacobi map induces an isomorphism between the Hodge-de Rham sequences of X and A (cf. [12, Proposition III.2.1, Lemma III.9.5.]). The same applies to the conjugate Hodge-de Rham sequences. Moreover, A is ordinary, and thus the natural inclusions:

$$H^0(A, \Omega_{A/k}) \rightarrow H_{dR}^1(A/k), \quad H^1(A, \mathcal{O}_A)' \rightarrow H_{dR}^1(A/k)$$

induce an isomorphism $H_{dR}^1(A/k) \cong H^0(A, \Omega_{A/k}) \oplus H^1(A, \mathcal{O}_A)'$ (cf. [23, §2.1]). This isomorphism is clearly functorial and thus is an isomorphism of $k[G]$ -modules. The remaining statement is clear. $\square$

Note in particular that Corollary 5.9 implies that ordinary curves admit only weakly ramified group actions. This follows also from the Deuring–Shafarevich formula (cf. [20]).

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