Correction to: Homological multiplicities in representation theory of p -adic groups

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1 Correction of [1, §6]

Regrettably, [1, Lemma 6.4] is incorrect, even though the statements of the theorems in [1] are correct with no modification required. However, [1, Theorem 6.1], whose proof relied upon the wrong [1, Lemma 6.4], requires a different proof which we provide here. We note that the conclusion of [1, Lemma 6.4] holds true for the case of symmetric pairs and, more generally, in the factorizable case (see Chapter 9 of [4]). The rest of [1] requires no modification.

A corrected version of the original paper [1] will appear on the Arxiv.

For the proof of Theorem [1, Theorem 6.1] we require a few preparations. We employ the notation of [1] and, specifically, those of [1, §6].

Lemma 1.1 *Let a discrete countable group Γ act freely on a locally compact space Y . Assume that $\Gamma \backslash Y$ is Hausdorff. Let $\phi : Y \rightarrow \Gamma \backslash Y$ be the quotient map. Then there is an open nonempty set $U \subset \Gamma \backslash Y$ such that $\phi^{-1}(U) \cong \Gamma \times U$ as a Γ -space.*

Proof Let $E = \{(x, y) \in Y \times Y \mid \exists g \in \Gamma \text{ such that } y = gx\}$ be the equivalence relation induced by the action $a : \Gamma \times Y \rightarrow Y$. By the assumption E is closed in $Y \times Y$, hence locally compact. We have $\Gamma \cdot \Delta Y = E$, thus, by Baire's category theorem, there is a $g \in \Gamma$ such that $g\Delta Y$ is not nowhere dense in E , therefore neither is ΔY nowhere dense in E . Since ΔY is closed in $Y \times Y$, it is also closed in E . Consequently, the interior of ΔY in E is not empty, thus we have a non-empty open set $V \subset Y$ such that ΔV is open in E . As a result, we have

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some open $W \subset Y \times Y$ such that $W \cap E = \Delta V$. Let $x \in \Delta V$. Let $A, B \subset Y$ be open sets such that $x \in A \times B \subset W$. Let $C := A \cap B$. We then get $x \in C \times C \subset W$ as well as $C \times C \cap E = \Delta C$. Let $U := \phi(C)$. The map $\phi|_C : C \rightarrow U$ is an open bijection, thus a homeomorphism. Let ψ be its inverse. Consider the composition

$$\Gamma \times U \xrightarrow{id \times \psi} \Gamma \times C \xrightarrow{a|_{\Gamma \times C}} \phi^{-1}(U).$$

This is a Γ -equivariant open surjection. Since the action of Γ of Y is free this composition is also injective. Thus it is bijective, and gives the required homeomorphism. $\square$

Corollary 1.2 *Let S be a locally compact group. Let Y be a transitive locally compact S -space. Let a discrete countable group Γ act freely on Y , commuting with the action of S . Assume that $\Gamma \backslash Y$ is Hausdorff. Then:*

1. *The map $\phi : Y \rightarrow \Gamma \backslash Y$ is a locally trivial fiber bundle.*
2. *If, in addition, Y is totally disconnected and countable at infinity, then ϕ is a trivial fiber bundle.*

Proof 1. Since the action of Γ commutes with the action of S on Y , the latter descends to an action on $\Gamma \backslash Y$. By Lemma 1.1 we have a non-empty $U \subset \Gamma \backslash Y$ such that $\phi^{-1}(U) \cong \Gamma \times U$ as a Γ -space. Thus for any $g \in S$, we have $\phi^{-1}(gU) \cong \Gamma \times gU$ as a Γ -space. This implies the local triviality of ϕ , since the action of S on $\Gamma \backslash Y$ is transitive.

2. The space Y has a countable basis consisting of open compact sets. Their images in the quotient $\Gamma \backslash Y$ form a countable basis for the topology of $\Gamma \backslash Y$ consisting of open compact sets. Therefore any open cover of $\Gamma \backslash Y$ has a refinement which is disjoint. This, together with the previous item, proves the assertion. $\square$

Corollary 1.3 *Let S be an l -group and let Y be an l -space with a transitive action of S . Let Γ be a discrete countable group acting freely on Y and commuting with the action of S . Assume that Y is countable at infinity and that $\Gamma \backslash Y$ is Hausdorff. Let $\mathcal{L}$ be an Γ -equivariant sheaf on Y . Then $S(Y, \mathcal{L})$ is a free $\mathcal{H}(\Gamma)$ -module.*

Proof By previous corollary (Corollary 1.2(2)), $Y \cong \Gamma \backslash Y \times \Gamma$ as a Γ space. Thus there is a vector bundle $\mathcal{L}'$ on $\Gamma \backslash Y$ such that $\mathcal{L} \cong \phi^*(\mathcal{L}')$, where $\phi : Y \rightarrow \Gamma \backslash Y$ is the quotient map. This implies that $S(Y, \mathcal{L}) \cong \mathcal{H}(\Gamma) \otimes S(\Gamma \backslash Y, \mathcal{L}')$. $\square$

Lemma 1.4 *Let $\phi : S_1 \rightarrow S_2$ be a surjective continuous homomorphism of l -groups. Assume that S_1 is countable at infinity. Then ϕ is open.*

Proof It is enough to show that, for any compact open subgroup $K < S_1$, the group $\phi(K)$ is open. Note that S_1/K is countable. We get $S_2 = \bigcup_i h_i \phi(K)$ for some sequence $(h_i) \subset S_2$. By Baire's category theorem we get i such that $h_i \phi(K)$ has non-empty interior. Hence, $\phi(K)$ has non-empty interior. Since $\phi(K)$ is a group, the fact that $\phi(K)$ has non-empty interior implies that it is open. $\square$

To guarantee the Hausdorff property we will use:

Lemma 1.5 *Let $\Lambda_{\mathcal{Z}}$ be as in [1, Lemma 6.3]. Let $\Gamma < \Lambda_{\mathcal{Z}}$, $\mathbf{H} < \mathbf{G}$ and put $H := \mathbf{H}(F)$. Assume that $\Gamma \cap H$ is trivial. Then $H\Gamma \subset G$ is closed.*

Proof Let $\mathbf{T} := \Gamma \otimes_{\mathbb{Z}} \mathbb{G}_m$. We have a natural homomorphism of algebraic groups $\psi : \mathbf{T} \rightarrow Z(\mathbf{G})$. It gives rise to a homomorphism $\alpha : \mathbf{H} \times \mathbf{T} \rightarrow \mathbf{G}$. We will now show that α has

a finite kernel. Let $\mathbf{K}$ be its kernel and $\mathbf{K}'$ be its projection to $\mathbf{T}$. If $\mathbf{K}$ is infinite, then so is $\mathbf{K}'$ and hence $\mathbf{K}'$ has a split non-trivial subtorus $\mathbf{T}'$. Let $m : \mathbb{G}_m \rightarrow \mathbf{T}'$ be a nontrivial homomorphism. Such a homomorphism gives a non-trivial element in $\Gamma \cap H$, contradicting our assumption. We thereby conclude that α has a finite kernel.

Let $Q := \text{Im}(\alpha_F)$ where $\alpha_F : H \times T \rightarrow G$ is the map induced by α on F -points. By [3, Theorem A.2], Q is constructible. Since $\alpha_F(T)$ is central, Q is a subgroup, and is therefore closed. By Lemma 1.4 the map α_F when considered as a map $H \times T \rightarrow Q$ is open. Therefore it remains to be shown that $\alpha_F^{-1}(H\Gamma)$ is closed. This follows from the finiteness of $K(F)$ and the equality

$$\alpha_F^{-1}(H\Gamma) = \mathbf{K}(F)(H \times \Gamma).$$

□

Lemma 1.6 *Let S be an l -group that is countable at infinity and ρ be an irreducible compact representation of S . Let $V \in \mathcal{M}(S)$. Then:*

1. *The map $ev : \rho \otimes \text{Hom}_S(\rho, V) \rightarrow V$ is an embedding, and there exists a unique S -submodule $L \subset V$ such that $\text{Im}(ev) \oplus L = V$.*
2. *If additionally a discrete group Γ acts on V commuting with the action of S , then $\text{Hom}_S(\rho, V)$ can be embedded as a direct summand of V as a Γ -module.*

Proof 1. By [2, Theorem 7] we can decompose $V = (\bigoplus W_i) \oplus L$ with $W_i \cong \rho$ and $\rho \notin JH(L)$. This implies that ev is an embedding and its image is $\bigoplus W_i$. This gives the existence of L . For its uniqueness, assume that we have a decomposition $\text{Im}(ev) \oplus L' = V$. This gives us a projection $p : V \rightarrow \text{Im}(ev)$. Consider $p|_L : L \rightarrow \text{Im}(ev)$. We get $p|_L = 0$, which implies that $L \subset L'$, and thus that $L = L'$.

2. Note that the map ev intertwines the action of Γ . By the previous item we can decompose $V \cong \rho \otimes \text{Hom}_S(\rho, V) \oplus L$. The uniqueness of L implies that it is Γ -invariant. Choosing a vector in ρ , we get an embedding of $\text{Hom}_S(\rho, V)$ to $\rho \otimes \text{Hom}_S(\rho, V)$ as a direct summand. □

Proof of [1, Theorem 6.1]

Step 1: Analysis of X as a $\Lambda_{\mathcal{Z}}$ -space.

We define a filtration $\Lambda_{\mathcal{Z}}^0 \subset \Lambda_{\mathcal{Z}}^1 \subset \Lambda_{\mathcal{Z}}$ that will allow us to analyze the action of $\Lambda_{\mathcal{Z}}$ on X . By [1, Lemma 6.3.] we decompose $G^1 = G^0 \times \Lambda_{\mathcal{Z}}$. Let $\Lambda_{\mathcal{Z}}^0 := H \cap \Lambda_{\mathcal{Z}}$ and $\Lambda_{\mathcal{Z}}^1 = HG^0 \cap \Lambda_{\mathcal{Z}}$. Note that for any $x \in X$, we have $\Lambda_{\mathcal{Z}}^1 \cdot x \subset G^0 \cdot x$.

Up to replacing groups by their finite index subgroups, the filtration $\Lambda_{\mathcal{Z}}^0 \subset \Lambda_{\mathcal{Z}}^1 \subset \Lambda_{\mathcal{Z}}$ splits. More precisely, denote

$$\Gamma_0 = \Lambda_{\mathcal{Z}}^0.$$

We can find subgroups $\Gamma_1, \Gamma_2 \subset \Lambda_{\mathcal{Z}}$ such that

$$\Gamma_0 \oplus \Gamma_1 := \Gamma_0 \times \Gamma_1 \subset \Lambda_{\mathcal{Z}}^1$$

is of finite index and

$$\Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 := \Gamma_0 \times \Gamma_1 \times \Gamma_2 \subset \Lambda_{\mathcal{Z}}$$

is of finite index.

Denote $G^{1,f} := (\Gamma_0 \times \Gamma_1 \times \Gamma_2)G^0 \subset G^1$. Note that $G^{1,f} \subset G$ is of finite index.

Let $g_1, \dots, g_k \in G$ be representatives of the cosets $G/G^{1,f}$ and let

$$X^0 := G^0 \cdot \{g_1, \dots, g_k\}.$$

Taking into account the fact that G^0 is normal in G , we get

$$X \cong X^0 \times \Gamma_2$$

as $G^{1,f} \cong G^0 \times \Gamma_0 \times \Gamma_1 \times \Gamma_2$ -spaces. Here, the action of $G^0 \times \Gamma_1$ is on the component X^0 , the action of Γ_2 is on the component Γ_2 , and the action of Γ_0 is trivial. Thus we have an isomorphism of $\Gamma_0 \times \Gamma_1 \times \Gamma_2$ representations:

$$\mathcal{F}_{X,\mathcal{L}}((G, \rho)) = \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X, \mathcal{L})) \cong \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0})) \otimes \mathbb{C}[\Gamma_2]. \quad (1)$$

Here Γ_0 acts on the right-hand side by a character, Γ_1 acts on the component $\mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0}))$, and Γ_2 acts on the component $\mathbb{C}[\Gamma_2]$.

Step 2: Proof that for any $G^0 \times \Gamma_1$ equivariant line bundle $\mathcal{M}$ on X^0 , the $\mathbb{C}[\Gamma_1]$ -module $\mathcal{S}(X^0, \mathcal{M})$ is free.

We will prove this using Corollary 1.3 and Lemma 1.5. It is sufficient to show that $\mathcal{S}(G^0 g_i, \mathcal{M}|_{G^0 g_i})$ is a free $\mathbb{C}[\Gamma_1]$ -module for any $1 \leq i \leq k$. Observe that Γ_1 acts freely on $G^0 g_i$. Thus by Corollary 1.3 it is sufficient to show that $G^0 g_i / \Gamma_1$ is Hausdorff. Furthermore, since $G^0 g_i / \Gamma_1 \subset X / \Gamma_1$ it is sufficient to show that the latter is Hausdorff. In other words, it must be shown that $H\Gamma_1 \subset G$ is closed, which follows from Lemma 1.5.

Step 3: Proof that $\mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0}))$ is a locally free $\mathbb{C}[\Gamma_1]$ -module.

We will deduce this from the previous step using Lemma 1.6.

Since Γ_1 acts on ρ by a character, we can twist the action of Γ_1 on $\mathcal{L}|_{X^0}$ to get a (G_0, Γ_1) -equivariant line bundle $\mathcal{L}'$ on X^0 such that

$$\mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0})) \cong \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}'))$$

as Γ_1 -modules. Here, the action of Γ_1 on the RHS is on the values only. Let $\rho = \rho_1 \oplus \dots \oplus \rho_k$ be the decomposition of ρ to irreducible representations of G_0 . It is enough to show that for all i , $\mathrm{Hom}_{G^0}(\rho_i, \mathcal{S}(X^0, \mathcal{L}'))$ is a locally free $\mathbb{C}[\Gamma_1]$ -module, when the action of $\mathbb{C}[\Gamma_1]$ is on the values only. By Lemma 1.6(2) the space $\mathrm{Hom}_{G^0}(\rho_i, \mathcal{S}(X^0, \mathcal{L}'))$ can be embedded as a direct summand of $\mathcal{S}(X^0, \mathcal{L}')$. The latter is free (as shown above), hence $\mathrm{Hom}_{G^0}(\rho_i, \mathcal{S}(X^0, \mathcal{L}'))$ is a locally free $\mathbb{C}[\Gamma_1]$ -module.

Step 4: The end of the proof.

We will deduce the theorem from the previous step using [1, Lemma 6.2.].

Let $L := \mathrm{Spec}(\mathbb{C}[\Gamma_0 \times \Gamma_1 \times \Gamma_2])$. The map $\pi : \mathfrak{X}_G = \mathrm{Spec}(\mathbb{C}[G/G^0]) \rightarrow L$ is a finite étale morphism. Using Equation (1) together with the previous step, we conclude that $\pi_*(\mathcal{F}_{X,\mathcal{L}}((G, \rho)))$ is the direct image of a free sheaf on a coset of $\mathrm{Spec}(\mathbb{C}[\Gamma_1 \times \Gamma_2])$.

Therefore [1, Lemma 6.2.] implies the assertion of the Theorem. $\square$

2 Correction of [1, Appendix B]

[1, Lemma B.2] is wrong as stated. However it holds true under the further assumption that either K is flat or N is projective. Corollary [1, Corollary B.4] have to be changed accordingly. The rest of the Appendix holds without changes.

In the following we provide a corrected version of the lemma, its proof and a discussion of its use in [1, Appendix B].

Lemma 2.1 *Let A and B be associative unital Noetherian algebras. Let N be a finitely generated left A -module, K be a left B -module, and M be an (A, B) -bi-module. Let*

$$b : \text{Hom}_A(N, M) \times K \rightarrow \text{Hom}_A(N, M \otimes_B K)$$

be the B -bilinear map given by

$$b(\phi, k)(n) = \phi(n) \otimes k.$$

Let

$$\beta : \text{Hom}_A(N, M) \otimes_B K \rightarrow \text{Hom}_A(N, M \otimes_B K)$$

be the corresponding morphism.

- (1) *if K is a flat B -module, then β is an isomorphism.*
- (2) *if N is a projective A -module, then β is an isomorphism.*

Proof We first consider the case where K is flat. Since N is finitely generated and A is Noetherian we can find a finite presentation

$$A^n \rightarrow A^m \rightarrow N \rightarrow 0 \quad (2)$$

of N . Applying the functor $\text{Hom}_A(\bullet, M)$ to (2), we obtain the following exact sequences:

$$0 \rightarrow \text{Hom}_A(N, M) \rightarrow M^m \rightarrow M^n.$$

Since K is flat we get an exact sequence

$$0 \rightarrow \text{Hom}_A(N, M) \otimes_B K \rightarrow M^m \otimes_B K \rightarrow M^n \otimes_B K. \quad (3)$$

Applying the functor $\text{Hom}_A(\bullet, M \otimes_B K)$ to (2), we also get

$$0 \rightarrow \text{Hom}_A(N, M \otimes_B K) \rightarrow (M \otimes_B K)^m \rightarrow (M \otimes_B K)^n.$$

Equivalently,

$$0 \rightarrow \text{Hom}_A(N, M \otimes_B K) \rightarrow M^m \otimes_B K \rightarrow M^n \otimes_B K \quad (4)$$

is exact. Consider now the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(N, M) \otimes_B K & \longrightarrow & M^m \otimes_B K & \longrightarrow & M^n \otimes_B K \\ & & \downarrow \beta & & \downarrow id & & \downarrow id \\ 0 & \longrightarrow & \text{Hom}_A(N, M \otimes_B K) & \longrightarrow & M^m \otimes_B K & \longrightarrow & M^n \otimes_B K \end{array}$$

Applying the 5-Lemma we deduce that β is an isomorphism as claimed.

To prove the second item, we use the projectivity and finite generation of N to write $A^n = N \oplus N'$, for some integer n and an A -module N' . Define

$$\beta' : \text{Hom}_A(N', M) \otimes_B K \rightarrow \text{Hom}_A(N', M \otimes_B K)$$

in a similar fashion to β . It is easy to see that $\beta \oplus \beta' : M^n \otimes_B K \rightarrow (M \otimes_B K)^n$ is an isomorphism. This imply that β is an isomorphism. $\square$

2.1 The usage of [1, Lemma B.2] in [1, Appendix B]

[1, Lemma B.2] is used in two places in [1]: [1, Lemma B.5] and [1, Corollary B.4].

In [1, Lemma B.5] we use [1, Lemma B.2] for the case where B is $\mathcal{H}(G^0, K)$ and K is $\mathcal{H}(G, K)$. It is easy to see that the module K is a free B -module and hence it is flat. Thus, this case is covered by the first item of Lemma 2.1.

For the validity of [1, Corollary B.4] we have to add the assumption that V is projective. With this further assumption [1, Corollary B.4] is true and is indeed a consequence of the second item of Lemma 2.1 (that is, the corrected version of [1, Lemma B.2] provided above).

In [1, Appendix B] we use [1, Corollary B.4] only in Proposition [1, Lemma B.6]. There we apply [1, Corollary B.4] with $V = \text{ind}_{M^0}^M(\rho|_{M^0})$ where ρ is a cuspidal representation of M and $M^0 < M$ the subgroup generated by compact subgroups of M . In this case, by a result of Harish-Chandra, the module V is indeed projective and the conclusion is valid.

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