A singular eigenvalue problem for the Dirichlet (p, q) -Laplacian

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Abstract

We consider a parametric nonlinear, nonhomogeneous Dirichlet problem driven by the (p, q) -Laplacian with a reaction involving a singular term plus a superlinear reaction which does not satisfy the Ambrosetti–Rabinowitz condition. The main goal of the paper is to look for positive solutions and our approach is based on the use of variational tools combined with suitable truncations and comparison techniques. We prove a bifurcation-type theorem describing in a precise way the dependence of the set of positive solutions on the parameter λ . Moreover, we produce minimal positive solutions and determine the monotonicity and continuity properties of the minimal positive solution map.

Keywords (p, q) -Laplacian · Singular term · Nonlinear regularity · Nonlinear maximum principle · Truncation · Bifurcation-type theorem · Minimal positive solution

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper, we study the following singular eigenvalue problem for the Dirichlet (p, q) -Laplacian

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda[u(z)^{-\eta} + f(z, u(z))] & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (P_\lambda)$$

with $\lambda > 0$, $0 < \eta < 1$ and $1 < q < p$.

For $r \in (1, \infty)$, by Δ_r we denote the r -Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \text{for all } u \in W^{1,r}(\Omega).$$

In (P_λ) , we have the sum of two such operators. So, in problem (P_λ) , the differential operator is nonhomogeneous and this is a source of difficulties in the study of (P_λ) . Boundary value problems driven by a combination of two or more operators of different nature [(such as (p, q) -equations)], arise in many mathematical models of physical processes. One of the first such models was introduced by Cahn-Hilliard [6] describing the process of separation of binary alloys. Other applications can be found in Zakharov [36] (on plasma physics), in Benci-D'Avenia-Fortunato-Pisani [4] (on quantum physics), in Cherfils-Il'Yasov [7] (on reaction-diffusion systems) and in Bahrouni-Rădulescu-Repovš [2] (on transonic flow problems).

In the reaction of (P_λ) , $\lambda > 0$ is a parameter, $u \mapsto u^{-\eta}$ with $0 < \eta < 1$ is a singular term and $f(z, x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable on Ω and for a.a $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). We assume that for a.a $z \in \Omega$, $f(z, \cdot)$ is $(p-1)$ -superlinear near $+\infty$. However, this superlinearity of the perturbation $f(z, \cdot)$ is not formulated using the very common in the literature Ambrosetti-Rabinowitz condition (the AR-condition, for short), see Ref. [1]. Instead, we employ a less restrictive condition which incorporates in our framework also superlinear nonlinearities with "slower" growth near $+\infty$ which fail to satisfy the AR-condition. The main goal of the paper is to explore the existence of a positive solution to (P_λ) . Using variational tools from the critical point theory together with truncations and comparison techniques, we show that (P_λ) has a continuous spectrum. More precisely, we prove a bifurcation-type theorem, producing a critical parameter value $\lambda^* > 0$ such that

- for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two positive solutions;
- for $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution;
- for all $\lambda > \lambda^*$, problem (P_λ) has no positive solution.

Moreover, we show that for every $\lambda \in \mathcal{L} := (0, \lambda^*)$ problem (P_λ) admits a minimal positive solution u_λ^* , and establish the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

Our work here extends that of Lü-Xie [25], who considered equations driven by the p -Laplacian only and $f(z, x) = x^{r-1}$ with $p < r < p^*$ (recall $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$ the critical Sobolev exponent corresponding to p). In [25], the authors did not prove the precise dependence of the set of positive solutions on the parameter $\lambda > 0$, that is, they did not prove a bifurcation-type theorem as described above and they did not produce the minimum positive solution.

Other type of eigenvalue problems for the (p, q) -Laplacian, but with no singular terms, can be found in Bobkov-Tanaka [5], Papageorgiou-Rădulescu-Repovš [27],

Papageorgiou-Vetro-Vetro [31], Tanaka [35], Zeng-Bai-Gasiński-Winkert [37,38] and the references therein. Elliptic problems with singular terms can be found in Ghergu-Rădulescu [15], Crandall-Rabinowitz-Tartar [9], Papageorgiou-Winkert [32], Bartušek-Fujimoto [3], Gasiński-Papageorgiou [14], Cîrstea-Ghergu-Rădulescu [8], Liu-Motreanu-Zeng [24], Dupaigne-Ghergu-Rădulescu [11], Papageorgiou-Vetro-Vetro [33]. A more detailed bibliography can be found in the book of Ghergu-Rădulescu [18].

2 Mathematical background, hypotheses and auxiliary results

The main spaces that will be used in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p, \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \mid u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\},$$

which has nonempty interior given by

$$\text{int}C_+ = \left\{ u \in C_+ \mid u(z) > 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial u}{\partial n} \Big|_{\partial\Omega} < 0 \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. We will also use another open cone in $C^1(\overline{\Omega})$, namely, the cone

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) \mid u(z) > 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial u}{\partial n} \Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Given $u, v \in W_0^{1,p}(\Omega)$ with $u \leq v$, we set

$$\begin{aligned} [u, v] &= \{h \in W_0^{1,p}(\Omega) \mid u(z) \leq h(z) \leq v(z) \text{ for a.a } z \in \Omega\}, \\ u &= \{h \in W_0^{1,p}(\Omega) \mid u(z) \leq h(z) \text{ for a.a } z \in \Omega\}. \end{aligned}$$

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u = u^+ - u^-, |u| = u^+ + u^- \text{ and } u^\pm \in W_0^{1,p}(\Omega).$$

We say that $S \subseteq W_0^{1,p}(\Omega)$ is "downward directed", if for every pair $(u_1, u_2) \in S \times S$, we can find $u \in S$ such that $u \leq u_1$ and $u \leq u_2$. Given $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 < h_2$, if for each $K \subseteq \Omega$ compact there exists a constant $c_K > 0$ such that

$$0 < c_K \leq h_2(z) - h_1(z) \text{ for a.a } z \in K.$$

It is obvious that if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 < h_2$.

With X a Banach space and $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R} \text{ is bounded and } (1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty,$$

admits a strongly convergent subsequence.

Also by K_φ we denote the critical set of $\varphi(\cdot)$, that is, $K_\varphi = \{u \in X \mid \varphi'(u) = 0\}$.

For every $r \in (1, \infty)$, by $A_r: W_0^{1,r}(\Omega) \rightarrow W_0^{1,r}(\Omega)^* = W^{-1,r'}(\Omega)$ ($\frac{1}{r} + \frac{1}{r'} = 1$) we denote the nonlinear map defined by

$$\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{r-2} (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,r}(\Omega).$$

The following properties of $A_r(\cdot)$ are well-known (see for example, Gasiński-Papageorgiou [13], Problem 2.192, p. 279).

Proposition 1 *The map $A_r: W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle A_r(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega).$$

The hypotheses on the perturbation $f(z, x)$ are following:

$\underline{H}$: $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a $z \in \Omega$ and

- (i) $f(z, x) \leq \alpha(z)[1 + x^{r-1}]$ for a.a $z \in \Omega$, all $x \geq 0$, with $\alpha \in L^\infty(\Omega)_+$ and $p < r < p^*$;
- (ii) if $F(z, x) = \int_0^x f(z, s) ds$, then $\lim_{x \rightarrow +\infty} \frac{F(z, x)}{x^p} = +\infty$ uniformly for a.a $z \in \Omega$;
- (iii) if $e(z, x) = \left[1 - \frac{p}{1-\eta}\right] x^{1-\eta} + f(z, x)x - pF(z, x)$, then there exists $\beta \in L^1(\Omega)_+$ such that

$$e(z, x) \leq e(z, y) + \beta(z) \text{ for a.a } z \in \Omega, \text{ all } 0 \leq x \leq y;$$

- (iv) there exist $\delta > 0$ and $\tau \in (1, q)$ such that

$$c_0 x^{\tau-1} \leq f(z, x) \text{ for a.a } z \in \Omega, \text{ all } x \in [0, \delta], \text{ with } c_0 > 0,$$

and for all $s > 0$, we have

$$0 < m_s \leq f(z, x) \text{ for a.a } z \in \Omega, \text{ all } x \geq s;$$

- (v) for every $\rho > 0$, there exists $\widehat{E}_\rho > 0$ such that for a.a $z \in \Omega$, the function

$$x \mapsto f(z, x) + \widehat{E}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 2 Since our goal is to find positive solutions for problem (P_λ) and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z, x) = 0 \text{ for a.a } z \in \Omega, \text{ all } x \leq 0. \quad (1)$$

Hypotheses $H(ii)$ and (iii) imply that

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for a.a } x \in \Omega,$$

that is, for a.a $z \in \Omega$ the perturbation $f(z, \cdot)$ is $(p - 1)$ -superlinear. Often in the literature superlinear problems are treated by using the AR-condition. In our case, on account of (1), we will state a unilateral version of this condition. According to the AR-condition, there exist $\mu > p$ and $M > 0$ such that

$$0 < \mu F(z, x) \leq f(z, x)x \quad \text{for a.a } z \in \Omega, \quad \text{all } x \geq M, \quad (2)$$

$$0 < \operatorname{ess\,inf}_{\Omega} F(\cdot, M) \quad (3)$$

(see Ambrosetti–Rabinowitz [1]). Integrating (2) and using (3), we obtain the following weaker condition

$$\begin{aligned} c_1 x^\mu &\leq F(z, x) \quad \text{for a.a } z \in \Omega, \quad \text{all } x \geq M, \quad \text{some } c_1 > 0 \\ \Rightarrow c_1 x^{\mu-1} &\leq f(z, x) \quad \text{for a.a } z \in \Omega, \quad \text{all } x \geq M. \end{aligned}$$

So, the AR-condition restricts $f(z, \cdot)$ to have at least $(\mu - 1)$ -polynomial growth near $+\infty$. In contrast, the quasimonotonicity condition that we use in this work (see hypothesis $H(iii)$), does not impose such a restriction on the growth of $f(z, \cdot)$ and permits also the consideration of superlinear nonlinearities with slower growth near $+\infty$ (see the examples below). Besides, hypothesis $H(iii)$ is a slight extension of a condition used by Li–Yang[23]. There are convenient ways to verify $H(iii)$. So, the hypothesis $H(iii)$ holds, if we can find $M > 0$ such that for a.a $z \in \Omega$

$$\begin{aligned} x \mapsto \frac{x^{-\eta} + f(z, x)}{x^{p-1}} &\text{ is nondecreasing on } [M, +\infty) \\ \text{or } x \mapsto e(z, x) &\text{ is nondecreasing on } [M, +\infty). \end{aligned}$$

Hypothesis $H(iv)$ implies the presence of a concave term near zero, while hypothesis $H(v)$ is a one sided local Hölder condition. It is satisfied, if for a.a $x \in \Omega$, $f(z, \cdot)$ is differentiable and for every $\rho > 0$ we can find $\widehat{c}_\rho > 0$ such that

$$-\widehat{c}_\rho x^{p-1} \leq f'_x(z, x)x \quad \text{for a.a } z \in \Omega, \quad \text{all } 0 \leq x \leq \rho.$$

Example 3 Consider the following functions (for the sake of simplicity, we drop the z -dependence):

$$\begin{aligned} f_1(x) &= \begin{cases} x^{\tau-1} & \text{if } 0 \leq x \leq 1 \\ x^{r-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < \tau < q < p < r < p^* \\ f_2(x) &= \begin{cases} x^{\tau-1} & \text{if } 0 \leq x \leq 1 \\ x^{p-1} \ln x + x^{s-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < \tau < q < p, 1 < s < p \end{aligned}$$

(see (1)). Both functions satisfy hypotheses H , but only $f_1(\cdot)$ satisfies the AR-condition.

As always by a solution of problem (P_λ) , we mean a "weak solution", namely, a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\langle A_p(u), h \rangle + \langle A_q(u), h \rangle = \int_{\Omega} \lambda [u^{-\eta} + f(z, u)] h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

The difficulty that we encounter in the analysis of problem (P_λ) is that the energy (Euler) function of the problem $\varphi_\lambda: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \lambda \int_{\Omega} \left[\frac{1}{1-\eta} (u^+)^{1-\eta} + F(z, u^+) \right] dz$$

for all $u \in W_0^{1,p}(\Omega)$, is not C^1 (due to the singular term). So, we can not use the minimax methods of critical point theory directly on $\varphi_\lambda(\cdot)$. We have to find ways to bypass the singularity and to deal with C^1 -functionals.

On account of hypotheses $H(i)$ and (iv) , we can find $c_2 > 0$ such that

$$c_0 x^{\tau-1} - c_2 x^{r-1} \leq f(z, x) \text{ for a.a } x \in \Omega, \text{ all } x \geq 0. \quad (4)$$

This unilateral growth estimate on $f(z, \cdot)$ leads to the following auxiliary Dirichlet (p, q) -equation

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda [c_0 u(z)^{\tau-1} - c_2 u(z)^{r-1}] & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u > 0, & \text{in } \Omega, \end{cases} \quad (Q_\lambda)$$

with $\lambda > 0$ and $1 < \tau < q < p < r < p^*$.

Proposition 4 *For every $\lambda > 0$, problem (Q_λ) admits a unique positive solution $\underline{u}_\lambda \in \text{int}C_+$ and $\underline{u}_\lambda \rightarrow 0$ in $C_0^1(\bar{\Omega})$ as $\lambda \rightarrow 0^+$.*

Proof First we prove the existence of a positive solution. To this end, let $\psi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional defined by

$$\psi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q + \lambda \frac{c_2}{r} \|u^+\|_r^r - \lambda \frac{c_0}{\tau} \|u^+\|_\tau^\tau \text{ for all } u \in W_0^{1,p}(\Omega).$$

Since $1 < \tau < q < p < r$, it is clear that $\psi_\lambda(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\underline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(\underline{u}_\lambda) = \min \left[\psi_\lambda(u) \mid u \in W_0^{1,p}(\Omega) \right]. \quad (5)$$

Recall that $1 < \tau < q < p < r$, if $u \in \text{int}C_+$ and $t \in (0, 1)$ is small, we have

$$\psi_\lambda(tu) < 0 \Rightarrow \psi_\lambda(\underline{u}_\lambda) < 0 = \psi_\lambda(0) \text{ (see (5))} \Rightarrow \underline{u}_\lambda \neq 0.$$

Using (5) again, we have

$$\psi'_\lambda(\underline{u}_\lambda) = 0,$$

i.e.,

$$\langle A_p(\underline{u}_\lambda), h \rangle + \langle A_q(\underline{u}_\lambda), h \rangle = \lambda \int_\Omega c_0 (\underline{u}_\lambda^+)^{\tau-1} h \, dz - \lambda \int_\Omega c_2 (\underline{u}_\lambda^+)^{r-1} h \, dz \quad (6)$$

for all $h \in W_0^{1,p}(\Omega)$. In (6), we choose $h = \underline{u}_\lambda^- \in W_0^{1,p}(\Omega)$. Then

$$\|\underline{u}_\lambda^-\|^p \leq 0 \Rightarrow \underline{u}_\lambda \geq 0 \text{ and } \underline{u}_\lambda \neq 0.$$

Therefore, it holds

$$-\Delta_p \underline{u}_\lambda(z) - \Delta_q \underline{u}_\lambda(z) = \lambda [c_0 \underline{u}_\lambda(z)^{\tau-1} - c_2 \underline{u}_\lambda(z)^{r-1}] \text{ for a.a } z \in \Omega. \quad (7)$$

But, Theorem 7.1, p. 286, of Ladyzhenskaya-Ural'tseva [20] implies that $\underline{u}_\lambda \in L^\infty(\Omega)$. Then, the nonlinear regularity theory of Lieberman [22] says that $\underline{u}_\lambda \in C_+ \setminus \{0\}$. Moreover, from (7) we have

$$\Delta_p \underline{u}_\lambda(z) + \Delta_q \underline{u}_\lambda(z) \leq \lambda c_2 \|\underline{u}_\lambda\|_\infty^{r-p} \underline{u}_\lambda(z)^{p-1} \text{ for a.a } z \in \Omega.$$

Hence, the nonlinear maximum principle of Pucci-Seerin [34] (pp. 111, 120) implies that $\underline{u}_\lambda \in \text{int}C_+$.

Next we show the uniqueness of this positive solution. Suppose that $\hat{\underline{u}}_\lambda \in W_0^{1,p}(\Omega)$ is another positive solution of (Q_λ) . As above, we show that $\hat{\underline{u}}_\lambda \in \text{int}C_+$. We introduce the integral functional $j: L^1(\Omega) \rightarrow \mathbb{R} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) := \begin{cases} \frac{1}{p} \|Du^{1/\tau}\|_p^p + \frac{1}{q} \|Du^{1/\tau}\|_q^q & \text{if } u \geq 0 \text{ and } u^{1/\tau} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 1 (and its proof) of Diaz-Saa [10], we have that the integral functional $j(\cdot)$ is convex (recall that $1 < \tau < q < p$).

Let $\text{dom } j = \{u \in L^1(\Omega) \mid j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). We let $h = \underline{u}_\lambda^\tau - \hat{\underline{u}}_\lambda^\tau \in C_0^1(\bar{\Omega})$. Since $\underline{u}_\lambda, \hat{\underline{u}}_\lambda \in \text{int}C_+$, for $|t| < 1$ small, we have

$$\underline{u}_\lambda^\tau + th \in \text{dom } j \text{ and } \hat{\underline{u}}_\lambda^\tau + th \in \text{dom } j$$

(see Papageorgiou-Radulescu-Repovš [28], Proposition 4.1.22, p. 274). Then we have the Gâteaux differentiability of $j(\cdot)$ at $\underline{u}_\lambda^\tau$ and at $\hat{\underline{u}}_\lambda^\tau$ in the direction h , respectively. Moreover, using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [28], Corollary 1.5.17, p. 35), we have

$$\begin{aligned} j'_\lambda(\underline{u}_\lambda^\tau)(h) &= \frac{1}{\tau} \int_\Omega \frac{-\Delta_p \underline{u}_\lambda - \Delta_q \underline{u}_\lambda}{\underline{u}_\lambda^{\tau-1}} h \, dz = \int_\Omega \lambda [c_0 - c_2 \underline{u}_\lambda^{r-\tau}] h \, dz, \\ j'_\lambda(\hat{\underline{u}}_\lambda^\tau)(h) &= \frac{1}{\tau} \int_\Omega \frac{-\Delta_p \hat{\underline{u}}_\lambda - \Delta_q \hat{\underline{u}}_\lambda}{\hat{\underline{u}}_\lambda^{\tau-1}} h \, dz = \int_\Omega \lambda [c_0 - c_2 \hat{\underline{u}}_\lambda^{r-\tau}] h \, dz. \end{aligned}$$

Whereas, the convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$0 \leq \lambda c_2 \int_\Omega [\hat{\underline{u}}_\lambda^{r-\tau} - \underline{u}_\lambda^{r-\tau}] (\underline{u}_\lambda^\tau - \hat{\underline{u}}_\lambda^\tau) \, dz \Rightarrow \underline{u}_\lambda = \hat{\underline{u}}_\lambda.$$

This proves the uniqueness of the positive solution $\underline{u}_\lambda \in \text{int}C_+$ of problem (Q_λ) .

For every $\lambda > 0$, we have

$$\begin{aligned} \|\underline{u}\|^p &\leq \lambda \hat{c} \|\underline{u}\|^\tau \text{ for some } \hat{c} > 0 \Rightarrow \|\underline{u}_\lambda\|^{p-\tau} \leq \lambda \hat{c}, \\ &\Rightarrow \underline{u}_\lambda \rightarrow 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } \lambda \rightarrow 0^+. \end{aligned}$$

Then, the nonlinear regularity theorem of Lieberman [22] and the compact embedding of $C_0^{1,\alpha}(\bar{\Omega}) := C^{1,\alpha}(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ ($0 < \alpha < 1$) into $C_0^1(\bar{\Omega})$, imply that

$$\underline{u}_\lambda \rightarrow 0 \text{ in } C_0^1(\bar{\Omega}) \text{ as } \lambda \rightarrow 0^+.$$

This completes the proof of the proposition. $\square$

Next we consider another auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda u_\lambda(z)^{-\eta} + 1 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u > 0, & \text{in } \Omega, \end{cases} \quad (N_\lambda)$$

with $\lambda > 0$, $0 < \eta < 1$ and $1 < q < p$.

Proposition 5 For every $\lambda > 0$, problem (N_λ) has a unique solution $\bar{u}_\lambda \in \text{int}C_+$ and we can find $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ it holds

$$\underline{u}_\lambda \leq \bar{u}_\lambda.$$

Proof Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \bar{\Omega}$. Lemma 14.16, p. 335, of Gilbarg-Trudinger [17] says that we can find $\delta_0 > 0$ such that $\hat{d} \in C^2(\Omega_{\delta_0})$ with $\Omega_{\delta_0} = \{z \in \bar{\Omega} \mid \hat{d}(z) < \delta_0\}$. It follows that $\hat{d} \in \text{int}C_+$ and so by Proposition 4.1.22, p. 274 of Papageorgiou-Rădulescu-Repovš [28], we can find $c_3 = c_3(\underline{u}_\lambda) > 0$ and $c_4 = c_4(\underline{u}_\lambda) > 0$ such that

$$c_3\hat{d} \leq \underline{u}_\lambda \leq c_4\hat{d}. \quad (8)$$

Then we can apply Theorem B.1 of Giacomoni-Saoudi [16] (see also Lieberman [22]) to produce a unique solution $\bar{u}_\lambda \in C_+ \setminus \{0\}$ to problem (N_λ) . In fact the nonlinear maximum principle of Pucci-Serern [34] (pp. 111, 120) implies that $\bar{u}_\lambda \in \text{int}C_+$.

Next we show that there exists $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, we have $\underline{u}_\lambda \leq \bar{u}_\lambda$. Acting on (N_λ) with $\bar{u}_\lambda \in \text{int}C_+$, we obtain

$$\begin{aligned} \|D\bar{u}_\lambda\|_p^p &\leq \lambda \int_{\Omega} \underline{u}_\lambda^{1-\eta} \frac{\bar{u}_\lambda}{\underline{u}_\lambda} dz \\ &\leq \lambda c_5 \int_{\Omega} \frac{\bar{u}_\lambda}{\hat{d}} dz \text{ for some } c_5 > 0 \text{ (see Proposition 4 and (8))} \\ &\leq \lambda c_6 \|D\bar{u}_\lambda\|_p \text{ (by Hardy's inequality).} \end{aligned}$$

So, we have $\{\bar{u}_\lambda\}_{\lambda \in (0,1]} \subseteq W_0^{1,p}(\Omega)$ is bounded. Then, from [12] and Lemma A.6 of Giacomoni-Saoudi [16] (see also Ladyzhenskaya-Ural'tseva [20] Theorem 7.1, p. 286), we obtain that

$$\{\bar{u}_\lambda\}_{\lambda \in (0,1]} \subseteq L^\infty(\Omega) \text{ is bounded.} \quad (9)$$

On account of (9) and hypothesis $H(i)$, we can find $\lambda_0 \in (0, 1]$ such that

$$\lambda f(z, \bar{u}_\lambda(z)) \leq 1 \text{ for a.a } z \in \Omega \text{ all } 0 < \lambda \leq \lambda_0. \quad (10)$$

Then, for $\lambda \in (0, \lambda_0]$, we consider the Carathéodory function $k_\lambda(z, x)$ defined by

$$k_\lambda(z, x) = \begin{cases} \lambda[c_0(x^+)^{\tau-1} - c_2(x^+)^{r-1}] & \text{if } x \leq \bar{u}_\lambda(z), \\ \lambda[c_0\bar{u}_\lambda(z)^{\tau-1} - c_2\bar{u}_\lambda(z)^{r-1}] & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \quad (11)$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s) ds$ and consider the C^1 -functional $\delta_\lambda: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\delta_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} K_\lambda(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega). \quad (12)$$

Evidently $\delta_\lambda(\cdot)$ is coercive (see (9)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\delta_\lambda(\tilde{u}_\lambda) = \min \left[\delta_\lambda(u) \mid u \in W_0^{1,p}(\Omega) \right] < 0 = \delta_\lambda(0) \text{ (since } 1 < \tau < q < p < r), \quad (13)$$

$$\Rightarrow \tilde{u}_\lambda \neq 0. \quad (14)$$

From (12) and (13), we have

$$\langle A_p(\tilde{u}_\lambda), h \rangle + \langle A_q(\tilde{u}_\lambda), h \rangle = \int_{\Omega} k_\lambda(z, \tilde{u}_\lambda) h dz \text{ for all } h \in W_0^{1,p}(\Omega). \quad (15)$$

In (15), first, we choose $h = -\tilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$ to obtain $\tilde{u}_\lambda \geq 0$ and $\tilde{u}_\lambda \neq 0$ (see (14)). Next, in (15), we take $h = [\tilde{u}_\lambda - \bar{u}_\lambda]^+ \in W_0^{1,p}(\Omega)$ to find

$$\begin{aligned} & \langle A_p(\tilde{u}_\lambda), (\tilde{u}_\lambda - \bar{u}_\lambda)^+ \rangle + \langle A_q(\tilde{u}_\lambda), (\tilde{u}_\lambda - \bar{u}_\lambda)^+ \rangle \\ &= \int_\Omega \lambda [c_0 \bar{u}_\lambda^{\tau-1} - c_2 \bar{u}_\lambda^{\tau-1}] (\tilde{u}_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (11))} \\ &\leq \int_\Omega \lambda f(z, \bar{u}_\lambda) (\tilde{u}_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (4))} \\ &\leq \int_\Omega [\lambda \underline{u}_\lambda^{-\eta} + 1] (\tilde{u}_\lambda - \bar{u}_\lambda)^+ dz \text{ for } 0 < \lambda \leq \lambda_0 \text{ (see (10))} \\ &= \langle A_p(\bar{u}_\lambda), (\tilde{u}_\lambda - \bar{u}_\lambda)^+ \rangle + A_q(\bar{u}_\lambda), (\tilde{u}_\lambda - \bar{u}_\lambda)^+ \rangle \\ &\Rightarrow \tilde{u}_\lambda \leq \bar{u}_\lambda. \end{aligned}$$

So, we have proved that

$$\tilde{u}_\lambda \in [0, \bar{u}_\lambda], \quad \tilde{u}_\lambda \neq 0. \quad (16)$$

From (11), (15), (16) and Proposition 4, we infer that

$$\tilde{u}_\lambda = \underline{u}_\lambda \Rightarrow \underline{u}_\lambda \leq \bar{u}_\lambda \text{ for all } 0 < \lambda \leq \lambda_0 \text{ (see (16)).}$$

This completes the proof of the proposition. $\square$

Remark 6 From the above proof we have $\underline{u}_\lambda^{-\eta} h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$, while from the proof of the Lemma in Lazer-McKenna [21], we have that $\underline{u}_\lambda^{-\eta} \in L^1(\Omega)$.

3 Positive solutions

We introduce the following two sets

$$\mathcal{L} := \{\lambda > 0 \mid \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$

and S_λ the set of positive solutions to problem (P_λ) .

First, we show the nonemptiness of $\mathcal{L}$.

Proposition 7 *If hypotheses H hold, then $\mathcal{L} \neq \emptyset$.*

Proof Let $\lambda_0 > 0$ be as postulated by Proposition 5, and let $\lambda \in (0, \lambda_0]$. We have

$$\underline{u}_\lambda \leq \bar{u}_\lambda \quad \text{and} \quad \lambda f(z, \bar{u}_\lambda(z)) \leq 1 \text{ for a.a } z \in \Omega \quad (17)$$

(see Proposition 5 and its proof). We introduce the following truncation of the reaction of the problem (P_λ)

$$g_\lambda(z, x) = \begin{cases} \lambda [\underline{u}_\lambda(z)^{-\eta} + f(z, \underline{u}_\lambda(z))] & \text{if } x < \underline{u}_\lambda(z), \\ \lambda [x^{-\eta} + f(z, x)] & \text{if } \underline{u}_\lambda(z) \leq x \leq \bar{u}_\lambda(z), \\ \lambda [\bar{u}_\lambda(z)^{-\eta} + f(z, \bar{u}_\lambda(z))] & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \quad (18)$$

This is a Carathéodory function. We set $G_\lambda(z, x) = \int_0^x g_\lambda(z, s) ds$ and consider the functional $\hat{\varphi}_\lambda: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega G_\lambda(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Then, $\hat{\varphi}_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ (see Papagerogiou-Smyrlis [30], Proposition 3). From (18), we see that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}\hat{\varphi}_\lambda(u_\lambda) &= \min [\hat{\varphi}_\lambda(u) \mid u \in W^{1,p}(\Omega)], \\ \Rightarrow \hat{\varphi}'_\lambda(u_\lambda) &= 0, \\ \Rightarrow \langle A_p(u_\lambda), h \rangle + \langle A_q(u_\lambda), h \rangle &= \int_\Omega g_\lambda(z, u_\lambda) h \, dz \text{ for all } h \in W_0^{1,p}(\Omega).\end{aligned}\quad (19)$$

In (19), first, we choose $h = (u_\lambda - \bar{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$ to get

$$\begin{aligned}\langle A_p(u_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle + \langle A_q(u_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle \\ = \int_\Omega \lambda [\bar{u}_\lambda^{-\eta} + f(z, \bar{u}_\lambda)] (u_\lambda - \bar{u}_\lambda)^+ \, dz \text{ (see (18))} \\ \leq \int_\Omega [\lambda \underline{u}_\lambda^{-\eta} + 1] (u_\lambda - \bar{u}_\lambda)^+ \, dz \text{ (see (17))} \\ = \langle A_p(\bar{u}_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle + \langle A_q(\bar{u}_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle \text{ (see Proposition 5)} \\ \Rightarrow u_\lambda \leq \bar{u}_\lambda.\end{aligned}$$

Next, in (19), we take $h = (\underline{u}_\lambda - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. It finds

$$\begin{aligned}\langle A_p(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \rangle + \langle A_q(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \rangle \\ = \int_\Omega \lambda [\underline{u}_\lambda^{-\eta} + f(z, \underline{u}_\lambda)] (\underline{u}_\lambda - u_\lambda)^+ \, dz \text{ (see (18))} \\ \geq \int_\Omega \lambda f(z, \underline{u}_\lambda) (\underline{u}_\lambda - u_\lambda)^+ \, dz \\ \geq \int_\Omega \lambda [c_0 \underline{u}_\lambda^{\tau-1} - c_2 \underline{u}_\lambda^{r-1}] (\underline{u}_\lambda - u_\lambda)^+ \, dz \text{ (see (4))} \\ = \langle A_p(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \rangle + \langle A_q(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+ \rangle \text{ (see Proposition 4)} \\ \Rightarrow \underline{u}_\lambda \leq u_\lambda.\end{aligned}$$

So, we have proved that

$$u_\lambda \in [\underline{u}_\lambda, \bar{u}_\lambda] \Rightarrow u_\lambda \in S_\lambda \text{ (see (18) and (19))} \Rightarrow (0, \lambda_0] \subseteq \mathcal{L} \neq \emptyset.$$

This completes the proof of the proposition. $\square$

Proposition 8 *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$ and $S_\lambda \subseteq \text{int}C_+$.*

Proof Let $u \in S_\lambda$. Reasoning as in the last part of the proof of Proposition 5 (see the part of the proof from (11) and below), replacing $\bar{u}_\lambda$ with u (see (11)), we show that $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$. Finally, $S_\lambda \subseteq \text{int}C_+$ follows from [16] (Theorem B.1, regularity theory) and from [34] (pp. 111 and 120, nonlinear maximum principle). $\square$

Next, we prove a structural property of the set $\mathcal{L}$, namely, we show that $\mathcal{L}$ is an interval.

Proposition 9 *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $\mu \in (0, \lambda)$, then $\mu \in \mathcal{L}$.*

Proof Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq \text{int}C_+$ (see Proposition 8). We consider the following Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \theta c_0 u(z)^{\tau-1} - \lambda c_2 u(z)^{r-1} & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u > 0, & \text{in } \Omega, \end{cases} \quad (H_\theta)$$

with $0 < \theta \leq \lambda$ and $1 < \tau < q < p < r$. Reasoning as in the proof of Proposition 4, via the direct method of the calculus of variations, we show that for every $\theta \in (0, \lambda]$ problem (H_θ) admits a unique solution $\tilde{u}_\theta \in \text{int}C_+$ and also we have that $\tilde{u}_\theta^{-\eta} \in L^1(\Omega)$ (see [21]). In addition, if $0 < \theta_1 < \theta_2 \leq \lambda$, then since $\theta_1 c_0 x^{\tau-1} - \lambda c_2 x^{r-1} \leq \theta_2 c_0 x^{\tau-1} - \lambda c_2 x^{r-1}$ for all $x \geq 0$, we have that $\tilde{u}_{\theta_1} \leq \tilde{u}_{\theta_2}$. Note that $\tilde{u}_\lambda = u_\lambda \in \text{int}C_+$, we have

$$\tilde{u}_\mu \leq u_\lambda \leq u_\lambda \quad (\text{see Proposition 8}).$$

Therefore, we can define the following truncation of the reaction in problem (P_μ)

$$\gamma_\mu(z, x) = \begin{cases} \mu[\tilde{u}_\mu(z)^{-\eta} + f(z, \tilde{u}_\mu(z))] & \text{if } x < \tilde{u}_\mu(z), \\ \mu[x^{-\eta} + f(z, x)] & \text{if } \tilde{u}_\mu(z) \leq x \leq u_\lambda(z), \\ \mu[u_\lambda(z)^{-\eta} + f(z, u_\lambda(z))] & \text{if } u_\lambda(z) < x. \end{cases} \quad (20)$$

This is a Carathéodory function. We set $\Gamma_\mu(z, x) = \int_0^x \gamma_\mu(z, s) ds$ and consider the C^1 -functional $\Sigma_p: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Sigma_p(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega \Gamma_\mu(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (20), it is clear that $\Sigma_\mu(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\Sigma_\mu(u_\mu) = \min \left[\Sigma_\mu(u) \mid u \in W_0^{1,p}(\Omega) \right] \Rightarrow \Sigma'_\mu(u_\mu) = 0,$$

that is,

$$\langle A_p(u_\mu), h \rangle + \langle A_q(u_\mu), h \rangle = \int_\Omega \gamma_\mu(z, u_\mu) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \quad (21)$$

In (21), first, we choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ to get

$$\begin{aligned} & \langle A_p(u_\mu), (u_\mu - u_\lambda)^+ \rangle + \langle A_q(u_\mu), (u_\mu - u_\lambda)^+ \rangle \\ &= \int_\Omega \mu[u_\lambda^{-\eta} + f(z, u_\lambda)](u_\mu - u_\lambda)^+ dz \quad (\text{see (20)}) \\ &\leq \int_\Omega \lambda[u_\lambda^{-\eta} + f(z, u_\lambda)](u_\mu - u_\lambda)^+ dz \quad (\text{since } \mu < \lambda) \\ &= \langle A_p(u_\lambda), (u_\lambda - u_\mu)^+ \rangle + \langle A_q(u_\lambda), (u_\lambda - u_\mu)^+ \rangle \\ &\Rightarrow u_\mu \leq u_\lambda. \end{aligned}$$

Next, in (21), we take $h = (\tilde{u}_\mu - u_\mu)^+ \in W_0^{1,p}(\Omega)$ to find

$$\begin{aligned}
 & \langle A_p(u_\mu), (\tilde{u}_\mu - u_\mu)^+ \rangle + \langle A_q(u_\mu), (\tilde{u}_\mu - u_\mu)^+ \rangle \\
 &= \int_{\Omega} \mu [\tilde{u}_\mu^{-\eta} + f(z, \tilde{u}_\mu)] (\tilde{u}_\mu - u_\mu)^+ dz \text{ (see (20))} \\
 &\geq \int_{\Omega} \mu f(z, \tilde{u}_\mu) (\tilde{u}_\mu - u_\mu)^+ dz \\
 &\geq \int_{\Omega} [\mu c_0 \tilde{u}_\mu^{\tau-1} - \mu c_2 \tilde{u}_\mu^{r-1}] (\tilde{u}_\mu - u_\mu)^+ dz \text{ (see (4))} \\
 &\geq \int_{\Omega} [\mu c_0 \tilde{u}_\mu^{\tau-1} - \lambda c_2 \tilde{u}_\mu^{r-1}] (\tilde{u}_\mu - u_\mu)^+ dz \text{ (since } \mu < \lambda) \\
 &= \langle A_p(\tilde{u}_\mu), (\tilde{u}_\mu - u_\mu)^+ \rangle + \langle A_q(\tilde{u}_\mu), (\tilde{u}_\mu - u_\mu)^+ \rangle \\
 &\quad \text{(since } \tilde{u}_\mu \in \text{int}C_+ \text{ is the unique solution of } (H_\mu)) \\
 &\Rightarrow \tilde{u}_\mu \leq u_\mu.
 \end{aligned}$$

So, we have proved that

$$u_\mu \in [\tilde{u}_\mu, u_\lambda] \Rightarrow u_\mu \in S_\mu \subseteq \text{int}C_+, \text{ (see (20), (21)) and so } \mu \in \mathcal{L}.$$

This completes the proof of the proposition. $\square$

Remark 10 The following observation is a byproduct of the above proof:

- if $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \text{int}C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \text{int}C_+$ such that

$$u_\mu \leq u_\lambda.$$

However, in the next proposition, we improve the observation.

Proposition 11 *If hypotheses H hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \text{int}C_+$ and $\mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \text{int}C_+$ such that*

$$u_\lambda - u_\mu \subseteq \text{int}C_+.$$

Proof From Proposition 9 and Remark 10, we know that $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \text{int}C_+$ such that $u_\mu \leq u_\lambda$. Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{E}_\rho > 0$ be as postulated by hypothesis $H(v)$. We have

$$\begin{aligned}
 & -\Delta_p u_\mu(z) - \Delta_q u_\mu(z) + \lambda \widehat{E}_\rho u_\mu(z)^{p-1} - \lambda u_\mu(z)^{-\eta} \\
 & \leq \mu f(z, u_\mu(z)) + \lambda \widehat{E}_\rho u_\mu(z)^{p-1} \quad (\text{since } u_\mu \in S_\mu \subseteq \text{int}C_+) \\
 & = \lambda [f(z, u_\mu(z)) + \widehat{E}_\rho u_\mu(z)^{p-1}] - (\lambda - \mu) f(z, u_\mu(z)) \\
 & \leq \lambda [f(z, u_\lambda(z)) + \widehat{E}_\rho u_\lambda(z)^{p-1}] \quad (\text{recall } f \geq 0, \mu < \lambda \text{ and see hypothesis } H(v)) \\
 & = -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \lambda \widehat{E}_\rho u_\lambda(z)^{p-1} - \lambda u_\lambda(z)^{-\eta}, \tag{22}
 \end{aligned}$$

due to $u_\lambda \in S_\lambda \subseteq \text{int}C_+$. On account of condition $H(iv)$, we have

$$0 < (\lambda - \mu) f(\cdot, u_\mu(\cdot)) \quad (\text{recall } u_\mu \in \text{int}C_+).$$

Then, from (22) and Proposition 7 of Papageorgiou-Rădulescu-Repovš [29], we conclude that

$$u_\lambda - u_\mu \in \text{int}C_+.$$

This completes the proof of the proposition. $\square$

Let $\lambda^* = \sup \mathcal{L}$. The following proposition reveals that λ^* is finite.

Proposition 12 *If hypotheses H hold, then $\lambda^* < +\infty$.*

Proof On account of hypotheses $H(i)$, (ii) , (iii) , we can find $\widehat{\lambda} > 0$ such that

$$x^{p-1} \leq \widehat{\lambda} f(z, x) \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geq 0. \quad (23)$$

Let $\lambda > \widehat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then, we can find $u_\lambda \in S_\lambda \subseteq \text{int} C_+$. Consider $\Omega_0 \subset \subset \Omega$ with C^2 -boundary $\partial\Omega_0$. We set $m_0 = \min_{\overline{\Omega}_0} u_\lambda > 0$, and for $\delta \in (0, 1)$ small we set $m_0^\delta = m_0 + \delta$. Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{E}_\rho > 0$ be as postulated by hypothesis $H(v)$. We have

$$\begin{aligned} & -\Delta_p m_0^\delta - \Delta_q m_0^\delta + \lambda \widehat{E}_\rho (m_0^\delta)^{p-1} - \lambda (m_0^\delta)^{-\eta} \\ & \leq \lambda \widehat{E}_\rho m_0^{p-1} + \chi(\delta) \quad (\text{with } \chi(\delta) \rightarrow 0^+ \text{ as } \delta \rightarrow 0^+) \\ & \leq [\lambda \widehat{E}_\rho + 1] m_0^{p-1} + \chi(\delta) \\ & \leq \widehat{\lambda} f(z, m_0) + \lambda \widehat{E}_\rho m_0^{p-1} + \chi(\delta) \quad (\text{see (23)}) \\ & = \lambda [f(z, m_0) + \widehat{E}_\rho m_0^{p-1}] - (\lambda - \widehat{\lambda}) f(z, m_0) + \chi(\delta) \quad (\text{since } \lambda > \widehat{\lambda}) \\ & \leq \lambda [f(z, u_\lambda) + \widehat{E}_\rho u_\lambda^{p-1}] \quad \text{for } \delta \in (0, 1) \text{ small enough,} \end{aligned}$$

where we have used the hypotheses $H(iv)$, (v) and the fact, $\chi(\delta) \rightarrow 0^+$ as $\delta \rightarrow 0^+$, i.e., for $\delta \in (0, 1)$ small enough, it has

$$0 < c_7 \leq (\lambda - \widehat{\lambda}) f(z, m_0) - \chi(\delta) \quad \text{for a.a. } z \in \Omega_0.$$

Besides, it holds

$$\lambda [f(z, u_\lambda) + \widehat{E}_\rho u_\lambda^{p-1}] = -\Delta_p u_\lambda - \Delta_q u_\lambda + \lambda \widehat{E}_\rho u_\lambda^{p-1} - \lambda u_\lambda^{-\eta} \quad \text{for a.a. } z \in \Omega_0. \quad (24)$$

Then, from (24) and Proposition 6 of Papageorgiou-Rădulescu-Repovš [29], we have that

$$u_\lambda - m_0^\delta \in D_+(\Omega_0) \quad \text{for } \delta \in (0, 1) \text{ small enough,}$$

which contradicts with the definition of m_0 . Consequently, it holds $0 < \lambda^* \leq \widehat{\lambda} < \infty$. $\square$

Therefore we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*]. \quad (25)$$

Proposition 13 *If hypotheses H hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has least two positive solutions*

$$u_0, \hat{u} \in \text{int} C_+ \text{ with } u_0 \leq \hat{u} \text{ and } u_0 \neq \hat{u}.$$

Proof Let $0 < \lambda < \theta < \lambda^*$. From (25), we have that $\lambda, \theta \in \mathcal{L}$. On account of Proposition 11, we can find $u_0 \in S_\lambda \subseteq \text{int} C_+$ and $u_\theta \in S_\theta \subseteq \text{int} C_+$ such that

$$u_\theta - u_0 \in \text{int} C_+. \quad (26)$$

From Proposition 8, we know that $\underline{u}_\lambda \leq u_0$, hence $u_0^{-\eta} \in L^1(\Omega)$. Additionally, we introduce the Carathéodory function $\widehat{w}_\lambda(z, x)$ defined by

$$\widehat{w}_\lambda(z, x) = \begin{cases} \lambda [u_0(z)^{-\eta} + f(z, u_0(z))] & \text{if } x < u_0(z), \\ \lambda [x^{-\eta} + f(z, x)] & \text{if } u_0(z) \leq x < u_\theta(z), \\ \lambda [u_\theta(z)^{-\eta} + f(z, u_\theta(z))] & \text{if } u_\theta(z) \leq x. \end{cases} \quad (27)$$

We set $\widehat{W}_\lambda(z, x) = \int_0^x \widehat{w}_\lambda(z, s) ds$ and consider the C^1 -functional $\widehat{\mu}_\lambda: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\mu}_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega \widehat{W}_\lambda(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

In addition, we introduce another Carathéodory function $w_\lambda(z, x)$ defined by

$$w_\lambda(z, x) = \begin{cases} \lambda[u_0(z)^{-\eta} + f(z, u_0(z))] & \text{if } x \leq u_0(z), \\ \lambda[x^{-\eta} + f(z, x)] & \text{if } u_0(z) < x. \end{cases} \quad (28)$$

We set $W_\lambda(z, x) = \int_0^x w_\lambda(z, s) ds$ and consider the C^1 -functional $\mu_\lambda: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mu_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega W_\lambda(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (27) and (28) that

$$\widehat{\mu}_\lambda(u)|_{[0, u_\theta]} = \mu_\lambda(u)|_{[0, u_\theta]} \text{ and } \widehat{\mu}'_\lambda(u)|_{[0, u_\theta]} = \mu'_\lambda(u)|_{[0, u_\theta]}. \quad (29)$$

Moreover, using (27), (28) and the nonlinear regularity theory, we can show that

$$K_{\widehat{\mu}_\lambda} \subseteq [u_0, u_\theta] \cap \text{int}C_+ \quad (30)$$

$$K_{\mu_\lambda} \subseteq [u_0] \cap \text{int}C_+. \quad (31)$$

From (31), it is clear that we can assume that

$$K_{\mu_\lambda} \text{ is finite.} \quad (32)$$

Otherwise, we already have an infinity of positive smooth solutions of (P_λ) bigger than u_0 (see (31) and (28)) and so we are done. In addition, we can also assume that

$$K_{\mu_\lambda} \cap [u_0, u_\theta] = \{u_0\}. \quad (33)$$

Otherwise, from (31) and (28), we see that there is a second positive smooth solution bigger than u_0 and so we are done. From (27), it is clear that $\widehat{\mu}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, there exists $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \widehat{\mu}_\lambda(\tilde{u}_0) &= \min \left[\widehat{\mu}_\lambda(u) \mid u \in W_0^{1,p}(\Omega) \right] \\ &\Rightarrow \tilde{u}_0 \in K_{\widehat{\mu}_\lambda} \subseteq [u_0, u_\theta] \cap \text{int}C_+ \text{ (see (30))}, \\ &\Rightarrow \tilde{u}_0 = u_0 \text{ (see (29) and (30))}. \end{aligned}$$

Then, (26) and (29) imply that

$$\begin{aligned} u_0 \in \text{int}C_+ \text{ is a local } C_0^1(\overline{\Omega}) - \text{minimizer of } \mu_\lambda(\cdot), \\ \Rightarrow u_0 \in \text{int}C_+ \text{ is a local } W_0^{1,p}(\Omega) - \text{minimizer of } \mu_\lambda(\cdot), \end{aligned} \quad (34)$$

where we have used Proposition 2.12 of Papageorgiou-Rădulescu [26]. From (34), (30) and Theorem 5.7.6, p.449, of Papageorgiou-Rădulescu-Repovš [28], we are able to find $\rho \in (0, 1)$ small such that

$$\mu_\lambda(u_0) < \inf [\mu_\lambda(u) \mid \|u - u_0\| = \rho] = m_\lambda. \quad (35)$$

If $u \in \text{int}C_+$, then on account of hypothesis $H(ii)$ we have

$$\mu_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \quad (36)$$

Claim. The function $\mu_\lambda(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\mu_\lambda(u_n)| \leq c_7 \text{ for some } c_7 > 0, \text{ all } n \in \mathbb{N}, \quad (37)$$

$$(1 + \|u_n\|)\mu'_\lambda(u_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \quad (38)$$

From (38), we have

$$\left| \langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle - \int_\Omega w_\lambda(z, u_n) h \, dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad (39)$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$. In (39), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ to get

$$\begin{aligned} \|Du_n^-\|_p^p + \|Du_n^-\|_q^q &\leq c_8 \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N} \\ \Rightarrow \{u_n^-\}_{n \geq 1} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned} \quad (40)$$

Next, in (39), we take $h = u_n^+ \in W_0^{1,p}(\Omega)$ to yield

$$-\|Du_n^+\|_p^p - \|Du_n^+\|_q^q + \int_\Omega \lambda[(u_n^+)^{-\eta} + f(z, u_n^+)]u_n^+ \, dz \leq c_9 \quad (41)$$

for some $c_9 > 0$, all $n \in \mathbb{N}$ (see (31)) and recall that $u_0^{-\eta} \in L^1(\Omega)$. By virtue of (37), (40) and (28), we have

$$\|Du_n^+\|_p^p + \frac{p}{q}\|Du_n^+\|_q^q - \int_\Omega \lambda \left[\frac{p}{1-\eta}(u_n^+)^{1-\eta} + pF(z, u_n^+) \right] dz \leq c_{10} \quad (42)$$

for some $c_{10} > 0$, all $n \in \mathbb{N}$. Note that $q < p$, we add (41) and (42) to find

$$\int_\Omega \lambda e(z, u_n^+) \, dz \leq c_{11} \quad (43)$$

for some $c_{11} > 0$, all $n \in \mathbb{N}$ (see hypothesis $H(\text{iii})$).

Suppose that $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. We may assume that

$$\|u_n^+\| \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (44)$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. This means that $\|y_n\| = 1$, and $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^r(\Omega) \text{ with } y \geq 0. \quad (45)$$

First, we assume that $y \neq 0$. Let $\Omega_+ = \{z \in \Omega \mid y(z) > 0\}$. Then, $|\Omega_+|_N > 0$, (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$), and we have

$$\begin{aligned} u_n^+(z) &\rightarrow +\infty \text{ for a.a } z \in \Omega_+, \text{ as } n \rightarrow \infty \\ \Rightarrow \frac{F(z, u_n^+(z))}{\|u_n^+\|^p} &= \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \rightarrow +\infty \text{ for a.a } z \in \Omega_+ \text{ (see } H(\text{ii})) \\ \Rightarrow \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz &\rightarrow +\infty \text{ (by Fatou's lemma),} \\ \Rightarrow \int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz &\rightarrow +\infty \text{ (since } F \geq 0). \end{aligned} \quad (46)$$

From (37), (40) and (28), we have for each $n \in \mathbb{N}$

$$\begin{aligned} \int_{\Omega} \lambda \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz &\leq \frac{1}{p} \|Dy_n\|_p^p + \frac{1}{q} \frac{1}{\|u_n^+\|^{p-q}} \|Dy_n\|_q^q + c_{13} \text{ for some } c_{13} > 0, \\ &\Rightarrow \lambda \int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz \leq c_{14} \text{ for some } c_{14} > 0. \end{aligned} \quad (47)$$

Comparing (46) and (47), we have a contradiction.

Next, we assume that $y \equiv 0$. Let $k > 0$ and set $v_n = (pk)^{1/p} y_n$, $n \in \mathbb{N}$. Then, we have

$$v_n \rightarrow 0 \text{ in } L^r(\Omega) \text{ (see (45))} \Rightarrow \int_{\Omega} F(z, v_n) dz \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (48)$$

Consider the C^1 -functional $\tilde{\mu}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{\mu}_\lambda(u) = \frac{1}{p} \|Du\|_p^p - \int_{\Omega} W_\lambda(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is not difficult to prove that

$$\tilde{\mu}_\lambda(u) \leq \mu_\lambda(u) \text{ for all } u \in W_0^{1,p}(\Omega). \quad (49)$$

For each $n \in \mathbb{N}$, let $t_n \in [0, 1]$ be such that

$$\tilde{\mu}_\lambda(t_n u_n^+) = \max \left[\tilde{\mu}_\lambda(tu_n^+) \mid 0 \leq t \leq 1 \right]. \quad (50)$$

From (44), we can find $n_0 \in \mathbb{N}$ such that

$$0 < \frac{(pk)^{\frac{1}{p}}}{\|u_n^+\|} \leq 1 \text{ for all } n \geq n_0. \quad (51)$$

Then for $n \geq n_0$, from (50) and (51), we can find a constant $c_{15} > 0$ satisfying

$$\begin{aligned} \tilde{\mu}_\lambda(t_n u_n^+) &\geq \tilde{\mu}_\lambda(v_n) \\ &= k - \int_{\Omega} \lambda F(z, v_n) dz - \lambda (pk)^{\frac{1-\eta}{p}} c_{15} \text{ (recall } \|y_n\| = 1) \\ &= \left[1 - \frac{\lambda p}{(pk)^{\frac{p+\eta-1}{p}}} c_{15} \right] k - \int_{\Omega} \lambda F(z, v_n) dz. \end{aligned}$$

Also, we can take $k_0 > 0$ such that

$$1 > \frac{\lambda p}{(pk)^{\frac{p+\eta-1}{p}}} \text{ for all } k > k_0.$$

So, from (48), it has

$$\tilde{\mu}_\lambda(t_n u_n^+) \geq c_{16} k$$

for some $c_{16} > 0$, all $k > k_0$, i.e.,

$$\tilde{\mu}_\lambda(t_n u_n^+) \rightarrow +\infty \text{ as } n \rightarrow \infty. \quad (52)$$

However, (37), (40) and (49) point out

$$\tilde{u}_\lambda(0) = 0 \text{ and } \tilde{u}_\lambda(u_n^+) \leq c_{17} \text{ for some } c_{17} > 0, \text{ all } n \in \mathbb{N}. \quad (53)$$

From (52) and (53), it follows that

$$t_n \in (0, 1) \text{ for all } n \geq n_2. \quad (54)$$

Then, from (54) and (50), we have

$$\begin{aligned} 0 &= t_n \frac{d}{dt} \tilde{\mu}_\lambda(tu_n)|_{t=t_n} = \langle \tilde{\mu}'_\lambda(t_n u_n^+), t_n u_n^+ \rangle \text{ (by the chain rule)} \\ &= \|D(t_n u_n^+)\|_p^p - \int_\Omega w_\lambda(z, t_n u_n^+)(t_n u_n^+) dz \text{ for all } n \geq n_2. \end{aligned}$$

This means that

$$\begin{aligned} \|D(t_n u_n^+)\|_p^p &= \int_\Omega w_\lambda(z, t_n u_n^+)(t_n u_n^+) dz \\ &\leq \int_\Omega \lambda[(t_n u_n^+)^{1-\eta} + f(z, t_n u_n^+)t_n u_n^+] dz + c_{18} \text{ for some } c_{18} > 0 \text{ (see (28))} \\ &= \int_\Omega \lambda \left[e(z, t_n u_n^+) - p \left(\frac{1}{1-\eta} (t_n u_n^+)^{1-\eta} + F(z, t_n u_n^+) \right) \right] dz + c_{18} \\ &\leq \int_\Omega \lambda \left[e(z, u_n^+) - p \left(\frac{1}{1-\eta} (t_n u_n^+)^{1-\eta} + F(z, t_n u_n^+) \right) \right] dz + c_{19} \end{aligned}$$

for all $n \geq n_2$ with some $c_{19} > 0$, where we have used (54) and hypothesis $H(\text{iii})$. So, we have

$$\tilde{\mu}_\lambda(t_n u_n^+) \leq c_{20} \text{ for some } c_{20} > 0, \text{ all } n \geq n_2 \text{ (see (28) and (43))}. \quad (55)$$

We compare (55) and (52) to have a contradiction. This proves that

$$\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded} \Rightarrow \{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (40))}.$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightarrow u \in L^r(\Omega). \quad (56)$$

In (39), we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$, and use (56) to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u_n), u_n - u \rangle] &= 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u), u_n - u \rangle] &\leq 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0 \text{ (see (56))}, \\ \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) &\text{ (see Proposition 1)}. \end{aligned}$$

Therefore $\mu_\lambda(\cdot)$ satisfies the C-condition. This proves the claim.

Then, (35), (36) and the claim, permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\mu_\lambda} \subseteq [u_0] \cap \text{int}C_+ \text{ (see (34))}, \quad m_\lambda \leq \mu_\lambda(\hat{u}) \text{ (see (35))}. \quad (57)$$

From (57), (28) and (35), we infer that $\hat{u} \in S_\lambda$, $u_0 \leq \hat{u}$ and $\hat{u} \neq u_0$. $\square$

Moreover, we check the admissibility of the critical parameter value λ^* .

Proposition 14 *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof Let $\{\lambda_n\} \subset (0, \lambda^*)$ be such that $\lambda_n \uparrow \lambda^*$. We have $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{L}$ (see Proposition 11). From the proof of Proposition 13 and keeping the notation introduced there, we are able to find $u_n \in S_{\lambda_n} \subseteq \text{int}C_+$ such that

$$\begin{aligned} \mu_{\lambda_n}(u_n) &= \frac{1}{p} \|Du_n\|_p^p + \frac{1}{q} \|Du_n\|_q^q - \lambda_n \int_{\Omega} [u_n^{1-\eta} + f(z, u_n)u_n] dz \quad (\text{see (31)}) \\ &= \frac{1}{p} \|Du_n\|_p^p + \frac{1}{q} \|Du_n\|_q^q - \|Du_n\|_p^p - \|Du_n\|_q^q \quad (\text{since } u_n \in S_{\lambda_n}) \\ &= \left[\frac{1}{p} - 1 \right] \|Du_n\|_p^p + \left[\frac{1}{q} - 1 \right] \|Du_n\|_q^q < 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (58)$$

Recall that $u_n \in S_{\lambda_n} \subseteq \text{int}C_+$, $n \in \mathbb{N}$. So, we have

$$\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle = \int_{\Omega} [\lambda_n u_n^{-\eta} h + f(z, u_n)h] dz \quad (59)$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$. From (58), (59) and reasoning as in the Claim in the proof of Proposition 13, we obtain that at least for a subsequence, we have

$$u_n \rightarrow u_* \text{ in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty. \quad (60)$$

We know that $\tilde{u}_{\lambda_1} \leq u_n$ for all $n \in \mathbb{N}$ (see the proof of Proposition 11). Therefore, from (60), we see that $u_* \neq 0$ and $u_*^{-\eta} h \leq \tilde{u}_{\lambda_1}^{-\eta} h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$. In (59), we pass to the limit as $n \rightarrow \infty$ and use (60) to admit

$$\begin{aligned} \langle A_p(u_*), h \rangle + \langle A_q(u_*), h \rangle &= \int_{\Omega} \lambda^* [u_*^{-\eta} + f(z, u_*)] h dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \\ \Rightarrow u_* &\in S_{\lambda^*} \subseteq \text{int}C_+ \text{ and so } \lambda^* \in \mathcal{L}. \end{aligned}$$

This completes the proof of the proposition. $\square$

We have proved that

$$\mathcal{L} = (0, \lambda^*].$$

So, summarizing our findings in this section, we can state the following bifurcation-type theorem.

Theorem 15 *If hypotheses H hold, then there exists $\lambda^* > 0$ such that*

- (a) *for every $\lambda \in (0, \lambda^*)$, problem (P_{λ}) has at least two positive solutions $u_0, \hat{u} \in \text{int}C_+$ with $u_0 \leq \hat{u}$ and $u_0 \neq \hat{u}$;*
- (b) *for $\lambda = \lambda^*$, problem (P_{λ}) has at least one positive solution $u_* \in \text{int}C_+$;*
- (c) *for every $\lambda > \lambda^*$, problem (P_{λ}) has no positive solutions.*

In the next section, we produce minimal positive solution and study the properties of the corresponding minimal positive solution map.

4 Minimal positive solutions

In this section, we show that for every $\lambda \in \mathcal{L}$ problem (P_{λ}) has a minimal positive solution $u_{\lambda}^* \in \text{int}C_+$ (that is, $u_{\lambda}^* \leq u$ for all $u \in S_{\lambda}$) and also examine the monotonicity and continuity properties of the map $\lambda \mapsto u_{\lambda}^*$.

Proposition 16 *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) admits a smallest positive solution $u^* \in S_\lambda \subseteq \text{int}C_+$, that is, $u_\lambda^* \leq u$ for all $u \in S_\lambda$.*

Proof From Proposition 19 of Papagetorgiou-Rădulescu-Repovš [29], we know that S_λ is downward directed. Then, invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [19], we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_\lambda$ such that

$$\inf_{n \geq 1} u_n = \inf S_\lambda.$$

From Proposition 5, we know that $\underline{u}_\lambda \leq u_n$ for all $n \geq 1$ and $\underline{u}_\lambda^{-\eta} \in L^1(\Omega)$ (see $\underline{u}_\lambda \in \text{int}C_+$). Therefore, we have

$$\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle = \lambda_n \int_{\Omega} [u_n^{-\eta} + f(z, u_n)] h \, dz \text{ for all } h \in W_0^{1,p}(\Omega), \quad (61)$$

$$\underline{u} \leq u_n \leq u_1 \quad (62)$$

for all $n \in \mathbb{N}$. Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (61) and using (62) and hypothesis $H(i)$, we see that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, using the monotonicity of the sequence $\{u_n\}_{n \geq 1}$ and the $(S)_+$ -property of $A_p(\cdot)$ (see Proposition 1), as in the proof of Proposition 13 (see the Claim), we obtain

$$u_n \rightarrow u_\lambda^* \text{ in } W_0^{1,p}(\Omega). \quad (63)$$

Then, passing to the limit as $n \rightarrow \infty$ in (61) and using (63) and (62), we conclude that

$$u_\lambda^* \in S_\lambda \subseteq \text{int}C_+, \quad u_\lambda^* = \inf S_\lambda.$$

This completes the proof of the proposition. $\square$

Next we examine the properties of the map $\mathcal{L} \ni \lambda \mapsto u_\lambda^* \in \text{int}C_+ \subseteq C_0^1(\overline{\Omega})$.

Proposition 17 *If hypotheses H hold, then the minimal positive solution map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) *strictly increasing, that is,*

$$0 < \mu < \lambda \leq \lambda^* \Rightarrow u_\lambda^* - u_\mu^* \in \text{int}C_+;$$

(b) *left continuous.*

Proof (a) From Proposition 11, we can find $u_\mu \in S_\mu \subseteq \text{int}C_+$ such that

$$u_\lambda^* - u_\mu \in \text{int}C_+ \Rightarrow u_\lambda^* - u_\mu^* \in \text{int}C_+ \text{ (since } u_\mu^* \leq u_\mu \text{)}.$$

(b) Suppose $\lambda_n \rightarrow \lambda^- \leq \lambda^*$ as $n \rightarrow \infty$. Then, from Proposition 8 and assertion (a), we have

$$\underline{u}_{\lambda_1} \leq u_{\lambda_n}^* \leq u_{\lambda^*}^* \in \text{int}C_+ \Rightarrow \{u_{\lambda_n}^*\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Then, the nonlinear regularity theory of Lieberman [22] implies that there exist $\alpha \in (0, 1)$ and $c_{21} > 0$ such that

$$u_{\lambda_n}^* \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\Omega) \cap C_0^1(\overline{\Omega}) \text{ and } \|u_{\lambda_n}^*\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_{21} \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and the monotonicity of $\{u_{\lambda_n}^*\}_{n \geq 1}$ (see part (a)) imply that

$$u_{\lambda_n}^* \rightarrow \tilde{u}_\lambda^* \text{ in } C_0^1(\overline{\Omega}). \quad (64)$$

Suppose that $\tilde{u}_\lambda^* \neq u_\lambda^*$. Then, we can find $z_0 \in \Omega$ such that

$$u_\lambda^*(z_0) < \tilde{u}_\lambda^*(z_0) \Rightarrow u_\lambda^*(z_0) < u_{\lambda_n}^*(z_0) \text{ for all } n \geq n_0 \text{ (see (64))}$$

and this contradicts with part (a). Therefore, $\tilde{u}_\lambda^* = u_\lambda^*$, so, $\lambda \mapsto u_\lambda^*$ is left continuous. $\square$

Summarizing, we can state the following theorem about minimal positive solutions for problem (P_λ) .

Theorem 18 *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in S_\lambda \subseteq \text{int}C_+$ and the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is strictly increasing and left continuous.*

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