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High perturbations of quasilinear problems with double criticality

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Abstract
This paper is concerned with the qualitative analysis of solutions to the following class of quasilinear problems

$$\begin{cases} -\Delta_\Phi u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Delta_\Phi u = \operatorname{div}(\varphi(x, |\nabla u|)\nabla u)$ and $\Phi(x, t) = \int_0^{|t|} \varphi(x, s)s\, ds$ is a generalized N-function. We assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain that contains two open regions Ω_N, Ω_p with $\overline{\Omega}_N \cap \overline{\Omega}_p = \emptyset$. The features of this paper are that $-\Delta_\Phi u$ behaves like $-\Delta_N u$ on Ω_N and $-\Delta_p u$ on Ω_p, and that the growth of $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is like that of $e^{\alpha|t|^{\frac{N}{N-1}}}$ on Ω_N and as $|t|^{p^*-2}t$ on Ω_p when $|t|$ is large enough. The main result establishes the existence of solutions in a suitable Musielak–Sobolev space in the case of high perturbations with respect to the values of a positive parameter.

Keywords Variational methods · Quasilinear problems · Musielak–Sobolev space

Mathematics Subject Classification 35A15 · 35J62 · 46E30

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1 Introduction

In this paper we study the existence of solutions for the following class of quasilinear problems

$$\begin{cases} -\Delta_\Phi u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $\Delta_\Phi u = \operatorname{div}(\varphi(x, |\nabla u|)\nabla u)$ is the Φ-Laplace operator, where $\Phi(x,t) = \int_0^{|t|} \varphi(x,s)s\,ds$, $\varphi : \Omega \times [0,+\infty) \to [0,+\infty)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy some hypothesis that will be mentioned later on.

Before proceeding further, let us go through some known results associated to the Φ-Laplace equations. In the recent past, the study of such equations concerning the existence theory has been a research topic of considerable interest. This nonhomogeneous differential operator extends the standard p-Laplace operator, the variable exponent p-Laplace operator, the weighted p-Laplace operator, and the p, q-Laplace operator.

When Φ is independent of x, solutions of problem (P) are investigated in the Orlicz–Sobolev space [40], and we refer the reader to Alves et al. [4], Alves et al. [5], Fukagai et al. [26], Carvalho et al. [13], Fukagai and Narukawa [27], Harjulehto and Hästö [32], and their references for the study of such PDEs. When Φ also depends on x, we are led to study the problems in variable exponent Sobolev spaces [22,36] or in Musielak–Sobolev spaces [17,33,38,40]. Differential equations in variable exponent Sobolev spaces have been studied extensively in the last years, most part of them involving the $p(x)$-Laplacian operator, see Alves and Barreiro [2], Alves and Ferreira [3], Alves and Souto [6], Alves and Rădulescu [7], Chabrowski and Fu [16], Fan and Zhang [24], Fan [25], Rădulescu and Repovš [41] and the references therein. However, differential equations in general Musielak–Sobolev spaces have been studied very little, see for instance, Azroul et al. [8], Benkirane and Sidi El Vally [11], Fan [23], Liu and Zhao [37], Wang and Liu [43] and the references therein.

In the present paper we will apply some recent results involving Musielak–Sobolev spaces to study the existence of nontrivial solutions for problem (P).

We now state our main hypotheses on the functions Φ and φ:

(φ_1) For each $x \in \Omega$, $\varphi(x, .)$ is a C^1 function in the interval $(0,+\infty)$.
(φ_2) $\varphi(x,t)$, $\partial_t(\varphi(x,t)t) > 0$, for $x \in \Omega$ and $t > 0$.
(φ_3) There exist $1 < p < N < q < p^*$ such that

$$p \leq \frac{\varphi(x,|t|)|t|^2}{\Phi(x,|t|)} \leq q, \quad \text{for } x \in \Omega \text{ and } t \neq 0.$$

Using some ideas developed by Fukagai et al. [26], we can show that if φ satisfies conditions $(\varphi_1)-(\varphi_3)$, then Φ is a generalized N-function.

The complementary function $\widetilde{\Phi}$ associated with Φ is given by the Legendre transformation, that is,

$$\widetilde{\Phi}(x,s) = \max_{t \geq 0}\{st - \Phi(x,t)\}, \quad x \in \Omega \text{ and } s \in \mathbb{R}. \tag{1.1}$$

The functions Φ and $\widetilde{\Phi}$ are complement of each other and $\widetilde{\Phi}$ is also a generalized N-function.

Hereafter, we also assume that for some constant d_1,

(φ_4) $\inf_{x\in\Omega} \Phi(x,1) = d_1 > 0$.

(φ_5) For each $t_0 \neq 0$, there is $c_0 > 0$ such that

$$\frac{\Phi(x,t)}{t} \geq c_0 \quad \text{and} \quad \frac{\tilde{\Phi}(x,t)}{t} \geq c_0 \qquad \text{for } t \geq t_0 \text{ and } x \in \Omega.$$

The conditions (φ_1)−(φ_5) are very important in our approach, because they permit to conclude that both the Musielak–Orlicz space $L^{\Phi}(\Omega)$ and the Musielak–Sobolev space $W^{1,\Phi}(\Omega)$ are reflexive and separable Banach spaces; see Sect. 2 for more details.

Next, we will state more conditions on the function φ. Hereafter, we will suppose that there are three smooth domains $\Omega_N, \Omega_q, \Omega_p \subset \Omega$ with nonempty interior such that

$$\Omega = \Omega_N \cup \Omega_q \cup \Omega_p$$

and there is $\delta > 0$ such that

$$(\overline{\Omega_N})_\delta \cap (\overline{\Omega_p})_\delta = \emptyset.$$

Hereafter, if $A \subset \Omega$, we denote by A_δ to be the δ-neighbourhood of A restricted to Ω, that is,

$$A_\delta = \{x \in \Omega \, : \, \text{dist}\,(x, A) < \delta\}.$$

Associated with the sets Ω_N, Ω_q and Ω_p, we will consider three continuous functions $\eta_N, \eta_q, \eta_p : \overline{\Omega} \to [0,1]$ satisfying:

$$\eta_N(x) = 1, \quad \forall x \in \overline{\Omega_N},$$
$$\eta_p(x) = 1, \quad \forall x \in \overline{\Omega_p},$$

and

$$\begin{aligned} &\eta_q(x) = 1, \quad \forall x \in \Omega_q = \Omega \backslash \overline{(\Omega_N \cup \Omega_p)}, \\ &\eta_N(x) = 0, \quad \forall x \in (\overline{\Omega_N})^c_\delta, \quad \eta_p(x) = 0, \quad \forall x \in (\overline{\Omega_p})^c_\delta, \\ &\eta_q(x) > 0, \quad \forall x \in (\overline{\Omega_q})_\delta, \quad \eta_q(x) = 0, \quad \forall x \in (\overline{\Omega_q})^c_\delta \end{aligned}$$

and for some positive constant c_4,

$$\eta_q(x) \leq c_4 \text{dist}(x, \partial(\Omega_q)_\delta \cap \Omega_p)^l, \quad \forall x \in \overline{\Omega_p} \cap (\Omega_q)_\delta,$$

where $l > q$ and $\text{dist}(x, \partial(\Omega_q)_\delta \cap \Omega_p) = \inf\{|x - y| \, : \, y \in \partial(\Omega_q)_\delta \cap \Omega_p\}$.

We assume that the continuous function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ has one of the following forms:

$$f(x,t) = \lambda \eta_N(x)|t|^{\beta-2} t e^{\alpha |t|^{\frac{N}{N-1}}} + \tilde{\eta}_q(x) g(x,t) + \eta_p(x)|t|^{p^*-2}t, \quad \forall\,(x,t) \in \Omega \times \mathbb{R}, \tag{f_1}$$

or

$$\begin{aligned} f(x,t) = \eta_N(x)|t|^{\beta-2} t e^{\alpha |t|^{\frac{N}{N-1}}} + \tilde{\eta}_q(x) g(x,t) + \eta_p(x)(\lambda |t|^{r-2}t \\ + |t|^{p^*-2}t), \quad \forall\,(x,t) \in \Omega \times \mathbb{R}, \end{aligned} \tag{f_2}$$

where λ is a positive parameter, $\alpha > 0$, $p^* > r > q > N > p > \frac{N}{2}, \beta > q$, where $p^* = \frac{Np}{N-p}$, $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\eta}_q : \bar{\Omega} \to [0,1]$ are continuous functions such that

$$\tilde{\eta}_q(x) = 1, \quad \forall x \in \Omega_q = \Omega \backslash \overline{(\Omega_N \cup \Omega_p)}$$

and

$$\tilde{\eta}_q(x) = 0, \quad \forall\, x \in (\overline{\Omega_q})^c_{\delta/2}.$$

Related to the function g, we assume the following conditions

$$g(x,t) = o(|t|^{q_1-1}), \quad \text{as } t \to 0, \text{ uniformly in } x \in (\overline{\Omega_q})_{\delta/2} \tag{g_1}$$

for some $q_1 > q$ and there is $\theta > q$ such that

$$0 < \theta G(x,t) \le g(x,t)t, \quad \forall\, x \in (\overline{\Omega}_q)_{\delta/2} \tag{g_2}$$

where $G(x,t) = \int_0^t g(x,s)\,ds$, for $t \in \mathbb{R}$.

With these notations, we are ready to mention our last conditions on φ. If f is the form (f_1), we assume for each $t > 0$ the following:

(φ_6) $\varphi(x,t) \ge t^{N-2}$, for $x \in \Omega_N$ and $c_1 t^{N-2} \ge \varphi(x,t)$, $x \in \Omega_N \setminus \overline{(\Omega_q)_\delta}$.

(φ_7) $\varphi(x,t) \ge \tau_1(x)t^{q-2}$, for $x \in (\Omega_q)_\delta$ where $\tau_1 : \overline{\Omega} \to \mathbb{R}$ is a continuous function satisfying:

$$\tau_1(x) > 0, \quad \forall x \in (\Omega_q)_\delta \qquad \text{and} \qquad \tau_1(x) = 0, \quad \forall x \in ((\Omega_q)_\delta)^c.$$

(φ_8) $\tau_2(x)t^{q-2} + c_2 t^{p-2} \ge \varphi(x,t) \ge t^{p-2}$, $x \in \Omega_p$ where $\tau_2 : \overline{\Omega_p} \to \mathbb{R}$ is a nonnegative continuous function satisfying:

$$\tau_2(x) \le c_3 \text{dist}(x, \partial(\Omega_q)_\delta \cap \Omega_p)^s, \quad \forall x \in \overline{\Omega_p} \cap (\Omega_q)_\delta$$

for some $s > q$ and

$$\tau_2(x) = 0, \quad \forall x \in \overline{\Omega_p} \setminus \overline{(\Omega_q)_\delta},$$

for some constants $c_i > 0$ with $i = 1, 2, 3$.

Now, if f is the form (f_2) we make a little adjustment in the condition (φ_6) of the following way:

(φ_6) $\varphi(x,t) \ge t^{N-2}$, for $x \in \Omega_N$.

As a model of a function that satisfies the conditions $(\varphi_1) - (\varphi_8)$ is the function $\varphi : \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\varphi(x,t) = \eta_N(x)t^{N-2} + \eta_q(x)t^{q-2} + \eta_p(x)t^{p-2}, \quad \forall\, (x,t) \in \Omega \times [0,+\infty) \tag{1.2}$$

and so,

$$\Phi(x,t) = \frac{\eta_N(x)}{N}|t|^N + \frac{\eta_q(x)}{q}|t|^q + \frac{\eta_p(x)}{p}|t|^p, \quad \forall\, (x,t) \in \Omega \times \mathbb{R}. \tag{1.3}$$

The reader is invited to observe that according to model (1.3), the operator Δ_Φ has different behaviors in the region Ω, it behaves like Δ_p in one region and Δ_N in another disjoint region, where the nonlinearity f behaves like $|t|^{p^*-2}t$ and $e^{|u|^{\frac{N}{N-1}}}$ respectively, and so, the problem (P) has double criticality. This type of phenomena is very interesting, because we will work in the same problem with two types of nonlinearity that bring to the problem a lost of compactness, and in this case, we need to control these terms by doing simultaneously two different types of estimates. More precisely, in the present paper we will apply the Concentration Compactness Lemma due to Lions in $W^{1,p}(\Omega_p)$ found in Medeiros [21, Lemma 3.1], to get good estimate involving the integrals with the function $|t|^{p^*}$, while we will use a version of the Trudinger–Moser inequality in $W^{1,N}(\Omega_N)$ by Cianchi [18], see Lemma 3.3, to obtain a control in the integrals involving the exponential growth. One difficulty that appears in our study is that we do not know if the trace of the functions on $\partial\Omega_p$ and $\partial\Omega_N$ are zero, hence we must use results that are applied in the study of problem with Neumann boundary conditions. We believe that this is the first article where this type of doubly criticality is studied in the literature.

An important fact that we would like to point out is that our study is strongly related to the double-phase problems that have received a special attention in the last years. As mentioned in [7], the study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point that has been continued by Mingione et al. [10,19,20], Bahrouni et al. [9], Cencelj et al. [14], Gasiński and Winkert [29,30], Papageorgiou et al. [39], Zhang and Rădulescu [45], etc. These contributions are in relationship with the work of Zhikov [46,47], which describe the behavior of phenomena arising in nonlinear elasticity. In fact, variational problems with nonstandard integrands were introduced at the beginning of the 1980's and were studied in the context of averaging and the Lavrent'ev phenomenon. Zhikov provided models for strongly anisotropic materials in the context of homogenisation. In particular, he considered the following model functional

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|Du|^p + a(x)|Du|^q)dx, \quad 0 \le a(x) \le L, \; 1 < p < q, \tag{1.4}$$

where the modulating coefficient $a(x)$ dictates the geometry of the composite made of two differential materials, with hardening exponents p and q, respectively. In our case, the functions $\eta_N(x)$, $\eta_p(x)$ and $\eta_q(x)$ work like function $a(x)$ in the papers due to Zhikov.

Our main result establishes the existence of solutions to problem (P) in the case of high perturbations, that is, for large values of the positive parameter λ.

Theorem 1.1 *Assume* (g_1), (g_2) *and* $(\varphi_1)-(\varphi_8)$. *Then, if either* (f_1) *or* (f_2) *holds, there exists* $\lambda^* > 0$ *such that problem* (P) *has a nontrivial solution for all* $\lambda \ge \lambda^*$.

The proof of Theorem 1.1 is done via Variational Methods, more precisely we have used the mountain pass theorem without (PS) condition found in Willem [44] to establish our main results, although we face several difficulties. As mentioned above, due to the exponential critical behavior, we establish several auxiliary results (Lemmas 3.4, 3.5 and Corollary 3.6) of Moser-Trudinger type which captures the nonzero Dirichlet boundary value Sobolev functions and become very useful in our setting. To handle the critical exponent term, we use a Lions concentration compactness principle (Lemma 3.1) for the nonzero Dirichlet boundary value Sobolev functions.

This paper is organised as follows. In Sect. 2, we make a brief review about the Musielak–Orlicz and Musielak–Sobolev spaces, while in Sect. 3 we discuss some technical results that are crucial to overcome the lost of compactness involving the terms with critical growth and exponential critical growth. Finally, in Sect. 4, we prove our main result.

2 A brief review about the Musielak–Sobolev spaces

In this section, we recall some results on Musielak–Orlicz and Musielak–Sobolev spaces. For more details we refer to [17,23,32,38] and their references.

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $\Phi(x,t) = \int_0^{|t|} \varphi(x,s)s\,ds$ be a generalized N-function, that is, for each $t \in \mathbb{R}$, the function $\Phi(.,t)$ is measurable, and for a.e. $x \in \Omega$, the function $\Phi(x,.)$ is an N-function. For the reader's convenience, we recall that a continuous function $A : \mathbb{R} \to [0,+\infty)$ is an N-function if

(i) A is convex.
(ii) $A = 0 \Leftrightarrow t = 0$.
(iii) $\lim_{t\to 0} \frac{A(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{A(t)}{t} = +\infty$.
(iv) A is even.

The Musielak–Orlicz space $L^{\Phi}(\Omega)$ is defined by

$$L^{\Phi}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \exists\, \tau > 0 \text{ such that } \int_{\Omega} \Phi\left(x, \frac{|u|}{\tau}\right) dx < +\infty \right\}$$

endowed with the Luxemburg norm

$$|u|_{\Phi} = \inf\left\{ \lambda > 0 \,\middle|\, \int_{\Omega} \Phi\left(x, \frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

We say that an N-function Φ satisfies the *weak Δ_2-condition*, denote by $\Phi \in \Delta_2$, if there are $K > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi(x, 2t) \leq K\Phi(x, t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R},$$

When $h = 0$, we say that Φ satisfies the Δ_2-condition. Arguing as in [40, Theorem 4.4.4], it follows that Φ satisfies the Δ_2-condition if, and only if,

$$\sup_{(x,t)\in\Omega\times(0,+\infty)} \frac{\varphi(x, |t|)|t|^2}{\Phi(x, |t|)} < +\infty.$$

Moreover, an important inequality involving Φ and its complementary function $\tilde{\Phi}$ (see (1.1)) is a Young's type inequality given by

$$st \leq \Phi(x, s) + \widetilde{\Phi}(x, t), \quad x \in \Omega \text{ and } \forall s, t \geq 0. \tag{2.1}$$

Using the above inequality, it is possible to prove a Hölder type inequality, that is,

$$\left| \int_{\Omega} uv dx \right| \leq 2\|u\|_{\Phi} \|v\|_{\widetilde{\Phi}} \quad \forall\, u \in L^{\Phi}(\Omega) \text{ and } \forall\, v \in L^{\widetilde{\Phi}}(\Omega).$$

Arguing as in [26], if (φ_3) holds, we derive that

$$\frac{q}{q-1} \leq \frac{\tilde{\varphi}(x, |t|)|t|^2}{\tilde{\Phi}(x, |t|)} \leq \frac{p}{p-1}, \quad x \in \Omega \text{ and } t \neq 0,$$

where

$$\tilde{\Phi}(x, t) = \int_0^{|t|} \tilde{\varphi}(x, s)s\, ds,$$

and

$$\tilde{\varphi}(x, s) = \sup\{t \,:\, \varphi(x, t)t \leq s\}, \quad x \in \overline{\Omega} \text{ and } s \geq 0.$$

Hence, if (φ_3) holds, we have $\tilde{\Phi}$ also satisfies the Δ_2-condition.

Arguing as in [26, Lemma A2], it is possible to prove that Φ and $\tilde{\Phi}$ satisfy the following inequality

$$\tilde{\Phi}(x, \varphi(x, t)t) \leq \Phi(x, 2t), \quad x \in \Omega \text{ and } t \geq 0. \tag{2.2}$$

The condition (φ_3) is very interesting, because following the ideas of [26, Lemmas 2.1 and 2.5], it is possible to prove the following: Setting the functions

$$\xi_0(t) = \min\{t^p, t^q\}, \quad \xi_1(t) = \max\{t^p, t^q\}, \quad \xi_3(t) = \min\{t^{\frac{p}{p-1}}, t^{\frac{q}{q-1}}\} \quad \text{and}$$
$$\xi_4(t) = \max\{t^{\frac{p}{p-1}}, t^{\frac{q}{q-1}}\},$$

we have

$$\xi_0(s)\Phi(x,t) \le \Phi(x,st) \le \xi_1(s)\Phi(x,t) \quad \text{for } s, t \ge 0, \tag{2.3}$$
$$\xi_0(|u|_\Phi) \le \int_\Omega \Phi(x,|u|)\,dx \le \xi_1(|u|_\Phi) \quad \text{for } u \in L^\Phi(\Omega), \tag{2.4}$$
$$\xi_3(s)\tilde{\Phi}(x,t) \le \tilde{\Phi}(x,st) \le \xi_4(s)\tilde{\Phi}(x,t) \quad \text{for } s, t \ge 0, \tag{2.5}$$

and

$$\xi_3(|u|_{\tilde{\Phi}}) \le \int_\Omega \tilde{\Phi}(x,|u|)\,dx \le \xi_4(|u|_{\tilde{\Phi}}) \quad \text{for } u \in L^{\tilde{\Phi}}(\Omega). \tag{2.6}$$

The Musielak–Sobolev space $W^{1,\Phi}(\Omega)$ can be defined by

$$W^{1,\Phi}(\Omega) = \left\{u \in L^\Phi(\Omega) \,\Big|\, |\nabla u| \in L^\Phi(\Omega)\right\}$$

with the norm

$$\|u\|_{1,\Phi} = |u|_\Phi + |\nabla u|_\Phi\,.$$

The conditions $(\varphi_1)-(\varphi_5)$ ensure that the spaces $L^\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are reflexive and separable Banach spaces, for more details see [23, Propositions 1.6 and 1.8]. In what follows, $W_0^{1,\Phi}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W_0^{1,\Phi}(\Omega)$ with respect to the above norm. Moreover, $\|u\| = |\nabla u|_\Phi$ is a norm in $W_0^{1,\Phi}(\Omega)$, and if $(\varphi_1)-(\varphi_5)$ holds, by [31, Lemma 5.7], $\| \ \|$ is equivalent to the norm $\|u\|_{1,\Phi}$ in $W_0^{1,\Phi}(\Omega)$.

As a consequence of (2.4) we have the lemma below that will be used later on.

Proposition 2.1 *The functional $\rho : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by*

$$\rho(u) = \int_\Omega \Phi(x,|\nabla u|)\,dx, \tag{2.7}$$

has the following properties:

(i) *If $\|u\| \ge 1$, then $\|u\|^p \le \rho(u) \le \|u\|^q$.*
(ii) *If $\|u\| \le 1$, then $\|u\|^q \le \rho(u) \le \|u\|^p$.*

In particular, $\rho(u) = 1$ if and only if $\|u\| = 1$ and if $(u_n) \subset W_0^{1,\Phi}(\Omega)$, then $\|u_n\| \to 0$ if and only if $\rho(u_n) \to 0$.

Remark 1 For the functional $\xi : L^\Phi(\Omega) \to \mathbb{R}$ given by

$$\xi(u) = \int_\Omega \Phi(x,|u|)\,dx,$$

the conclusion of Proposition 2.1 also holds, for example, if $(u_n) \subset L^\Phi(\Omega)$, then $|u_n|_\Phi \to 0$ if and only if $\xi(u_n) \to 0$.

From the definition of $W^{1,\Phi}(\Omega)$ and properties of Φ, we have the continuous embedding

$$W^{1,\Phi}(\Omega) \hookrightarrow W^{1,q}((\Omega_q)_\omega)$$

for all $\omega \in (0, \delta)$ and the compact embedding

$$W^{1,q}((\Omega_q)_\delta) \hookrightarrow C(\overline{(\Omega_q)_\omega}),$$

because $q > N$, from where it follows that

$$W^{1,\Phi}(\Omega) \hookrightarrow C(\overline{(\Omega_q)_\omega}), \tag{2.8}$$

is compact, which is crucial in our approach.

Next we would like to state our last result found in [23, Theorem 2.2], which says the operator $-\Delta_\Phi : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ belongs to the Class (S_+).

Lemma 2.2 *Assume the conditions* $(\varphi_1)-(\varphi_8)$. *If* $u_n \rightharpoonup u$ *in* $W_0^{1,\Phi}(\Omega)$ *and*

$$\lim_{n\to+\infty} \int_\Omega \langle \varphi(x, |\nabla u_n|)\nabla u_n, \nabla u_n - \nabla u\rangle\, dx = 0,$$

then $u_n \to u$ *in* $W_0^{1,\Phi}(\Omega)$.

3 Some technical results

The main goal of this section is to recall and prove some technical results that are crucial in the proof of our main result. Since we are going to work with double criticality, which involves the exponential critical growth and the critical growth p^*, the next two results are crucial in our approach. The first one is a Concentration Compactness Lemma due to Lions for $W^{1,p}(\Theta)$ explored in Medeiros [21], where $\Theta \subset \mathbb{R}^N$ is a smooth bounded domain .

Lemma 3.1 *Let* (u_n) *be a sequence in* $W^{1,p}(\Theta)$ *with* $1 < p < N$ *and* $u_n \rightharpoonup u$ *in* $W^{1,p}(\Theta)$. *If*

(i) $|\nabla u_n|^p \to \mu$ *weakly-* in the sense of measure,*
and
(ii) $|u_n|^{p^*} \to \nu$ *weakly-* in the sense of measure,*

then for at most a countable index set J, *we have*

$$\begin{cases} (a) & \nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j},\ \nu_j \geq 0. \\ (b) & \mu \geq |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j},\ \mu_j \geq 0. \\ (c) & \text{If } x_j \in \Theta, \text{ then } S_p \nu_j^{\frac{p}{p^*}} \leq \mu_j. \\ (d) & \text{If } x_j \in \partial\Theta, \text{ then } \frac{S_p}{2^{p/N}} \nu_j^{\frac{p}{p^*}} \leq \mu_j, \end{cases}$$

where $p^* = \frac{Np}{N-p}$ *and* S_p *denotes the best constant of the embedding* $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ *given by*

$$S_p = \inf_{\substack{u \in D^{1,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\, dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*}\, dx\right)^{\frac{p}{p^*}}}. \tag{3.1}$$

The proof of the above lemma follows by combining the arguments explored in Struwe [42, Chapter I, Section 4] and the following Cherrier's inequality [15] below.

Lemma 3.2 *Let $\Theta \subset \mathbb{R}^N$ be a smooth bounded domain and $p \in (1, N)$. Then for each $\tau > 0$, there is $M_\tau > 0$ such that*

$$\left[\frac{S_p}{2^{\frac{p}{N}}} - \tau\right] \|u\|^p_{L^{p^*}(\Theta)} \leq \|\nabla u\|^p_{L^p(\Theta)} + M_\tau \|u\|^p_{L^p(\Theta)}, \quad \forall u \in W^{1,p}(\Theta).$$

The second result that we would like to point out is a version of Trundiger–Moser inequality in $W^{1,N}(\Theta)$ due to Cianchi [18, Theorem 1.1].

Lemma 3.3 *Let $\Theta \subset \mathbb{R}^N$ be a smooth bounded domain for $N \geq 2$ and $u \in W^{1,N}(\Theta)$. Then, there is a constant $C(\Theta) > 0$ such that*

$$\int_\Theta e^{\alpha_N \left(\frac{|u-u_\Theta|}{\|\nabla u\|_{L^N(\Theta)}}\right)^{N'}} dx \leq C(\Theta), \tag{3.2}$$

where $N' = \frac{N}{N-1}$, $u_\Theta = \frac{1}{|\Theta|}\int_\Theta u\,dx$ is the mean value of u in Θ, $\alpha_N = N\left(\frac{w_N}{2}\right)^{\frac{1}{N}}$ and w_N is the volume of sphere S^{N-1}. The integral on the left-hand of (3.2) is finite for each $u \in W^{1,N}(\Theta)$ even if α_N is replaced by any other small positive number, but no inequality of type (3.2) can hold with a large constant in the place of α_N.

From Lemma 3.3, for each $u \in W^{1,N}(\Theta)$, we have

$$e^{t|u|^{N'}} \in L^1(\Theta), \quad \forall t \geq 0. \tag{3.3}$$

For the reader interested in Trudinger–Moser inequality for functions in $W^{1,N}(\Theta)$, we would like to cite the papers due to Adimurthi and Yadava [1], Kaur and Sreenadh [35] and their references.

As a consequence of Lemma 3.3, we have the following two results.

Lemma 3.4 *Given $t > 1$ and $\alpha > 0$, there is $r \in (0, 1)$ and $C = C(t, r, N) > 0$ such that*

$$\sup\left\{\int_\Theta e^{t\alpha|u|^{N'}} dx \; : \; u \in W^{1,N}(\Theta), \; \|\nabla u\|_{L^N(\Theta)} \leq r \quad and \quad \|u\|_{L^1(\Theta)} \leq r\right\} \leq C. \tag{3.4}$$

Proof Note that if $u \in W^{1,N}(\Theta)$, we have

$$\int_\Theta e^{t\alpha|u|^{N'}} dx \leq e^{t2^{N'}\alpha|u_\Theta|^{N'}} \int_\Theta e^{t2^{N'}\alpha|u-u_\Theta|^{N'}} dx.$$

Since

$$|u_\Theta| \leq \frac{1}{|\Theta|}\int_\Theta |u|\,dx \leq \frac{r}{|\Theta|}$$

it follows that

$$\begin{aligned}\int_\Theta e^{t\alpha|u|^{N'}} dx &\leq K \int_\Theta e^{t2^{N'}\alpha\|\nabla u\|^{N'}_{L^N(\Theta)}\left(\frac{|u-u_\Theta|}{\|\nabla u\|_{L^N(\Theta)}}\right)^{N'}} dx \\ &\leq K \int_\Theta e^{t2^{N'}\alpha r^{N'}\left(\frac{|u-u_\Theta|}{\|\nabla u\|_{L^N(\Theta)}}\right)^{N'}} dx,\end{aligned}$$

where $K = e^{t2^{N'}\alpha\left(\frac{r}{|\Theta|}\right)^{N'}}$. Fixing r of such way that $t2^{N'}\alpha r^{N'} \leq \alpha_N$, the result follows by employing Lemma 3.3. $\square$

Lemma 3.5 *Let $\alpha > 0$ and $(u_n) \subset W^{1,N}(\Theta)$ be a sequence satisfying $\|\nabla u_n\|^{N'}_{L^N(\Theta)} \leq \frac{\tau}{2^{N'}}\frac{\alpha_N}{\alpha}$ and $\|u_n\|_{L^1(\Theta)} \leq M$ for some $\tau \in (0,1)$ and $M > 0$. Then, there is $t > 1$ with $t \approx 1$ such that*

$$\sup_{n\in\mathbb{N}} \int_\Theta e^{t\alpha|u_n|^{N'}}\,dx < +\infty. \tag{3.5}$$

Hence, the sequence $f_n(x) = e^{\alpha|u_n(x)|^{N'}}$ is a bounded sequence in $L^t(\Theta)$.

Proof Arguing as in Lemma 3.4, we get

$$\int_\Theta e^{t\alpha|u|^{N'}}\,dx \leq K\int_\Theta e^{t2^{N'}\alpha\|\nabla u_n\|^{N'}_{L^N(\Theta)}\left(\frac{|u_n-(u_n)_\Theta|}{\|\nabla u_n\|_{L^N(\Theta)}}\right)^{N'}}\,dx,$$

where $K = e^{t2^{N'}\alpha\left(\frac{M}{|\Theta|}\right)^{N'}}$ and so,

$$\int_\Theta e^{t\alpha|u|^{N'}}\,dx \leq K\int_\Theta e^{t\tau\alpha_N\left(\frac{|u_n-(u_n)_\Theta|}{\|\nabla u_n\|_{L^N(\Theta)}}\right)^{N'}}\,dx.$$

As $\tau \in (0,1)$, we can take $t > 1$ with $t \approx 1$ of such way that $t\tau \in (0,1)$, and the result follows again by using Lemma 3.3. ◻

As a consequence of Lemma 3.5, we have the corollary below.

Corollary 3.6 *Let $(u_n) \subset W^{1,N}(\Theta)$ be a sequence as in Lemma 3.5. If $u_n(x) \to u(x)$ a.e. in Θ, then $f_n \rightharpoonup f$ in $L^t(\Theta)$ where $f(x) = e^{\alpha|u(x)|^{N'}}$, that is,*

$$\int_\Theta f_n\varphi\,dx \to \int_\Theta f\varphi\,dx, \quad \forall\varphi \in L^{t'}(\Theta),$$

where $\frac{1}{t} + \frac{1}{t'} = 1$.

Our next result will help us to conclude that the energy functional associated with problem (P) is $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$. Since it follows as in Bezerra do Ó, Medeiros and Severo [12, Proposition 1], we will omit its proof.

Lemma 3.7 *Let $(u_n) \subset W^{1,N}(\Theta)$ be a sequence such that $u_n \to u$ in $W^{1,N}(\Theta)$ for some $u \in W^{1,N}(\Theta)$. Then, for some subsequence, still denoted by itself, there is $v \in W^{1,N}(\Theta)$ such that:*

(i) *$u_n(x) \to u(x)$ a.e. in Θ.*
(ii) *$|u_n(x)| \leq v(x)$ a.e. in Θ for all $n \in \mathbb{N}$.*

The energy functional $I : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ associated to problem (P) is given by

$$I(u) = \int_\Omega \Phi(x, |\nabla u|)\,dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,t) = \int_0^t f(x,s)\,ds,\ t \in \mathbb{R}$.

Lemma 3.8 *The functional I belongs to $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and*

$$I'(u)v = \int_\Omega \varphi(x, |\nabla u|)\nabla u\nabla v\,dx - \int_\Omega f(x,u)v\,dx, \quad \forall u, v \in W_0^{1,\Phi}(\Omega).$$

Proof In what follows we will only do the proof by supposing that f is of the type (f_1), because the type (f_2) can be done of a similar way. Note that functional I can be written of the form

$$I(u) = \Psi_0(u) - \Psi_1(u) - \Psi_2(u) - \Psi_3(u)$$

where

$$\Psi_0(u) = \int_\Omega \Phi(x, |\nabla u|)\,dx,$$
$$\Psi_1(u) = \int_{(\Omega_q)_{\delta/2}} F(x, u)\,dx,$$
$$\Psi_2(u) = \lambda \int_{\Omega_N \setminus (\Omega_q)_{\delta/2}} F_1(x, u)\,dx,$$

where $F_1(x, t) = \int_0^t |s|^{\beta-2} s e^{\alpha|s|^{N'}}\,ds$, and

$$\Psi_3(u) = \frac{1}{p^*} \int_{\Omega_p \setminus (\Omega_q)_{\delta/2}} |u|^{p^*}\,dx.$$

Since for each $x \in \Omega$, we have $\Phi(x, .) \in C^1([0, +\infty), [0, +\infty))$, a well known argument ensures that $\Psi_0 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with

$$\Psi_0'(u)v = \int_\Omega \varphi(x, |\nabla u|)\nabla u \nabla v\,dx, \quad \forall u, v \in W_0^{1,\Phi}(\Omega).$$

Now, by $(\varphi_6)-(\varphi_8)$, we know that the space $W_0^{1,\Phi}(\Omega)$ is continuously embedded into $C(\overline{(\Omega_q)_{\delta/2}})$, $W^{1,\Phi}(\Omega_N \setminus (\Omega_q)_{\delta/2})$ and $W^{1,\Phi}(\Omega_p \setminus (\Omega_q)_{\delta/2})$. Therefore, it is easy to prove that the functionals Ψ_1, Ψ_2 and Ψ_3 also belong to $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with

$$\Psi_1'(u)v = \int_{(\Omega_q)_{\delta/2}} f(x, u)v\,dx, \quad \forall\, u, v \in W_0^{1,\Phi}(\Omega),$$
$$\Psi_2'(u)v = \lambda \int_{\Omega_N \setminus (\Omega_q)_{\delta/2}} |u|^{\beta-2} u e^{\alpha|u|^{N'}} v\,dx, \quad \forall\, u, v \in W_0^{1,\Phi}(\Omega)$$

and

$$\Psi_3'(u)v = \int_{\Omega_p \setminus (\Omega_q)_{\delta/2}} |u|^{p^*-2} uv\,dx, \quad \forall u, v \in W_0^{1,\Phi}(\Omega).$$

This proves the desired result. Here, Lemma 3.7 plays an important rule in the proof that Ψ_2 belongs to $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ □

Next, our goal is to prove that I satisfies the mountain pass geometry and the well known (PS) condition.

Lemma 3.9 *The functional I satisfies the mountain pass geometry for $\lambda \geq 1$, that is,*

(a) There are $r, \rho > 0$ such that

$$I(u) \geq \rho \quad \text{for } \|u\| = r.$$

(b) There is $\psi \in W_0^{1,\Phi}(\Omega) \setminus \overline{B}_r(0)$, independent of $\lambda \geq 1$, such that $I(\psi) < 0$.

Proof In what follows we will assume that f is of the type (f_1), because if (f_2) holds the argument is similar. In fact when f is of the type (f_2) the result follows for any $\lambda > 0$. As in the proof of Lemma 3.8, we are going to write I of the form

$$I(u) = \int_\Omega \Phi(x, |\nabla u|)\, dx - \Psi_1(u) - \Psi_2(u) - \Psi_3(u), \quad \forall\, u \in W_0^{1,\Phi}(\Omega).$$

The embedding (2.8) together with the definition of f and (g_1) ensures that if r is small, we have

$$\int_{(\Omega_q)_{\delta/2}} |F(x,u)|\, dx \leq C \int_{(\Omega_q)_{\delta/2}} (|u|^{q_1} + |u|^\beta + |u|^{p^*})\, dx, \quad \text{for } \|u\| = r,$$

for some positive constant C and $q_1 > q$. Here, we have used the fact that β, $p^* > q$. Thus,

$$\Psi_1(u) \leq C(\|u\|^{q_1} + \|u\|^\beta + \|u\|^{p^*}) \tag{3.6}$$

for some $C > 0$.

From definition of Ψ_2, f, (3.3) and Hölder inequality, we get

$$\Psi_2(u) \leq \lambda \left(\int_{\Omega_N} |u|^{2\beta}\, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_N} e^{2\alpha |u|^{N'}}\, dx \right)^{\frac{1}{2}}.$$

Fixing $\|u\| = r$ with r small enough, the Lemma 3.4 guarantees that

$$\sup \left\{ \int_{\Omega_N} e^{t\alpha |u|^{N'}}\, dx \ :\ \|u\| \leq r \right\} \leq C.$$

Hence

$$\Psi_2(u) \leq C|u|^\beta_{L^{2\beta}(\Omega_N)} \leq C_1 \|u\|^\beta. \tag{3.7}$$

Now, a direct argument shows that

$$\Psi_3(u) \leq C_2 \|u\|^{p^*}. \tag{3.8}$$

From (3.7) and (3.8),

$$I(u) \geq \int_\Omega \Phi(x, |\nabla u|)\, dx - C\|u\|^\beta - C_1\|u\|^{q_1} - C_2\|u\|^{p^*}, \quad \text{for } \|u\| = r.$$

Now, applying Proposition 2.1(ii) for r small enough, we find

$$I(u) \geq \|u\|^q - C\|u\|^\beta - C_3\|u\|^{q_1} - C_4\|u\|^{p^*}, \quad \text{for } \|u\| = r.$$

Now, (a) follows by using the fact that $\beta, q_1, p^* > q$.

In order to prove (b), as $\lambda \geq 1$, note that

$$f(x,t) \geq |t|^{\beta-2}t, \qquad \forall x \in \Omega_N \backslash \overline{(\Omega_q)_\delta} \quad \text{and} \quad t \geq 0.$$

From this, fixing a nonnegative function $w \in C_0^\infty(\Omega_N \backslash \overline{(\Omega_q)_\delta}) \backslash \{0\}$ and $t > 0$ we find

$$I(tw) \leq \frac{t^N c_1}{N} \int_{\Omega_N} |\nabla w|^N\, dx - \frac{t^\beta}{\beta} \int_{\Omega_N} |w|^\beta\, dx.$$

As $\beta > N$,

$$I(tw) \to -\infty \quad \text{when } t \to +\infty,$$

and so, (b) follows with $\psi = tw$ and t being large enough. □

In the sequel, we denote by d the mountain pass level associated with I, that is,

$$d = \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t)) \geq \rho > 0$$

where

$$\Gamma = \left\{ \gamma \in C([0,1], W_0^{1,\Phi}(\Omega)) \, : \, \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = \psi \right\},$$

and ψ was given in Lemma 3.9.

By using the mountain pass theorem found in Willem [44, Theorem 1.15], there is a $(PS)_d$ sequence $(u_n) \subset W_0^{1,\Phi}(\Omega)$ for I, that is,

$$I(u_n) \to d \quad \text{and} \quad I'(u_n) \to 0 \qquad \text{as } n \to +\infty. \tag{3.9}$$

Lemma 3.10 *The sequence (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$.*

Proof Setting $\chi = \min\{\theta, \beta, p^*\} > q$, it follows by definition of f that

$$0 < \chi F(x,t) \leq f(x,t)t, \quad \forall (x,t) \in \Omega \times (\mathbb{R}\backslash\{0\}), \tag{3.10}$$

which says that f satisfies the famous Ambrosetti–Rabinowitz condition. Since (u_n) is a $(PS)_d$ sequence for I, there are $C_1, C_2 > 0$ such that

$$I(u_n) - \frac{1}{\chi} I'(u_n)u_n \leq C_1 + C_2\|u_n\|, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

From definition of I and (φ_3),

$$\begin{aligned} I(u_n) - \frac{1}{\chi} I'(u_n)u_n &\geq \int_\Omega \Phi(x, |\nabla u_n|)\,dx - \frac{1}{\chi}\int_\Omega \varphi(x, |\nabla u_n|)|\nabla u_n|^2\,dx \\ &\geq \left(1 - \frac{q}{\chi}\right)\int_\Omega \Phi(x, |\nabla u_n|)\,dx. \end{aligned}$$

Therefore,

$$\left(1 - \frac{q}{\chi}\right)\int_\Omega \Phi(x, |\nabla u_n|)\,dx \leq C_1 + C_2\|u_n\|, \quad \forall n \in \mathbb{N}.$$

If $\|u_n\| \geq 1$, then Proposition 2.1(i) leads to

$$\left(1 - \frac{q}{\chi}\right)\|u_n\|^p \leq C_1 + C_2\|u_n\|, \quad \forall n \in \mathbb{N},$$

from where it follows the boundedness of (u_n), finishing the proof. □

Since $W_0^{1,\Phi}(\Omega)$ is reflexive and $(u_n) \subset W_0^{1,\Phi}(\Omega)$ is a bounded sequence, we assume that for some subsequence, still denoted by itself, there is $u \in W_0^{1,\Phi}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\Phi}(\Omega),$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in } \Omega.$$

Lemma 3.11 *There is $\lambda^* > 1$, such that for $\lambda \geq \lambda^*$, it holds*

$$d < \left(1 - \frac{q}{\chi}\right)\min\left\{\frac{1}{N}\left(\frac{\alpha_N}{2^{N'}\alpha}\right)^{N-1}, \frac{1}{p}S_p^{\frac{N}{p}}\right\},$$

where $\chi = \min\{\theta, \beta, p^*\}$.

Proof Taking a nonnegative function $\psi \in C_0^\infty(\Omega_N \backslash \overline{(\Omega_q)_\delta}) \backslash \{0\}$ and $t > 0$ as in the proof of Lemma 3.9, we obtain

$$I(t\psi) \leq \frac{t^N c_1}{N} \int_{\Omega_N} |\nabla \psi|^N \, dx - \frac{\lambda t^\beta}{\beta} \int_{\Omega_N} |\psi|^\beta \, dx.$$

A direct computation gives

$$\max_{t \in [0,+\infty)} I(t\psi) \leq \frac{1}{\lambda^{\frac{N}{\beta-N}}} \left(\frac{1}{N} - \frac{1}{\beta} \right) \frac{\left(c_1 \|\nabla \psi\|^N_{L^N(\Omega_N)} \right)^{\frac{\beta}{\beta-N}}}{\left(\|\psi\|^\beta_{L^\beta(\Omega_N)} \right)^{\frac{N}{\beta-N}}}.$$

Therefore, fixing the path $\gamma_1(s) = s\psi$ for $s \in [0, 1]$, we have $\gamma_1 \in \Gamma$, and so,

$$d \leq \max_{s \in [0,1]} I(\gamma_1(s)) \leq \max_{t \in [0,+\infty)} I(t\psi) \leq \frac{1}{\lambda^{\frac{N}{\beta-N}}} \left(\frac{1}{N} - \frac{1}{\beta} \right) \frac{\left(c_1 \|\nabla \psi\|^N_{L^N(\Omega_N)} \right)^{\frac{\beta}{\beta-N}}}{\left(\|\psi\|^\beta_{L^\beta(\Omega_N)} \right)^{\frac{N}{\beta-N}}}.$$

Now, choosing $\lambda^* > 0$ of such way that for all $\lambda \geq \lambda^*$, we have

$$\frac{1}{\lambda^{\frac{N}{\beta-N}}} \left(\frac{1}{N} - \frac{1}{\beta} \right) \frac{\left(c_1 \|\nabla \psi\|^N_{L^N(\Omega_N)} \right)^{\frac{\beta}{\beta-N}}}{\left(\|\psi\|^\beta_{L^\beta(\Omega_N)} \right)^{\frac{N}{\beta-N}}} < \left(1 - \frac{q}{\chi} \right) \min \left\{ \frac{1}{N} \left(\frac{\alpha_N}{2^{N'}\alpha} \right)^{N-1}, \frac{1}{p} S_p^{\frac{N}{p}} \right\}.$$

Therefore,

$$d < \left(1 - \frac{q}{\chi} \right) \min \left\{ \frac{1}{N} \left(\frac{\alpha_N}{2^{N'}\alpha} \right)^{N-1}, \frac{1}{p} S_p^{\frac{N}{p}} \right\}, \quad \forall \lambda \geq \lambda^*,$$

which shows the desired result. □

Corollary 3.12 *The sequence (u_n) satisfies*

$$\limsup_{n \to +\infty} \|\nabla u_n\|^{\frac{N}{N-1}}_{L^N(\Omega_N)} < \frac{\alpha_N}{2^{N'}\alpha}.$$

Then, without lost of generality, we can assume that there is $\tau \in (0, 1)$ such that

$$\|\nabla u_n\|^{\frac{N}{N-1}}_{L^N(\Omega_N)} \leq \frac{\tau \alpha_N}{2^{N'}\alpha}, \quad \forall n \in \mathbb{N}.$$

Proof First of all, we must recall that

$$I(u_n) - \frac{1}{\chi} I'(u_n)u_n = d + o_n(1)\|u_n\| + o_n(1),$$

from where it follows that

$$\begin{aligned} d + o_n(1)\|u_n\| + o_n(1) &\geq \int_\Omega \left((\Phi(x, |\nabla u_n|) - \frac{1}{\chi} \varphi(x, |\nabla u_n|)|\nabla u_n|^2 \right) dx \\ &\geq \frac{1}{N} \left(1 - \frac{q}{\chi} \right) \int_{\Omega_N} |\nabla u_n|^N \, dx. \end{aligned}$$

Hence, by Lemma 3.11,

$$\limsup_{n\to+\infty}\frac{1}{N}\left(1-\frac{q}{\chi}\right)\int_{\Omega_N}|\nabla u_n|^N\,dx \le d < \min\left(1-\frac{q}{\chi}\right)\left\{\frac{1}{N}\left(\frac{\alpha_N}{2^{N'}\alpha}\right)^{N-1},\frac{1}{p}S_p^{\frac{N}{p}}\right\}$$

leading to

$$\limsup_{n\to+\infty}\int_{\Omega_N}|\nabla u_n|^N\,dx < \left(\frac{\alpha_N}{2^{N'}\alpha}\right)^{N-1},$$

which proves the lemma. □

Lemma 3.13 *The functional I verifies the $(PS)_d$ condition.*

Proof In what follows, we will assume that f is of the type (f_1). Moreover, let us set

$$P_n=\int_\Omega\langle\varphi(x,|\nabla u_n|)\nabla u_n,\nabla u_n-\nabla u\rangle\,dx,$$

that is,

$$P_n=I'(u_n)u_n+\int_\Omega f(x,u_n)u_n\,dx-I'(u_n)u-\int_\Omega f(x,u_n)u\,dx.$$

Consequently

$$P_n=\int_\Omega f(x,u_n)u_n\,dx-\int_\Omega f(x,u_n)u\,dx+o_n(1).$$

From the definition of f together with embedding (2.8),

$$\begin{aligned}
\lim_{n\to+\infty}\int_\Omega\tilde{\eta}_q(x)g(x,u_n)u_n\,dx &= \lim_{n\to+\infty}\int_\Omega\tilde{\eta}_q(x)g(x,u_n)u\,dx\\
&=\int_\Omega\tilde{\eta}_q(x)g(x,u)u\,dx,\\
\lim_{n\to+\infty}\int_{\Omega\backslash\Omega_N}\eta_N(x)|u_n|^\beta e^{\alpha|u_n|^{N'}}\,dx &= \int_{\Omega\backslash\Omega_N}\eta_N(x)|u|^\beta e^{\alpha|u|^{N'}}\,dx,\\
\lim_{n\to+\infty}\int_{\Omega\backslash\Omega_N}\eta_N(x)|u_n|^{\beta-2}u_nue^{\alpha|u_n|^{N'}}\,dx &= \int_{\Omega\backslash\Omega_N}\eta_N(x)|u|^\beta e^{\alpha|u|^{N'}}\,dx,\\
\lim_{n\to+\infty}\int_{\Omega\backslash\Omega_p}\eta_p(x)|u_n|^{p^*}\,dx &= \int_{\Omega\backslash\Omega_p}\eta_p(x)|u|^{p^*}\,dx,
\end{aligned}$$

and

$$\lim_{n\to+\infty}\int_{\Omega\backslash\Omega_p}\eta_p(x)|u_n|^{p^*-2}u_nu\,dx=\int_{\Omega\backslash\Omega_p}\eta_p(x)|u|^{p^*}\,dx.$$

Consequently

$$\begin{aligned}
P_n=\lambda\int_{\Omega_N}|u_n|^\beta e^{\alpha|u_n|^{N'}}\,dx-\lambda\int_{\Omega_N}|u_n|^{\beta-2}u_nue^{\alpha|u_n|^{N'}}\,dx+\int_{\Omega_p}|u_n|^{p^*}\,dx\\
-\int_{\Omega_p}|u_n|^{p^*-2}u_nu\,dx\,dx+o_n(1).
\end{aligned}$$

By Corollary 3.12, the sequence (u_n) satisfies

$$\|\nabla u_n\|_{L^N(\Omega_N)}^{N'} \leq \frac{\tau\alpha_N}{2^{N'}\alpha}, \quad \forall n \in \mathbb{N},$$

for some $\tau \in (0, 1)$. Employing Corollary 3.6, there is $t > 1$ and $t \approx 1$ such that the sequence $h_n(x) = e^{\alpha|u_n(x)|^{N'}}$ is weakly convergent to $h(x) = e^{\alpha|u(x)|^{N'}}$ in $L^t(\Omega_N)$, that is,

$$\int_{\Omega_N} h_n\varphi\, dx \to \int_{\Omega_N} h\varphi\, dx, \quad \forall\varphi \in L^{t'}(\Omega_N), \tag{3.12}$$

where $t' = \frac{t}{t-1}$. As

$$|u_n|^\beta \to |u|^\beta \quad \text{in } L^{t'}(\Omega_N)$$

it follows that

$$\int_{\Omega_N} h_n|u_n|^\beta\, dx \to \int_{\Omega_N} h|u|^\beta\, dx,$$

that is,

$$\int_{\Omega_N} |u_n|^\beta e^{\alpha|u_n|^{N'}}\, dx \to \int_{\Omega_N} |u|^\beta e^{\alpha|u|^{N'}}\, dx.$$

Now, using the fact that

$$|u_n|^{\beta-2}u_n u \to |u|^\beta \quad \text{in } L^{t'}(\Omega_N)$$

we also derive that

$$\int_{\Omega_N} |u_n|^{\beta-2}u_n u e^{\alpha|u_n(x)|^{N'}}\, dx \to \int_{\Omega_N} |u|^{\beta-2}u u e^{\alpha|u(x)|^{N'}}\, dx.$$

The above analysis ensures that

$$\lim_{n\to+\infty}\int_{\Omega_N} |u_n|^\beta e^{\alpha|u_n(x)|^{N'}}\, dx = \lim_{n\to+\infty}\int_{\Omega_N} |u_n|^{\beta-2}u_n u e^{\alpha|u_n(x)|^{N'}}\, dx = \int_{\Omega_N} |u|^\beta e^{\alpha|u|^{N'}}\, dx,$$

and then,

$$P_n = \int_{\Omega_p} |u_n|^{p^*}\, dx - \int_{\Omega_p} |u_n|^{p^*-2}u_n u\, dx + o_n(1).$$

By [34, Lemma 4.8],

$$\lim_{n\to+\infty}\int_{\Omega_p} |u_n|^{p^*-2}u_n u\, dx = \int_{\Omega_p} |u|^{p^*}\, dx,$$

then

$$P_n = \int_{\Omega_p} |u_n|^{p^*}\, dx - \int_{\Omega_p} |u|^{p^*}\, dx + o_n(1).$$

Now, we are going to use the Concentration Compactness Lemma 3.1 to the sequence $(u_n) \subset W^{1,p}(\Omega_p)$. From (φ_7), for each open ball $B \subset (\Omega_q)_\delta$ we have that the embedding $W^{1,\Phi}(\Omega) \hookrightarrow C(\overline{B})$ is compact, then as (u_n) is a bounded (PS) for I, it is possible to prove that for some subsequence there holds

$$\int_B \langle\varphi(x, |\nabla u_n|)\nabla u_n, \nabla u_n - \nabla u\rangle\, dx \to 0.$$

Since from $(\varphi_6)-(\varphi_8)$, the embedding $W^{1,\Phi}(B) \hookrightarrow L^{\Phi}(B)$ is compact, the last limit together with the Δ_2-condition implies that

$$u_n \to u \quad \text{in } W^{1,\Phi}(B).$$

Now, recalling that the embedding $W^{1,\Phi}(B) \hookrightarrow W^{1,p}(B)$ is continuous, we derive that

$$u_n \to u \quad \text{in } W^{1,p}(B),$$

from where it follows that $x_i \in \overline{\Omega_p} \backslash (\Omega_q)_\delta$ for all $i \in J$. Now, our goal is proving that J must be a finite set. Have this in mind, we will consider $J = J_1 \cup J_2$ where

$$J_1 = \{i \in J \,:\, x_i \in \overline{\Omega_p} \backslash \overline{(\Omega_q)_\delta}\}$$

and

$$J_2 = \{i \in J \,:\, x_i \in \partial(\Omega_q)_\delta \cap \Omega_p\}.$$

If $i \in J_1$, the condition (φ_8) says that $c_2 t^{p-2} \geq \varphi(x,t) \geq t^{p-2}$ for $x \in \overline{\Omega_p} \backslash \overline{(\Omega_q)_\delta}$. This fact permits to repeat the same arguments explored in [28, Lemma 2.3] to conclude that J_1 is finite. Now, if $i \in J_2$, the situation is more subtle and we must be careful. In what follows let us consider $\tilde{\psi} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\tilde{\psi} \equiv 1 \quad \text{on } B(0,1) \qquad \text{and} \qquad \tilde{\psi} \equiv 0 \quad \text{on } B(0,2)^c.$$

For each $\epsilon > 0$, we set

$$\psi(x) = \tilde{\psi}((x - x_i)/\epsilon), \quad \forall x \in \mathbb{R}^N.$$

Since (u_n) is a bounded sequence in $W^{1,\Phi}(\Omega)$, the sequence (ψu_n) is also bounded in $W^{1,\Phi}(\Omega)$, and so, $I'(u_n)\psi u_n = o_n(1)$. Hence,

$$\int_\Omega \varphi(x, |\nabla u_n|)\nabla u_n \nabla(\psi u_n)\, dx = \int_\Omega \tilde{\eta}_q(x) g(x, u_n)\psi u_n \, dx + \int_\Omega \eta_p(x)|u_n|^{p^*}\psi\, dx + o_n(1).$$

Now, given $\xi > 0$, the Young's inequality (2.1) combined with (2.2) and Δ_2-condition gives

$$\int_\Omega |\varphi(x, |\nabla u_n|)|\nabla u_n||u_n||\nabla\psi|\, dx \leq \xi \int_\Omega \Phi(x, |\nabla u_n|)\, dx + C_\xi \int_\Omega \Phi(x, |\nabla\psi||u_n|)\, dx,$$

for some $C_\xi > 0$. Note that by (φ_8),

$$\int_\Omega \Phi(x, |\nabla\psi||u_n|)\, dx \leq C_1 \left(\int_{B(x_i,2\epsilon)} |\nabla\psi|^p |u_n|^p \, dx + \int_{B(x_i,2\epsilon)} \tau_2(x)|\nabla\psi|^q |u_n|^q \, dx \right).$$

By Hölder inequality

$$\limsup_{n\to+\infty} \int_{B(x_i,2\epsilon)} |u_n|^p |\nabla\psi|^p \, dx \leq C_2 \left(\int_{B(x_i,2\epsilon)} |u|^{p^*}\, dx \right)^{\frac{N-p}{N}}$$

from where it follows that

$$\lim_{\epsilon\to 0} \left[\limsup_{n\to+\infty} \int_{B(x_i,2\epsilon)} |u_n|^p |\nabla\psi|^p \, dx \right] \leq \lim_{\epsilon\to 0} C_2 \left(\int_{B(x_i,2\epsilon)} |u|^{p^*}\, dx \right)^{\frac{N-p}{N}} = 0. \quad (3.13)$$

Arguing as above, we also have

$$\limsup_{n\to+\infty}\int_{\Omega}\tau_2(x)|u_n|^q|\nabla\psi|^q\,dx\le\left(\int_{B(x_i,2\epsilon)}\left|\tau_2^{\frac{1}{q}}(x)\nabla\psi\right|^{\frac{qp^*}{p^*-q}}\,dx\right)^{\frac{p^*-q}{p^*}}\left(\int_{B(x_i,2\epsilon)}|u|^{p^*}\,dx\right)^{\frac{q}{p^*}}.$$

By change of variable,

$$\begin{aligned}\int_{B(x_i,2\epsilon)}\left|\tau_2^{\frac{1}{q}}(x)\nabla\psi\right|^{\frac{qp^*}{p^*-q}}\,dx&=\left(\frac{1}{\epsilon}\right)^{\frac{qp^*}{p^*-q}}\int_{B(0,2)}\left|\tau_2^{\frac{1}{q}}(\epsilon x+x_i)\nabla\tilde{\psi}\right|^{\frac{qp^*}{p^*-q}}\,dx\\&\le C_5\left(\frac{1}{\epsilon}\right)^{\frac{qp^*}{p^*-q}}\int_{B(0,2)}\left|\tau_2^{\frac{1}{q}}(\epsilon x+x_i)\right|^{\frac{qp^*}{p^*-q}}\,dx.\end{aligned}$$

Since $x_i\in\partial(\Omega_q)_\delta\cap\Omega_p$, it follows that

$$\tau_2(\epsilon x+x_i)\le c_3\epsilon^s|x|^s$$

and

$$\int_{B(x_i,2\epsilon)}\left|\tau_2^{\frac{1}{q}}(x)\nabla\psi\right|^{\frac{qp^*}{p^*-q}}\,dx\le C_6\epsilon^{\frac{(s-q)p^*}{p^*-q}}.$$

As $s>q$, it follows that

$$\lim_{\epsilon\to0}\left[\limsup_{n\to+\infty}\int_{\Omega}\tau_2(x)|u_n|^q|\nabla\psi|^q\,dx\right]=0.\tag{3.14}$$

Now, the boundedness of (u_n) in $W^{1,\Phi}(\Omega)$ together with Proposition 2.1, (3.13) and (3.14) ensures that

$$\lim_{\epsilon\to0}\left[\limsup_{n\to+\infty}\int_{\Omega}|\varphi(x,|\nabla u_n|)|\nabla u_n||u_n||\nabla\psi|\,dx\right]\le\xi C,$$

for some $C>0$. Since $\xi>0$ is arbitrary, we can deduce that

$$\lim_{\epsilon\to0}\left[\limsup_{n\to+\infty}\int_{\Omega}|\varphi(x,|\nabla u_n|)|\nabla u_n||u_n||\nabla\psi|\,dx\right]=0.$$

The last limit together with the fact that $\varphi(x,t)\ge t^{p-2}$ for $x\in\Omega_p$ permit to conclude as in [28, Lemma 2.3], that J_2 is also finite. Consequently, J is a finite set. However, in order to conclude the proof of the lemma, we need to show that J is in fact an empty set. Seeking by a contradiction, assume that there is $i\in J$. In this case, the argument explored in [28] also says for us that

$$\nu_i\ge S_p^{\frac{N}{p}}.$$

Hence, by Lemma 3.1(d),

$$\mu_i\ge S_p^{\frac{N}{p}}.$$

As $|\nabla u_n|^p\to\mu$ weakly-* in the sense of measure, we have

$$\liminf_{n\to+\infty}\int_{\Omega_p}|\nabla u_n|^p\,dx\ge\mu_i$$

and so,

$$\liminf_{n\to+\infty}\int_{\Omega_p}|\nabla u_n|^p\,dx \geq S_p^{\frac{N}{p}}.$$

Now, using once more the equality

$$I(u_n)-\frac{1}{\chi}I'(u_n)u_n = d+o_n(1)\|u_n\|+o_n(1),$$

we get

$$d+o_n(1)\|u_n\|+o_n(1)\geq \frac{1}{p}\left(1-\frac{q}{\chi}\right)\int_{\Omega_p}|\nabla u_n|^p\,dx.$$

Taking the limit of $n\to+\infty$, we find the inequality below

$$d\geq \frac{1}{p}\left(1-\frac{q}{\chi}\right)S_p^{\frac{N}{p}}$$

that contradicts the Lemma 3.11, showing that $J=\emptyset$. Thereby, by Lemma 3.1(a), $\nu=|u|^{p^*}$ and

$$\int_{\Omega_p}|u_n|^{p^*}\,dx\to\int_{\Omega_p}|u|^{p^*}\,dx,$$

implying that $P_n=o_n(1)$, that is,

$$\lim_{n\to+\infty}\int_{\Omega}\langle\varphi(x,|\nabla u_n|)\nabla u_n,\nabla u_n-\nabla u\rangle\,dx=0.$$

Now, it is enough to apply Lemma 2.2 to finish the proof. □

4 Proof of the main result

Proof of Theorem 1.1 completed First of all, we recall that Lemmas 3.9 and 3.13 showed that the energy functional I satisfies the mountain pass geometry and the $(PS)_d$ condition on space $W_0^{1,\Phi}(\Omega)$. Hence, there is a nontrivial critical point $u\in W_0^{1,\Phi}(\Omega)$ of I such that

$$u_n\to u \quad \text{in } W_0^{1,\Phi}(\Omega),$$

and so,

$$I(u)=d>0 \quad \text{and} \quad I'(u)=0,$$

finishing the proof. □

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