Two-step solvable SKT shears

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Abstract

We use the shear construction to construct and classify a wide range of two-step solvable Lie groups admitting a left-invariant SKT structure. We reduce this to a specification of SKT shear data on Abelian Lie algebras, and which then is studied more deeply in different cases. We obtain classifications and structure results for $\mathfrak{g}$ almost Abelian, for derived algebra $\mathfrak{g}'$ of codimension 2 and not J -invariant, for $\mathfrak{g}'$ totally real, and for $\mathfrak{g}'$ of dimension at most 2. This leads to a large part of the full classification for two-step solvable SKT algebras of dimension six.

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1 Introduction

Recent years have seen a rising interest in non-Kähler Hermitian manifolds (M, g, J) . The most studied case is *strong Kähler with torsion* (SKT) geometry, which may be characterized by $\partial\bar{\partial}\omega = 0$, where ω is the associated fundamental two-form. An equivalent description is obtained if one uses the Bismut connection ∇^B on (M, g, J) , which is the unique connection with $\nabla^B g = 0$, $\nabla^B J = 0$ and the associated torsion tensor T^B totally skew-symmetric (after lowering one index). Then $T^B = -J^*d\omega$ and (M, g, J) is SKT if and only if $dT^B = 0$.

The latter property is used in physics in the context of supersymmetric theories, cf. for example [15, 17, 21]. These structures also appear in generalized Kähler geometry [3, 6, 10, 15], and in two complex dimensions they are “standard” in the sense of Gauduchon [14], i.e. any conformal class of any Hermitian metric on a compact complex surface admits an SKT metric. Moreover, any compact Lie group possesses a left-invariant SKT structure, see e.g. [18].

However, in higher dimensions, the existence of an SKT metric is not automatically guaranteed anymore. For example, looking at the invariant case on nilpotent Lie groups, only 4 out of the 34 six-dimensional nilpotent Lie algebras admit (non-Kähler) SKT structures [11]. Also the eight-dimensional nilpotent case has been classified in [7], and in both cases all nilpotent SKT Lie algebras are two-step nilpotent, leading to the question whether this is a general feature in arbitrary dimensions [13].

A more general class of Lie algebras that so far only has partially been studied, is the solvable one. Here, a full classification has only been achieved in dimension four by the second author in cooperation with T. B. Madsen [18]. This reflects the much more complicated classification of solvable Lie algebras. In fact, in dimension six, the classification of all solvable Lie algebras is already very long and complicated [20, 23] and in higher dimensions there is no classification at all, nor is there hope that there will be a full list in the future. Hence, so far, only classifications of solvable SKT Lie algebras under severe extra conditions in dimensions six and higher are known. For example, [9] classifies the six-dimensional solvable Lie algebras admitting an SKT structure and a non-zero holomorphic $(3, 0)$ -form, and in [8] it is shown that no complex solvable Lie group may admit an invariant SKT structure. Moreover, [4] provides a classification of the almost Abelian Lie algebras admitting an SKT structure, cf. also [10] for a more explicit list in the six-dimensional case.

In this article, we determine, for various natural classes of two-step solvable (almost Hermitian) Lie algebras, the ones which are SKT. This includes, in particular, also a new proof for the classification in the almost Abelian case leading to a more compact description of the almost Abelian Lie algebras admitting an SKT structure in Corollary 4.7. Moreover, our results eventually lead to a classification of all but one class of six-dimensional two-step solvable Lie algebras admitting an SKT structure in Theorem 7.1.

To obtain the classification results, we apply the shear construction introduced in our joint paper [12]. This construction is a generalisation of the twist construction invented by the second-named author in [22]. The twist looks at double fibrations of the form $M \leftarrow P \rightarrow W$ of two principal A -bundles with commuting A -actions, A being a connected Abelian Lie group, and transfers A -invariant tensor fields from M to W by requiring that their pullbacks to W agree on the horizontal distribution in P . The principal A -bundles and all other data may be obtained by “twist data” on M and the twist is well-adapted to nilpotent Lie groups and nilmanifolds in the sense that any nilmanifold may be constructed from a torus by several twists increasing the nilpotent step length of the associated nilpotent Lie group by one with each step. Note that the second author used the twist construction in the above mentioned

paper [22] to construct new examples of compact simply-connected SKT manifolds out of well-known ones.

The shear construction now replaces the group actions of the twist by flat connections and the principal bundles by certain types of submersions and is well-adapted to solvable Lie groups in the sense that any simply-connected solvable Lie group may be obtained from $\mathbb{R}^n$ by consecutive shears increasing the solvable step length by one. Again, the shear construction can be completely encoded by “shear data” on M and so any two-step solvable SKT Lie algebra can be obtained by one shear from the standard flat (left-invariant) Kähler structure on $\mathbb{R}^n$ and appropriate shear data on $\mathbb{R}^n$.

Let us now give a more detailed overview of the results in this paper and outline its organisation.

In Sect. 2, we recall the shear construction on arbitrary Lie algebras and give a full description of the possible shear data on Abelian Lie algebras.

Section 3 first derives the general conditions for shear data on $\mathbb{R}^{2n}$ to shear the flat Kähler structure to an SKT-structure. For this, we decompose $\mathfrak{g}$ into $\mathfrak{g}'_J \oplus \mathfrak{g}'_r \oplus J\mathfrak{g}'_r \oplus U_J$, where $\mathfrak{g}'_J := \mathfrak{g}' \cap J\mathfrak{g}'$, $\mathfrak{g}'_r$ is the orthogonal complement of $\mathfrak{g}'_J$ in $\mathfrak{g}'$ and U_J is the orthogonal complement of $\mathfrak{g}'_J \oplus \mathfrak{g}'_r \oplus J\mathfrak{g}'_r$ in $\mathfrak{g}$. Afterwards, we derive some general consequences on the different components of the shear data with respect to this splitting. In particular, we show that $\text{ad}(J\mathfrak{g}'_r)$ preserves $\mathfrak{g}'_J$ and is diagonalisable over the complex on this subspace.

Section 4 deals with the case $U_J = 0$, which is equivalent to $\mathfrak{g}' \oplus J\mathfrak{g}' = \mathfrak{g}$. We derive the general form of a two-step solvable SKT Lie algebra of this form and then deal individually with the cases $\text{codim}(\mathfrak{g}') = 1$ and $\text{codim}(\mathfrak{g}') = 2$. The first case is the almost Abelian one and we give a full classification of all SKT almost Abelian Lie algebras in Theorem 4.5 as well as an easy characterisation of the almost Abelian Lie algebras admitting an SKT structure in Corollary 4.7. The second case may equivalently be described as $\mathfrak{g}'$ being of codimension two and $\mathfrak{g}'$ not being J -invariant. We classify all such two-step solvable SKT Lie algebras in Theorem 4.10. In both cases, we note that some of the resulting algebras are unimodular.

In Sect. 5, we consider the case $\mathfrak{g}'_J = \{0\}$, so $\mathfrak{g}'$ is totally real. Note that then the complex structure of an SKT structure is Abelian and classify such SKT Lie algebras in Theorem 5.5 up to some minor remaining equation. In particular, we show that then the Lie algebra is of the form $\text{raff}_{\mathbb{R}} \oplus \mathfrak{h}$ for some nilpotent Lie algebra $\mathfrak{h}$, $r := \dim([\mathfrak{g}', J\mathfrak{g}'])$. If $[\mathfrak{g}', J\mathfrak{g}']$ is of codimension ℓ at most two in $\mathfrak{g}'$, we are also able to solve the remaining equation and $\mathfrak{g}$ is isomorphic to $\text{raff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-2r}$ ($\ell = 0$), $\text{raff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}^{2n-2r-3}$ ($\ell = 1$) or to $\text{raff}_{\mathbb{R}} \oplus 2\mathfrak{h}_3 \oplus \mathbb{R}^{2n-2r-6}$ ($\ell = 2$), respectively. We use our results to obtain a classification of all SKT Lie algebras with one-dimensional commutator or with two-dimensional commutator which is not J -invariant. The corresponding Lie algebras are isomorphic to $\text{aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-2}$, $\mathfrak{h}_3 \oplus \mathbb{R}^{2n-3}$ or to $2\text{aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-4}$, $\text{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}^{2n-5}$, $2\mathfrak{h}_3 \oplus \mathbb{R}^{2n-6}$, respectively.

Section 6 determines all SKT Lie algebras with $\dim(\mathfrak{g}') = 2$ and $\mathfrak{g}'$ being J -invariant in Theorem 6.5. The associated Lie algebra is isomorphic either to $\mathfrak{t}'_{3,0} \oplus \mathbb{R}^{2n-3}$ or to $\mathfrak{h} \oplus \mathbb{R}^{2n-2k}$ with a six- or eight-dimensional (i.e. $k \in \{3, 4\}$) nilpotent SKT Lie algebra. Hence, we obtain a full list of these Lie algebras using the results in [11] and [7].

Finally, in Sect. 7 we put our previous results together, solve one additional case in Theorem 7.5 (for $\dim(\mathfrak{g}) = 6$, $\dim(\mathfrak{g}') = 3$, $\mathfrak{g}' \neq \mathfrak{g}'_r$) and so are able to give a full classification of six-dimensional two-step solvable Lie algebras admitting an SKT structure in Theorem 7.1 up to one particular case, where we write down the resulting equations.

2 Two-step solvable algebras

The general shear construction [12], is motivated by and specialises to constructions of Lie groups H from left-invariant data on a Lie group G of the same dimension. The construction is set-up so that if G is solvable, then H is also solvable with in general step length one greater than that of G . The construction may be understood purely at the algebraic level, using left-invariant shear data, as follows.

Let $\mathfrak{g}$ be the Lie algebra of G . Suppose $\mathfrak{a}_G$ and $\mathfrak{a}_P$ are Abelian Lie algebras of the same dimension and we have $(\xi, a, \omega, \gamma, \eta)$, where $\gamma \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_G)$, $\eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P)$, $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$ a Lie algebra monomorphism, $a: \mathfrak{a}_G \rightarrow \mathfrak{a}_P$ a Lie algebra isomorphism and a two-form $\omega \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$ with values in $\mathfrak{a}_P$. Then $(\xi, a, \omega, \gamma, \eta)$ defines left-invariant *shear data* on $\mathfrak{g}$ if and only if $\xi(\mathfrak{a}_G)$ is an ideal in $\mathfrak{g}$ and the following conditions are satisfied:

- (Sa) $\xi^* \omega = 0$,
- (Sb) $d\omega + \eta \wedge \omega = 0$,
- (Sc) $\eta = a\gamma a^{-1} - (\xi \lrcorner \omega)a^{-1}$,
- (Sd) $\gamma(\xi Y) = 0$ for all $Y \in \mathfrak{a}_G$,
- (Se) $\xi \circ \gamma(X) = \text{ad}(X) \circ \xi$ for all $X \in \mathfrak{g}$, and
- (Sf) $d\eta + \eta \wedge \eta = 0$.

The *shear* H of G then has Lie algebra $\mathfrak{h} = \mathfrak{p}/\tilde{\xi}(\mathfrak{a}_G)$, where $\mathfrak{p}$ is the extension of $\mathfrak{g}$ by $\mathfrak{a}_P$ via the two-form ω and $\tilde{\xi}: \mathfrak{a} \rightarrow \mathfrak{p}$ is a lift of ξ . More precisely, as a vector space $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$ with $\mathfrak{a}_P$ an Abelian ideal, $[X, Y]_{\mathfrak{p}} = [X, Y]_{\mathfrak{g}} - \omega(X, Y)$, $[X, Z]_{\mathfrak{p}} = \eta(X)Z$. The lift $\tilde{\xi}$ is given as $\tilde{\xi} + \rho \circ a$, where $\rho: \mathfrak{a}_P \rightarrow \mathfrak{p}$ is the inclusion and $\tilde{\xi}: \mathfrak{a}_G \rightarrow \mathfrak{p}$ is the horizontal lift of ξ with respect to the flat connection on $E = G \times \mathfrak{a}_G$ defined by γ .

The above construction and data simplifies considerably if we start with G Abelian, as we will now explain. So take $\mathfrak{g} = \mathbb{R}^N$ for some N . Then as ξ is injective, we may assume that $\mathfrak{a} = \mathfrak{a}_G$ is a Lie subalgebra of $\mathfrak{g} = \mathbb{R}^N$, so an arbitrary subspace, and that $\xi = \text{inc}$ is the inclusion. Next, since a is invertible, we may take $\mathfrak{a}_P = \mathfrak{a}$ and $a = \text{id}_{\mathfrak{a}}$, the identity map. It turns out that given the form ω the remaining pieces of shear data γ and η are uniquely determined. To explain this, let U be a vector space complement to $\mathfrak{a}$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{a} + U.$$

Then

$$\Lambda^2 \mathfrak{g}^* = \Lambda^2 \mathfrak{a}^* \oplus U^* \wedge \mathfrak{a}^* \oplus \Lambda^2 U^*,$$

and we correspondingly decompose

$$\omega = \omega_{-1} + \omega_0 + \omega_1,$$

with $\omega_{-1} \in \Lambda^2 \mathfrak{a}^* \otimes \mathfrak{a}$, $\omega_0 \in (U^* \wedge \mathfrak{a}^*) \otimes \mathfrak{a} \subset U^* \otimes \text{End } \mathfrak{a}$ and $\omega_1 \in \Lambda^2 U^* \otimes \mathfrak{a}$.

Lemma 2.1 *For $\mathfrak{g} = \mathbb{R}^N$ Abelian, $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega = \omega_{-1} + \omega_0 + \omega_1, \gamma, \eta)$ defines left-invariant shear data if and only if $\omega_{-1} = 0$, $\gamma = 0$, $\eta = -\omega_0$ and*

$$\mathcal{A}(\omega(\omega(\cdot, \cdot), \cdot)) = 0, \quad (2.1)$$

where $\mathcal{A}$ is the anti-symmetrisation map over all arguments.

Proof We need to check conditions (Sa)–(Sf). The first condition (Sa) is exactly $\omega_{-1} = 0$. Now as $\mathfrak{g}$ is Abelian condition (Se), says $\text{inc} \circ \gamma(X) = \text{ad}(X) \circ \text{inc} = 0$ for all $X \in \mathfrak{g} = \mathbb{R}^N$, so $\gamma = 0$. This implies condition (Sd) is automatically satisfied and condition (Sc) is equivalent

to $\eta = -\text{inc} \lrcorner \omega = -\omega_0 \in U^* \otimes \text{End } \mathfrak{a}$. Considering condition (Sb), we get $0 = d\omega + \eta \wedge \omega = -\omega_0 \wedge \omega$. But if we now regard ω_0 as an element of $(U^* \wedge \mathfrak{a}^*) \otimes \mathfrak{a} \subseteq \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$, then this condition reads $\mathcal{A}(\omega_0(\omega(\cdot, \cdot), \cdot)) = 0$. As $\omega_1(\omega(\cdot, \cdot), \cdot) = 0$, this is equivalent to (2.1).

Finally, we need to consider condition (Sf), i.e. $0 = d\eta + \eta \wedge \eta = \eta \wedge \eta$. If we regard ω_0 as an element of $(U^* \wedge \mathfrak{a}^*) \otimes \mathfrak{a} \subseteq \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$, then this condition is equivalent to $\mathcal{A}(\omega_0(\omega_0(\cdot, \cdot), \cdot)) = 0$, which is automatic under (2.1). $\square$

As all two-step solvable algebras are obtained as shears of Abelian algebras [12, Proposition 2.5], we see that shear data for such a two-step algebra consists of $\mathfrak{g} = \mathbb{R}^N$, a subalgebra $\mathfrak{a} \leq \mathfrak{g}$ and $\omega = \omega_0 + \omega_1 \in (U^* \wedge \mathfrak{a}^* + \Lambda^2 U^*) \otimes \mathfrak{a}$ satisfying (2.1).

Definition 2.2 We call $(\mathfrak{a}, \omega)$ *two-step shear data* on $\mathfrak{g} = \mathbb{R}^N$, with $\mathfrak{a}$ the *shearing algebra* and ω the *shearing form*. If $\omega \in (U^* \wedge \mathfrak{a}^* + \Lambda^2 U^*) \otimes \mathfrak{a}$ does not necessarily satisfy (2.1), then we call $(\mathfrak{a}, \omega)$ *pre-shear data*.

3 Data for two-step solvable SKT algebras

The shear construction becomes a useful computational tool in the presence of geometric structures. In general, provided they satisfy an invariance property with respect to α_G , they may be lifted horizontally to $\mathfrak{p}$ and then descend to the shear $\mathfrak{h} = \mathfrak{p}/\xi^* \mathfrak{a}_G$; starting with a tensor α , we denote the resulting tensor on $\mathfrak{h}$ by $\alpha_{\mathfrak{h}}$ and say that α and $\alpha_{\mathfrak{h}}$ are $\mathcal{H}$ -related, where $\mathcal{H}$ refers the horizontal distribution of a certain connection on $\mathfrak{p}$. In [12], we provided a number of simple formulae for computing various derived properties of $\alpha_{\mathfrak{h}}$ in terms of properties of α on $\mathfrak{g}$ and the shear data.

Let $(\mathfrak{a}, \omega)$ be two-step shear data on $\mathfrak{g} = \mathbb{R}^N$ and assume that $N = 2n$ for some $n \in \mathbb{N}$. We equip $\mathfrak{g} = \mathbb{R}^{2n}$ with the standard flat Kähler structure $(g, J, \sigma = g(J \cdot, \cdot))$. We will determine the conditions for such a shear $\mathfrak{h}$ to be SKT (strong Kähler with torsion) if we start with the standard structure on $\mathbb{R}^{2n}$.

First recall, that an SKT structure on an algebra $\mathfrak{h}$ consists of a Hermitian structure $(g_{\mathfrak{h}}, J_{\mathfrak{h}}, \sigma_{\mathfrak{h}})$, in particular $J_{\mathfrak{h}}$ is integrable, such that the torsion three-form $c_{\mathfrak{h}} = -J_{\mathfrak{h}}^* d\sigma_{\mathfrak{h}}$ is closed. This then defines a left-invariant SKT structure on the group H , and conversely each such left-invariant structure arises this way. Note that any Kähler structure is SKT, and that any almost Hermitian structure on the Abelian algebra $\mathfrak{g} = \mathbb{R}^{2n}$ is Kähler.

Lemma 3.1 *Let $(\mathfrak{a}, \omega)$ be two-step shear data on $\mathfrak{g} = \mathbb{R}^{2n}$. Let (g, J, σ) be a flat Kähler structure on $\mathfrak{g} = \mathbb{R}^{2n}$. Then the induced almost Hermitian geometry $(g_{\mathfrak{h}}, J_{\mathfrak{h}}, \sigma_{\mathfrak{h}})$ on the shear $\mathfrak{h}$ is SKT if and only if the following two equations hold on $\mathfrak{g}$:*

$$J^* \omega = \omega - J \circ J \cdot \omega, \quad (3.1)$$

$$\mathcal{A}\left(g(J^* \omega(\cdot, \cdot), \omega(\cdot, \cdot)) + 2g(J^* \omega(\omega(\cdot, \cdot), \cdot), \cdot)\right) = 0, \quad (3.2)$$

where $(J \cdot \omega)(X, Y) = -\omega(JX, Y) - \omega(X, JY)$, and $\mathcal{A}$ is anti-symmetrisation over all arguments. Furthermore, any two-step solvable SKT algebra $\mathfrak{h}$ may be obtained as a shear of a flat Kähler structure on $\mathfrak{g} = \mathbb{R}^{2n}$ in this way.

Proof As $\gamma = 0$, the invariance conditions in [12, Sects. 3.2–3.3] are just that the tensors are left-invariant on G . The condition for $J_{\mathfrak{h}}$ to be integrable is equivalent to (3.1) by [12, Proposition 3.13], since J is integrable. If $\alpha \in \Lambda^k \mathfrak{g}^*$, then as $\mathfrak{g}$ is Abelian, we have $d\alpha = 0$. It follows from [12, Corollary 3.11], that $d\alpha_{\mathfrak{h}} \in \Lambda^{k+1} \mathfrak{h}^*$ is $\mathcal{H}$ -related to the differential form

$d_{\mathfrak{h}}\alpha := -(\xi \lrcorner \alpha) \wedge \omega \in \Lambda^{k+1}\mathfrak{g}^*$, where $\xi = \text{inc}: \mathfrak{a} \rightarrow \mathfrak{g}$ acts on the values of ω . In other words, $d_{\mathfrak{h}}\alpha = -\mathcal{A}(\omega(\cdot, \cdot), \cdot, \dots, \cdot)$. We thus find the three-form on $c_{\mathfrak{g}}$ on $\mathfrak{g}$ that is $\mathcal{H}$ -related to the torsion three-form $c_{\mathfrak{h}} = -J_{\mathfrak{h}}^* d\sigma_{\mathfrak{h}}$ on $\mathfrak{h}$ is given by

$$c_{\mathfrak{g}} = -J^* d_{\mathfrak{h}}\sigma = J^*((\xi \lrcorner \sigma) \wedge \omega) = -J^*\omega \wedge \xi^{\flat} = -\mathcal{A}(g(J^*\omega(\cdot, \cdot), \cdot, \cdot)).$$

Now the differential $dc_{\mathfrak{h}}$ is $\mathcal{H}$ -related to $d_{\mathfrak{g}}c_{\mathfrak{g}}$ is given by

$$\begin{aligned} d_{\mathfrak{g}}c_{\mathfrak{g}} &= -\omega \wedge (\xi \lrcorner c_{\mathfrak{g}}) = -\mathcal{A}(c_{\mathfrak{g}}(\omega(\cdot, \cdot), \cdot, \cdot)) \\ &= \frac{1}{3}\mathcal{A}\left(g(J^*\omega(\cdot, \cdot), \omega(\cdot, \cdot)) + 2g(J^*\omega(\omega(\cdot, \cdot), \cdot, \cdot))\right). \end{aligned}$$

So $dc_{\mathfrak{h}} = 0$ is Eq. (3.2).

Conversely, if $\mathfrak{h}$ is two-step solvable, then there is some left-invariant shear to $\mathfrak{g} = \mathbb{R}^{2n}$, where $2n = \dim \mathfrak{h}$, by [12, Proposition 2.5]. Now the SKT structure on $\mathfrak{h}$ induces is $\mathcal{H}$ -related to an almost Hermitian structure on $\mathfrak{g}$. But $\mathfrak{g}$ is Abelian, so this structure is Kähler, and the structure on $\mathfrak{h}$ comes from the above construction. $\square$

Definition 3.2 Two-step shear data $(\mathfrak{a}, \omega)$ satisfying (3.1) and (3.2) will be called *two-step SKT shear data*.

Notation We set

$$\begin{aligned} v_1 &:= v_1(\omega) := \mathcal{A}\left(g(J^*\omega(\cdot, \cdot), \omega(\cdot, \cdot))\right) \in \Lambda^4\mathfrak{g}^*, \\ v_2 &:= v_2(\omega) := \mathcal{A}\left(g(J^*\omega(\omega(\cdot, \cdot), \cdot, \cdot))\right) \in \Lambda^4\mathfrak{g}^*, \\ v &:= v(\omega) := v_1(\omega) + 2v_2(\omega). \end{aligned}$$

for two-step shear data $(\mathfrak{a}, \omega)$. Note that (3.2) holds if and only if $v = 0$.

Summarizing, to find two-step solvable SKT algebras we need to solve Eqs. (2.1), (3.1) and (3.2). To do this, let us reformulate the conditions using information from both the data $(\mathfrak{a}, \omega)$ and the Kähler structure (g, J, σ) on $\mathfrak{g} = \mathbb{R}^{2n}$. For $V \leq \mathfrak{g}$ a linear subspace, we write

$$V_J = V \cap JV \quad \text{and} \quad V_r = (V_J)^{\perp} \cap V.$$

We say that V is *complex* if $V = V_J$ or *totally real* for $V = V_r$. Putting $U := J\mathfrak{a}_r \oplus (\mathfrak{a} \oplus J\mathfrak{a}_r)^{\perp}$, we thus have

$$\mathfrak{g} = \mathfrak{a} \oplus U = \mathfrak{a}_J \oplus \mathfrak{a}_r \oplus U_J \oplus U_r.$$

We note that

$$U_r = J\mathfrak{a}_r, U_J = (\mathfrak{a} \oplus J\mathfrak{a}_r)^{\perp}.$$

In this setting, it is very natural to decompose also $\omega_0 \in (U^* \wedge \mathfrak{a}^*) \otimes \mathfrak{a}$ and $\omega_1 \in \Lambda^2 U^* \otimes \mathfrak{a}$ further, using the decompositions $\mathfrak{a} = \mathfrak{a}_J \oplus \mathfrak{a}_r$, $U = U_J \oplus U_r$ and the associated dual decompositions. We then write $(\omega_0)_r^J$ for the $U_r^* \otimes \mathfrak{a}_J^* \otimes \mathfrak{a}_r$ -component of ω_0 , etc. Similarly, we write $(\omega_1)_r^J$ for the $U_J^* \otimes U_r^* \otimes \mathfrak{a}_r \cong U_J^* \wedge U_r^* \otimes \mathfrak{a}_r$ -part of ω_1 , etc.

It turns out to be useful to consider the endomorphism $(\omega_0)(Y, \cdot) \in \text{End}(\mathfrak{a})$ for $Y \in U_r$. More conveniently, we define

$$A_X := -\omega_0(JX, \cdot) \in \text{End}(\mathfrak{a}), \quad \text{for } X \in \mathfrak{a}_r, \quad (3.3)$$

and decompose this further as

$$\begin{aligned} A_X &= F_X + G_X + H_X + K_X \\ &\in \text{End}(\mathfrak{a}_r) + \text{Hom}(\mathfrak{a}_J, \mathfrak{a}_r) + \text{Hom}(\mathfrak{a}_r, \mathfrak{a}_J) + \text{End}(\mathfrak{a}_J), \end{aligned} \quad (3.4)$$

so $F_X = -(\omega_0)_{rr}^r(JX, \cdot)$, etc. Associated to F_X and H_X , we have bilinear maps

$$\begin{aligned} f: \mathfrak{a}_r \otimes \mathfrak{a}_r &\rightarrow \mathfrak{a}_r & f(X, Y) &:= F_X(Y) \\ h: \mathfrak{a}_r \otimes \mathfrak{a}_r &\rightarrow \mathfrak{a}_J & h(X, Y) &:= H_X(Y) \end{aligned}$$

for $X, Y \in \mathfrak{a}_r$.

Lemma 3.3 *The shearing form ω satisfies the two-step integrability equation (3.1) if and only if*

- (i) $G_X = 0$ for all $X \in \mathfrak{a}_r$,
- (ii) $[J, K_X] = 0$ for all $X \in \mathfrak{a}_r$,
- (iii) $f \in \mathfrak{a}_r^* \otimes \mathfrak{a}_r^* \otimes \mathfrak{a}_r$ is symmetric in its $\mathfrak{a}_r^*$ -factors,
- (iv) $(\omega_1)_{rr}^r = 0$,
- (v) $J^*(\omega_1)_{rr}^J = -2J \circ \mathcal{A}(h)$,
- (vi) $(\omega_0)_{Jr}^r = J^*(\omega_1)_{Jr}^r$,
- (vii) $(\omega_1)_{Jr}^J(JX, JY) - J((\omega_1)_{Jr}^J(X, JY)) = (\omega_0)_{Jr}^J(X, Y) + J((\omega_0)_{Jr}^J(JX, Y))$ for all $X \in U_J$ and all $Y \in \mathfrak{a}_r$,
- (viii) $(\omega_0)_{JJ}^r$ is of type $(1, 1)$ and $J^*(\omega_0)_{JJ}^J = (\omega_0)_{JJ}^J - J(J \cdot (\omega_0)_{JJ}^J)$, and
- (ix) $(\omega_1)_{JJ}^r$ is of type $(1, 1)$ and $J^*(\omega_1)_{JJ}^J = (\omega_1)_{JJ}^J - J(J \cdot (\omega_1)_{JJ}^J)$.

Proof First of all, observe that if (3.1) holds for $X, Y \in \mathfrak{g} = \mathbb{R}^{2n}$, then this equation is also true when evaluated on the pairs (JX, Y) , (X, JY) and (JX, JY) . Surely, all tensors in (3.1) are additionally anti-symmetric. Moreover, as $\omega|_{\Lambda^2 \mathfrak{a}^*} = 0$, (3.1) is trivially satisfied for $X, Y \in \mathfrak{a}_J$, as then both sides of that equation are zero. Hence, we are left with the following cases, which correspond directly to the conditions of the Lemma: (a) $X \in \mathfrak{a}_r$, $Y \in \mathfrak{a}_J$ gives (i) and (ii), (b) $X, Y \in \mathfrak{a}_r$ gives (iii), (iv) and (v), (c) $X \in \mathfrak{a}_r$, $Y \in U_J$ leads to (vi) and (vii), (d) $X \in \mathfrak{a}_J$, $Y \in U_J$ gives (viii), and finally (e) $X, Y \in U_J$ leads to (ix). $\square$

Using Lemma 3.3 (i) and (iv), $\mathfrak{a}_J \perp (\mathfrak{a}_r \oplus U_r)$ and $U_J \perp \mathfrak{a}$, one observes

Lemma 3.4 *Let $(\mathfrak{a}, \omega)$ be two-step shear data. Then*

- (i) $v_1(\omega) = 0$ on $\Lambda^4 \mathfrak{a} + \Lambda^4(\mathfrak{a}_J \oplus U_r) + \Lambda^3 \mathfrak{a}_J \wedge \mathfrak{g}$,
- (ii) $v_2(\omega) = 0$ on $\Lambda^4 \mathfrak{a} + \Lambda^4(\mathfrak{a}_J \oplus U_r) + \Lambda^3 \mathfrak{a}_r \wedge U_r + \Lambda^4 U_J$, and
- (iii) $v(\omega) = 0$ on $\Lambda^4 \mathfrak{a} + \Lambda^4(\mathfrak{a}_J \oplus U_r)$.

Next, let us look at (2.1). We observe that, evaluated on $X_1, X_2 \in U$, $Y \in \mathfrak{a}$, the equation is satisfied if and only if the endomorphisms $\omega_0(X_1, \cdot)$, $\omega_0(X_2, \cdot)$ of $\mathfrak{a}$ commute. Thus, (2.1) is true on $\Lambda^2 U_r \wedge \mathfrak{a}$ if and only if for all $X_1, X_2 \in \mathfrak{a}_r$ we have $[A_{X_1}, A_{X_2}] = 0$, which is equivalent to $[K_{X_1}, K_{X_2}] = 0$, $K_{X_1} H_{X_2} + H_{X_1} F_{X_2} = K_{X_2} H_{X_1} + H_{X_1} F_{X_1}$ and $[F_{X_1}, F_{X_2}] = 0$ for all $X_1, X_2 \in \mathfrak{a}_r$, the latter equation also may be reformulated, due to f being symmetric, as $f(f(\cdot, \cdot), \cdot)$ being totally symmetric.

The other parts of (2.1) are more difficult and we will evaluate them explicitly below in the different cases that we are going to consider in more detail. Thus, let us look at (3.2), i.e. at $v(\omega)$. Here, the $\Lambda^2 \mathfrak{a}_J \wedge \mathfrak{a}_r \wedge U_r$ -part of $v(\omega)$ gives equations which solely contain different components of A_X . A straightforward computation using the fact that f is symmetric, that K_X commutes with J and all other $K_{X'}$ gives us the second part of

Lemma 3.5 *Let $\mathfrak{a}$ be a subspace of $\mathbb{R}^{2n}$, let $\omega \in \Lambda^2(\mathbb{R}^{2n})^* \otimes \mathfrak{a}$ with $\omega|_{\Lambda^2 \mathfrak{a}^*} = 0$ fulfilling (3.1) and let (g, J) be the standard Kähler structure on $\mathbb{R}^{2n}$. Then, with the decompositions and notations from above, the two-form ω satisfies (2.1) on $\Lambda^2 U_r \wedge \mathfrak{a}$ and (3.2) on $\Lambda^2 \mathfrak{a}_J \wedge \mathfrak{a}_r \wedge U_r$ if and only if for all $X, W \in \mathfrak{a}_r$ all of the following conditions are valid*

$$\begin{aligned} [K_X, K_W] &= 0, & [F_X, F_W] &= 0, \\ K_X H_W + H_X F_W &= K_W H_X + H_W F_X \end{aligned} \quad (3.5)$$

and

$$K_X^T K_W + K_X K_W + K_{f(X, W)} \quad (3.6)$$

is g -antisymmetric. Here, the superscript “ T ” denotes the g -transpose of an endomorphism of $\mathfrak{a}_J$.

Before we specialize to the different cases, we will use (3.5) and (3.6) to simplify the form of the endomorphisms K_X of $\mathfrak{a}_J$ with respect to a certain basis of $\mathfrak{a}_J$.

For this, note that by Lemma 3.3(ii) these endomorphisms may be considered as complex endomorphisms on the complex vector space $(\mathfrak{a}_J, J)$. From now on, we will often consider $\mathfrak{a}_J$ as a complex vector space and note that the scalar multiplication satisfies $(a + ib)X = aX + bJX$ for $a, b \in \mathbb{R}$ and $X \in \mathfrak{a}_J$. Even more, we will identify the real Hermitian vector space $(\mathfrak{a}_J, g, J)$ with the complex unitary vector space $(\mathfrak{a}_J, g + i\omega)$.

The rest of this section aims at showing that all complex endomorphisms K_X are simultaneously complex diagonalizable by a unitary basis. For this, note that Lemma 3.5 tells us that all these endomorphisms commute and so we need to show that all these endomorphisms are diagonalizable. This will eventually follow from (3.6) and for this it turns out to be useful to find a good basis $(v_1, \dots, v_m)$ of $\mathfrak{a}_r$ in which $f(v_i, v_i)$ is not too complicated.

Lemma 3.6 *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space of dimension n and let $f: S^2 V \rightarrow V$ be linear. Then there exists an orthonormal basis $(v_1, \dots, v_n)$ of V with $f(v_i, v_i) \in \text{span}(v_1, \dots, v_i)$ for all $i = 1, \dots, n$.*

Proof The proof is by induction on n . The case $n = 1$ holds trivially.

For $n > 1$, we first show that there is some unit vector $v_1 \in V$ with $f(v_1, v_1) \in \text{span}(v_1)$. If there is some $v_1 \in V \setminus \{0\}$ with $f(v_1, v_1) = 0$, we are done. Otherwise, let $S^{n-1} = \{v \in V : \|v\| = 1\}$ be the unit sphere and consider the function $F: S^{n-1} \rightarrow S^{n-1}$, $F(v) := f(v, v)/\|f(v, v)\|$. It suffices to show that F has a fixed point. Since any fixed point-free map from S^{n-1} to S^{n-1} has degree equal to $(-1)^n$, it is enough to show that F has even degree. If F is not surjective, then it is homotopic to a constant map and so has degree 0, and we are done. When F is surjective, Sard's Theorem gives the existence of a regular value w of F . At each point v of $F^{-1}(w)$ the differential of F has full rank, so F is a diffeomorphism in a neighbourhood of v . This implies that $F^{-1}(w)$ is discrete in S^{n-1} and so finite. But then the local degree $\deg(F|_v)$ of F at v is 1 or -1 and we conclude that $\deg(F) = \sum_{v \in F^{-1}(w)} \deg(F|_v) \equiv |F^{-1}(w)| \equiv 0 \pmod{2}$, since $v \in F^{-1}(w)$ gives $-v \in F^{-1}(w)$ too.

So there is some unit vector $v_1 \in V$ with $f(v_1, v_1) \in \text{span}(v_1)$. Then take the orthogonal complement W of $\text{span}(v_1)$ in V . Applying the induction hypothesis to the map $h := \pi_W \circ f|_{S^2 W}: S^2 W \rightarrow W$, where $\pi_W: V \rightarrow W$ is orthogonal projection onto W , yields an orthonormal basis $v_2, \dots, v_n$ with $h(v_i, v_i) \in \text{span}(v_2, \dots, v_i)$. This gives $f(v_i, v_i) \in \text{span}(v_1, \dots, v_i)$ for all $i = 2, \dots, n$, and the result follows. $\square$

Looking at (3.6), one may imagine that the following lemma will be useful in proving that all K_X are complex diagonalizable:

Lemma 3.7 *If $P \in \mathbb{C}^{n \times n}$ has $G(P) = P^2 + P^*P + aP$ skew-Hermitian for some $a \in \mathbb{R}$, then P is diagonalisable via a unitary transformation and the real parts μ of the eigenvalues of P satisfy $\mu(2\mu + a) = 0$.*

Proof Write $P = H + S$, where $H = (P + P^*)/2$ is Hermitian, $S = (P - P^*)/2$ is skew-Hermitian. Then $G(P) = 2H^2 + (HS + SH) + (HS - SH) + aH + aS$. Noting that $HS + SH$ is skew-Hermitian and $[H, S] = HS - SH$ is Hermitian, we see that the Hermitian part of $G(P)$ is $2H^2 + aH + [H, S]$. By applying a unitary transformation, we may assume that H is diagonal, i.e. $H = \text{diag}(\mu_1, \dots, \mu_n)$ for certain $\mu_1, \dots, \mu_n \in \mathbb{R}$.

Let now $G(P)$ be skew-Hermitian. Then $0 = 2H^2 + aH + [H, S]$. But $2H^2 + aH = \text{diag}(\mu_1(2\mu_1 + a), \dots, \mu_n(2\mu_n + a))$ and $[H, S]$ has no diagonal component. Hence, both terms have to vanish individually. In particular, S commutes with H and so preserves the eigenspaces of H . But S is also diagonalisable, with eigenvalues in $i\mathbb{R}$, and the result follows. $\square$

From this, we get

Proposition 3.8 *Let (α, ω) be two-step shear data such that the shear is SKT. Then there is a unitary basis $(Y_1, \dots, Y_m)$ of α_J and $\alpha_1, \dots, \alpha_m \in \alpha_r^* \otimes \mathbb{C}$ such that*

$$K_X(Y_i) = \alpha_i(X) \cdot Y_i \quad (3.7)$$

for all $X \in \alpha_r$.

Proof Let $m := \dim(\alpha_r)$. By Lemma 3.6 and Lemma 3.3(iii), there is a basis $(X_1, \dots, X_m)$ of α_r with $f(X_i, X_i) \in \text{span}(X_1, \dots, X_i)$. Set $K_i := K_{X_i}$ for $i = 1, \dots, m$ and note that K_i is a complex endomorphism of the complex vector space (α_J, J) by Lemma 3.3(ii). We need to show that $K_1, \dots, K_m$ are simultaneously complex diagonalizable. For this, note that these endomorphisms commute by (3.5) in Lemma 3.5. We show inductively that $K_1, \dots, K_m$ are simultaneously complex diagonalizable for all $m = 1, \dots, n$.

For $m = 1$, this follows directly from (3.6) in Lemmas 3.5, 3.7 and $f(X_1, X_1) \in \text{span}(X_1)$.

Now let us assume that we have shown that $K_1, \dots, K_m$ are simultaneously complex diagonalizable for some $s \in \{1, \dots, m-1\}$. As K_{m+1} commutes with $K_1, \dots, K_s$, it preserves the common complex eigenspaces of $K_1, \dots, K_s$ and so we may restrict to such an eigenspace, i.e. we may assume that $K_i = \lambda_i \text{id}_{\alpha_J}$ for some $\lambda_i \in \mathbb{C}$, $i = 1, \dots, s$. Decompose now $K_i = H_i + S_i$ into its Hermitian part H_i and skew-Hermitian part S_i for $i = 1, \dots, m$. We may find a unitary basis $(Y_1, \dots, Y_k)$ of (α_J, g, J) , $k := \dim_{\mathbb{C}}(\alpha_J)$ such that $H_{s+1} = \text{diag}(\mu_1, \dots, \mu_k)$ for certain $\mu_1, \dots, \mu_k \in \mathbb{R}$ with respect to that basis. By (3.6) in Lemma 3.5, the Hermitian part of $K_{s+1}^* K_{s+1} + K_{s+1} K_{s+1} + \sum_{i=1}^{s+1} a_i K_i$ is zero, where $a_1, \dots, a_{s+1} \in \mathbb{R}$ are such that $f(X_{s+1}, X_{s+1}) = \sum_{i=1}^{s+1} a_i X_i$. This Hermitian part is given by $2H_{s+1}^2 + [H_{s+1}, S_{s+1}] + \sum_{i=1}^s a_1 \text{Re}(\lambda_i) \text{id}_{\alpha_J} + a_{s+1} H_{s+1}$. Since all terms are diagonal in the unitary basis $(Y_1, \dots, Y_k)$, apart from $[H_{s+1}, S_{s+1}]$, which is off-diagonal, we get $[H_{s+1}, S_{s+1}] = 0$. But S_{s+1} is complex diagonalizable and so H_{s+1} and S_{s+1} are simultaneously complex diagonalizable. Hence, $K_{s+1} = H_{s+1} + S_{s+1}$ is complex diagonalizable as well, which shows the statement for $s+1$ and so by induction also for $s = m$. $\square$

It is now tempting to assume that certain parts of these decompositions vanish and this is what we will now do.

4 Two-step SKT shears with totally real complement

In this case, $U_J = 0$, ω_0 is fully given by (3.3) and $\omega_1 = (\omega_1)_{rr}^J + (\omega_1)_{rr}^r$. Now by Lemma 3.3 (iv) and (iv), we have $(\omega_1)_{rr}^r = 0$ and $(\omega_1)_{rr}^J = -2J \circ \mathcal{A}(h)$. So here everything is already determined by A as in (3.3). If we decompose $h = h_1 + h_2$ with $h_1 \in S^2 \mathfrak{a}_r^* \otimes \mathfrak{a}_J$, $h_2 \in \Lambda^2 \mathfrak{a}_r^* \otimes \mathfrak{a}_J$, then $J^*(\omega_1)_{rr}^J = -2ih_2$.

Next, looking at the results in Lemma 3.3 and in Proposition 3.8 and using a complex basis $Y_1, \dots, Y_m$ of $\mathfrak{a}_J$ as in Proposition 3.8, (3.5) may be rewritten as $(f(\cdot, \cdot), \cdot) \in S^3 \mathfrak{a}_r^* \otimes \mathfrak{a}_r$ and

$$\alpha^i(X_1)h^i(X_2, X_3) - \alpha^i(X_2)h^i(X_1, X_3) = h^i(X_2, f(X_1, X_3)) - h^i(X_1, f(X_2, X_3))$$

for all $X_1, X_2, X_3 \in \mathfrak{a}_r$, where $h = \sum_{i=1}^r h^i Y_i$ with $h^1, \dots, h^m \in \mathfrak{a}_r^* \otimes \mathfrak{a}_r^* \otimes \mathbb{C}$. Imposing these conditions, (2.1) is valid (i.e. our shear defines a Lie algebra) if and only if $\sum_{j=1}^3 \omega(JX_j, \omega(JX_{j+1}, JX_{j+2})) = 0$ for all $X_1, X_2, X_3 \in \mathfrak{a}_r$, which is easily seen to be equivalent to

$$\alpha_i \wedge h_2^i = 0$$

for all $i = 1, \dots, m$, where we have written $h_2 = \sum_{i=1}^m h_2^i Y_i$ with $h_2^i \in \Lambda^2 \mathfrak{a}_r^* \otimes \mathbb{C}$, $i = 1, \dots, m$. Note that the latter condition is equivalent to $\alpha_i = 0$ or $h_2^i = \alpha_i \wedge \beta_i$ for some $\beta_i \in \mathfrak{a}_r^* \otimes \mathbb{C}$. Summarizing, we need to satisfy the following equations for all $i = 1, \dots, m$ and all $X_1, X_2, X_3 \in \mathfrak{a}_r$:

$$\begin{aligned} \alpha_i(X_1)h^i(X_2, X_3) - \alpha_i(X_2)h^i(X_1, X_3) &= h^i(X_2, f(X_1, X_3)) - h^i(X_1, f(X_2, X_3)), \\ \alpha_i \wedge h_2^i &= 0 \end{aligned} \tag{4.1}$$

Summarizing, all two-step solvable Lie algebras which are shears of the Abelian Lie algebra with $U_J = 0$, i.e. which fulfil $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$ if we assume that ω is surjective, are of the following form:

Definition 4.1 Let $\mathfrak{a}_J$ be a m -dimensional complex vector space with basis $Y_1, \dots, Y_m$, $\mathfrak{a}_r$ be a k -dimensional real vector space. Moreover, set $U_r := \mathfrak{a}_r$ and let $I_0: \mathfrak{a}_r \rightarrow U_r$ be a linear isomorphism. Next, let $f \in S^2 \mathfrak{a}_r^* \otimes \mathfrak{a}_r$, $h_1 \in S^2 \mathfrak{a}_r^* \otimes \mathfrak{a}_J$, $h_2 \in \Lambda^2 \mathfrak{a}_r^* \otimes \mathfrak{a}_J$ and $\alpha_1, \dots, \alpha_m \in \mathfrak{a}_r^* \otimes \mathbb{C}$ be such that $f(f(\cdot, \cdot), \cdot) \in S^3 \mathfrak{a}_r^* \otimes \mathfrak{a}_r$ and let $g_0 \in S^2 \mathfrak{a}_r^*$ and $\tau_0 \in \Lambda^2 \mathfrak{a}_r^*$ be given. Set $h := h_1 + h_2$ and define h_1^i and h_2^i for $i = 1, \dots, m$ as above.

Then put

$$\mathfrak{g} := \mathfrak{a}_J \oplus \mathfrak{a}_r \oplus U_r \tag{4.2}$$

as vector spaces and endow it with the anti-symmetric bilinear map $[\cdot, \cdot]$ uniquely defined by

$$\begin{aligned} [Y_i, Y_j] &= 0, \quad [X, Y_i] = 0, \quad [I_0 X, Y_i] = \alpha_i(X) \cdot Y_i, \\ [I_0 X, W] &= f(X, W) + h(X, W), \quad [I_0 X, I_0 W] = -2ih_2(X, W) \end{aligned} \tag{4.3}$$

for $i, j \in \{1, \dots, m\}$, $X, W \in \mathfrak{a}_r$, extended complex linearly if one of the arguments is in $\mathfrak{a}_J$. Next, define an almost complex structure I on $\mathfrak{g}$ by

$$I(Y_j + X + I_0 Z) := iY_j - Z + I_0 X \tag{4.4}$$

for all $j \in \{1, \dots, m\}$ and all $X, Z \in \mathfrak{a}_r$ and a Euclidean metric $\tilde{g}$ on $\mathfrak{g}$ such that $Y_1, \dots, Y_m, iY_1, \dots, iY_m$ is a $\tilde{g}$ -orthonormal basis of (the realification of) $\mathfrak{a}_J$, $\mathfrak{a}_r \oplus U_r$ is $\tilde{g}$ -orthogonal to $\mathfrak{a}_J$ and

$$\tilde{g}(X, W) = \tilde{g}(JX, JW) = g_0(X, W), \quad \tilde{g}(JX, W) = -\tilde{g}(X, JW) = \tau_0(X, W) \quad (4.5)$$

for all $X, W \in \mathfrak{a}_r$.

We write $\mathfrak{g}_{f,h,\alpha}$ for the triple $(\mathfrak{g}, \tilde{g}, I)$ or only for $\mathfrak{g}$, where $\alpha := (\alpha_1, \dots, \alpha_m)$ and we suppress the dependence on the other tensors.

The discussions above and in the last subsection give us:

Proposition 4.2 *Let $\mathfrak{g}_{f,h,\alpha}$ be as in Definition 4.3. If the defining data satisfies (4.1), then $\mathfrak{g}_{f,h,\alpha}$ is an almost Hermitian Lie algebra with integrable almost complex structure. Moreover, any two-step solvable SKT Lie algebra $(\mathfrak{g}, g, J)$ with $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$ is isomorphic to some $\mathfrak{g}_{f,h,\alpha}$.*

Actually, we can say a tiny bit more which will help us classifying all two-step solvable SKT Lie algebras $(\mathfrak{g}, g, J)$ with $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$ and $\dim(\mathfrak{g}'_r) \leq 2$.

For this, recall that the proof of Lemma 3.6 yields some $v_1 \in V \setminus \{0\}$ such that either $f(v_1, v_1) = 0$ or $f(v_1, v_1) = v_1$. In fact, the latter case is always true if we additionally have that $\text{im}(f) = V$ and $f(f(\cdot, \cdot), \cdot) \in S^3 V^* \otimes V$. Even more, $f(v_1, \cdot) = \text{id}_V$:

Proposition 4.3 *Let V be an n -dimensional real vector space and $f \in S^2 V^* \otimes V$ be such that $f(f(\cdot, \cdot), \cdot) \in S^3 V^* \otimes V$ and $\text{im}(f) = V$. Then there exists a $v \in V$ with $f(v, \cdot) = \text{id}_V$.*

Proof The symmetry assumptions on f imply that the endomorphisms $f_v = f(v, \cdot)$, $v \in V$, commute and form a closed algebra: $f_u \circ f_v = f_{f(u,v)}$. It is enough to show that id_V is in the algebra A which is the complex linear span of $\{f_v : v \in V\}$, since $\text{id}_V = \sum_{i=1}^k a_i f_{v_i}$ with $a_i \in \mathbb{C}$ implies $\text{id}_V = \sum_{i=1}^k \text{Re}(a_i) f_{v_i}$, as f_{v_i} are real. Extending f complex linearly, it is enough to prove the assertion on the complexification $V \otimes \mathbb{C}$.

Now the operators f_v , $v \in V \otimes \mathbb{C}$, commute, so $V \otimes \mathbb{C}$ is a direct sum of subspaces W , on which each f_v has only a single eigenvalue. The assumption that f is surjective on the real space V , implies that it is also surjective on $V \otimes \mathbb{C}$. So for a given summand W there is $u(w) \in V \otimes \mathbb{C}$ for which $f_{u(w)}$ has a non-zero eigenvalue λ . It follows there is a linear combination a of these $f_{u(w)}$ that has all eigenvalues non-zero. Now consider the characteristic polynomial $\Delta_a(t) = \det(a - t \text{id}_V) = c + p_a(t)$, where $c = \det(a) \neq 0$ and $p_a(t)$ is a polynomial in t without constant term. By the Cayley-Hamilton theorem, $0 = \Delta_a(a) = c \text{id}_V + p_a(a)$, so $\text{id}_V = -p_a(a)/c$ which is an element of the algebra A and hence has the form f_w for some $w \in V \otimes \mathbb{C}$. By the remarks above we have $\text{id}_V = f_v$ for $v = \text{Re}(w)$, as required. $\square$

Proposition 4.3 yields:

Corollary 4.4 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable Lie algebra with $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$. Then there exist some $X \in \mathfrak{g}'_r$ and $h_1, h_2 \in \text{End}(\mathfrak{g}'_r, \mathfrak{g}'_J)$ such that for all $W \in \mathfrak{g}'_r$ we have*

$$[JX, W] = W + h_1(W), \quad [JW, X] = W + h_2(W).$$

In particular, $\mathfrak{g}$ is not nilpotent.

Note that Proposition 4.3 does not give us any information on the endomorphism $f(w, \cdot)$ for $w \notin \text{span}(v)$ besides $f(w, v) = w$. This is one of the reasons why we are restricting in the following to the case $\dim(\mathfrak{g}'_r) \leq 2$. Another reason is that surely (4.1) gets easier in this case, in particular, the second equation in (4.1) is void in this case.

4.1 Almost Abelian algebras

An *almost Abelian* Lie algebra is one with a codimension one Abelian ideal and the SKT structures on such Lie algebras of dimension $2n$ are precisely those which may be obtained by a shear from the flat Kähler SKT Lie algebra $(\mathbb{R}^{2n}, g_0, J_0, \sigma_0)$ with $\dim(\mathfrak{a}) = 2n - 1$ or, equivalently, with $U_J = 0$ and $\dim(\mathfrak{a}_r) = 1$. Note that a characterization of those almost Abelian Lie algebras which are SKT has already been given in [4], cf. also [10] for a more detailed list in the six-dimensional case.

We prove this characterization by a different approach and even solve the remaining equations in [4] completely. Note further that, in contrast to all other cases in this paper, we will *not* assume here that $\operatorname{im}(\omega) = \mathfrak{a}$, i.e. we also do not necessarily have $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$. Still, one easily sees that we may use all the results from above apart from Corollary 4.4, where we use the mentioned identity in the proof.

Choose $X \in \mathfrak{a}_r$ of norm one. Then $f(X, X) = aX$ for some $a \in \mathbb{R}$ and $h(X, X) = \sum_{i=1}^{n-1} w_i Y_i$ for certain $w_1, \dots, w_{n-1} \in \mathbb{C}$. Moreover, $\alpha_i(X) = z_i$ for $z_i \in \mathbb{C}, i = 1, \dots, n - 1$. Note that here also the first equation in (4.1) is void in total and so we only still need to impose (3.2), i.e. $\nu(\omega) = 0$. But by Lemma 3.4 and for dimensional reasons, $\nu(\omega)$ only has a component in $\Lambda^2 \mathfrak{a}_J^* \wedge \mathfrak{a}_r^* \wedge U_r^*$. The corresponding equation is given in (3.6) and is equivalent to

$$\bar{z}_i z_i + z_i^2 + a z_i = -\overline{\bar{z}_i z_i + z_i^2 + a z_i} \quad (4.6)$$

This may be written as $\operatorname{Re}(z_i)(2\operatorname{Re}(z_i) + a) = 0$. So $\operatorname{Re}(z_i) \in \{0, -a/2\}$ for any $i = 1, \dots, n - 1$. Thus, we have obtained:

Theorem 4.5 *Let $(\mathfrak{g}, g, J)$ be a $2n$ -dimensional almost Abelian SKT Lie algebra. Then $(\mathfrak{g}, g, J)$ is isomorphic to $\mathfrak{g}_{f,h,\alpha}$ with $\mathfrak{a}_r = \operatorname{span}(X)$ for a unit vector X , $h(X, X) = \sum_{i=1}^{n-1} w_i Y_i$, $f(X, X) = aX$ and $\alpha(X) = \sum_{i=1}^{n-1} z_i Y_i$ for certain $a \in \mathbb{R}, z = (z_1, \dots, z_{n-1})$, $w = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$ with $\operatorname{Re}(z_i) \in \{0, -a/2\}$ for $i = 1, \dots, n - 1$. Conversely, all such $\mathfrak{g}_{f,h,\alpha}$ are almost Abelian SKT Lie algebras.*

Remark 4.6 So any almost Abelian SKT Lie algebra $(\mathfrak{g}, g, J)$ with codimension one Abelian ideal $\mathfrak{a}$ has a real basis $Y_1, \dots, Y_{n-1}, iY_1, \dots, iY_{n-1}, X, JX$, where $Y_1, \dots, Y_{n-1}$ is a complex basis of $\mathfrak{a}_J$ and X spans $\mathfrak{a}_r$, such that $Y_1, \dots, Y_{n-1}, iY_1, \dots, iY_{n-1}, X$ is an orthonormal system and there is some $m \in \{0, \dots, n - 1\}$ so the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension of the bracket to $\mathfrak{a}_J$) are given by

$$[JX, Y_j] = \left(-\frac{a}{2} + ib_j\right)Y_j, \quad [JX, Y_k] = ib_k Y_k, \quad [JX, X] = aX + \sum_{i=1}^m w_i Y_i$$

for $j = 1, \dots, m, k = m + 1, \dots, n - 1$ and certain $a \in \mathbb{R}, b = (b_1, \dots, b_n) \in \mathbb{R}^{n-1}$, $w = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$. For definiteness, we take $a \neq 0$ if $m > 0$.

Observe that here the unimodular cases are exactly those with $m = 1$ or $a = 0$. Note that $\mathfrak{g}' \oplus J\mathfrak{g}' = \mathfrak{g}$ holds if $a \neq 0$ and $b_j \neq 0$ for all $j = m + 1, \dots, n - 1$.

If $\mathfrak{g}' \oplus J\mathfrak{g}' \neq \mathfrak{g}$, then, in contrast to Corollary 4.4, $\mathfrak{g}$ may be nilpotent. However, this is only possible if $m = n - 1, a = 0$ and $b_j = 0$ for all $j = 1, \dots, n$. In that case, $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^{2n-3}$.

Since two almost Abelian Lie algebras $\mathbb{R}^{2n-1} \rtimes_{f_1} \mathbb{R}$ and $\mathbb{R}^{2n-1} \rtimes_{f_2} \mathbb{R}$ are isomorphic if and only if f_1 is conjugated to f_2 up to a non-zero multiple, Theorem 4.5 implies

Corollary 4.7 *A $2n$ -dimensional almost Abelian Lie algebra $\mathbb{R}^{2n-1} \rtimes_f \mathbb{R}$ admits an SKT structure if and only if*

- (i) *either f is diagonalizable over the complex numbers and there is one real eigenvalue $a \in \mathbb{R}$ such that the other complex eigenvalues come in $n-1$ pairs $(z_i, \bar{z}_i)$ with $\operatorname{Re}(z_i) \in \{0, -a/2\}$, $i = 1, \dots, n-1$,*
- (ii) *or the complex Jordan normal form of f has one Jordan block of size 2 with eigenvalue 0 and all other Jordan blocks are of size 1 with eigenvalues $ib_1, -ib_1, \dots, ib_{n-2}, -ib_{n-2}$ for certain real numbers $b_1, \dots, b_{n-2} \in \mathbb{R}$.*

In the six-dimensional case, Corollary 4.7 yields, in agreement with [10], that the following almost Abelian Lie algebras admit an SKT-structure, where we refer to Table 1 for the Lie brackets of the mentioned Lie algebras:

Corollary 4.8 *Let $\mathfrak{g}$ be a six-dimensional almost Abelian Lie algebra. Then $\mathfrak{g}$ admits an SKT structure if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

$$\begin{aligned} & \mathfrak{g}_{6,1}^{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}, \mathfrak{g}_{6,8}^{a, -\frac{a}{2}, -\frac{a}{2}, -\frac{a}{2}}, \mathfrak{g}_{6,8}^{a, -\frac{a}{2}, -\frac{a}{2}, 0}, \mathfrak{g}_{6,11}^{a, -\frac{a}{2}, -\frac{a}{2}, s}, \mathfrak{g}_{6,11}^{a, 0, -\frac{a}{2}, s}, \\ & \mathfrak{g}_{6,11}^{a, -\frac{a}{2}, 0, s}, \mathfrak{g}_{6,11}^{a, 0, 0, s}, \mathfrak{g}_{5,14}^0 \oplus \mathbb{R}, \mathfrak{g}_{5,17}^{0,0,s} \oplus \mathbb{R}, \mathfrak{r}_{4,-1/2,-1/2} \oplus \mathbb{R}^2, \\ & \mathfrak{r}_{4,a,-\frac{a}{2}}' \oplus \mathbb{R}^2 \ (a > 0), \mathfrak{r}_{4,a,0}' \oplus \mathbb{R}^2 \ (a > 0), \mathfrak{r}_{3,0}' \oplus \mathbb{R}^3, \mathfrak{h}_3 \oplus \mathbb{R}^3, \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^4, \mathbb{R}^6. \end{aligned}$$

4.2 Commutator ideal not complex and of codimension two

Note that if $\mathfrak{g}'$ is of codimension two, the condition $\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{g}$ is equivalent to $\mathfrak{g}'$ being non-complex.

By Proposition 4.3, we may choose an orthonormal basis X_1, X_2 of $\mathfrak{a}_{\mathfrak{r}}$ such that $f(X_1, \cdot) = a \operatorname{id}_{\mathfrak{a}_{\mathfrak{r}}}$ for some $a > 0$ and such that $f(X_2, X_2) = b_1 X_1 + b_2 X_2$ for certain $b_1, b_2 \in \mathbb{R}$ with $b_2 \geq 0$.

We set $z_i := \alpha_i(X_1)$, $w_i := \alpha_i(X_2) \in \mathbb{C}$ for $i = 1, \dots, n-2$ and first look at (3.6). As in the last subsection, this equation, applied to $X = W = X_1$, gives us $\operatorname{Re}(z_i) \in \{0, -a/2\}$. Next, applying (3.6) to $X = X_1$, $W = X_2$, we obtain

$$\operatorname{Re}(w_i)(2\operatorname{Re}(z_i) + a) = 0,$$

so that $\operatorname{Re}(w_i) = 0$ if $\operatorname{Re}(z_i) = 0$. Otherwise, the above equation is automatically fulfilled. Finally, if $X = W = X_2$, (3.6) yields

$$2\operatorname{Re}(w_i)^2 + b_2\operatorname{Re}(w_i) + b_1\operatorname{Re}(z_i) = 0.$$

If $\operatorname{Re}(z_i) = 0$ and so also $\operatorname{Re}(z_i) = 0$, this equation is automatically satisfied. Otherwise, $\operatorname{Re}(w_i)$ has to be a solution of the quadratic equation

$$2x^2 + b_2x - \frac{1}{2}b_1a = 0,$$

which has a solution if $b_2^2 + 4b_1a \geq 0$. So our parameters a, b_1, b_2 have to fulfil this inequality if for some $i \in \{1, \dots, n-2\}$ we have $\operatorname{Re}(z_i) \neq 0$.

Next, we look at the first equation in (4.1), which yields

$$z_i h^i(X_2, X_1) - w_i h^i(X_1, X_1) = a h^i(X_2, X_1) - a h^i(X_1, X_2),$$

$$z_i h^i(X_2, X_2) - w_i h^i(X_1, X_2) = a h^i(X_2, X_2) - b_1 h^i(X_1, X_1) - b_2 h^i(X_1, X_2).$$

Since $z_i - a \neq 0$ due to $\operatorname{Re}(z_i) \in \{0, -a/2\}$ and $a \neq 0$, these equations are equivalent to

$$\begin{aligned} h^i(X_2, X_1) &= \frac{w_i}{z_i - a} h^i(X_1, X_1) - \frac{a}{z_i - a} h^i(X_1, X_2), \\ h^i(X_2, X_2) &= -\frac{b_1}{z_i - a} h^i(X_1, X_1) + \frac{w_i - b_2}{z_i - a} h^i(X_1, X_2). \end{aligned} \quad (4.7)$$

Summarizing the discussion up to this point, we have obtained:

Lemma 4.9 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable SKT Lie algebra with $\mathfrak{g}'$ being non-complex and of codimension two. Then $(\mathfrak{g}, g, J)$ is isomorphic to $\mathfrak{g}_{f,h,\alpha}$ with X_1, X_2 being an orthonormal basis of $\mathfrak{a}_r$, f satisfying*

$$f(X_1, X_1) = aX_1, \quad f(X_1, X_2) = f(X_2, X_1) = aX_2, \quad f(X_2, X_2) = b_1X_1 + b_2X_2 \quad (4.8)$$

for certain $a > 0$, $b_2 \geq 0$, h fulfilling (4.7) and for all $i = 1, \dots, n-2$, and $z_i, w_i \in \mathbb{C}$ defined by

$$\alpha_i(X_1) = z_i, \quad \alpha(X_2) = w_i$$

satisfy either

- (i) $\operatorname{Re}(z_i) = \operatorname{Re}(w_i) = 0$, or
- (ii) $\operatorname{Re}(z_i) = -a/2$ and $2\operatorname{Re}(w_i)^2 + b_2\operatorname{Re}(w_i) - \frac{1}{2}b_1a = 0$, where this case may only occur if $b_1 \geq -b_2^2/(4a)$.

Moreover, with this data, $\mathfrak{g}_{f,h,\alpha}$ is an almost Hermitian Lie algebra with integrable almost complex structure I .

We still need to solve (3.2) and remark that we already solved it on $\Lambda^2\mathfrak{a}_J \wedge \mathfrak{a}_r \wedge U_r$. Using Lemma 3.4 and for dimensional reasons, we only have to consider this equation on $\Lambda^2\mathfrak{a}_r \wedge \Lambda^2U_r$, $\mathfrak{a}_J \wedge \Lambda^2\mathfrak{a}_r \wedge U_r$ and $\mathfrak{a}_J \wedge \mathfrak{a}_r \wedge \Lambda^2U_r$. We first look at the latter two equations. Let $Y \in \mathfrak{a}_J$ and $k \in \{1, 2\}$ be given. Since $J^*v_1(\omega) = v_1(\omega)$, we have $v_1(\omega)(Y, X_1, X_2, JX_k) = -v_1(\omega)(iY, JX_1, JX_2, X_k)$ and so $v(\omega)(Y, X_1, X_2, JX_k) = 0$ and $v(\omega)(iY, X_k, JX_1, JX_2) = 0$ are equivalent to

$$\begin{aligned} v_1(Y, X_1, X_2, JX_k) + 2v_2(Y, X_1, X_2, JX_k) &= 0, \\ v_2(Y, X_1, X_2, JX_k) &= -v_2(iY, JX_1, JX_2, X_k). \end{aligned}$$

In the following, we use that $\omega(\mathfrak{a}, \mathfrak{a}) = 0$, $\omega(U_r, \mathfrak{a}_J) \subseteq \mathfrak{a}_J$, $\omega(U_r, U_r) \subseteq \mathfrak{a}_J$, $\mathfrak{a}_J \perp \mathfrak{a}_r$ and the fact that f is symmetric and $f(f(\cdot, \cdot), \cdot) \in S^3\mathfrak{a}_r^* \otimes \mathfrak{a}_r$ to simplify these equations and first look at the second equation. We have

$$\begin{aligned} v_1(\omega)(X_1, X_2, JX_k, Y) &= \frac{1}{12} g(-h(f(X_k, X_1), X_2) + h(X_2, f(X_k, X_1))) \\ &\quad + K_{X_2}(h(X_k, X_1)) + h(f(X_k, X_2), X_1) \\ &\quad - h(X_1, f(X_k, X_2)) - K_{X_1}(h(X_k, X_2)), iY) \end{aligned}$$

and

$$\begin{aligned} &-v_2(\omega)(JX_1, JX_2, X_k, iY) \\ &= \frac{1}{12} g(K_{X_k}(h(X_2, X_1) - h(X_1, X_2)) \\ &\quad + h(f(X_k, X_2), X_1) - h(f(X_k, X_1), X_2), iY), \end{aligned}$$

i.e. $v_2(\omega)(Y, X_1, X_2, JX_k) = -v_2(\omega)(iY, JX_1, JX_2, X_k)$ for all $Y \in \mathfrak{a}_J$ is equivalent to

$$\begin{aligned} & -h(f(X_k, X_1), X_2) + h(X_2, f(X_k, X_1)) + K_{X_2}(h(X_k, X_1)) + h(f(X_k, X_2), X_1) \\ & - h(X_1, f(X_k, X_2)) - K_{X_1}(h(X_k, X_2)) \\ & = K_{X_k}(h(X_2, X_1) - h(X_1, X_2)) + h(f(X_k, X_2), X_1) - h(f(X_k, X_1), X_2). \end{aligned}$$

However, by (3.5), we have

$$\begin{aligned} & -h(f(X_k, X_1), X_2) + h(X_2, f(X_k, X_1)) + K_{X_2}(h(X_k, X_1)) \\ & + h(f(X_k, X_2), X_1) - h(X_1, f(X_k, X_2)) - K_{X_1}(h(X_k, X_2)) \\ & = -h(f(X_k, X_1), X_2) + h(X_k, f(X_2, X_1)) + K_{X_k}(h(X_2, X_1)) \\ & + h(f(X_k, X_2), X_1) - h(X_k, f(X_1, X_2)) - K_{X_k}(h(X_1, X_2)) \\ & = K_{X_k}(h(X_2, X_1) - h(X_1, X_2)) + h(f(X_k, X_2), X_1) - h(f(X_k, X_1), X_2), \end{aligned}$$

i.e. $v_2(\omega)(Y, X_1, X_2, JX_k) = -v_2(\omega)(iY, JX_1, JX_2, X_k)$ is automatically satisfied.

Next, we consider $v_1(Y, X_1, X_2, JX_k) + 2v_2(Y, X_1, X_2, JX_k) = 0$ and note that

$$\begin{aligned} 6v_1(X_1, X_2, JX_k, Y) & = g(K_{X_2}^T(h(X_k, X_1)) - K_{X_1}^T(h(X_k, X_2)) \\ & + K_{X_k}^T(h(X_2, X_1) - h(X_1, X_2)), iY). \end{aligned}$$

Thus, using that $v_2(X_1, X_2, JX_k, Y) = -v_2(\omega)(JX_1, JX_2, X_k, iY)$, one sees that $v_1(X_1, X_2, JX_k, Y) + 2v_2(X_1, X_2, JX_k, Y) = 0$ for all $Y \in \mathfrak{a}_J$ is equivalent to

$$\begin{aligned} 0 & = (K_{X_k} + K_{X_k}^T)(h(X_2, X_1) - h(X_1, X_2)) + h(f(X_k, X_2), X_1) \\ & - h(f(X_k, X_1), X_2) + K_{X_2}^T(h(X_k, X_1)) - K_{X_1}^T(h(X_k, X_2)). \end{aligned}$$

We look at these equations componentwise and for $k = 1, 2$. For $k = 1$, we obtain

$$0 = (2\operatorname{Re}(z_i) + a)(h^i(X_2, X_1) - h^i(X_1, X_2)) - \bar{z}_i h^i(X_1, X_2) + \bar{w}_i h^i(X_1, X_1), \quad (4.9)$$

whereas for $k = 2$, we get

$$\begin{aligned} 0 & = b_1 h^i(X_1, X_1) - 2\operatorname{Re}(w_i) h^i(X_1, X_2) \\ & + (2\operatorname{Re}(w_i) + b_2 + \bar{w}_i) h^i(X_2, X_1) - (\bar{z}_i + a) h^i(X_2, X_2). \end{aligned} \quad (4.10)$$

Now either $\operatorname{Re}(z_i) = 0$ or $\operatorname{Re}(z_i) = -a/2$.

In the first case, (4.9) gives us

$$\begin{aligned} h^i(X_2, X_1) & = -\frac{\bar{w}_i}{a} h^i(X_1, X_1) + \frac{\bar{z}_i + a}{a} h^i(X_1, X_2) \\ & = \frac{w_i}{a} h^i(X_1, X_1) + \frac{a - z_i}{a} h^i(X_1, X_2) \end{aligned}$$

Together with (4.7), we obtain

$$\frac{w_i}{z_i - a} h^i(X_1, X_1) - \frac{a}{z_i - a} h^i(X_1, X_2) = \frac{w_i}{a} h^i(X_1, X_1) + \frac{a - z_i}{a} h^i(X_1, X_2)$$

i.e.

$$z_i h^i(X_1, X_2) = w_i h^i(X_1, X_1)$$

where we have used that $2a - z_i \neq 0$.

So we have to discuss the two subcases $z_i \neq 0$ and $z_i = 0$. First for $z_i \neq 0$: Here we get $h^i(X_2, X_1) = \frac{w_i}{z_i} h^i(X_1, X_1)$ and so, inserting into (4.7), we obtain

$$h^i(X_2, X_1) = \frac{w_i}{z_i} h^i(X_1, X_1) = h^i(X_1, X_2)$$

and

$$h^i(X_2, X_2) = \frac{w_i^2 - b_2 w_i - b_1 z_i}{z_i(z_i - a)} h^i(X_1, X_1),$$

and one checks that (4.9) and (4.10) are also satisfied then.

Second, for $z_i = 0$: Then $w_i h^i(X_1, X_1) = 0$ and the first equation in (4.7) gives us $h^i(X_2, X_1) = h^i(X_1, X_2)$ and (4.9) is automatically satisfied. We now show $w_i \neq 0$, because $\text{im}(\omega) = \mathfrak{a}$.

Suppose $w_i = 0$. Since $\omega(JX_1, Y_i) = -z_i Y_i$, $\omega(JX_2, Y_i) = -w_i Y_i$ for all $i = 1, \dots, n-2$, we may ignore those i where $z_i \neq 0$ or $w_i \neq 0$, i.e. we may assume that $\omega(JX_i, \mathfrak{a}_J) = 0$ globally for all $i = 1, 2$ and $\dim_{\mathbb{R}}(\mathfrak{a}_J) \geq 2$. But then we have $\omega(JX_1, X_2) = \omega(JX_2, X_1)$ and so $\omega(JX_1, JX_2) = 0$. Hence, $\omega(JX_1, X_1)$, $\omega(JX_1, X_2)$, $\omega(JX_2, X_2)$ have to span the at least four-dimensional real vector space $\mathfrak{a} = \mathfrak{a}_J \oplus \mathfrak{a}_r$, a contradiction.

Thus, $w_i \neq 0$ and so we must have $h^i(X_1, X_1) = 0$. But then $h^i(X_2, X_2) = \frac{b_2 - w_i}{a} h^i(X_1, X_2)$ and again (4.10) is satisfied.

Finally, we need to consider the case $\text{Re}(z_i) = -a/2$, i.e. $2\text{Re}(z_i) + a = 0$. Then (4.9) is equivalent to $h^i(X_1, X_2) = \frac{\bar{w}_i}{z_i} h^i(X_1, X_1)$ and the first equation in (4.7) yields

$$h^i(X_2, X_1) = \frac{w_i \bar{z}_i - a \bar{w}_i}{|z_i|^2 - a \bar{z}_i} h^i(X_1, X_1)$$

whereas the second equation in (4.7) gives us

$$h^i(X_2, X_2) = \frac{|w_i|^2 - b_1 \bar{z}_i - b_2 \bar{w}_i}{|z_i|^2 - a \bar{z}_i} h^i(X_1, X_1)$$

Finally, we need to consider (3.2) on $\Lambda^2 \mathfrak{a}_r \wedge \Lambda^2 U_r$. Here, we get

$$\begin{aligned} 6\nu_1(\omega)(X_1, X_2, JX_1, JX_2) \\ = 2a(a - b_1) + \|h(X_1, X_2) - h(X_2, X_1)\|^2 + \|h(X_1, X_2)\|^2 \\ + \|h(X_2, X_1)\|^2 - 2g(h(X_1, X_1), h(X_1, X_2)), \end{aligned}$$

whereas

$$\begin{aligned} 12\nu_1(\omega)(X_1, X_2, JX_1, JX_2) \\ = -g(\omega(J\omega(X_1, JX_1), X_2), X_2) + g(\omega(J\omega(X_1, JX_2), X_1), X_2) \\ + g(\omega(J\omega(X_2, JX_1), X_2), X_1) - g(\omega(J\omega(X_2, JX_2), X_1), X_1) \\ = g(f(f(X_1, X_1), X_2), X_2) - g(f(f(X_2, X_1), X_1), X_2) \\ - g(f(f(X_1, X_2), X_2), X_1) + g(f(f(X_2, X_2), X_1), X_1) \\ = 0, \end{aligned}$$

since $f(f(\cdot, \cdot), \cdot)$ is totally symmetric in its arguments. Hence, (3.2) is true on $\Lambda^2 \mathfrak{a}_r \wedge \Lambda^2 U_r$ if and only if

$$\begin{aligned} 2a(a - b_1) + \|h(X_1, X_2) - h(X_2, X_1)\|^2 + \|h(X_1, X_2)\|^2 \\ + \|h(X_2, X_1)\|^2 - 2g(h(X_1, X_1), h(X_1, X_2)) = 0. \end{aligned} \quad (4.11)$$

Unfortunately, (4.11) seems not to be solvable in a nice and short form in general.

Summarizing, we have obtained:

Theorem 4.10 *Let $(\mathfrak{g}, g, J)$ be a two-step solvable $2n$ -dimensional SKT Lie algebra with $\mathfrak{g}'$ being non-complex and of codimension two. Then $(\mathfrak{g}, g, J)$ is isomorphic to $\mathfrak{g}_{f,h,\alpha}$ with (X_1, X_2) being an orthonormal basis of $\mathfrak{a}_r$, $f(X_1, X_j) = aX_j$ for $j = 1, 2$ and $f(X_2, X_2) = b_1X_1 + b_2X_2$ for certain $a > 0$, $b_1 \in \mathbb{R}$, $b_2 \geq 0$ and $\alpha(X_1) = \sum_{i=1}^{n-2} z_i Y_i$, $\alpha(X_2) = \sum_{i=1}^{n-2} w_i Y_i$ for $(z_1, \dots, z_{n-2}), (w_1, \dots, w_{n-2}) \in \mathbb{C}^{n-2}$ such that (4.11) is valid and for all $i = 1, \dots, n-2$ we have one of the following three cases:*

(i) $\operatorname{Re}(z_i) = 0$, $\operatorname{Re}(w_i) = 0$, $z_i \neq 0$ and

$$h^i(X_2, X_1) = h^i(X_1, X_2) = \frac{w_i}{z_i} h^i(X_1, X_1),$$

$$h^i(X_2, X_2) = \frac{w_i^2 - b_2 w_i - b_1 z_i}{z_i(z_i - a)} h^i(X_1, X_1),$$

(ii) $z_i = 0$, $\operatorname{Re}(w_i) = 0$, $w_i \neq 0$ and

$$h^i(X_1, X_1) = 0, \quad h^i(X_2, X_1) = h^i(X_1, X_2), \quad h^i(X_2, X_2) = \frac{b_2 - w_i}{a} h^i(X_1, X_2),$$

(iii) $\operatorname{Re}(z_i) = -a/2$, $\operatorname{Re}(w_i)$ is a solution of $2x^2 + b_2x - \frac{b_1}{2}a = 0$ and

$$h^i(X_2, X_1) = \frac{w_i \bar{z}_i - a \bar{w}_i}{|z_i|^2 - a \bar{z}_i} h^i(X_1, X_1),$$

$$h^i(X_2, X_2) = \frac{|w_i|^2 - b_1 \bar{z}_i - b_2 \bar{w}_i}{|z_i|^2 - a \bar{z}_i} h^i(X_1, X_1),$$

where this case may only occur if $b_1 \geq -b_2^2/(4a)$.

Conversely, all such $\mathfrak{g}_{f,h,\alpha}$ are two-step solvable SKT Lie algebras with $\mathfrak{g}'$ being non-complex and of codimension two.

Remark 4.11 In Theorem 4.10, we may always choose $h \equiv 0$. Then (4.11) is valid if and only if $b_1 = a$ and so $a = b_1 \geq -b_2^2/(4a)$, i.e. $4a^2 \geq -b_2^2$, is true. Thus, also case (iii) may occur. Moreover, setting $b := b_2$, the solutions of $2x^2 + bx - (a^2/2) = 2x^2 + b_2x - (b_1/2)a = 0$ are given by $x = (-b \pm \sqrt{b^2 + 4a^2})/4$.

In this case, there are $0 \leq r_1 \leq r_2 \leq r_3 \leq n-2$, $c_{r_1}, \dots, c_{n-1}, d_1, \dots, d_{n-1}$ such that the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension to $\mathfrak{a}_J$) are given by

$$[JX_2, Y_j] = i d_j Y_j, \quad j = 1, \dots, r_1,$$

$$[JX_1, Y_k] = i c_k Y_k, \quad [JX_2, Y_k] = i d_k Y_k, \quad k = r_1 + 1, \dots, r_2,$$

$$[JX_1, Y_\ell] = \left(-\frac{a}{2} + i c_\ell\right) Y_\ell, \quad [JX_2, Y_\ell] = \left(\frac{-b + \sqrt{b^2 + 4a^2}}{4} + i d_\ell\right) Y_\ell,$$

$$\ell = r_2 + 1, \dots, r_3,$$

$$[JX_1, Y_m] = \left(-\frac{a}{2} + i c_m\right) Y_m, \quad [JX_2, Y_m] = \left(\frac{-b - \sqrt{b^2 + 4a^2}}{4} + i d_m\right) Y_m,$$

$$m = r_3 + 1, \dots, n-2,$$

$$[JX_1, X_1] = aX_1, \quad [JX_1, X_2] = aX_2, \quad [JX_2, X_1] = aX_2, \quad [JX_2, X_2] = aX_1 + bX_2,$$

with $a > 0$, $b \geq 0$. The Lie algebra is unimodular precisely when $r_2 = n-4$ and $r_3 = n-3$.

5 Commutator ideal totally real

Here, we look at two-step SKT shear data (α, ω) with totally real α , i.e. $\alpha = \alpha_r$ and $\alpha_J = 0$, and with $\text{im}(\omega) = \alpha$. We note that then the shear Lie algebra $\mathfrak{g}$ has a totally real commutator ideal $\mathfrak{g}'$, i.e. we have $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$.

Notation We simplify the notation and, for $Y \in U_J$, define the endomorphisms $\alpha_Y, \beta_Y \in \text{End}(\alpha)$ by

$$\alpha_Y(X) := \omega(Y, X) = (\omega_0)_{Jr}^r(Y, X), \quad \beta_Y(X) := \omega(Y, JX) = (\omega_1)_{Jr}^r(Y, JX)$$

for $X \in \alpha$. Moreover, we set

$$v := (\omega_1)_{JJ} = (\omega_1)_{JJ}^r \in \Lambda^2 U_J^* \otimes \alpha.$$

Moreover, we also use some notation introduced previously without further notification.

We observe the following:

Proposition 5.1 *Let (α, ω) be pre-shear data with α being totally real. Then (α, ω) is two-step SKT shear data if and only if $f \in S^2 \alpha^* \otimes \alpha$, $(\omega_1)_{rr}^r = 0$, $\beta_Y = \alpha_{JY}$ for all $Y \in U_J$, v is of type $(1, 1)$ and*

$$\mathcal{A}\left(\omega(\omega(\cdot, \cdot), \cdot)\right) = 0, \quad \mathcal{A}\left(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot))\right) = 0. \quad (5.1)$$

Proof All the conditions but (5.1) follow from Lemma 3.3.

Next, note that $J^*\omega$ and ω both have values in $\alpha = \alpha_r$ while $J \circ J.\omega$ has values in $J\alpha = J\alpha_r = U_r$. Hence, (3.1) is equivalent $J^*\omega = \omega$, since this equation says that ω is of type $(1, 1)$ and so $J.\omega = 0$. Thus, $\omega(J\omega(\cdot, \cdot), J\cdot) = \omega(\omega(\cdot, \cdot), \cdot)$, i.e. $v_2(\omega) = \mathcal{A}\left(\omega(\omega(\cdot, \cdot), \cdot)\right) = 0$, and $v_1(\omega) = \mathcal{A}\left(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot))\right)$. Hence, (2.1) and (3.2) are satisfied if and only if

$$\mathcal{A}\left(\omega(\omega(\cdot, \cdot), \cdot)\right) = 0, \quad \mathcal{A}\left(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot))\right) = 0.$$

□

Remark 5.2 Note that $J^*\omega = \omega$ implies that the induced almost complex structure on the shear Lie algebra is Abelian.

We first look at the exact form of f . For this, observe that $f \in S^2 \alpha^* \otimes \alpha$ and $f(f(\cdot, \cdot), \cdot) \in S^3 \alpha^* \otimes V$. Moreover, note that the second equation in (5.1) on $U_r^* \otimes U_r^* \otimes \alpha^* \otimes \alpha^*$ is equivalent to $g(f(\cdot, \cdot), f(\cdot, \cdot)) \in (\alpha^*)^{\otimes 4}$ being totally symmetric.

Such an f can always be brought into a certain “normal form”:

Lemma 5.3 *Let (V, g) be an m -dimensional Euclidean vector space and $f \in S^2 V^* \otimes V$ such that both $f(f(\cdot, \cdot), \cdot) \in S^2 V^* \otimes V^* \otimes V$ and $g(f(\cdot, \cdot), f(\cdot, \cdot)) \in S^2 V^* \otimes S^2 V^*$ are totally symmetric in their V^* -factors. Set $V_1 := \text{im}(f|_{S^2 \text{im}(f)})$, $V_2 := V_1^\perp \cap \text{im}(f)$ and $r := \dim(V_1)$. Then there exist an orthonormal basis $(X_1, \dots, X_r)$ of V_1 , non-zero $\lambda_1, \dots, \lambda_r \in \mathbb{R} \setminus \{0\}$ and a complementary subspace V_3 of $V_1 \oplus V_2$ in V which is orthogonal to V_2 satisfying the following conditions:*

- (i) $f(X_a, X_b) = \delta_{ab} \lambda_a X_a$ for all $a, b \in \{1, \dots, r\}$,
- (ii) $f(V_2 \oplus V_3, V_1 \oplus V_2) = 0$

(iii) $f(V_3, V_3) = V_2$.

Proof Let $W = \text{im}(f)$ be the image of f . Then our assumptions imply

$$g(f(\cdot, \cdot), \cdot)|_{W^{\otimes 3}} \in S^3 W^*,$$

i.e. is totally symmetric, since $g(f(\cdot, f(\cdot, \cdot)), f(\cdot, \cdot)) \in S^5(V^*)$. Hence, $f_X := f(X, \cdot)|_W \in \text{End}(W)$ is self-adjoint with respect to g for all $X \in \tilde{W}$. Now the symmetry assumption implies that all the endomorphisms f_X commute. It follows that W has a common orthonormal basis $X_1, \dots, X_m$ of eigenvectors for these endomorphisms. Label these vectors so that $f(X_i, X_i) \neq 0$ for $i = 1, \dots, r$. As $f_{X_a}(X_b) = f(X_a, X_b) = f_{X_b}(X_a) \in \text{span}(X_a) \cap \text{span}(X_b)$ for all $a, b \in \{1, \dots, m\}$, we find that $f(X_a, X_b)$ is only non-zero for $a = b \leq r$. Thus we get $W = V_1 \oplus V_2$ orthogonally, with $V_1 = \text{im}(f|_{S^2 W}) = \text{span}(X_1, \dots, X_r)$ and $V_2 = \text{span}(X_{r+1}, \dots, X_m)$, giving (i) and $f(V_2, V_1 \oplus V_2) = 0$ in (ii).

Note that for $Y \in V_2$ and $Z \in V$, we have

$$g(f(Y, Z), f(Y, Z)) = g(f(Y, Y), f(Z, Z)) = 0,$$

which implies that $f(Y, Z) = f(Z, Y) = 0$ for any $Z \in V$. Next, let $H: W^\perp \rightarrow V$, $H(Y) := Y - \sum_{i=1}^r \frac{1}{\lambda_i} f(Y, X_i)$ and note that H is an injective linear map with $f(H(Y), X_i) = 0$ for all $i = 1, \dots, r$. Hence, $V_3 := H(W^\perp)$ is a complement of W in V with $f(V_3, W) = 0$, and so (ii) holds. Note that V_3 is not necessarily orthogonal to V_1 but it is orthogonal to V_2 . However, we get

$$g(X_i, f(Y, Z)) = \frac{1}{\lambda_i} g(f(X_i, X_i), f(Y, Z)) = \frac{1}{\lambda_i} g(f(X_i, Y), f(X_i, Z)) = 0$$

for all $i = 1, \dots, r$ and all $Y, Z \in V_3$, which implies that $f(Y, Z) \in V_1^\perp \cap (\text{im}(f)) = V_2$, as claimed in (iii). $\square$

In our case, i.e. the case of two-step shear data (α, ω) with totally real α , we get $\alpha_2 = \{0\}$ in Lemma 5.3 and even more the following is true:

Proposition 5.4 *Let (α, ω) be two-step SKT shear data with totally real α . Then, with the notation from the beginning of this subsection, we have $\alpha = \alpha_1 \oplus \alpha_3$, an orthonormal basis $X_1, \dots, X_r$ of α_1 , $r := \dim(\alpha_1)$, $\lambda_1, \dots, \lambda_r \in \mathbb{R} \setminus \{0\}$ and one-forms $\mu_1, \dots, \mu_r$ such that*

$$f(X_i, X_j) = \delta_{ij} \lambda_i X_i, \quad f(\alpha_3, \alpha) = 0, \quad \alpha_Y(X_i) = \mu_i(Y) X_i, \quad \alpha_Y(\alpha_3) = 0 \quad (5.2)$$

for all $i, j \in \{1, \dots, r\}$.

Proof Let $\alpha_1, \alpha_2, \alpha_3, r, s, (X_1, \dots, X_r), \lambda_1, \dots, \lambda_r \in \mathbb{R} \setminus \{0\}$ be as in Lemma 5.3 and let us first look at $\alpha \in U_J^* \otimes \text{End}(\alpha)$.

Then the first condition in (5.1) gives us that $\alpha_Y \in \text{End}(\alpha)$ commutes with $f_X := f(X, \cdot)$ for any $X \in \alpha$. In particular, α_Y preserves the eigenspaces of all f_{X_i} and so $\text{span}(X_i)$ for all $i = 1, \dots, r$ as well as $\alpha_2 \oplus \alpha_3$, which is the common zero eigenspace of all $f_{X_i}, i = 1, \dots, r$. In particular, there are $\mu_1, \dots, \mu_r \in U_J^*$ with $\alpha_Y(X_i) = \mu_i(Y) X_i$ for all $i = 1, \dots, r$.

Next, look at the second equation in (5.1) and insert into that equation (X, JX, Y, JY) with $X \in \alpha_2$. As $f(X, X) = 0$, one gets

$$\begin{aligned} 0 &= \mathcal{A}(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot)))(X, JX, Y, JY) \\ &= -g(\omega(X, Y), \omega(JX, JY)) + g(\omega(X, JY), \omega(JX, Y)) \\ &= -g(\alpha_Y(X), \beta_{JY}(X)) + g(\alpha_{JY}(X), \beta_Y(JX)) = -\|\alpha_Y(X)\|^2 - \|\alpha_{JY}(X)\|^2, \end{aligned}$$

i.e. $\alpha_Y(X) = 0$ and so $\alpha_Y(\mathfrak{a}_2) = 0$, where here and in the following we leave out irrelevant normalization factors coming from anti-symmetrization map $\mathcal{A}$.

Now take $X, W \in \mathfrak{a}_3$. Then $0 = [f_X, \alpha_Y](W) = f_X(\alpha_Y(W))$ since $f_X(W) \in \mathfrak{a}_2 \subseteq \ker(\alpha_Y)$. Hence, if we denote by $\mathfrak{a}'_3$ the common kernel of all $f_X|_{\mathfrak{a}_3}$, $X \in \mathfrak{a}_3$, then $\alpha_Y(\mathfrak{a}_3) \subseteq \mathfrak{a}_2 \oplus \mathfrak{a}'_3$. We are going to show that $\mathfrak{a}'_3 = \mathfrak{a}_3$, which implies $f_X = 0$ for all $X \in \mathfrak{a}_3$ and so $\mathfrak{a}_2 = 0$ by condition (iii) in Lemma 5.3.

For this, let $X \in \mathfrak{a}'_3$ and $Y \in U_J$. Again, we have $f(X, X)$ and so

$$\begin{aligned} 0 &= \mathcal{A}(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot)))(X, JX, Y, JY) \\ &= -g(\alpha_Y(X), \beta_{JY}(X)) + g(\alpha_{JY}(X), \beta_Y(JX)) = -\|\alpha_Y(X)\|^2 - \|\alpha_{JY}(X)\|^2, \end{aligned}$$

i.e. $\alpha_Y(\mathfrak{a}'_3) = \{0\}$, and so $\alpha_Y(\mathfrak{a}_2 \oplus \mathfrak{a}'_3) = \{0\}$.

Next let $X \in \mathfrak{a}_3$, $Y, Z \in U_J$ be given. Then (5.1) gives us

$$\begin{aligned} 0 &= \mathcal{A}(\omega(\omega(\cdot, \cdot), \cdot))(JX, Y, Z) = f(X, v(Y, Z)) - \alpha_Y(\alpha_{JZ}(X)) + \alpha_Z(\alpha_{JY}(X)) \\ &= f(X, v(Y, Z)) \end{aligned}$$

since $\alpha_{JZ}(X), \alpha_{JY}(X) \in \mathfrak{a}_2 \oplus \mathfrak{a}'_3$.

But this implies $v(Y, Z) \in \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}'_3$ for any $Y, Z \in U_J$. As the images of f , α_Y and β_Z are all $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}'_3$ for any $Y, Z \in U_J$, we have $\text{im}(\omega) \subseteq \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}'_3$. However, by assumption, $\text{im}(\omega) = \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$, and so $\mathfrak{a}'_3 = \mathfrak{a}_3$. As said above, this implies $\mathfrak{a}_2 = \{0\}$. Note that now also $0 = \alpha_Y(\mathfrak{a}'_3) = \alpha_Y(\mathfrak{a}_3)$ for any $Y \in U_J$, which finishes the proof. $\square$

We write now $v = \sum_{i=1}^r v_i X_i + \tilde{v}$ with $v_1, \dots, v_r \in \Lambda^2 U_J^*$ and $\tilde{v} \in \Lambda^2 U_J^* \otimes \mathfrak{a}_3$. Then, applying the first part of (5.1) to (JX_i, Y, Z) with $i \in \{1, \dots, r\}$ and $Y, Z \in U_J$, one obtains

$$\begin{aligned} 0 &= \mathcal{A}(\omega(\omega(\cdot, \cdot), \cdot))(JX_i, Y, Z) = f(X_i, v(Y, Z)) - (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i \\ &= (\lambda_i v_i - (\mu_i \wedge \mu_i \circ J))(Y, Z)X_i, \end{aligned}$$

so $v_i = (\mu_i \wedge \mu_i \circ J)/\lambda_i$.

Next, we show that the splitting $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_3$ is g -orthogonal. For this, we insert (JX_i, X_i, Y, Z) with $i \in \{1, \dots, r\}$ and $Y, Z \in U_J$ into the second equation in (5.1) and get

$$\begin{aligned} 0 &= \mathcal{A}(g(\omega(\cdot, \cdot), \omega(\cdot, \cdot)))(JX_i, X_i, Y, Z) \\ &= g(f(X_i, X_i), v(Y, Z)) + g(\alpha_{JY}(X_i), \alpha_Z(X_i)) - g(\alpha_{JZ}(X_i), \alpha_Y(X_i)) \\ &= (\mu_i \wedge \mu_i \circ J)(Y, Z) + \lambda_i g(X_i, \tilde{v}(Y, Z)) - (\mu_i \wedge \mu_i \circ J)(Y, Z) \\ &= \lambda_i g(X_i, \tilde{v}(Y, Z)), \end{aligned}$$

thus $\tilde{v}(Y, Z) \perp_g \mathfrak{a}_1$ for all $Y, Z \in U_J$. As by assumption $\text{im}(\omega) = \mathfrak{a}$ and so $\mathfrak{a}_3$ has to be spanned by $\text{im}(\tilde{v})$, we obtain $\mathfrak{a}_1 \perp_g \mathfrak{a}_3$. Now one checks that (5.1) is valid with the exception of the second equation on $\Lambda^4 U_J^*$, which is equivalent to $\sum_{i=r+1}^m \tilde{v}_i \wedge \tilde{v}_i = 0$ if we write $\tilde{v} = \sum_{i=r+1}^m \tilde{v}_i X_i$ for an orthonormal basis $X_{r+1}, \dots, X_m$ of $\mathfrak{a}_3$ with $\tilde{v}_{r+1}, \dots, \tilde{v}_m \in \Lambda^2 U_J^*$. Summarizing, we have obtained:

Theorem 5.5 *Let (g, g, J) be a $2n$ -dimensional almost Hermitian two-step solvable Lie algebra with totally real m -dimensional commutator ideal $\mathfrak{g}'$.*

Then (g, g, J) is SKT if and only if there exists an orthonormal basis $(X_1, \dots, X_m)$ of $\mathfrak{g}'$, $r \in \{0, \dots, m\}$, $\lambda_1, \dots, \lambda_r \in \mathbb{R} \setminus \{0\}$, $\mu_1, \dots, \mu_r \in U_J^$ with $U_J := (\mathfrak{g}' \oplus J\mathfrak{g}')^\perp$ and*

$\tilde{v}_{r+1}, \dots, \tilde{v}_m \in \Lambda^2 U_J^*$ such that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[JX_i, X_i] = \lambda_i X_i, \quad [Y, X_i] = \mu_i(Y)X_i, \quad [Y, JX_i] = -\mu_i(JY)X_i, \\ [Y, Z] = \sum_{i=1}^r \frac{1}{\lambda_i} (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i + \sum_{j=r+1}^m \tilde{v}_j(Y, Z)X_j,$$

for $i = 1, \dots, r$, $Y, Z \in U_J$, and such that for $\tilde{v} := \sum_{i=r+1}^m \tilde{v}_i X_i$ we have $\sum_{i=r+1}^m \tilde{v}_i \wedge \tilde{v}_i = 0$ and $\text{im}(\tilde{v}) = \text{span}(X_{r+1}, \dots, X_m)$.

If this is the case, we have $\mathfrak{g} \cong r \text{ aff}_{\mathbb{R}} \oplus \mathfrak{h}$ for some $2(n-r)$ -dimensional nilpotent Lie algebra $\mathfrak{h}$.

Remark 5.6 Observe that by Theorem 5.5 and in stark contrast to the Lie algebras considered in Sect. 4, a unimodular two-step solvable Lie algebra with totally real commutator ideal is automatically nilpotent. Moreover, we then have $[\mathfrak{g}', \mathfrak{g}] = 0$, i.e. $\mathfrak{g}$ is two-step nilpotent. This confirms the conjecture by Enrietti, Fino and Vezzoni [7, 13] that any nilpotent SKT Lie algebra is two-step nilpotent in the particular case of a nilpotent, two-step solvable SKT Lie algebra with totally real commutator ideal.

If $m = r$ in Theorem 5.5, then $\tilde{v}$ does not occur anymore and we have the following result.

Corollary 5.7 Let $(\mathfrak{g}, g, J)$ be a $2n$ -dimensional almost Hermitian two-step solvable Lie algebra with totally real m -dimensional commutator ideal $\mathfrak{g}'$ such that $[\mathfrak{g}', J\mathfrak{g}'] = \mathfrak{g}'$.

Then $(\mathfrak{g}, g, J)$ is SKT if and only if there exists an orthonormal basis $(X_1, \dots, X_m)$ of $\mathfrak{g}'$, $\lambda_1, \dots, \lambda_m \in \mathbb{R} \setminus \{0\}$, $\mu_1, \dots, \mu_m \in U_J^*$ with $U_J := (\mathfrak{g}' \oplus J\mathfrak{g}')^\perp$, such that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[JX_i, X_i] = \lambda_i X_i, \quad [Y, X_i] = \mu_i(Y)X_i, \quad [Y, JX_i] = -\mu_i(JY)X_i, \\ [Y, Z] = \sum_{i=1}^m \frac{1}{\lambda_i} (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i,$$

for $i = 1, \dots, m$, $Y, Z \in U_J$. If this is the case, we have $\mathfrak{g} \cong m \text{ aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-2m}$.

For $n = m$ in Corollary 5.7 we get:

Corollary 5.8 A $2n$ -dimensional almost Hermitian two-step solvable Lie algebra $(\mathfrak{g}, g, J)$ with totally real commutator ideal $\mathfrak{g}'$ of dimension n is SKT if and only if there exists an orthonormal basis $(X_1, \dots, X_n)$ of $\mathfrak{g}'$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R} \setminus \{0\}$ such that the only non-zero Lie brackets (up to anti-symmetry) are given by $[JX_i, X_i] = \lambda_i X_i$ for $i = 1, \dots, n$

Moreover, we then have $\mathfrak{g} \cong n \text{ aff}_{\mathbb{R}}$.

For $r < m$, we still need to solve $\sum_{i=r+1}^m \tilde{v}_i \wedge \tilde{v}_i = 0$ and $\text{im}(\tilde{v}) = \text{span}(X_{r+1}, \dots, X_m)$ in Theorem 5.5. This is not too difficult for $r \in \{m-2, m-1\}$, and so we will do that in the following. For smaller r , solving these equations seems to be more involved, cf. Remark 5.11.

Let us start with $r = m-1$. Then $\tilde{v}_m \wedge \tilde{v}_m = 0$, $\text{im}(\tilde{v}) = \text{span}(X_m)$ and $\tilde{v}_m$ being a $(1, 1)$ -form implies $\tilde{v}_m = \alpha \wedge J^* \alpha$ for some $\alpha \in U_J^* \setminus \{0\}$.

Next, consider the case $r = m-2$. Then we need to solve the equation $\tilde{v}_{m-1}^2 + \tilde{v}_m^2 = 0$, which is done by the next lemma.

Lemma 5.9 *Let V be a $2n$ -dimensional vector space with complex structure J and let real $v_1, v_2 \in \Lambda^{1,1}V^*$ be given with $v_1^2 + v_2^2 = 0$. Then, up to a rotation in $\text{span}(v_1, v_2)$, there are $\alpha, \beta \in V^*$ such that $v_1 = \alpha \wedge J^*\alpha$ and $v_2 = \beta \wedge J^*\beta$.*

Proof First of all, we show that if $v_1^2 + v_2^2 = 0$ implies then both v_1 and v_2 has rank at most four. For this, assume contrary, without loss of generality, that v_1 has rank at least six. Then we may find a decomposition $V = V_1 \oplus V_2$ with $\dim(V_1) = 4$ such that $v_1 = \omega_1 + \tau_1$ with $\omega_1 \in \Lambda^2 V_1^*$ being of rank four and $0 \neq \tau_1 \in \Lambda^2 V_2^*$. We may decompose $\tilde{\mu}_2 = \omega_2 + \rho_2 + \tau_2$ with $\omega_2 \in \Lambda^2 V_1^*$, $\rho_2 \in V_1^* \wedge V_2^*$ and $\tau_2 \in \Lambda^2 V_2^*$. Looking at the $\Lambda^4 V_1^*$ -component and the $\Lambda^3 V_1^* \wedge V_2^*$ -component of $v_1^2 + v_2^2 = 0$, we obtain

$$\omega_1^2 + \omega_2^2 = 0, \quad \omega_2 \wedge \rho_2 = 0$$

Due to $\omega_1^2 \neq 0$, we must have $\omega_2 \neq 0$ and so the second equation implies $\rho_2 = 0$. But then the $\Lambda^2 V_1^* \wedge \Lambda^2 V_2^*$ -component of $v_1^2 + v_2^2 = 0$ gives us

$$\omega_1 \wedge \tau_1 + \omega_2 \wedge \tau_2 = 0.$$

As $\omega_2^2 = -\omega_1^2$, the two-forms ω_1, ω_2 cannot be linearly dependent and so we must have $\tau_1 = \tau_2 = 0$, a contradiction. Hence, both v_1 and v_2 have rank at most four. If they both have rank four, then they must have the same kernel since if X is in the kernel of say v_1 , then $0 = X \lrcorner (v_1^2 + v_2^2) = 2(X \lrcorner v_2) \wedge v_2$, which forces $X \lrcorner v_2 = 0$ as well. Moreover, it cannot be that exactly one of the two-form ω_2, ω_1 has rank four so the only other option is that both have rank two. So in this case, we have $v_1 = \alpha \wedge J^*\alpha$ and $v_2 = \beta \wedge J^*\beta$ for $\alpha, \beta \in U_J^*$.

Let us now come back to the case that both v_1 and v_2 have rank four, and so have common support. Setting $\Omega := v_1^2$, we have $v_2^2 = -\Omega$ and $v_1 \wedge v_2 = a\Omega$ for some $a \in \mathbb{R}$. As the matrix $\begin{pmatrix} 1 & a \\ a & -1 \end{pmatrix}$ is symmetric, we may apply an orthogonal transformation to $\text{span}(X_1, X_2)$ such that with respect to the new basis we get $a = 0$. But then $(\frac{1}{\sqrt{2}}v_1 \pm \frac{1}{\sqrt{2}}v_2)^2 = 0$ and so by applying another orthogonal transformation to $\text{span}(v_1, v_2)$, we may also here assume that both v_1 and v_2 have rank two. Hence, we always can rotate the pair (v_1, v_2) to get $v_1 = \alpha \wedge J^*\alpha$ and $v_2 = \beta \wedge J^*\beta$ for certain $\alpha, \beta \in V^*$ $\square$

Coming back to our case, we thus have $\tilde{v}_{m-1} = \alpha \wedge J^*\alpha$ and $\tilde{v}_m = \beta \wedge J^*\beta$ for certain $\alpha, \beta \in U_J^*$. Additionally, we know that we must have $\text{im}(v) = \text{span}(X_{m-1}, X_m)$. But so $\tilde{v}_{m-1}, \tilde{v}_m$ have to be $\mathbb{R}$ -linearly independent, which implies that α, β are $\mathbb{C}$ -linearly independent. Hence, we have obtained:

Theorem 5.10 *Let $(\mathfrak{g}, g, J)$ be a $2n$ -dimensional almost Hermitian two-step solvable Lie algebra with totally real m -dimensional commutator ideal $\mathfrak{g}'$ such that $[\mathfrak{g}', J\mathfrak{g}']$ has codimension $\ell \in \{1, 2\}$ in $\mathfrak{g}'$. Moreover, set $U_J := (\mathfrak{g}' \oplus J\mathfrak{g}')^\perp$.*

- (a) *If $\ell = 1$, then $(\mathfrak{g}, g, J)$ is SKT if and only if there exists an orthonormal basis $(X_1, \dots, X_m)$ of $\mathfrak{g}'$, $\lambda_1, \dots, \lambda_{m-1} \in \mathbb{R} \setminus \{0\}$, $\mu_1, \dots, \mu_{m-1}, \alpha \in U_J^*$ with $\alpha \neq 0$ such that the only non-zero Lie brackets (up to anti-symmetry) are given by*

$$\begin{aligned} [JX_i, X_i] &= \lambda_i X_i, \quad [Y, X_i] = \mu_i(Y)X_i, \quad [Y, JX_i] = -\mu_i(JY)X_i, \\ [Y, Z] &= \sum_{i=1}^{m-1} \frac{1}{\lambda_i} (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i + (\alpha \wedge J^*\alpha)(Y, Z)X_m, \end{aligned}$$

for $i = 1, \dots, m-1$, $Y, Z \in U_J$. In this case, we have $\mathfrak{g} \cong (m-1)\text{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}^{2n-2m-1}$.

- (b) If $\ell = 2$, then $(\mathfrak{g}, g, J)$ is SKT if and only if there exists an orthonormal basis $(X_1, \dots, X_m)$ of $\mathfrak{g}'$, $\lambda_1, \dots, \lambda_{m-2} \in \mathbb{R} \setminus \{0\}$, $\mu_1, \dots, \mu_{m-2}, \alpha_1, \alpha_2 \in U_J^*$ with α_1, α_2 being $\mathbb{C}$ -linearly independent such that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[JX_i, X_i] = \lambda_i X_i, [Y, X_i] = \mu_i(Y)X_i, [Y, X_i] = -\mu_i(JY)X_i,$$

$$[Y, Z] = \sum_{i=1}^{m-2} \frac{1}{\lambda_i} (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i + \sum_{i=1}^2 (\alpha_i \wedge J^* \alpha_i)(Y, Z)X_{m-2+i},$$

for $i = 1, \dots, m-2$, $Y, Z \in U_J$. In this case, we have $\mathfrak{g} \cong (m-2) \operatorname{aff}_{\mathbb{R}} \oplus 2\mathfrak{h}_3 \oplus \mathbb{R}^{2n-2m-2}$.

Remark 5.11 Note that for higher codimensions ℓ it is not clear whether Lemma 5.9 generalizes in the natural way to more than two $(1, 1)$ -forms v_i . So one is left with the description from Theorem 5.5 for these cases. Note further that even if all v_i are decomposable, still the nilpotent Lie algebra $\mathfrak{h}$ from Theorem 5.5 is, in general, not equal to a sum of the form $r \operatorname{aff}_{\mathbb{R}} \oplus \ell \mathfrak{h}_3 \oplus \mathbb{R}^{2(n-r)-3s}$ as in the case of $\ell \in \{1, 2\}$.

For example, if $\ell = 3$, take $\mathbb{C}$ -linearly independent $\alpha_1, \alpha_2 \in U_J$ and set $v_{m-2} := \alpha_1 \wedge J^* \alpha_1$, $v_{m-1} := \alpha_2 \wedge J^* \alpha_2$, $v_m := (\alpha_1 + \alpha_2) \wedge J^*(\alpha_1 + \alpha_2)$. Then $\mathfrak{h} \cong r \operatorname{aff}_{\mathbb{R}} \oplus (37D) \oplus \mathbb{R}^{2(n-r)-7}$ with the seven-dimensional nilpotent Lie algebra $37D$ from [16].

Now all Lie algebras up to dimension two are solvable and the only nilpotent Lie algebras up to this dimension are the Abelian ones. As the commutator ideal $\mathfrak{g}'$ of a solvable Lie algebra is nilpotent, one sees that any Lie algebra with one-dimensional or two-dimensional commutator ideal is two-step solvable. Now surely if the dimension of $\mathfrak{g}'$ is one, $\mathfrak{g}'$ is totally real and the codimension of $[\mathfrak{g}', J\mathfrak{g}']$ in $\mathfrak{g}'$ is at most one. Similarly, if the dimension of $\mathfrak{g}'$ is two and $\mathfrak{g}'$ is totally real, then the codimension of $[\mathfrak{g}', J\mathfrak{g}']$ in $\mathfrak{g}'$ is at most two. Hence, Corollary 5.7 and Theorem 5.10 imply

Corollary 5.12 Let $(\mathfrak{g}, g, J)$ be a $2n$ -dimensional almost Hermitian Lie algebra.

- (a) If $\dim(\mathfrak{g}') = 1$, then $(\mathfrak{g}, g, J)$ is an SKT Lie algebra if and only if there exists $0 \neq X \in \mathfrak{g}'$ of norm one such that

- (i) there exists $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mu \in U_J^*$ so that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$= \lambda X, [Y, X] = \mu(Y)X, [Y, JX] = -\mu(JY)X \quad \text{for } Y \in U_J,$$

$$[Y, Z] = \frac{1}{\lambda} (\mu \wedge \mu \circ J)(Y, Z)X \quad \text{for } Y, Z \in U_J$$

or

- (ii) there exists $\alpha \in U_J^* \setminus \{0\}$ so that the only non-zero Lie bracket is given by

$$[Y, Z] = (\alpha \wedge J^* \alpha)(Y, Z)X$$

for $Y, Z \in U_J$.

In the first case, we have $\mathfrak{g} \cong \operatorname{aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-2}$ whereas in the second case we have $\mathfrak{g} \cong \mathfrak{h}_3 \oplus \mathbb{R}^{2n-3}$.

- (b) If $\dim(\mathfrak{g}') = 2$ and $\mathfrak{g}'$ is totally real, then $(\mathfrak{g}, g, J)$ is an SKT Lie algebra if and only if there exists an orthonormal basis (X_1, X_2) of $\mathfrak{g}'$ such that

- (i) there exist $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\mu_1, \mu_2 \in U_J^*$ so that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[JX_i, X_i] = \lambda X, [Y, X_i] = \mu(Y)X_i, [Y, JX_i] = -\mu(JY)X_i \text{ for } i = 1, 2, Y \in U_J,$$

$$[Y, Z] = \sum_{i=1}^2 \frac{1}{\lambda_i} (\mu_i \wedge \mu_i \circ J)(Y, Z)X_i \text{ for } Y, Z \in U_J,$$

- (ii) there exist $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mu, \alpha \in U_J^*, \alpha \neq 0$ so that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[JX_1, X_1] = \lambda X_1, [Y, X_1] = \mu(Y)X_1, [Y, JX_1] = -\mu(JY)X_1 \text{ for } Y \in U_J,$$

$$[Y, Z] = \frac{1}{\lambda} (\mu \wedge \mu \circ J)(Y, Z)X_1 + (\alpha \wedge J^*\alpha)(Y, Z)X_2 \text{ for } Y, Z \in U_J$$

or

- (iii) there exist $\mathbb{C}$ -linearly independent $\alpha, \beta \in U_J^*$ so that the only non-zero Lie bracket is given by

$$[Y, Z] = (\alpha \wedge J^*\alpha)(Y, Z)X_1 + (\beta \wedge J^*\beta)(Y, Z)X_2$$

for $Y, Z \in U_J$.

In the first case, we have $\mathfrak{g} \cong 2\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-4}$, in the second case, we have $\mathfrak{g} \cong \mathfrak{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}^{2n-5}$ and in the third case, we have $\mathfrak{g} \cong 2\mathfrak{h}_3 \oplus \mathbb{R}^{2n-6}$.

Remark 5.13 Note that by [1, Proposition 2.2], a $2n$ -dimensional Lie algebras $\mathfrak{g}$ with one-dimensional commutator ideal is either isomorphic to $\mathfrak{h}_{2k+1} \oplus \mathbb{R}^{2(n-k)-1}$ or $\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^{2n-2}$ for some $k \in \mathbb{R}$ and that all these Lie algebras admit an Abelian complex structure J . So we have shown that on $\mathfrak{h}_{2k+1} \oplus \mathbb{R}^{2(n-k)-1}$ with $k \geq 2$ any of these Abelian complex structures cannot come from an SKT structure on $\mathfrak{g}$.

Corollary 5.14 A six-dimensional two-step solvable Lie algebra $\mathfrak{g}$ admits an SKT structure with totally real $\mathfrak{g}'$ if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:

$$\mathbb{R}^6, \mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}^4, \mathfrak{h}_3 \oplus \mathbb{R}^3, 2\mathfrak{aff}_{\mathbb{R}} \oplus \mathbb{R}, \mathfrak{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}, \mathfrak{h}_3 \oplus \mathfrak{h}_3, 3\mathfrak{aff}_{\mathbb{R}}.$$

Remark 5.15 In [1], all six-dimensional Lie algebras with an Abelian complex structure have been classified. All these Lie algebra are two-step solvable since this is true in general for Lie algebras endowed with an Abelian complex structure. There are in total 8 (including the Abelian Lie algebra $\mathbb{R}^6$) six-dimensional Lie algebras and 12 single and 1 two-parameter family of six-dimensional non-nilpotent Lie algebras admitting an Abelian complex structure. By our results, 3 out of these 8 nilpotent and 4 out of the 12 single non-solvable admits Abelian complex structures coming from an SKT structure with totally real commutator ideal. Note that such an Abelian complex structure does not exist on any of the non-nilpotent Lie algebras in the two-parameter family.

6 Commutator ideal complex of real-dimension two

Here, we consider two-step SKT shear data (ω, α) with α being complex and of dimension two. Note that we then have $\alpha = \alpha_J, \alpha_r = 0, U_r = J\alpha_r = 0$ and so $U = U_J$.

We first show that $\omega(Z, \cdot)|_{\alpha}$ acts on α by multiplication with an imaginary number. The proof will use a useful interpretation of the second part of Lemma 3.3(viii) in dimension 2.

Lemma 6.1 Suppose $V = \mathbb{R}^2$, with the standard complex structure J . If $A_1, A_2 \in \mathbb{R}^{2 \times 2}$ satisfy (a) $[A_1, A_2] = 0$ and (b) $[A_2, J] = J[A_1, J]$, then both A_1 and A_2 commute with J .

Proof Decompose $A_i = A_i^J + A_i^{J-}$ with $JA_i^J = A_i^J J$ and $JA_i^{J-} = -A_i^{J-} J$. Then $\dim_{\mathbb{R}} V = 2$, implies $[A_1^J, A_2^J] = 0$, so (a) implies the J -anti-invariant parts A_i^{J-} commute. These parts are symmetric matrices, so can be simultaneously diagonalised. However, (b) gives $2A_2^{J-}J = [A_2, J] = J[A_1, J] = 2A_1^{J-}$, which for diagonal matrices implies $A_i^{J-} = 0$. $\square$

Lemma 6.2 Let $(\omega, \mathfrak{a})$ be two-step SKT shear data with $\mathfrak{a}$ complex and of dimension two. Then there exists $\alpha \in U^*$ such that $\omega(Z, X) = \alpha(Z)JX$ for any $Z \in U, X \in \mathfrak{a}$.

Proof Fix $Z \in U$ of norm one and set $A_1 := \omega(Z, \cdot)|_{\mathfrak{a}} \in \text{End}(\mathfrak{a})$, $A_2 := \omega(JZ, \cdot)|_{\mathfrak{a}} \in \text{End}(\mathfrak{a})$. Inserting (Z, JZ, X_1, X_2) , for $X_1, X_2 \in \mathfrak{a}$, into (3.2), we obtain

$$0 = JA_2^t A_2 + A_2^t A_2 J + JA_1^t A_1 + A_1^t A_1 J + A_1^t J A_1^J + A_1 J A_1 + A_2^t J A_2^J + A_2 J A_2^J \quad (6.1)$$

Equation (2.1) implies $[A_1, A_2] = 0$ and Lemma 3.3(viii) gives $[A_2, J] = J[A_1, J]$, so by Lemma 6.1 both A_i commute with J . Considering $\mathfrak{a}$ as a complex one-dimensional space, we identify A_i with $z_i \in \mathbb{C}$. As J becomes multiplication by i and transpose is complex conjugation, Eq. (6.1) is equivalent to

$$0 = |z_1|^2 + |z_2|^2 + z_1^2 + z_2^2 = \text{Re}(z_1)^2 + \text{Re}(z_2)^2,$$

so $\text{Re}(z_1) = 0 = \text{Re}(z_2)$, giving the claimed statement. $\square$

Remark 6.3 Let $(\mathfrak{a}, \omega)$ be pre-shear data with $\mathfrak{a}$ being complex and of dimension two such that there is some $\alpha \in U^*$ with $\omega(Z, X) = \alpha(Z)JX$ for any $Z \in U, Y \in \mathfrak{a}$. Then (2.1) is fulfilled on $\mathfrak{a} \wedge \Lambda^2 U$, (3.1) is satisfied on $\mathfrak{a} \wedge U$ and (3.2) is fulfilled on $\Lambda^2 \mathfrak{a} \wedge \Lambda^2 U$.

Next, we are going to show that if $\alpha \neq 0$, then $v_i = \lambda_i \alpha \wedge J^* \alpha$ for some $\lambda_i \in \mathbb{R}, i = 1, 2$, where $v_1, v_2 \in \Lambda^2 U^*$ are uniquely defined by $v = \sum_{i=1}^2 v_i X_i$.

First of all, (2.1) is valid on $\Lambda^3 U$ if and only if $\alpha \wedge v_1 = 0, \alpha \wedge v_2 = 0$. Next, (3.2) on $\mathfrak{a} \wedge \Lambda^3 U$ gives us

$$\begin{aligned} 0 &= J^* v_2 \wedge \alpha + J^* \alpha \wedge v_2 + \alpha \wedge v_1 = J^* v_2 \wedge \alpha + J^* \alpha \wedge v_2, \\ 0 &= J^* v_1 \wedge \alpha + J^* \alpha \wedge v_1 + \alpha \wedge v_1 = J^* v_1 \wedge \alpha + J^* \alpha \wedge v_1. \end{aligned}$$

Now we may write $v_i = \alpha \wedge \beta_i$ for some $\beta_i \in U^*, i = 1, 2$, and then the just obtained equations give us $\alpha \wedge J^* \alpha \wedge \beta_i = 0$ for all $i = 1, 2$. Hence, if $\alpha \neq 0$, then $\beta_i \in \text{span}(\alpha, J^* \alpha)$, i.e. $v_i = \lambda_i \alpha \wedge J^* \alpha$ for some $\lambda_i \in \mathbb{R}, i = 1, 2$. In this case, one easily checks that then (2.1)–(3.2) are fulfilled and $\mathfrak{g} \cong \mathfrak{r}_{3,0}' \oplus \mathbb{R}^{2n-3}$.

Otherwise, $\alpha = 0$ and so $\mathfrak{g}$ is nilpotent. Then (3.1) evaluated on $\Lambda^2 U$ gives us $v_1 = \tau_1 + \text{Re}(\theta), v_2 = \tau_2 - \text{Im}(\theta)$ for certain $\tau_1, \tau_2 \in \Lambda^{1,1} U^*, \theta \in \Lambda^{2,0} U^*$. Then, we still have to solve (3.2) on $\Lambda^4 U^*$, which gives us

$$0 = J^* v_1 \wedge v_1 + J^* v_2 \wedge v_2 = (\tau_1 - \text{Re}(\theta)) \wedge (\tau_1 + \text{Re}(\theta)) + (\tau_2 - \text{Im}(\theta)) \wedge (\tau_2 + \text{Im}(\theta)),$$

which is equivalent to

$$\tau_1 \wedge \tau_1 + \tau_2 \wedge \tau_2 = \text{Re}(\theta) \wedge \text{Re}(\theta) + \text{Im}(\theta) \wedge \text{Im}(\theta) = \theta \wedge \bar{\theta}.$$

We will now show the following

Lemma 6.4 *Let (V, J) be a vector space with a complex structure. Suppose that $\omega \in \Lambda^{2,0}V$ and real $\sigma_i \in \Lambda^{1,1}V$ satisfy*

$$\sigma_1^2 + \sigma_2^2 = \omega \wedge \bar{\omega}. \quad (6.2)$$

Then the complex rank of ω is at most two and the common kernel of $\sigma_1, \sigma_2, \omega$ has codimension at most six.

Proof If $\omega = 0$, the result follows from Lemma 5.9.

Otherwise, there is a decomposition $V = U \oplus W$ with $\dim(U) = 4$ such that $\omega = \alpha \wedge \beta + \omega'$ with $\alpha, \beta \in \Lambda^{1,0}U^*$ $\mathbb{C}$ -linearly independent and $\omega' \in \Lambda^{2,0}W^*$. We then have $\Lambda^2 V^* = \Lambda^2 U^* + U^* \wedge W^* + \Lambda^2 W^*$ and correspondingly write elements of $\Lambda^2 V^*$ as $\gamma = \gamma^U + \gamma^m + \gamma^W$.

We first show that $\omega' = 0$, i.e. that ω has complex rank two. For this, put $a = \operatorname{Re} \alpha$, $b = \operatorname{Re} \beta$. We may use the action of $\operatorname{SL}(2, \mathbb{C})$ as $\operatorname{SO}_0(1, 3)$ on $\Lambda^{1,1}(\mathbb{C}^2)^*$ to get that $\sigma_j = p(a \wedge Ja + b \wedge Jb)$ for some $p \in \mathbb{R}$ and then may use the remaining $\operatorname{SU}(2)$ -action on $\Lambda_-^2 U^*$ to get

$$\sigma_2^U = qa \wedge Ja + rb \wedge Jb,$$

for some $q, r \in \mathbb{R}$. As $(\sigma_1^U)^2 + (\sigma_2^U)^2 = 4a \wedge Ja + b \wedge Jb$, we get $p^2 + qr = 0$. Thus $(p, q) \neq (0, 0)$ and we may rotate (σ_1, σ_2) in such a way that $\sigma_1 = p a \wedge Ja + q b \wedge Jb$ and $\sigma_2 = r b \wedge Jb$ for certain $p, q, r \in \mathbb{R}$ with $p > 0$.

We may now write $\sigma_j^m = a \wedge x_j + Ja \wedge Jx_j + b \wedge y_j + Jb \wedge Jy_j$ for some $x_j, y_j \in W$. As $\omega \wedge \bar{\omega}$ has no component in $\Lambda^3 U^* \wedge W^*$, we get $\sigma_1^U \wedge \sigma_1^m + \sigma_2^U \wedge \sigma_2^m = 0$. The $a \wedge Ja \wedge b$ term in this equation gives $py_1 = 0$, so $y_1 = 0$.

Now the contribution to $\Lambda^2 U^* \wedge \Lambda^2 W^*$ is $\sum_{j=1}^2 (2\sigma_j^U \wedge \sigma_j^W + (\sigma_j^m)^2) = (a - iJa) \wedge (b - iJb) \wedge \bar{\omega}' + \omega' \wedge (a + iJa) \wedge (b + iJb)$. Consider the terms multiplying $a \wedge b$. There is no contribution from σ_j^U nor σ_1^m , so we get $-x_2 \wedge y_2 = \operatorname{Re}(\omega')$. Next, consider the terms multiplying $a \wedge b$. In this case, there is again no contribution from σ_j^U and σ_1^m and we have $x_2 \wedge Jy_2 = \operatorname{Im}(\omega')$. So both $\operatorname{Re}(\omega')$ and $\operatorname{Im}(\omega')$ have rank two and, consequently, $\omega' \wedge \bar{\omega}' = 0$. Thus $\omega' = 0$ and ω has rank two. Moreover, $x_2 = 0$ or $y_2 = 0$. We look again at the equation $\sigma_1^U \wedge \sigma_1^m + \sigma_2^U \wedge \sigma_2^m = 0$. For $x_2 = 0$, the $b \wedge Jb \wedge a$ -term gives us $qx_1 = 0$ and so, since $q \neq 0$, that $x_1 = 0$ as well. For $y_2 = 0$, the same term gives us $qx_1 + rx_2 = 0$, i.e. $x_2 = -\frac{r}{q}x_1$.

Next, looking at the $\Lambda^4 U^*$ -terms, we have $(\sigma_1^W)^2 + (\sigma_2^W)^2 = 0$ and so Lemma 5.9 shows the existence of $c, e \in W^*$ such that $\sigma_1^W = c \wedge Jc$ and $\sigma_2^W = e \wedge Je$. Next, look at the $U^* \wedge \Lambda^3 W^*$ -terms, i.e. at the equation $\sigma_1^m \wedge \sigma_1^W + \sigma_2^m \wedge \sigma_2^W = 0$.

If $x_2 = 0$, then we saw above that $\sigma_1^m = 0$ and so we get $y_1 \wedge e \wedge Je = 0$, i.e. y_1, e are $\mathbb{C}$ -linearly dependent. Then the $a \wedge Ja$ -term in $\sum_{j=1}^2 (2\sigma_j^U \wedge \sigma_j^W + (\sigma_j^m)^2) = 0$ gives us $pc \wedge Jc = 0$, i.e. $c = 0$. Altogether, we see that the common kernel of σ_1, σ_2 in W is contained in the kernel of y_1, Jy_1, e, Je and this kernel has codimension at most two since y_1, e are $\mathbb{C}$ -linearly dependent.

Finally, let $y_2 = 0$. Then $x_2 = -\frac{r}{q}x_1$ and the equation $\sigma_1^m \wedge \sigma_1^W + \sigma_2^m \wedge \sigma_2^W = 0$ gives us $x_1 \wedge (c \wedge Jc - \frac{r}{q}e \wedge Je) = 0$. So either $x_1 = 0$ or c, e are $\mathbb{C}$ -linearly dependent. with $c \wedge Jc = \frac{r}{q}e \wedge Je$.

In the first case $x_2 = 0$ as well, i.e. we have $\sigma_1^m = 0, \sigma_2^m = 0$, and we get again from $\sum_{j=1}^2 2\sigma_j^U \wedge \sigma_j^W = \sum_{j=1}^2 (2\sigma_j^U \wedge \sigma_j^W + (\sigma_j^m)^2) = 0$ that $c = 0$ and the statement follows. Otherwise, c, e are $\mathbb{C}$ -linearly dependent with $c \wedge Jc = \frac{r}{q}e \wedge Je$ and the equation

$\sum_{j=1}^2 (2\sigma_j^U \wedge \sigma_j^W + (\sigma_j^m)^2) = 0$ is equivalent to $pc \wedge Jc - 2(1 + \frac{r^2}{q^2})x_1 \wedge Jx_1 = 0$. This shows that the span of c, Jc, e, Je, x_1, Jx_1 is at most two-dimensional and the statement follows. $\square$

The last lemma implies that the J -invariant kernel of $\tau_1, \tau_2 \in \Lambda^{1,1}U^*, \theta \in \Lambda^{2,0}U^*$ as above, i.e. satisfying

$$\tau_1 \wedge \tau_1 + \tau_2 \wedge \tau_2 = \theta \wedge \bar{\theta},$$

is of codimension at most 6 in U and this kernel corresponds to an Abelian SKT Lie algebra. Thus, we have obtained:

Theorem 6.5 *Let $(\mathfrak{g}, g, J)$ be an $2n$ -dimensional almost Hermitian Lie algebra such that $\mathfrak{g}'$ is two-dimensional and J -invariant. Then $(\mathfrak{g}, g, J)$ is SKT if and only if*

- (i) *there exists $\alpha \in U^* \setminus \{0\}$ such that the only non-zero Lie brackets (up to anti-symmetry) are given by*

$$[Y, X] = \alpha(Y)JX$$

for $X \in \mathfrak{g}', Y \in \mathfrak{g}'^\perp$ or

- (ii) *there exists an orthonormal basis (X, JX) of $\mathfrak{g}'$, $\tau_1, \tau_2 \in \Lambda^{1,1}(\mathfrak{g}'^\perp)^*$ and $\theta \in \Lambda^{2,0}(\mathfrak{g}'^\perp)^*$ with $\tau_1 \neq 0, \tau_2 \neq 0$ or $\theta \neq 0$ and*

$$\tau_1 \wedge \tau_1 + \tau_2 \wedge \tau_2 = \theta \wedge \bar{\theta}.$$

such that the only non-zero Lie brackets (up to anti-symmetry) are given by

$$[Y, Z] = (\tau_1(Y, Z) + \operatorname{Re}(\theta)(Y, Z))X + (\tau_2(Y, Z) - \operatorname{Im}(\theta)(Y, Z))JX$$

for $Y, Z \in \mathfrak{g}'^\perp$.

Moreover, in case (i), we have $\mathfrak{g} \cong \mathfrak{r}'_{3,0} \oplus \mathbb{R}^{2n-3}$, whereas in case (ii) $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}^{2n-2k}$ for a $2k$ -dimensional two-step nilpotent Lie algebra $\mathfrak{h}$ with $k \in \{3, 4\}$ and the splitting respects the SKT structure.

Remark 6.6 All algebras in Theorem 6.5 are unimodular; only those in case (ii) are nilpotent. All these cases may be explicitly described thanks to the classifications of nilpotent SKT algebras in dimensions 6 and 8 given in [11] and [7].

7 Six-dimensional two-step solvable SKT Lie algebras

In this section, we work towards classifying six-dimensional two-step solvable SKT Lie algebras $(\mathfrak{g}, g, J)$.

For this, let us first summarize what our previous results imply for this situation. First of all, Corollary 5.14 gives us a complete list in the cases when $\mathfrak{g}'$ is totally real. Next, Theorem 6.5 describes all the cases with $\mathfrak{g}'$ two-dimensional and complex, and Theorem 4.10, we determines cases with $\mathfrak{g}'$ four-dimensional but non-complex. Finally, Corollary 4.8 contains a classification of the cases where $\mathfrak{g}'$ has dimension 5.

So, in order to classify all six-dimensional two-step solvable SKT Lie algebras, we still have to determine those with three-dimensional non-totally real $\mathfrak{g}'$ and those with four-dimensional complex $\mathfrak{g}'$. Unfortunately, it seems to be too hard to solve the latter case

completely and we will just determine the relevant equations in Sect. 7.2 below. In contrast, the first case is treated completely in the next subsection and we note already now the following classification, that includes results from [11] for the nilpotent case and uses some of their notation.

Theorem 7.1 *Let $(\mathfrak{g}, g, J)$ be a six-dimensional two-step solvable SKT Lie algebra. Then either $\mathfrak{g}$ is isomorphic as a Lie algebra to one of the following Lie algebras*

$$\mathbb{R}^6, \quad \text{aff}_{\mathbb{R}} \oplus \mathbb{R}^4, \quad \text{aff}_{\mathbb{R}} \oplus \mathfrak{h}_3 \oplus \mathbb{R}, \quad 2\text{aff}_{\mathbb{R}} \oplus \mathbb{R}^2, \quad 3\text{aff}_{\mathbb{R}}, \quad \mathfrak{h}_3 \oplus \mathbb{R}^3, \quad 2\mathfrak{h}_3, \\ \mathfrak{t}'_{3,0} \oplus \mathbb{R}^3, \quad (0, 0, 0, 0, 12, 14 + 23), \quad (0, 0, 0, 0, 13 + 42, 14 + 23),$$

or an algebra in Corollary 4.8, or $(\mathfrak{g}, g, J)$ is isomorphic to one of the six-dimensional SKT Lie algebras in Theorem 4.10 or Theorem 7.5, or $\mathfrak{g}'$ is four-dimensional, J -invariant, and, for a unit vector $X \in (\mathfrak{g}')^\perp$, the endomorphisms $A_1 := \text{ad}(X)|_{\mathfrak{g}'}$ and $A_2 := \text{ad}(JX)|_{\mathfrak{g}'}$ satisfy (7.10).

Remark 7.2 Note that Theorem 7.1 implies the four-dimensional two-step Lie algebras admitting an SKT structure are, in the notation of [18],

$$\mathbb{R}^4, \quad \text{aff}_{\mathbb{R}} \oplus \mathbb{R}^2, \quad 2\text{aff}_{\mathbb{R}}, \quad \mathfrak{h}_3 \oplus \mathbb{R}, \quad \mathfrak{t}'_{3,0} \oplus \mathbb{R}, \quad \mathfrak{t}'_{4,\lambda,0}, \quad \mathfrak{t}_{4,-1/2,-1/2}, \quad \mathfrak{t}'_{4,2\lambda,-\lambda}.$$

agreeing with the results of that paper.

7.1 Three-dimensional commutator ideal not totally real

Here, we consider two-step SKT shear data $(\mathfrak{a}, \omega)$ with $\dim(\mathfrak{a}) = 3$ and $\dim(\mathfrak{a}_r) = 1$. Then $\mathfrak{a}_J$ and U_J are real two-dimensional, and U_r is real one-dimensional. Take orthonormal bases (Y, JY) of $\mathfrak{a}_J$, X of $\mathfrak{a}_r$ and (Z, JZ) of U_J . Then JX is a basis of U_r and (Y, JY, X) is an orthonormal basis of $\mathfrak{a}$. Setting $A := -\omega_0(JX, \cdot) \in \text{End}(\mathfrak{a})$, $B_1 := -\omega_0(Z, \cdot) \in \text{End}(\mathfrak{a})$ and $B_2 := \omega_0(JZ, \cdot) \in \text{End}(\mathfrak{a})$, Lemma 3.3(i) and (ii), give us with respect to the splitting $\mathfrak{a} = \mathfrak{a}_J \oplus \mathfrak{a}_r$

$$A = \begin{pmatrix} z & u \\ 0 & a \end{pmatrix}$$

for some $a \in \mathbb{R}$, $z, u \in \mathbb{C}$. Replacing X by $-X$ if necessary we may take $a \geq 0$. Equation (3.6) of Lemma 3.5, reduces to (4.6), and we conclude that $\text{Re}(z) \in \{0, -a/2\}$.

We now consider individually the cases $A = 0$ and $A \neq 0$.

Lemma 7.3 *With the above notation, if $A = 0$, then the basis (Z, JZ) of U_J may be rotated so that $\omega(X, \cdot) = \omega(JX, \cdot) = 0$, $\omega(Z, Y) = 0$, $\omega(JZ, Y) = ibY$, $b \in \mathbb{R}_{>0}$, and $\omega(Z, JZ) = qY + hX$, $h \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{C}$.*

Conversely, pre-shear data of this form is two-step SKT shear data with $\text{im}(\omega) = \mathfrak{a}$ and the obtained Lie algebra is almost Abelian and isomorphic to $\mathfrak{g}_{5,14}^0 \oplus \mathbb{R}$.

Proof Lemma 3.3(iv) and $a = 0$ give $\omega(X, JX) = 0$ and so

$$6 \nu_1(\omega)(X, JX, Z, JZ) \\ = -\|\omega(JX, JZ)\|^2 - \|\omega(JX, Z)\|^2 - \|\omega(X, JZ)\|^2 - \|\omega(X, Z)\|^2. \quad (7.1)$$

Moreover, in the current situation $\mathfrak{a} \perp U$, so we get

$$12 \nu_2(\omega)(X, JX, Z, JZ) = g(\omega(J\omega(JX, Z), Z), X) + g(\omega(J\omega(JX, JZ), JZ), X).$$

Now $\omega^r(J\omega(JX, JZ), JZ) = \omega^r(\omega(JX, JZ), Z)$ by (3.1) since the other terms in this equation land in U_r . Moreover, $\omega^r(\omega(JX, JZ), Z) = \omega^r(\omega(JX, Z), JZ)$ by (2.1) as $\omega(JX, \cdot)|_{\mathfrak{a}} = 0$. Then, $\omega^r(\omega(JX, Z), JZ) = -\omega^r(J\omega(JX, Z), Z)$, again by (3.1). But so $g(\omega(J\omega(JX, JZ), JZ), X) = -g(\omega(J\omega(JX, Z), Z), X)$, which is equivalent to $v_2(\omega)(X, JX, Z, JZ) = 0$. Thus, (3.2) implies (7.1) is zero, so

$$\omega(X, Z) = \omega(JX, Z) = \omega(X, JZ) = \omega(JX, JZ) = 0.$$

Hence, $\omega(X, \cdot) = \omega(JX, \cdot) = 0$.

Next, $B_i = \begin{pmatrix} C_i & 0 \\ y_i^T & 0 \end{pmatrix}$ with $C_i \in \mathbb{R}^{2 \times 2}$ and $y_i \in \mathbb{R}^2$, $i = 1, 2$. By (2.1), $[B_1, B_2] = 0$, so $[C_1, C_2] = 0$. Moreover, (3.1) gives us $[C_2, J] = J[C_1, J]$, Hence Lemma 6.1 implies that C_1 and C_2 commute with J . So write $c_i \in \mathbb{C}$ instead of $C_i \in \mathbb{R}^{2 \times 2}$ and consider also y_i as complex numbers.

Then we observe that

$$\begin{aligned} 6 v_1(\omega)(Y, iY, Z, JZ) \\ &= -\|\omega(iY, JZ)\|^2 - \|\omega(Y, JZ)\|^2 - \|\omega(iY, Z)\|^2 - \|\omega(Y, Z)\|^2 \\ &= -2|c_1|^2 - 2|c_2|^2 - |y_1|^2 - |y_2|^2 \end{aligned}$$

and, using $\omega(J\omega(W_1, W_2), W_3) = 0$ for all $W_1, W_2 \in \mathbb{R}^{2n}$, $W_3 \in \mathfrak{a}_J$, we get

$$\begin{aligned} 12 v_2(\omega)(Y, iY, Z, JZ) \\ &= -g(\omega(J\omega(Y, Z), Z), iY) - g(\omega(J\omega(Y, JZ), JZ), iY) \\ &\quad + g(\omega(J\omega(iY, JZ), JZ), Y) + g(\omega(J\omega(iY, Z), Z), Y) \\ &= -g(C_1 J C_1 Y + C_2 J C_2 Y, iY) + g(C_2 J C_2 iY + C_1 J C_1 iY, Y) \\ &= -2g((c_1^2 + c_2^2)Y, Y) = -2\operatorname{Re}(c_1^2) - 2\operatorname{Re}(c_2^2). \end{aligned}$$

Since $\operatorname{Re}(c_i^2) + |c_i|^2 = 2\operatorname{Re}(c_i)^2$ for $i = 1, 2$, Eq. (3.2) gives

$$0 = 6(v_1(\omega) + 2v_2(\omega))(Y, iY, Z, JZ) = -4\operatorname{Re}(c_1)^2 - 4\operatorname{Re}(c_2)^2 - |y_1|^2 - |y_2|^2.$$

Thus $\operatorname{Re}(c_1) = \operatorname{Re}(c_2) = y_1 = y_2 = 0$. One may now check that ω satisfies all of the Eqs. (2.1), (3.1) and (3.2) and that $\operatorname{im}(\omega) = \mathfrak{a}$ precisely when $(c_1, c_2) \neq (0, 0)$ and $b_2 \neq 0$. Rotating (Z, JZ) in such a way that $\omega(Z, Y) = 0$, we may assume that $c_1 = 0$ and $c_2 = ib$ for some $b \in \mathbb{R}_{>0}$. Writing $\omega(Z, JZ) = qY + hX$ with $q \in \mathbb{C}$ and $h \in \mathbb{R}$, the condition $\operatorname{im} \omega = \mathfrak{a}$ gives $h \neq 0$ and we have the statement. $\square$

Next, we consider the case $A \neq 0$. For this, let us write

$$\begin{aligned} \omega(JX, Z) &= w_1Y + f_1X, & \omega(JX, JZ) &= w_2Y + f_2X, \\ \omega(Z, JZ) &= qY + hX, \end{aligned} \tag{7.2}$$

for some $w_1, w_2, q \in \mathbb{C}$, $f_1, f_2, h \in \mathbb{R}$.

Lemma 7.4 *With the above notation, if $A = \begin{pmatrix} z & u \\ 0 & a \end{pmatrix} \neq 0$, then up to rotations in the $\operatorname{span}(Y, iY)$ - and $\operatorname{span}(Z, JZ)$ -planes, we have $u \in \mathbb{R}_{\geq 0}$ and either*

$$(i) \operatorname{Re}(z) = -a/2 \neq 0,$$

$$B_1 = \begin{pmatrix} ic_1 & v_1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\frac{b_2}{2} + ic_2 & v_2 \\ 0 & b_2 \end{pmatrix}$$

$$v_1 = -\frac{ic_1}{\frac{3}{2}a - ic}u, \quad v_2 = \frac{\frac{3}{2}b_2 - ic_2}{\frac{3}{2}a - ic}u,$$

$$z = -\frac{a}{2} + ic, f_1 = -b_2, f_2 = 0, h = b_2^2/a \text{ and } w_1 + iw_2 = i(v_1 + iv_2), \text{ or}$$

$$(ii) \operatorname{Re}(z) = 0, (a, c = \operatorname{Im}(z)) \neq (0, 0),$$

$$B_1 = \begin{pmatrix} ic_1 & v_1 \\ 0 & b_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} ic_2 & v_2 \\ 0 & b_2 \end{pmatrix}$$

$$v_1 = \frac{b_1 - ic_1}{a - ic}u, \quad v_2 = \frac{b_2 - ic_2}{a - ic}u,$$

$$z = ic, f_1 = -b_2, f_2 = b_1 \text{ and } w_1 + iw_2 = i(v_1 + iv_2),$$

for some $b_1, b_2, c_1, c_2, c \in \mathbb{R}$.

Conversely, pre-shear data $(\mathfrak{a}, \omega)$ with $\dim(\mathfrak{a}) = 3$ and $\mathfrak{a}$ not totally real with all these properties satisfies (3.1) everywhere, (2.1) on $\Lambda^2 U \wedge \mathfrak{a}$ and (3.2) on $\Lambda^2 \mathfrak{a}_J \wedge \Lambda^2 \mathbb{R}^6$.

Proof First we rotate the $\operatorname{span}(Y, iY)$ -plane to get $u \in \mathbb{R}_{\geq 0}$.

Note that $\operatorname{Re}(z) \in \{0, -a/2\}$, implies either $z - a \neq 0$ or $z = 0 = a$. Consider $j \in \{1, 2\}$. Since A and B_j commute, B_j preserves the image and kernel of $A - aI_3$. As $\operatorname{span}(e_1, e_2)$ is one of these spaces, the image when $z - a \neq 0$, the kernel when $z = 0 = a$, this space is preserved by B_j . Thus,

$$B_j = \begin{pmatrix} C_j & v_j \\ 0 & b_j \end{pmatrix}$$

with $C_j \in \mathbb{R}^{2 \times 2}$, $v_j \in \mathbb{R}^2 \cong \mathbb{C}$ and $b_j \in \mathbb{R}$. Now $[B_1, B_2] = 0$ gives $[C_1, C_2] = 0$, furthermore Lemma 3.3(viii) means we can apply Lemma 6.1 to get that C_j are J -invariant and may be identified with complex numbers. We may rotate (Z, JZ) so that $\operatorname{Re}(C_1) = 0$, i.e. $C_1 = ic_1$ for some $c_1 \in \mathbb{R}$.

Similarly to the proof of Lemma 7.3, we get

$$6v_1(\omega)(Y, iY, Z, JZ) = -2|C_1|^2 - 2|C_2|^2$$

and

$$\begin{aligned} 12v_2(\omega)(Y, iY, Z, JZ) &= -2\operatorname{Re}(C_1^2) - 2\operatorname{Re}(C_2^2) \\ &\quad + g(\omega(J(\omega(Z, JZ)), iY), iY) + g(\omega(J(\omega(Z, JZ)), Y), Y) \\ &= -2(\operatorname{Re}(C_1^2) + \operatorname{Re}(C_2^2) - h\operatorname{Re}(z)). \end{aligned}$$

Thus, (3.2) gives us

$$\begin{aligned} 0 &= 3v_1(\omega)(Y, iY, Z, JZ) + 6v_2(\omega)(Y, iY, Z, JZ) \\ &= -2\operatorname{Re}(C_1)^2 - 2\operatorname{Re}(C_2)^2 - h\operatorname{Re}(z) = -2\operatorname{Re}(C_2)^2 - h\operatorname{Re}(z), \end{aligned}$$

So $\operatorname{Re}(C_2) = 0$ if $\operatorname{Re}(z) = 0$, and $\operatorname{Re}(C_2)^2 = ah/4$ if $\operatorname{Re}(z) = -a/2$.

Next,

$$\begin{aligned} 6v_1(\omega)(Y, iY, X, Z) &= -g(\omega(iY, JX), \omega(iY, Z)) - g(\omega(Y, JX), \omega(Y, Z)) \\ &= -2\operatorname{Re}(z\overline{C_1}) \end{aligned}$$

and

$$\begin{aligned} 12v_2(\omega)(Y, iY, X, Z) &= -g(\omega(J\omega(Y, Z), JX), iY) + g(\omega(J\omega(iY, Z), JX), Y) \\ &\quad + g(\omega(J\omega(X, Z), iY), iY) + g(\omega(J\omega(X, Z), Y), Y) \\ &= -2\operatorname{Re}(zC_1) - 2b_1\operatorname{Re}(z), \end{aligned}$$

so that (3.2) gives us

$$\begin{aligned} 0 &= 6v(\omega)(Y, iY, X, Z) = -2\operatorname{Re}(z\overline{C_1}) - 2\operatorname{Re}(zC_1) - 2b_1\operatorname{Re}(z) \\ &= -2\operatorname{Re}(z)(2\operatorname{Re}(C_1) + b_1) = -2\operatorname{Re}(z)b_1. \end{aligned}$$

Hence, if $\operatorname{Re}(z) = -a/2 \neq 0$, then $b_1 = 0$. Similarly, we obtain

$$0 = 6v(\omega)(Y, iY, X, JZ) = -2\operatorname{Re}(z)(2\operatorname{Re}(C_2) + b_2),$$

and so $\operatorname{Re}(C_2) = -b_2/2$ if $\operatorname{Re}(z) = -a/2 \neq 0$, which implies $b_2^2/4 = \operatorname{Re}(C_2)^2 = ah/4$ and hence $h = b_2^2/a$.

Note that $v_1(\omega)(Y, iY, JX, Z) = v_1(\omega)(iY, Y, X, JZ) = -v_1(\omega)(Y, iY, X, JZ)$ and that

$$\begin{aligned} v_2(\omega)(Y, iY, JX, Z) &= g(\omega(J\omega(Y, JX), JZ), iY) - g(\omega(J\omega(iY, JX), JZ), Y) \\ &\quad + g(\omega(J\omega(JX, Z), iY), iY) + g(\omega(J\omega(JX, Z), Y), Y) \\ &= g(i\omega(\omega(Y, JX), JZ), iY) - g(i\omega(\omega(iY, JX), JZ), Y) \\ &\quad + g(\omega(J\omega^r(JX, Z), iY), iY) + g(\omega(J\omega^r(JX, Z), Y), Y) \\ &= g(\omega(J\omega(Y, JZ), JX), iY) - g(\omega(J\omega(iY, JZ), JX), Y) \\ &\quad - g(\omega(J\omega(X, JZ), iY), iY) - g(\omega(J\omega(X, JZ), Y), Y) \\ &= -v_2(\omega)(Y, iY, X, JZ), \end{aligned}$$

so that $v(\omega)(Y, iY, JX, Z) = -v(\omega)(Y, iY, X, JZ) = 0$. Similarly, the identity $v(\omega)(Y, iY, JX, JZ) = v(\omega)(Y, iY, X, Z) = 0$ follows. Thus, (3.2) is also valid on $\Lambda^2\mathfrak{a}_J \wedge U_r \wedge U_J$, and so on $\Lambda^2\mathfrak{a}_J \wedge \Lambda^2\mathbb{R}^6$.

Next, we notice that (3.1) is satisfied if and only if parts (vi) and (vii) of Lemma 3.3 hold. Now (vi) is equivalent to $f_1 = -b_2$ and $f_2 = b_1$ and (vii) is equivalent to

$$w_1 + iw_2 = i(v_1 + iv_2).$$

Moreover, by (2.1), we must have $[A, B_i] = 0$, which is equivalent to $zv_i + ub_i = C_iu + v_ia$, i.e. to $(z-a)v_i = (C_i - b_i)u$, and so to $v_i = \frac{C_i - b_i}{z-a}u$, $i = 1, 2$, if $z-a \neq 0$.

If $z-a = 0$, then $z = 0 = a$ and $u \neq 0$. As $C_j = ic_j$, we have $0 = (z-a)v_j = (ic_j - b_j)u$, giving $b_j = c_j = 0$ for $j = 1, 2$. This implies $\omega(X, Z), \omega(JX, Z), \omega(X, JZ), \omega(JX, JZ)$ all lie in $\mathfrak{a}_J$ and $\mathfrak{a}_J \subseteq \ker(\omega(JX, \cdot)) \cap \ker(\omega(Z, \cdot)) \cap \ker(\omega(JZ, \cdot))$. Using these properties, (2.1) implies

$$\begin{aligned} 0 &= \omega(JX, \omega(Z, JZ)) + \omega(Z, \omega(JZ, JX)) + \omega(JZ, \omega(JX, Z)) \\ &= \omega(JX, qY + hX) = uhY. \end{aligned}$$

Since $u \neq 0$, we get $h = 0$ and thus $\operatorname{im}(\omega) \subseteq \mathfrak{a}_J$, a contradiction. We conclude that $z-a \neq 0$, which is clear if $\operatorname{Re}(z) = -a/2 \neq 0$ but implies $a \neq 0$ or $\operatorname{Im}(z) \neq 0$ in the case $\operatorname{Re}(z) = 0$. $\square$

Summarizing our results to this point, we observe that we still have to impose (2.1) on $\Lambda^3U = U_r \wedge \Lambda^2U_J$ and (3.2) on $\mathfrak{a}_J \wedge \Lambda^3(\mathfrak{a}_r \oplus U_r \oplus U_J)$ and on $\Lambda^4(\mathfrak{a}_r \oplus U_r \oplus U_J) = \mathfrak{a}_r \wedge U_r \wedge \Lambda^2U_J$.

First of all, (2.1) evaluated on $U_r \wedge \Lambda^2 U_J$ yields

$$\begin{aligned} 0 &= \omega(JX, \omega(Z, JZ)) + \omega(Z, \omega(JZ, JX)) + \omega(JZ, \omega(JX, Z)) \\ &= A \cdot (q, h)^T - B_1(w_2, b_1)^T + B_2(w_1, -b_2), \end{aligned}$$

where we set $b_1 = 0$ if $\operatorname{Re}(z) = -a/2$. This equation is equivalent to

$$0 = zq + hu - ic_1 w_2 - b_1 v_1 + C_2 w_1 - b_2 v_2 \quad \text{and} \quad ah = \|b\|^2 \quad (7.3)$$

Note that for $\operatorname{Re}(z) = -a/2 \neq 0$, the second equation is automatically satisfied as $b_1 = 0$ and $h = b_2^2/a$.

Let us now first look at (3.2) on $\mathfrak{a}_J \wedge \mathfrak{a}_r \wedge U_r \wedge U_J$. For this, let $\tilde{Y} \in \mathfrak{a}_J$. Then

$$\begin{aligned} 6v_1(\omega)(\tilde{Y}, X, JX, Z) &= g(\omega(i\tilde{Y}, JX), \omega(JX, Z)) + g(\omega(i\tilde{Y}, JZ), \omega(X, JX)) \\ &\quad - g(\omega(JX, X), \omega(\tilde{Y}, Z)) - g(\omega(JX, JZ), \omega(\tilde{Y}, JX)) \\ &= g(iz\tilde{Y}, w_1 Y) + g(iC_2 \tilde{Y}, uY) + g(uY, ic_1 \tilde{Y}) - g(w_2 Y, z\tilde{Y}) \end{aligned}$$

and

$$\begin{aligned} 12v_2(\omega)(\tilde{Y}, X, JX, Z) &= -g(\omega(J\omega(X, JX), JZ), \tilde{Y}) - g(\omega(J\omega(JX, Z), JX), \tilde{Y}) \\ &\quad + g(\omega(J\omega(Z, X), X), \tilde{Y}) \\ &= -g(iuC_2 Y, \tilde{Y}) - g(aw_2 Y, \tilde{Y}) - g(iw_1 zY, \tilde{Y}) + g(b_1 uY, \tilde{Y}). \end{aligned}$$

Thus, $v(\omega)(\tilde{Y}, X, JX, Z) = 0$ for all $\tilde{Y} \in \mathfrak{a}_J$ is equivalent to

$$0 = -2i \operatorname{Re}(z)w_1 - 2i \operatorname{Re}(C_2)u - ic_1 u - w_2 \bar{z} - aw_2 + b_1 u. \quad (7.4)$$

Similarly, $v(\omega)(\tilde{Y}, X, JX, JZ) = 0$ for all $\tilde{Y} \in \mathfrak{a}_J$ is seen to be equivalent to

$$\begin{aligned} 0 &= -i\bar{z}w_2 + c_1 u + \bar{C}_2 u + \bar{z}w_1 - uc_1 + aw_1 - iw_2 z + b_2 u \\ &= -2i \operatorname{Re}(z)w_2 + \bar{C}_2 u + w_1 \bar{z} + aw_1 + b_2 u. \end{aligned} \quad (7.5)$$

Analogous computations give $v(\omega)(\tilde{Y}, X, Z, JZ) = 0$ for all $\tilde{Y} \in \mathfrak{a}_J$ if and only if

$$0 = 2iv_2 \operatorname{Re}(C_2) - 2iq \operatorname{Re}(z) + b_1 w_1 + b_2 w_2 + \bar{C}_2 w_2 - ic_1 w_1 \quad (7.6)$$

and that $v(\omega)(\tilde{Y}, JX, Z, JZ) = 0$ for all $\tilde{Y} \in \mathfrak{a}_J$ if and only if

$$0 = 2iw_2 \operatorname{Re}(C_2) - \bar{C}_2 v_2 + ic_1 v_1 + q\bar{z} - b_2 w_1 - hu + b_1 w_2 \quad (7.7)$$

Finally, we have to consider the equation $v(\omega)(X, JX, Z, JZ) = 0$. Using

$$\begin{aligned} \omega^r(J\omega(JX, JZ), JZ) &= \omega^r(\omega(JX, JZ), Z) \\ &= \omega^r(\omega(Z, JZ), JX) + \omega^r(\omega(JX, Z), JZ) \\ &= -\omega^r(J\omega(Z, JZ), X) - \omega^r(J\omega(JX, Z), Z), \end{aligned}$$

one easily sees that

$$v_2(\omega)(X, JX, Z, JZ) = 0.$$

Thus, $0 = v(\omega)(X, JX, Z, JZ) = v_1(\omega)(X, JX, Z, JZ)$, and a short computation yields that this equation is equivalent to

$$\begin{aligned} 0 &= 2g(\omega(X, JX), \omega(Z, JZ)) - \|\omega(X, Z)\|^2 - \|\omega(X, JZ)\|^2 \\ &\quad - \|\omega(JX, Z)\|^2 - \|\omega(JX, JZ)\|^2. \end{aligned} \quad (7.8)$$

Let us now consider individually the cases (a) $\operatorname{Re}(z) = 0$ and (b) $\operatorname{Re}(z) = -a/2 \neq 0$.

(a) $\operatorname{Re}(z) = 0$. Here, $\operatorname{Re}(C_2) = 0$ and so $C_2 = ic_2$ for some $c_2 \in \mathbb{R}$. Then (7.4) and (7.5) imply

$$w_2 = \frac{b_1 - ic_1}{a - ic}u = v_1, \quad w_1 = -\frac{b_2 - ic_2}{a - ic}u = -v_2, \quad (7.9)$$

which is consistent with $w_i + iw_2 = i(v_1 + iv_2)$. Moreover, (7.6) is equivalent to $(b_2 - ic_2)w_2 = -(b_1 - ic_1)w_1$, and so also automatically satisfied. Furthermore, the first equation in (7.3) and (7.7) are both easily seen to be equivalent to

$$\begin{aligned} 0 &= icq + hu - (b_1 + ic_1)v_1 - (b_2 + ic_2)v_2 = icq + hu - \frac{b_1^2 + b_2^2 + c_1^2 + c_2^2}{a - ic}u \\ &= icq + \frac{ha - ihc - b_1^2 - b_2^2 - c_1^2 - c_2^2}{a - ic}u = icq - \frac{ihc + c_1^2 + c_2^2}{a - ic}u. \end{aligned}$$

So if $c \neq 0$, we obtain

$$q = \frac{ich + c_1^2 + c_2^2}{c^2 + aci}u.$$

If $c = 0$, we would get $c_1^2 + c_2^2 = 0$, hence $c_1 = c_2 = 0$. However, this case cannot occur as then the only non-zero Lie brackets on the shear would be (up to anti-symmetry) $[Z, JZ]$ and

$$[X, JX] = \frac{b_1}{a}[X, Z] = \frac{b_1}{a}[JX, JZ] = \frac{b_2}{a}[X, JZ] = -\frac{b_2}{a}[JX, Z]$$

giving $\dim(\mathfrak{g}') \leq 2 < 3$, a contradiction.

Thus, $c \neq 0$ and we still have to satisfy (7.8). Due to $\omega(JX, JZ) = \omega(X, Z)$ and $\omega(JX, Z) = -\omega(X, JZ)$ and $ha = \|b\|^2$, this equation is

$$0 = 2\operatorname{Re}(q)u + 2ha - 2|v_1|^2 - 2b_1^2 - 2|v_2|^2 - 2b_2^2 = 2(\operatorname{Re}(q)u - |v_1|^2 - |v_2|^2),$$

so

$$\operatorname{Re}(q)u = |v_1|^2 + |v_2|^2 = \frac{b_1^2 + b_2^2 + c_1^2 + c_2^2}{a^2 + c^2}u^2.$$

On the other hand, $q = \frac{ich + c_1^2 + c_2^2}{c^2 + aci}u$ implies

$$\operatorname{Re}(q)u = \operatorname{Re}\left(\frac{ich + c_1^2 + c_2^2}{c^2 + aci}u\right)u = \frac{c_1^2 + c_2^2 + ah}{a^2 + c^2}u^2 = \frac{c_1^2 + c_2^2 + b_1^2 + b_2^2}{a^2 + c^2}u^2,$$

and so (7.8) is automatically satisfied.

(b) $\operatorname{Re}(z) = -a/2 \neq 0$. In this case, (7.4) and (7.5) are explicitly given by

$$iaw_1 - \left(\frac{a}{2} - ic\right)w_2 + i(b_2 - c_1)u = 0, \quad iaw_2 + \left(\frac{a}{2} - ic\right)w_1 + \left(\frac{b_2}{2} - ic_2\right)u = 0.$$

The solution is given by

$$\begin{aligned} w_1 &= -\frac{\left(\frac{a}{2} - ic\right)\left(\frac{b_2}{2} - ic_2\right) - a(b_2 - c_1)}{\left(\frac{a}{2} - ic\right)^2 - a^2}u, \\ w_2 &= i\frac{\left(\frac{a}{2} - ic\right)(b_2 - c_1) - a\left(\frac{b_2}{2} - ic_2\right)}{\left(\frac{a}{2} - ic\right)^2 - a^2}u, \end{aligned}$$

and $w_1 + iw_2 = i(v_1 + iv_2)$ holds.

Next, using $w_2 = v_1 + iv_2 + iw_1$, (7.7) is explicitly given by

$$\begin{aligned} 0 &= -ib_2w_2 + \left(\frac{b_2}{2} + ic_2\right)v_2 + ic_1v_1 - \left(\frac{a}{2} + ic\right)q - b_2w_1 - hu \\ &= -ib_2v_1 + b_2v_2 + \left(\frac{b_2}{2} + ic_2\right)v_2 + ic_1v_1 - \left(\frac{a}{2} + ic\right)q - hu \\ &= \left(\frac{3}{2}b_2 + ic_2\right)v_2 - i(b_2 - c_1)v_1 - \left(\frac{a}{2} + ic\right)q - hu, \end{aligned}$$

and so we do get

$$q = \frac{-i(b_2 - c_1)v_1 + \left(\frac{3}{2}b_2 + ic_2\right)v_2 - hu}{\frac{a}{2} + ic} = \frac{(c_1 - b_2)c_1 + \frac{9}{4}b_2^2 + c_2^2 - h\left(\frac{3}{2}a - ic\right)}{\left(\frac{a}{2} + ic\right)\left(\frac{3}{2}a - ic\right)}u.$$

A lengthy but straightforward computation shows that then the first equation in (7.3) and also (7.6) and (7.8) are satisfied.

Theorem 7.5 *Let $(\mathfrak{g}, g, J)$ be a six-dimensional almost Hermitian Lie algebra. Then $(\mathfrak{g}, g, J)$ is a two-step solvable SKT Lie algebra with $\mathfrak{g}'$ three-dimensional and not totally real if and only if $(\mathfrak{g}, g, J)$ admits a unitary (Y, iY, X, JX, Z, JZ) basis such that Y spans $\mathfrak{g}'_J$ as a complex vector space and X spans the real vector space $\mathfrak{g}'_r$ and the only non-zero Lie brackets (up to anti-symmetry and complex-linear extension to $\mathfrak{g}'_J$) are given by either*

$$(i) \quad [JZ, Y] = ibY, \quad [Z, JZ] = qY + hX \text{ for } b, h \in \mathbb{R} \setminus \{0\}, q \in \mathbb{C},$$

(ii)

$$= icY, \quad [Z, Y] = ic_1Y, \quad [JZ, Y] = ic_2Y,$$

$$[JX, X] = uY + aX, \quad [Z, X] = [JZ, JX] = -\frac{ic_1}{\frac{3}{2}a - ic}uY + b_1X,$$

$$[JZ, X] = -[Z, JX] = \frac{\frac{3}{2}b_2 - ic_2}{\frac{3}{2}a - ic}uY + b_2X, \quad [Z, JZ] = -\frac{ich + c_1^2 + c_2^2}{c^2 + aci}uY - hX$$

for certain $c \in \mathbb{R} \setminus \{0\}$, $b_1, b_2, c_1, c_2, h \in \mathbb{R}$, $a, u \in \mathbb{R}_{\geq 0}$ with $ha = b_1^2 + b_2^2$ and at least one of a, b_1, b_2, h non-zero, or

(iii)

$$= \left(-\frac{a}{2} + ic\right)Y, \quad [Z, Y] = ic_1Y, \quad [JZ, Y] = \left(-\frac{b_2}{2} + ic_2\right)Y,$$

$$[JX, X] = uY + aX, \quad [Z, X] = -\frac{ic_1}{\frac{3}{2}a - ic}uY,$$

$$[JZ, X] = \frac{\frac{3}{2}b_2 - ic_2}{\frac{3}{2}a - ic}uY + b_2X, \quad [Z, JZ] = -qY - hX,$$

$$[Z, JX] = -\frac{\left(\frac{a}{2} - ic\right)\left(\frac{b_2}{2} - ic_2\right) - a(b_2 - c_1)}{\left(\frac{a}{2} - ic\right)^2 - a^2}uY - b_2X,$$

$$[JZ, JX] = i\frac{\left(\frac{a}{2} - ic\right)(b_2 - c_1) - a\left(\frac{b_2}{2} - ic_2\right)}{\left(\frac{a}{2} - ic\right)^2 - a^2}uY$$

for $a \in \mathbb{R} \setminus \{0\}$, $b_2, c, c_1, c_2 \in \mathbb{R}$, $u \in \mathbb{R}_{\geq 0}$, with

$$h = \frac{b_2^2}{a}, \quad q = \frac{(c_1 - b_2)c_1 + \frac{9}{4}b_2^2 + c_2^2 - h\left(\frac{3}{2}a - ic\right)}{\left(\frac{a}{2} + ic\right)\left(\frac{3}{2}a - ic\right)}u.$$

Remark 7.6 Identifying the unimodular Lie algebras in Theorem 7.5, one observes that those in (i) and (iii) are always unimodular whereas those in (ii) are unimodular precisely when $a = b_1 = b_2 = 0$ (and then $h \neq 0$). Note that none of the Lie algebras in Theorem 7.5 can be nilpotent.

7.2 Four-dimensional complex commutator ideal

As announced at the beginning of Sect. 7, we only write down the equations in this case.

First of all, note that $\mathfrak{a} = \mathfrak{a}_J$, $U = U_J$ and take an orthonormal basis $Y_1, Y_2 = JY_1$ of U . Then ω is completely determined by $A_i := \omega_0(Y_i, \cdot) \in \text{End}(\mathfrak{a})$, $i = 1, 2$, and by $X := \omega_1(Y_1, Y_2) \in \mathfrak{a}$. Moreover, (2.1) reduces to $[A_1, A_2] = 0$ and (3.1) reduces to $[A_2, J] = J[A_1, J]$, i.e. $A_2^{J-} = JA_1^{J-}$, where the superscript $J-$ denotes the J -anti-invariant part of an endomorphism of $\mathfrak{a} = \mathfrak{a}_J$. Finally, (3.2) reduces to

$$JA_1^T A_1 + A_1^T A_1 J + JA_2^T A_2 + A_2^T A_2 J + A_1 J A_1 + A_1^T J A_1^T + A_2 J A_2 + A_2^T J A_2^T = 0.$$

where the transpose is taken with respect to the Riemannian metric $g|_{\mathfrak{a}}$ on $\mathfrak{a}$. Note that this is equivalent to $JA_1^T A_1 + A_1 J A_1 + JA_2^T A_2 + A_2 J A_2$ being symmetric. Summarizing, A_1 and A_2 have to satisfy the following equations:

$$\begin{aligned} [A_1, A_2] &= 0, \quad A_2^{J-} = JA_1^{J-}, \\ \sum_{i=1}^2 JA_i^T A_i + A_i^T A_i J + A_i J A_i + A_i^T J A_i^T &= 0. \end{aligned} \quad (7.10)$$

Writing $A_i = (A_i)_+ + (A_i)_-$, with $(A_i)_\varepsilon^T = \varepsilon(A_i)_\varepsilon$, for $\varepsilon \in \{\pm 1\}$, this decomposition commutes with the decomposition $A_i = A_i^J + A_i^{J-}$ and the final equation in (7.10) become the two equations

$$\begin{aligned} \sum_{i=1}^2 [(A_i)_+^J, (A_i)_+^{J-}] - [(A_i)_-^J, (A_i)_-^{J-}] &= 0, \\ \sum_{i=1}^2 [(A_i)_+^J, (A_i)_-^{J-}] - [(A_i)_-^J, (A_i)_+^{J-}] + 2((A_i)_+^J)^2 - 2((A_i)_-^{J-})^2 &= 0. \end{aligned}$$

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Table 1 Notation for certain Lie algebras

$\mathfrak{g}$	dim	Differentials	Conditions
$\mathfrak{r}'_{3,\lambda}$	3	$(0, \lambda.21 + 31, -21 + \lambda.31)$	$\lambda \geq 0$
$\mathfrak{r}_{4,\mu,\lambda}$	4	$(0, 21, \mu.31, \lambda.41)$	$0 < \lambda \leq \mu \leq 1$
$\mathfrak{r}'_{4,\mu,\lambda}$	4	$(0, \mu.21, \lambda.31 + 41, -31 + \lambda.41)$	$\mu > 0$
$\mathfrak{g}_{5,14}^{\alpha}$	5	$(0, 0, 21, \alpha.41 + 51, -41 + \alpha.51)$	$\alpha \geq 0$
$\mathfrak{g}_{5,17}^{\alpha,\beta,\gamma}$	5	$(0, \alpha.21 + 31, -21 + \alpha.31, \beta.41 + \gamma.51, -\gamma.41 + \alpha.51)$	$\alpha \geq 0, \gamma \neq 0$
$\mathfrak{g}_{6,1}^{\alpha,\beta,\gamma,\delta}$	6	$(0, 21, \alpha.31, \beta.41, \gamma.51, \delta.61)$	$0 < \delta \leq \gamma \leq \beta \leq \alpha \leq 1$
$\mathfrak{g}_{6,8}^{\alpha,\beta,\gamma,\delta}$	6	$(0, \alpha.21, \beta.31, \gamma.41, \delta.51 + 61, -51 + \delta.61)$	$0 < \gamma \leq \beta \leq \alpha $
$\mathfrak{g}_{6,11}^{\alpha,\beta,\gamma,\delta}$	6	$(0, \alpha.21, \beta.31 + 41, -31 + \beta.41, \gamma.51 + \delta.61, -\delta.51 + \gamma.61)$	$\alpha\delta \neq 0$
(37D)	7	$(0, 0, 0, 0, e^{12} + e^{34}, e^{13}, e^{24})$	—

Appendix A: Notation for particular Lie algebras

Table 1 contains a list of all indecomposable Lie algebras $\mathfrak{g}$ of various dimensions which occur in this article and which are not well-known, in contrast to the Abelian Lie algebra $\mathbb{R}$, the two-dimensional Lie algebra $\text{aff}_{\mathbb{R}}$ of affine motions of the real line and the $(2k+1)$ -dimensional Heisenberg Lie algebra $\mathfrak{h}_{2k+1}$. The names for Lie algebras in dimension 3 are taken from [5], in four dimensions from [2], in five dimensions from [19], in six dimensions from [20] and in seven dimensions from [16].

In all cases, the Lie bracket is encoded dually by the tuple of differentials $(de^1, \dots, de^n)$ for a basis $e^1, \dots, e^n$ of $\mathfrak{g}$ and we use Salamon's notation for this tuple. Moreover, in some cases, we do not give all the necessary conditions on the parameters to ensure that two Lie algebras in the corresponding parameter families are non-isomorphic.

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