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The spectral decomposition of $|\theta|^2$

Paul D. Nelson[1]

Abstract

Let θ be an elementary theta function, such as the classical Jacobi theta function. We establish a spectral decomposition and surprisingly strong asymptotic formulas for $\langle |\theta|^2, \varphi \rangle$ as φ traverses a sequence of Hecke-translates of a nice enough fixed function. The subtlety is that typically $|\theta|^2 \notin L^2$. Applications to the subconvexity, quantum variance and 4-norm problems are indicated.

Keywords Theta functions · Half-integral weight modular forms · Spectral decomposition

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1 Introduction

Let Γ be a congruence subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and let $\mathbb{H}$ denote the upper half-plane. For pair of *square-integrable* automorphic functions $\varphi_1, \varphi_2 \in L^2(\Gamma \backslash \mathbb{H})$, the Petersson inner product

$$\langle \varphi_1, \varphi_2 \rangle := \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{z \in \Gamma \backslash \mathbb{H}} \overline{\varphi_1}(z) \varphi_2(z) \, \frac{dx \, dy}{y^2} \tag{1}$$

may be written in terms of the inner products of φ against the constant function 1, the elements of an orthonormal basis $\mathcal{B}_{\mathrm{cusp}}$ for the space of cusp forms, and the unitary Eisenstein series $E_{\mathfrak{a}, 1/2+it}$ attached to the various cusps $\mathfrak{a}$ of Γ: with suitable normalization (see, e.g., [9], [10, §15], [6] for details),

$$
\begin{aligned}
\langle \varphi_1, \varphi_2 \rangle = \langle \varphi_1, 1 \rangle \langle 1, \varphi_2 \rangle &+ \sum_{\varphi \in \mathcal{B}_{\mathrm{cusp}}} \langle \varphi_1, \varphi \rangle \langle \varphi, \varphi_2 \rangle \\
&+ \sum_{\mathfrak{a}} \int_{t \in \mathbb{R}} \langle \varphi_1, E_{\mathfrak{a}, 1/2+it} \rangle \langle E_{\mathfrak{a}, 1/2+it}, \varphi_2 \rangle \, \frac{dt}{4\pi}.
\end{aligned}
\tag{2}
$$

✉ Paul D. Nelson
  paul.nelson@math.ethz.ch

[1] ETH Zurich, Department of Mathematics, Rämistrasse 101, 8092 Zurich, Switzerland

This formula may be used to establish (among other things) the equidistribution of the Hecke correspondences T_n on $\Gamma\backslash\mathbb{H}$ through estimates such as

$$\langle\varphi_1, T_n\varphi_2\rangle = \langle\varphi_1, 1\rangle\langle 1, \varphi_2\rangle + O(\tau(n)n^{-1/2+\vartheta}) \tag{3}$$

for unit vectors $\varphi_1, \varphi_2 \in L^2(\Gamma\backslash\mathbb{H})$. Here the Hecke operator T_n is normalized so that $T_n 1 = 1$ (thus for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $T_n f(z)$ is the *average* of $f((az+b)/d)$ over all factorizations $n = ad$ and integers $0 \le b < d$) and $\vartheta \in [0, 7/64]$ quantifies the strongest known bound [13] for the Hecke eigenvalues of the cusp forms and Eisenstein series occurring in (2). Given such a bound for Hecke eigenvalues, the estimate (3) follows from (2) and the Cauchy–Schwarz inequality.

Variations on (3) in which φ_1, φ_2 are *not* both square-integrable turn out to play a fundamental role in analytic number theory, extending far beyond the evident application to Hecke equidistribution. For instance, in the periods-based approach to the subconvexity problem on GL_2 following Venkatesh [31] and Michel–Venkatesh [18], the basic quantitative input is an analogue of (3) for

- $\varphi_1 = |E|^2$ the squared magnitude of a unitary Eisenstein series E and
- $\varphi_2 = |\Phi|^2$ that of a cusp form Φ,

so that φ_2 is rapidly-decaying but φ_1 fails to belong to L^2, or even to L^1. The inner products $\langle|E|^2, |\Phi|^2\rangle$ arise naturally after applying Cauchy–Schwarz to the integral representation $L(\Psi \times \Phi, 1/2) = \langle\Psi\Phi, E\rangle$ for the Rankin–Selberg L-function attached to a pair cusp forms Ψ, Φ; the magnitudes of such L-functions are in turn related to fundamental arithmetic equidistribution problems, such as those concerning the distribution of solutions to $x^2 + y^2 + z^2 = n$ (see for instance [17]). The standard Plancherel formula does not apply to $\langle|E|^2, |\Phi|^2\rangle$, and indeed, its "formal application" gives the wrong answer. There arises the need for a *regularized Plancherel formula* which the authors of [18, §4.3.8] develop in generality sufficient for their purposes.

This paper is concerned with another such variation. We will prove analogues (in fact, counterintuitive *strengthenings*) of the spectral decomposition (2) and the asymptotic formula (3) for

- $\varphi_1 := |\theta|^2$ the squared magnitude of an elementary theta function, such as the *Jacobi theta function*

$$\theta(z) := y^{1/4} \sum_{n\in\mathbb{Z}} \exp(2\pi i n^2 z), \quad z = x + iy \tag{4}$$

  on $\Gamma_0(4) \supseteq \Gamma$, and
- $\varphi_2 =: \varphi$ of sufficiently mild growth that the inner product $\langle|\theta|^2, \varphi\rangle$ converges absolutely; for instance, it will suffice to impose the growth condition

$$\varphi(z) \ll \mathrm{ht}(z)^{1/2-\delta} \tag{5}$$

  for some fixed $\delta > 0$, where the height function $\mathrm{ht} : \Gamma\backslash\mathbb{H} \to \mathbb{R}_+$ is defined by $\mathrm{ht}(z) := \max_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)$.

As we discuss below, a detailed analysis of such inner products $\langle|\theta|^2, \varphi\rangle$ is at the heart of each of the recent works [19–22], and also seems likely to be useful in further applications, motivating the focused discussion recorded here.

Unfortunately, $|\theta|^2 \notin L^2$, so the standard Plancherel formula does not apply to the inner products $\langle|\theta|^2, \varphi\rangle$. Indeed, it is immediate from (1) that a continuous function f on $\Gamma\backslash\mathbb{H}$

satisfying the asymptotic $|f(z)| \asymp \mathrm{ht}(z)^\beta$ near the cusps is absolutely integrable if and only if $\beta < 1$, while from (4) we see that $|\theta|^4(z) \gg \mathrm{ht}(z)$ near the cusp ∞. In this article, we develop a different, robust technique for decomposing and estimating such inner products. We focus for now on the following special case of the results obtained, postponing general statements to Sect. 2.

Theorem 1 *Let θ denote the Jacobi theta function* (4). *Fix a measurable function φ on $\Gamma \backslash \mathbb{H}$ satisfying the growth condition* (5).

Let n traverse a sequence of integers coprime to the level of Γ. Then

$$\langle |\theta|^2, T_n \varphi \rangle = \langle |\theta|^2, 1 \rangle \langle 1, \varphi \rangle + O(\tau(n) n^{-1/2}). \tag{6}$$

Curiously, the new bound (6) is *stronger* than the more straightforward estimate (3) in that $n^{-1/2}$ replaces $n^{-1/2+\vartheta}$. To put it another way, the strength of (6) is comparable to that of (3) together with the assumption of the (unsolved) Ramanujan conjecture $\vartheta = 0$. We will explain this surprise shortly.

Before doing so, we summarize our interest in studying inner products involving $|\theta|^2$. Our original motivation was that asymptotic formulas such as (6) turned out to be the global quantitative inputs to the method introduced and developed in [19–21] for attacking the *quantum variance* problem. That problem concerns the sums given by

$$\sum_{f \in \mathcal{F}} \langle |f|^2, \Psi_1 \rangle \langle \Psi_2, |f|^2 \rangle \tag{7}$$

for some "nice enough family of automorphic forms" $\mathcal{F}$ and fixed observables Ψ_1, Ψ_2, which we assume here for simplicity to be cuspidal. The asymptotic determination of (7) when $\mathcal{F}$ consists of the Maass forms of eigenvalue bounded by some parameter $T \to \infty$ may be understood as a fundamental problem in semiclassical analysis (see for instance [35, §15.6], [23, §4.1.3], [16,36], [19, §1]). The method of [19] uses the theta correspondence to relate the sums (7) to inner products roughly of the form

$$\langle |\theta|^2, h_1 \overline{h_2} \rangle, \tag{8}$$

where h_1, h_2 are half-integral weight Maass–Shimura–Shintani–Waldspurger lifts of Ψ_1, Ψ_2. The local data underlying the lifts depends rather delicately upon the family $\mathcal{F}$. It turns out that when $\mathcal{F}$ is "nice enough," the product $h_1 \overline{h_2}$ is essentially a *translate* of the product $\varphi := h_1^0 \overline{h_2^0}$ of some *fixed* lifts h_1^0, h_2^0. If that translate is induced by (for instance) the correspondence T_n, then (8) is of the form $\langle |\theta|^2, T_n \varphi \rangle$ and so estimates like (6) become relevant for determining the asymptotics of sums like (7).

Another motivation for the present study comes from the appearance of elementary theta functions in the Shimura integral representation [5,28]

$$L(\mathrm{sym}^2 \varphi, 1/2) \approx \int \varphi \theta \tilde{E} \tag{9}$$

for symmetric square L-functions on GL_2; here φ is a cusp form and $\tilde{E}$ is a suitable half-integral weight Eisenstein series, and $\approx$ denotes equality up to local factors. One also knows the cuspidal analogue of (9), namely

$$L(\mathrm{sym}^2 \varphi \otimes \Psi, 1/2) \approx |\int \varphi \theta h|^2 \tag{10}$$

for cusp forms Ψ with half-integral lift h (see Qiu [24, Thm 4.5]). An application of the Cauchy–Schwartz inequality to such integrals yields inner products involving $|\theta|^2$. By study-

ing those inner products using the results of this paper, a surprising implication concerning the subconvexity problem for (twisted) symmetric square L-functions was obtained in [22].

A third source of motivating applications may be obtained by summing (precise forms of) the identity (10) over either φ or h in an orthonormal basis. The LHS of the resulting identity is a moment of L-functions, while the RHS, by Parseval, is an inner product involving $|\theta|^2$. By applying the results of this paper to such inner products, one may hope to obtain summation formulas for certain moments of L-functions. Summing over φ yields the moments relevant for the quantum variance problem discussed above, while taking a suitably normalized sum over h yields a summation formula for twisted symmetric square L-functions to which one might hope to apply the techniques of [20]. We plan to pursue this idea separately.

For the applications motivating this work, it is indispensible to have more flexible forms of Theorem 1 in which:

- The squared magnitude $|\theta|^2$ is generalized to any product $\theta_1 \overline{\theta_2}$ of unary theta series θ_1, θ_2, such as those obtained by imposing congruence conditions in the summation defining θ.
- One allows greater variation than $\varphi \mapsto T_n \varphi$ for n coprime to the level of Γ. For instance, one would like to consider variation under the diagonal flow on $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$, or with respect to "Hecke operators" at primes dividing the level.
- The dependence of the error term upon φ and θ_1, θ_2 is quantified.

The main purpose of this article is to supply such flexible forms. Our results, to be formulated precisely in Sect. 2, are natural completions of Theorem 1: working over a fixed global field, we prove that the translates under the metaplectic group of a product of two elementary theta functions equidistribute with an essentially optimal rate and polynomial dependence upon all parameters. In quantifying the latter we make systematic use of the adelic Sobolev norms developed in [18, §2].

The regularized Plancherel formula of Michel–Venkatesh, which builds on a method of Zagier [34], does not seem to apply to the inner products $\langle |\theta|^2, \varphi \rangle$ considered here: that method would involve finding an Eisenstein series $\mathcal{E}$ with parameter to the right of the unitary axis for which $|\theta|^2 - \mathcal{E} \in L^2$, but it is easy to see from the expansion $|\theta|^2(z) \sim y^{1/2} + \cdots$ near the cusp ∞ that such an $\mathcal{E}$ does not exist. The singular nature of the parameter $1/2$ presents further difficulties if one tries to adapt that method. We are not aware of an adequate formal reduction (e.g., via a simple approximation argument or truncation) by which one may deduce flexible forms of estimates such as (6) directly from (2). The technique developed here is more direct, and specific to $|\theta|^2$. We illustrate it now briefly in the context of Theorem 1:

(1) Some careful but elementary manipulations (change of variables, Poisson summation, folding up, Mellin inversion, contour shift) to be explained in Sect. 3 give a pointwise expansion

$$|\theta|^2 = 1 + \sum_{\mathfrak{a}} \int_{t \in \mathbb{R}} c(\mathfrak{a}, t) E^*_{\mathfrak{a}, 1/2+it} \frac{dt}{2\pi} \qquad (11)$$

where $\mathfrak{a}$ traverses the cusps of $\Gamma_0(4)$,

$$E^*_{\mathfrak{a}, 1/2+it} := 2\xi(1 + 2it) E_{\mathfrak{a}, 1/2+it}, \quad \xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s),$$

and the complex coefficients $c(\mathfrak{a}, t)$ are explicit and uniformly bounded (and best described in the language of Theorem 2 below; see also [19, §9]). It seems worth recalling here that the standard unitary Eisenstein series $E_{\mathfrak{a}, 1/2+it}$, as given in the simplest

case $\mathfrak{a} = \infty$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by

$$E_s(z) := \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2-\{(0,0)\}: \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}} = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \cdots \qquad (12)$$

for $\mathrm{Re}(s) > 1$, vanishes like $O(t)$ as $t \to 0$, while $\xi(1+2it)$ has a simple pole at $t = 0$, so the normalized variant $E^*_{\mathfrak{a},1/2+it}$ is well-defined and $E^*_{\mathfrak{a},1/2}$ is not identically zero.

(2) From the expansion (11) we deduce first that $\langle |\theta|^2, 1 \rangle = 1$ and then that

$$\langle |\theta|^2, \varphi \rangle = \langle |\theta|^2, 1 \rangle \langle 1, \varphi \rangle + \sum_{\mathfrak{a}} \int_{t\in\mathbb{R}} c(\mathfrak{a}, t) \langle E^*_{\mathfrak{a},1/2+it}, \varphi \rangle \frac{dt}{2\pi}. \qquad (13)$$

(3) We deduce Theorem 1 in the expected way by replacing φ with $T_n\varphi$ and appealing to the consequence $\langle E^*_{\mathfrak{a},1/2+it}, T_n\varphi \rangle \ll (1+|t|)^{-100}\tau(n)n^{-1/2}$ of standard bounds for unitary Eisenstein series and the rapid decay of $\xi(1+2it)$.

The main analytic difference between the integrands on the RHS of (2) and (13) is in their $t \to 0$ behavior: for nice enough φ, φ_1, φ_2, the typical magnitude of the integrand in (2) is $\asymp |t|$ while that in (13) is $\asymp 1$. The more glaring difference between the two expansions is that (13) contains no cuspidal contribution. The improvement of (6) compared to (3) is now explained by noting as above that the Hecke eigenvalues of unitary Eisenstein series (unlike those of cusp forms) are known to satisfy bounds consistent with the Ramanujan conjecture $\vartheta = 0$.

Remark 1 As in [18, §4.3.8] or [34], one can define a regularized inner product $\langle |\theta|^2, \varphi \rangle_{\mathrm{reg}}$ whenever φ admits an asymptotic expansion near each cusp in terms of finite functions $y^\beta \log(y)^m$ with exponent of real part $\mathrm{Re}(\beta) \neq 1/2$. The regularization takes the form $\langle |\theta|^2, \varphi \rangle_{\mathrm{reg}} := \langle |\theta|^2, \varphi - \mathcal{E} \rangle$ for an auxiliary Eisenstein series $\mathcal{E}$. The difference $\varphi - \mathcal{E}$ satisfies the growth condition (5), so the results of this paper apply directly to such regularized inner products.

Remark 2 It follows in particular from (11) that

$$|\theta|^2 \text{ is orthogonal to every cusp form}, \qquad (14)$$

as does not appear to be widely known; this feature is crucial for the application to subconvexity pursued in [22]. By taking a residue in Shimura's integral [28], (14) is equivalent to the well-known fact that $L(\mathrm{ad}\varphi, s)$ is holomorphic at $s = 1$ for every cusp form φ; compare also with [12]. The proof given here is not directly dependent on such considerations.

Remark 3 In more high-level terms, our arguments amount to viewing $|\theta|^2$ as the restriction to the first factor of a theta kernel for $(\mathrm{SL}_2, \mathrm{O}_2)$, where O_2 denotes the orthogonal group of the split binary quadratic form $(x, y) \mapsto x^2 - y^2$; the expansion (11) then amounts to the (regularized) decomposition of that kernel with respect to the O_2-action. We may then understand (11) as asserting that cusp forms do not participate in the global theta correspondence with the split O_2, as should be well-known to experts.

This paper is organized as follows. In Sect. 2, we formulate our main results. In Sect. 3, we sketch a direct proof of the simplest special case stated above (Theorem 1). The general case is treated in Sect. 6 after some local (Sect. 4) and global (Sect. 5) preliminaries.

2 Statements of main results

We refer to Sect. 5 for detailed definitions in what follows and to [7,8,29,33] for general background.

Let k be a global field of characteristic $\neq 2$ with adele ring $\mathbb{A}$. Let $\psi : \mathbb{A}/k \to \mathbb{C}^{(1)}$ be a non-trivial additive character. Denote by $\mathrm{Mp}_2(\mathbb{A})$ the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$, $\mathcal{A}(\mathrm{SL}_2)$ (resp. $\mathcal{A}(\mathrm{Mp}_2)$) the space of automorphic forms on $\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})$ (resp. $\mathrm{SL}_2(k)\backslash\mathrm{Mp}_2(\mathbb{A})$), ω the Weil representation of $\mathrm{Mp}_2(\mathbb{A})$ attached to ψ and the dual pair $(\mathrm{Mp}_2, \mathrm{O}(1))$, $\omega_\psi \ni \phi \mapsto \theta_{\psi,\phi} \in \mathcal{A}(\mathrm{Mp}_2)$ the standard theta intertwiner parametrizing the elementary theta functions, $\mathcal{I}(\chi)$ the unitary induction to $\mathrm{SL}_2(\mathbb{A})$ of a unitary character $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^{(1)}$, and $\mathrm{Eis} : \mathcal{I}(\chi) \to \mathcal{A}(\mathrm{SL}_2)$ the standard Eisenstein intertwiner defined by averaging over $\left\{ \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix} \right) \right\} \backslash\mathrm{SL}_2(k)$ and analytic continuation along flat sections. Choose a Haar measure $d^\times y$ on $\mathbb{A}^\times$, hence on $\mathbb{A}^\times/k^\times$.

Using $d^\times y$, we define in Sect. 5.11 for all nontrivial unitary characters χ of $\mathbb{A}^\times/k^\times$ an $\mathrm{Mp}_2(\mathbb{A})$-equivariant map $I_\chi : \omega_\psi \otimes \overline{\omega_\psi} \to \mathcal{I}(\chi)$; it may be regarded as a restricted tensor product of local maps, regularized by the local factors of $L(\chi, 1)$. As we explain in Sect. 5.11, the composition $\mathrm{Eis} \circ I_\chi$ makes sense for all unitary χ.

Denote by $\int_{(0)}$ the integral over unitary characters χ of $\mathbb{A}^\times/k^\times$ with respect to the measure dual to $d^\times y$. Write simply $\int$ for an integral over $\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})$ with respect to Tamagawa measure, or equivalently, the probability Haar.

The map $\omega_\psi \ni \phi \mapsto \theta_{\psi,\phi}$ has kernel given by the subspace of odd functions, so we restrict attention to the even subspace $\omega_\psi^{(+)} = \{\phi \in \omega_\psi : \phi(x) = \phi(-x)\}$. (That subspace is reducible, but its further reduction is unimportant for us.)

Theorem 2 *Let $\phi_1, \phi_2 \in \omega_\psi$ We have the pointwise expansion: for $\sigma \in Mp_2(\mathbb{A})$,*

$$\theta_{\psi,\phi_1}(\sigma)\overline{\theta_{\psi,\phi_2}}(\sigma) = \int \theta_{\psi,\phi_1}\overline{\theta_{\psi,\phi_2}} + \int_{(0)} Eis(I_\chi(\phi_1, \overline{\phi_2}))(\sigma) \qquad (15)$$

Moreover, if $\phi_1, \phi_2 \in \omega_\psi^{(+)}$, then we have the inner product formula

$$\int \theta_{\psi,\phi_1}\overline{\theta_{\psi,\phi_2}} = 2\int_{\mathbb{A}} \phi_1\overline{\phi_2}. \qquad (16)$$

Finally, for $\varphi \in \mathcal{A}(SL_2)$ satisfying the growth condition (30) analogous to (5), we have the inner product expansion

$$\int \theta_{\psi,\phi_1}\overline{\theta_{\psi,\phi_2}}\varphi = \int \theta_{\psi,\phi_1}\overline{\theta_{\psi,\phi_2}} \int \varphi + \int_{(0)} \int Eis(I_\chi(\phi_1, \overline{\phi_2}))\varphi. \qquad (17)$$

Theorem 2 specializes to (11) upon taking $k = \mathbb{Q}$ and ϕ_1, ϕ_2 as in the example of Sect. 5.4. A special case of Theorem 2 was proved by us in [19, §10]; the proofs are presented differently, and their comparison may be instructive. The contribution from the trivial character χ to the RHS of (17) should be compared with the case of Siegel–Weil discussed in [4, §7.2].

More generally, suppose $\phi_1 \in \omega_\psi, \phi_2 \in \omega_{\psi'}$ for some nontrivial characters ψ, ψ' of $\mathbb{A}/k$. One can write $\psi'(x) = \psi(ax)$ for some $a \in k^\times$. If $a \in k^{\times 2}$, then $\omega_\psi \cong \omega_{\psi'}$, and so one can study $\int \theta_{\psi,\phi_1}\overline{\theta_{\psi',\phi_2}}$ using Theorem 2. If $a \notin k^{\times 2}$, one can prove (more easily) an analogue of Theorem 2 involving dihedral forms for the quadratic space $k^2 \ni (x, y) \mapsto x^2 - ay^2$; see Sect. 5.12. One finds in particular that

$$\int \theta_{\psi,\phi_1}\overline{\theta_{\psi',\phi_2}} = 0. \qquad (18)$$

Following [18, §2], we employ unitary Sobolev norms $\mathcal{S}_d$ on ω and automorphic Sobolev norms $\mathcal{S}_d^{\mathbf{X}}$ on $\mathcal{A}(\mathrm{SL}_2)$ (see Sects. 4.6, 5.3). Denote by $\Xi : \mathrm{SL}_2(\mathbb{A}) \to \mathbb{R}_{>0}$ the Harish–Chandra spherical function (Sect. 5.13).

Theorem 3 *There exists an integer d depending only upon the degree of k so that for nontrivial characters ψ, ψ' of $\mathbb{A}/k$ and $\phi_1 \in \omega_\psi$, $\phi_2 \in \omega_{\psi'}$, $\varphi \in \mathcal{A}(\mathrm{SL}_2)$ and $\sigma \in \mathrm{SL}_2(\mathbb{A})$, the right translate $\sigma\varphi(x) := \varphi(x\sigma)$ satisfies*

$$\int \theta_{\psi,\phi_1} \overline{\theta_{\psi',\phi_2}} \cdot \sigma\varphi = \int \theta_{\psi,\phi_1} \overline{\theta_{\psi',\phi_2}} \int \varphi + O(\Xi(\sigma)\mathcal{S}_d(\phi_1)\mathcal{S}_d(\phi_2)\mathcal{S}_d^{\mathbf{X}}(\varphi)). \quad (19)$$

The implied constant depends at most upon (k, ψ, ψ').

One should understand the conclusion of Theorem 3 as follows: if θ_1, θ_2 are a pair of essentially fixed elementary theta functions, φ is an essentially fixed automorphic form on SL_2 of sufficient decay, and $\sigma \in \mathrm{SL}_2(\mathbb{A})$ traverses a sequence that eventually escapes any fixed compact, then $\langle \theta_1\overline{\theta_2}, \sigma\varphi \rangle$ tends to $\langle \theta_1\overline{\theta_2}, 1 \rangle \langle 1, \varphi \rangle$ as rapidly as the Ramanujan conjecture would predict if $\theta_1\overline{\theta_2}$ were square-integrable, and with polynomial dependence on the parameters of the "essentially fixed" quantities. An inspection of the proof also reveals polynomial dependence upon the heights of the characters ψ, ψ'. The estimate (19) can be sharpened a bit at the cost of lengthening the argument (see e.g. Remark 6), but already suffices for our intended applications; the groundwork has been laid here for the pursuit of more specialized refinements should motivation arise.

Remark 4 We have already indicated that Theorem 1 follows by specializing Theorem 2 to (13). Alternatively, the T_{n^2} case of Theorem 1 may be recovered from Theorem 3 by taking $k = \mathbb{Q}$ and for σ the finite-adelic matrix $\mathrm{diag}(n, 1/n)$. One can deduce from (17) a more general form of (19) involving an extension of $\theta_{\phi_1}\overline{\theta_{\phi_2}}$ to the similitude group $\mathrm{PGL}_2(\mathbb{A})$ which then specializes to the general case of Theorem 1; it is not clear to us how best to formulate such an extension, and our immediate applications do not require it, so we omit it.

3 Sketch of proof in the simplest case

We sketch here the proof of the expansion (11) underlying the proof of Theorem 1. Let θ be as in (4). Set $z := x + iy$, $e(z) := e^{2\pi iz}$. Consider the Fourier expansion

$$|\theta|^2(z) = y^{1/2} \sum_{m,n \in \mathbb{Z}} e(m^2 z)\overline{e(n^2 z)}$$

$$= y^{1/2} \sum_{m,n \in \mathbb{Z}} e((m^2 - n^2)x) \exp(-2\pi(m^2 + n^2)y).$$

Change variables to $\mu := m - n$ and $\nu := m + n$, so that $m^2 - n^2 = \mu\nu$ and $m^2 + n^2 = (\mu^2 + \nu^2)/2$:

$$|\theta|^2(z) = y^{1/2} \sum_{\mu,\nu \in \mathbb{Z}} e(\mu\nu x) \exp(-\pi(\mu^2 + \nu^2)y) 1_{\mu \equiv \nu(2)}.$$

Detect the condition $1_{\mu \equiv \nu(2)}$ as $\frac{1}{2} \sum_{\xi=0,1}(-1)^{\xi \mu} e^{\pi i \xi \nu}$ and apply Poisson summation to (say) the ν sum:

$$|\theta|^2(z) = \frac{1}{2} \sum_{\xi=0,1} \sum_{\mu,\nu \in \mathbb{Z}} (-1)^{\xi \mu} \exp(-\pi((\mu x + \tfrac{\xi}{2} + \nu)^2/y + \mu^2 y)) \tag{20}$$

$$= \frac{1}{2} \sum_{\mu \in \mathbb{Z}, \nu \in \frac{1}{2}\mathbb{Z}} (-1)^{\mu\nu} \exp(-\pi((\mu x + \nu)^2/y + \mu^2 y)). \tag{21}$$

For simplicity, we now consider instead of $|\theta|^2(z)$ the closely related sum

$$E(z) := \sum_{\mu,\nu \in \mathbb{Z}} \exp(-\pi((\mu x + \nu)^2/y + \mu^2 y)) \tag{22}$$

obtained by stripping (21) of its 2-adic factors. By isolating the contribution of $(\mu, \nu) = (0, 0)$ and writing the remaining pairs (μ, ν) in the form $(\lambda c, \lambda d)$ for some unique up to sign nonzero integers λ, c, d with $\gcd(c, d) = 1$, so that $(\mu x + \nu)^2/y + \mu^2 y = |cz + d|^2 \lambda^2/y$, we obtain

$$E(z) = 1 + \sum_{\substack{(c,d) \in (\mathbb{Z}^2 - \{(0,0)\})/\{\pm 1\} \\ \gcd(c,d)=1}} f\left(\frac{y}{|cz + d|^2}\right)$$

with $f(y) := \sum_{\lambda \in \mathbb{Z}-\{0\}} \exp(-\pi \lambda^2/y)$. The function f decays rapidly as $y \to 0$ and satisfies $f(y) \sim y^{1/2}$ as $y \to \infty$. Its Mellin transform $\widetilde{f}(s) := \int_{y \in \mathbb{R}_+^\times} f(y) y^{-s} d^\times y$ is given for $\mathrm{Re}(s) > 1/2$ by

$$\widetilde{f}(s) = \int_{y \in \mathbb{R}_+^\times} \sum_{\lambda \in \mathbb{Z}-\{0\}} \exp(-\pi \lambda^2/y) y^{-s} d^\times y = 2\xi(2s).$$

By Mellin inversion, $f(y) = \int_{(2)} 2\xi(2s) y^s \frac{ds}{2\pi i}$. Thus $E(z) = 1 + \int_{(2)} E_s^*(z) \frac{ds}{2\pi i}$, where $E_s^* := 2\xi(2s)E_s$ with E_s as in (12). It is known that E_s vanishes to order one as $s \to 1/2$, while E_s^* is holomorphic for $\mathrm{Re}(s) \geq 1/2$ except for a simple pole at $s = 1$ of constant residue 1. Shifting contours, we obtain

$$E(z) = 2 + \int_{(1/2)} E_s^*(z) \frac{ds}{2\pi i}. \tag{23}$$

By standard bounds on $E_s^*(z)$ that take into account the rapid decay of the factor $\Gamma(s)$,

$$\int_{(1/2)} \int_{\Gamma \backslash \mathbb{H}} \mathrm{ht}(z)^{1/2-\delta} |E_s^*(z)| < \infty.$$

Therefore (23) holds not only pointwise but also weakly when tested against functions φ satisfying (5). In particular, $\langle E, 1 \rangle = \langle 2, 1 \rangle = 2$.

The expansion (11) follows from the above argument applied to $|\theta|^2(z)$ rather than $E(z)$, or alternatively, by specializing Theorem 2; see also [19, §9].

In passing to the general results of Sect. 2, we must keep track of how more complicated variants of the 2-adic factors in (21) affect the residue arising in the contour shift. This is ultimately achieved by the inversion formula for an adelic partial Fourier transform. We must also quantify everything; we do so crudely.

Remark 5 The proof sketched above and its generalization given below is the third that we have found. The following alternative arguments are possible, but less efficient:

(1) One can realize θ as the residue as $\varepsilon \to 0$ of a 1/2-integral weight Eisenstein series $\tilde{E}_{3/4+\varepsilon}$ and then subtract off a weight zero Eisenstein series $\mathcal{E}_{1+\varepsilon}$ to regularize the inner product $\langle \bar{\theta} \tilde{E}_{3/4+\varepsilon}, \varphi \rangle_{\mathrm{reg}} := \langle \bar{\theta} \tilde{E}_{3/4+\varepsilon} - \mathcal{E}_{1+\varepsilon}, \varphi \rangle$ following the scheme of [18, §4.3.5] with some necessary modifications; the hypotheses do not literally apply, but the method can be adapted with some work. One can then extract the residue of the regularized spectral expansion of this regularized inner product to obtain the required formula for $\langle |\theta|^2, \varphi \rangle$.

(2) As in the proof of the standard Plancherel formula (see e.g. [6,9]), one can reduce first to understanding $\langle |\theta|^2, \varphi \rangle$ when φ is an incomplete Eisenstein series, expand $\varphi = \int_{(2)} c(s) E_s \frac{ds}{2\pi i}$ as an integral of spectral Eisenstein series E_s, and then shift contours to the line $\mathrm{Re}(s) = 1/2$. Difficulty arises (especially in the generality of Theorem 2) when one wishes to compare the expansion so-obtained to the spectral coefficients $\langle E_{1/2+it}, \varphi \rangle$ of φ, which are given by $c(1/2 + it) + M(1/2 + it)c(1/2 - it)$ rather than $c(1/2 + it)$ (here $M(s) := \xi(2s)/\xi(2(1-s))$). The analogous difficulty in the proof of the standard Plancherel formula is addressed by the functional equation for the intertwining operators, of which some more complicated variants are required here. The present approach is more direct.

4 Local preliminaries

4.1 Generalities

In this section we work over a local field k of characteristic $\neq 2$. Let $\psi : k \to \mathbb{C}^{(1)}$ be a nontrivial character. Equip k with the Haar measure self-dual for the character ψ_2 defined by $\psi_2(x) := \psi(2x)$.

When k is non-archimedean, we denote by $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ its maximal ideal, and $q := \#\mathfrak{o}/\mathfrak{p}$ the cardinality of its residue field.

4.2 "The unramified case"

We use this phrase to mean specifically that k is non-archimedean, ψ is unramified, and the residue characteristic of k is $\neq 2$.

4.3 Conventions on multiplicative characters

We represent the group

$$\mathfrak{X}(k^\times) := \mathrm{Hom}(k^\times, \mathbb{C}^\times)$$

of continuous homomorphisms *additively*. Denote by y^χ the value taken by the character $\chi \in \mathfrak{X}(k^\times)$ at the element $y \in k^\times$, by $0 \in \mathfrak{X}(k^\times)$ the trivial character $y \mapsto y^0 := 1$, by α the absolute value character $y \mapsto y^\alpha := |y|$, and, for any complex number s, by $s\alpha$ the character $y \mapsto y^{s\alpha} := |y|^s$; thus

$$y^{\chi_1+\chi_2} = y^{\chi_1} y^{\chi_2}, \quad (y_1 y_2)^\chi = y_1^\chi y_2^\chi, \quad y^{-\chi} = (1/y)^\chi, \quad y^0 = 1.$$

Every χ may be written uniquely as $c\alpha + \chi_0$ for some $c \in \mathbb{R}$ and χ_0 unitary; $\mathrm{Re}(\chi) := c$ is called the *real part* of χ.

4.4 The metaplectic group

Denote by $\mathrm{Mp}_2(k)$ the metaplectic double cover of $\mathrm{SL}_2(k)$, defined using Kubota cocycles [15] as the set of all pairs $(\sigma, \zeta) \in \mathrm{SL}_2(k) \times \{\pm 1\}$ with the multiplication law $(\sigma_1, \zeta_1)(\sigma_2, \zeta_2) = (\sigma_1 \sigma_2, \zeta_1 \zeta_2 c(\sigma_1, \sigma_2))$ where

$$c(\sigma_1, \sigma_2) := \left(\frac{x(\sigma_1 \sigma_2)}{x(\sigma_1)}, \frac{x(\sigma_1 \sigma_2)}{x(\sigma_2)} \right), \quad x\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \right) := \begin{cases} d & \text{if } c = 0, \\ c & \text{if } c \neq 0 \end{cases} \qquad (24)$$

with $(,) : k^\times / k^{\times^2} \times k^\times / k^{\times^2} \to \{\pm 1\}$ the Hilbert symbol. As generators for $\mathrm{Mp}_2(k)$ we take for $a \in k^\times, b \in k$ and $\zeta \in \{\pm 1\}$ the elements

$$n(b) = \left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1 \right), \quad t(a) = \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1 \right),$$

$$w = \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1 \right), \quad \varepsilon(\zeta) = (1, \zeta)$$

which satisfy the relations $n(b_1)n(b_2) = n(b_1 + b_2)$, $t(a_1)t(a_2) = t(a_1 a_2)\varepsilon((a_1, a_2))$, $t(a)n(b) = n(a^2 b)t(a)$, $wt(a) = t(a^{-1})w$, $w^2 = t(-1)$ and (when $b \neq 0$) $wn(-b^{-1}) = n(b)t(b)wn(b)w^{-1}$. Identify functions on $\mathrm{SL}_2(k)$ with their pullbacks to $\mathrm{Mp}_2(k)$.

4.5 Principal congruence subgroups

Suppose for this subsection that k is non-archimedean. Set $K_{\mathrm{SL}_2} := \mathrm{SL}_2(\mathfrak{o})$. For $m \in \mathbb{Z}_{\geq 0}$, denote by $K_{\mathrm{SL}_2}[m] := K \cap (1 + \mathfrak{p}^m M_2(\mathfrak{o}))$ the mth principal congruence subgroup. Define a map

$$\sigma : K_{\mathrm{SL}_2} \to \mathrm{Mp}_2(k)$$

$$\sigma\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \right) := \begin{cases} (c, d)^{\nu(c)} & \text{if } c \neq 0, \\ 1 & \text{if } c = 0 \end{cases}$$

where ν denotes the normalized valuation on k. There exists an m_0, depending only upon $\nu(2)$, so that the restriction of σ to $K_{\mathrm{SL}_2}[m_0]$ is a homomorphism ([8, Prop 2.8], [14]) in fact, one may take $m_0 := 0$ in the unramified case (Sect. 4.2). In general, denote for $m \geq m_0$ by $K[m] := \sigma(K_{\mathrm{SL}_2}[m]) < \mathrm{Mp}_2(k)$ the image. It defines a filtration $K[m_0] \supset K[m_0 + 1] \supset \cdots$ of congruence subgroups of $\mathrm{Mp}_2(k)$. One has $w \in K[0]$ whenever $K[0]$ is defined and $n(b), t(a) \in K[m]$ for all $b, a \in \mathfrak{p}^m, 1 + \mathfrak{p}^m$ whenever $K[m]$ is defined.

For a smooth representation V of $\mathrm{Mp}_2(k)$ and $m \geq m_0$, denote by $V[m] := V^{K[m]}$ the subspace of $K[m]$-invariant vectors. For $m < m_0$, write $V[m] := \{0\}$. Thus $\{0\} = V[-1] \subseteq V[0] \subseteq V[1] \subseteq \cdots$ and $V = \cup V[m]$.

4.6 Sobolev norms

For each integer d and unitary admissible representation V of $\mathrm{Mp}_2(k)$, denote by $\mathcal{S}_d^V$ the Sobolev norm on V defined by the recipe of [18, §2]. Strictly speaking, that article considers the case of reductive groups and not their finite covers, but the definitions and results apply verbatim in our context (using the principal subgroups defined above in the non-archimedean case).

These norms have the shape $\mathcal{S}_d^V(v) := \|\Delta^d v\|$ for a positive self-adjoint operator Δ on V, whose definition is given the archimedean case by (essentially) $\Delta := 1 - \sum_{X \in \mathcal{B}(\mathrm{LieMp}_2(k))} X^2$ and in the non-archimedean case by multiplication by q^m on the orthogonal complement in $V[m]$ of $V[m-1]$. A number of useful properties of such norms ("axioms (S1a) through (S4d)") are established in [18, §2]; for the purposes of Sect. 4, we shall need only the following:

(S1b) *The distortion property.* There is a constant κ, depending only upon $\deg(k)$, so that for all $g, v \in \mathrm{Mp}_2(k)$, V, one has $\mathcal{S}_d^V(gv) \ll \|\mathrm{Ad}(g)\|^{\kappa d} \mathcal{S}_d^V(v)$

(S4d) *Reduction to the case of Δ-eigenfunctions.* If W is a normed vector space and $\ell : V \to W$ a linear functional with the property that $|\ell(v)| \le A \|\Delta^d v\|_V$ for each Δ-eigenfunction v, then $|\ell(v)| \le A' \mathcal{S}_{d'}^V(v)$ for all $v \in V$, where (A', d') depends only upon (A, d) and $\deg(k)$.

When the representation V is clear from context, we abbreviate $\mathcal{S}_d := \mathcal{S}_d^V$.

4.7 Conventions on implied constants

Implied constants in this section are allowed to depend upon (k, ψ) except in the unramified case (Sect. 4.2), in which implied constants are required to be depend at most upon $\deg(k)$. Similarly, we abbreviate $\mathcal{S} := \mathcal{S}_d$ when d (the *implied index*) may be chosen with the above dependencies. The purpose of this convention is to ensure that implied constants are uniform when (k, ψ) traverses the local components of analogous global data.

4.8 The Weil representation

For (V, q) a quadratic space over k, the Weil representation $\omega_{\psi, V}$ of $\mathrm{Mp}_2(k)$ on the Schwartz–Bruhat space $\mathcal{S}(V)$ is defined on the generators as follows: there is a quartic character $\chi_{\psi, V} : k^\times \to \mu_4 < \mathbb{C}^{(1)}$ and an eighth root of unity $\gamma_{\psi, V} \in \mu_8 < \mathbb{C}^{(1)}$, whose precise definitions are unimportant for our purposes (see [8,11,28] for details), so that

$$\omega_{\psi, V}(n(b))\phi(x) = \psi(bq(x))\phi(x), \quad \omega_{\psi, V}(t(a))\phi(x) = a^{\chi_{\psi, V} + (d/2)\alpha}\phi(ax),$$
$$\omega_{\psi, V}(w)\phi = \gamma_{\psi, V}\phi^\wedge, \quad \omega_{\psi, V}(\varepsilon(\zeta))\phi = \zeta^{\dim V}\phi$$

where $\phi^\wedge(y) := \int_V \phi(x)\psi(\langle x, y \rangle)\, dx$ with $\langle x, y \rangle := q(x + y) - q(x) - q(y)$ and the Haar measure dx normalized so that $((\phi)^\wedge)^\wedge(x) = \phi(-x)$. The assignment $V \mapsto \omega_{\psi, V}$ is compatible with direct sums in the sense that if $V = V' \oplus V''$, then $\omega_{\psi, V} = \omega_{\psi, V'} \otimes \omega_{\psi, V''}$ with respect to the dense inclusion $\mathcal{S}(V') \otimes \mathcal{S}(V'') \hookrightarrow \mathcal{S}(V)$. The complex conjugate representation is given by $\overline{\omega_{\psi, V}} \cong \omega_{\psi, V^-}$ where $V^- := (V, -q)$ denotes the quadratic space "opposite" to $V = (V, q)$ obtained by negating the quadratic form.

We are concerned here primarily, although not exclusively, with the case that V is the one-dimensional quadratic space $V_1 \cong k$ with the quadratic form $x \mapsto x^2$. In that case, we write simply $\omega_\psi := \omega_{V_1, \psi}$. Since ψ is fixed throughout Sect. 4, we accordingly abbreviate $\omega := \omega_\psi$. It is realized on the space $\mathcal{S}(k)$. Note that the Fourier transform $\phi \mapsto \phi^\wedge$ attached above to this space differs from the "usual one" by a factor of 2 in the argument of the phase, i.e., $\phi^\wedge(x) = \int_{y \in V_1} \phi(y)\psi(2xy)\, dy$. We normalized the Haar measure on k as we did in Sect. 4.1 so that the isomorphism $V_1 \cong k$ is measure-preserving and ω is unitary. In the unramified case, the space $\omega^{K[0]}$ of $K[0]$-invariant vectors in ω is one-dimensional and spanned by the characteristic function $1_\mathfrak{o}$ of the maximal order $\mathfrak{o}$ in k.

4.9 Basic estimates in the Weil representation

The following estimate is cheap, but adequate for us.

Lemma 4 *For $\phi \in \omega$, one has $\|\phi\|_{L^1} \ll \mathcal{S}(\phi)$ and $\|\phi\|_{L^\infty} \ll \mathcal{S}(\phi)$.*

Proof The L^∞ bound follows from the L^1-bound, Fourier inversion and the distortion property applied to the Weyl element w. We turn to the L^1-bound. In the real case $k = \mathbb{R}$, there is an element Z in the complexified Lie algebra of $\mathrm{Mp}_2(k)$ so that $\omega(Z)\phi(x) = x^2\phi(x)$. By Cauchy–Schwarz, the contribution to $\|\phi\|_{L^1}$ from the range $|x| \leq 1$ is bounded by $O(\|\phi\|)$ and that from the remaining range by $\int_{x\in k:|x|>1} |\phi(x)| = \int_{x\in k:|x|>1} |x|^{-2}|\omega(Z)\phi(x)| \ll \|\omega(Z)\phi\| \ll \mathcal{S}_1(\phi)$. The complex case is similar. In the non-archimedean case, it suffices by reduction to the case of Δ-eigenfunctions (S4d) to show for each $m \geq 0$ and $\phi \in \omega[m]$ that $\|\phi\|_{L^1} \ll q^{Am}\|\phi\|$ for some absolute A. The condition $\phi \in \omega[m]$ implies that for all $b \in \mathfrak{p}^m$, one has $\omega(n(b))\phi = \phi$, that is to say, $(\psi(bx^2) - 1)\phi(x) = 0$ for all $x \in k$. Therefore ϕ is supported on elements $x \in k$ satisfying the constraint $\psi(bx^2) = 1$ for all $b \in \mathfrak{p}^m$. That constraint implies $|x| \ll q^{m/2}$, and the set of elements satisfying it has volume $O(q^{m/2})$, so Cauchy–Schwarz gives as required that $\|\phi\|_{L^1} \ll q^{m/4}\|\phi\| \ll q^{O(m)}\|\phi\|$. □

4.10 Induced representations

Denote by $\mathcal{I}(\chi)$ the unitarily normalized induction to $\mathrm{SL}_2(k)$ of a character χ of $k^\times$, realized in its induced model as a space of functions $f : \mathrm{SL}_2(k) \to \mathbb{C}$ satisfying $f(n(x)t(y)g) = y^{\alpha+\chi}f(g)$ for all $x, y, g \in k, k^\times, \mathrm{SL}_2(k)$. When χ is unitary, $\mathcal{I}(\chi)$ is unitary with respect to the norm $\|f\| := (\int_{K_{\mathrm{SL}_2}} |f|^2)^{1/2}$ for $K_{\mathrm{SL}_2} \leq \mathrm{SL}_2(k)$ the standard maximal compact subgroup equipped with the probability Haar. When χ is unitary, $\mathcal{I}(\chi)$ is irreducible if and only if χ is not a non-trivial quadratic character.

4.11 Change of polarization

Recall from Sect. 4.8 that $V_1 \cong k$ is the one-dimensional quadratic space with the form $x \mapsto x^2$ underlying ω and $V_1^- \cong k$ that with $x \mapsto -x^2$ underlying $\overline{\omega}$. We abbreviate the tensor product of ω and $\overline{\omega}$ as $\omega^2 := \omega \otimes \overline{\omega}$; it is not in any literal sense the square of the representation ω, but we shall have no occasion to refer to the latter. Then ω^2 identifies with the Weil representation ω_{ψ,V_2} attached to the quadratic space $V_2 := V_1 \oplus V_1^- \cong k^2$ equipped with the form $(x_1, x_2) \mapsto x_1^2 - x_2^2$. The latter quadratic space is split, and so by a well-known procedure (see e.g. [25, §0, (VII)]) involving a change of polarization in the symplectic space $W \otimes V_2$ underlying the construction of ω_{ψ,V_2} (here W is the symplectic space for which $\mathrm{SL}_2(k) = \mathrm{Sp}(W)$), there exists an intertwiner $\mathcal{F} : \mathcal{S}(V_2) \to \mathcal{S}(W) \cong \mathcal{S}(k^2)$ under which the representation ω^2 on the source corresponds to a natural geometric action of $\mathrm{Mp}_2(k) = \mathrm{Mp}_2(W)$ on the target:

Denote by $V_s \cong k^2$ the standard split quadratic space with the form $(y_1, y_2) \mapsto y_1 y_2$. The map $\rho : V_s \cong k^2 \to V_2 \cong k^2$ given by $\rho(y_1, y_2) := \left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$ is an isometry: if $(x_1, x_2) = \rho(y_1, y_2)$, then $x_1^2 - x_2^2 = y_1 y_2$. Define for $\phi \in \omega \otimes \overline{\omega}$ the partial Fourier transform

$$\mathcal{F}\phi(y_1, y_2) := \int_{t\in k} \phi(\rho(y_1, t))\psi(y_2 t)\, dt = \int_{t\in k} \phi\left(\frac{y_1 + t}{2}, \frac{y_1 - t}{2}\right)\psi(y_2 t)\, dt.$$

Lemma 5 *In the unramified case, $\mathcal{F}1_{\mathfrak{o}^2} = 1_{\mathfrak{o}^2}$.*

Proof By direct calculation. □

Lemma 6 *For $\sigma \in Mp_2(k)$ and $\phi \in \omega \otimes \overline{\omega}$, one has $\mathcal{F}\omega^2(\sigma)\phi(y) = \mathcal{F}\phi(y\sigma)$. Here $y\sigma$ denotes the right multiplication of y by the image of σ in $SL_2(k)$.*

Proof See for instance Jacquet–Langlands [11, Prop 1.6] or Bump [3, Prop 4.8.7]. □

4.12 The local intertwiner

Let χ be a character of $k^\times$ with $\mathrm{Re}(\chi) > -1$. By Lemma 6, the map $I_\chi : \omega \otimes \overline{\omega} \to \mathcal{I}(\chi)$ defined by the convergent integral

$$I_\chi(\phi)(\sigma) = \int_{y \in k^\times} y^{\alpha + \chi} \mathcal{F}\phi(ye_2\sigma) \, d^\times y$$

is equivariant. The normalized local Tate integral $\chi \mapsto I_\chi(\phi)/L(\chi, 1)$ extends to an entire function of χ. We will ultimately only need to consider the range $\mathrm{Re}(\chi) \geq 0$.

Lemma 7 *Suppose we are in the unramified case (Sect. 4.2). Let χ be an unramified character of $k^\times$ with $\chi \neq -\alpha$. Then $I_\chi(1_\mathfrak{o} \otimes 1_\mathfrak{o})$ is the K-invariant vector taking the value $L(\chi, 1)$ at the identity.*

Proof Our assumptions imply by Lemma 5 that $\mathcal{F}\phi = 1_{\mathfrak{o}^2}$, so we conclude by the standard evaluation of unramified local Tate integrals. □

Proposition 8 *Suppose χ is unitary, so that $\mathcal{I}(\chi)$ is unitary. For each d there exists d' so that for all $\phi = \phi_1 \otimes \overline{\phi_2} \in \omega^2$,*

$$\mathcal{S}_d^{\mathcal{I}(\chi)}(I_\chi(\phi)) \ll \mathcal{S}_{d'}^\omega(\phi_1)\mathcal{S}_{d'}^\omega(\phi_2).$$

Proof By the equivariance of I_χ, we reduce to showing that $\|I_\chi(\phi)\| \ll \mathcal{S}(\phi_1)\mathcal{S}(\phi_2)$. The norm on $I_\chi(\phi)$ is given by integration over the maximal compact, so by the distortion property (S1b) and – once again – the equivariance of I_χ, we reduce to establishing the pointwise bound $I_\chi(\phi)(1) \ll \mathcal{S}(\phi)$. But

$$I_\chi(\phi)(1) \ll \int_{y \in k^\times} y^{\alpha + \chi} \mathcal{F}\phi(ye_2) \, d^\times y \ll \int_{x \in k} |\mathcal{F}\phi(0, x)| \, dx \ll \|\phi_1\|_{L^1}\|\phi_2\|_{L^1},$$

so we conclude by Lemma 4. □

5 Global preliminaries

5.1 Fields, groups, spaces, measures

Let k, $\mathbb{A}$, ψ be as in Sect. 2. In what follows, equip all discrete spaces with discrete measures and quotient spaces with quotient measures. Equip $\mathbb{A}$ with Tamagawa measure, so that $\mathrm{vol}(\mathbb{A}/k) = 1$. Fix an arbitrary Haar measure $d^\times y$ on $\mathbb{A}^\times$. Denote by $\mathfrak{p}$ a typical place of k. Then for all but finitely many $\mathfrak{p}$, the pair $(k_\mathfrak{p}, \psi_\mathfrak{p})$ will be in the "unramified case" (Sect. 4.2).

Denote by $\mathrm{Mp}_2(\mathbb{A})$ the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$; it is the set of pairs $(\sigma, \zeta) \in \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ with respect to the multiplication law $(\sigma_1, \zeta_1)(\sigma_2, \zeta_2) =$

$(\sigma_1\sigma_2, \zeta_1\zeta_2 c(\sigma_1, \sigma_2))$ where $c(\sigma_1, \sigma_2) := \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\sigma_{1,\mathfrak{p}}, \sigma_{2,\mathfrak{p}})$ is the product of local Kubota cocycles $c_{\mathfrak{p}}$ defined in 4.4. Identify functions on $\mathrm{SL}_2(\mathbb{A})$ with their ("non-genuine") pullbacks to the double cover $\mathrm{Mp}_2(\mathbb{A})$. Define $n(b), t(a) \in \mathrm{Mp}_2(\mathbb{A})$ for $b, a \in \mathbb{A}, \mathbb{A}^\times$ as in Sect. 4.4.

Denote by $P < \mathrm{SL}_2$ the upper-triangular subgroup and $U < P$ the strictly upper-triangular subgroup. Write $e_1 := (1, 0), e_2 := (0, 1)$. Equip $U(\mathbb{A}), \mathrm{SL}_2(\mathbb{A})$ with Tamagawa measures, so that the map $U(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A}) \ni \sigma \mapsto e_2\sigma \in \mathbb{A}^2$ is measure-preserving. Equip $P(\mathbb{A})$ with the left Haar measure compatible with the natural isomorphism $P(\mathbb{A})/U(\mathbb{A}) \cong \mathbb{A}^\times$ and the chosen Haar measure on $\mathbb{A}^\times$. Set $\mathbf{X} := \mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A})$ equipped with the Tamagawa measure (i.e., the probability Haar). We retain and adapt to the adelic setting the conventions of Sect. 4.3 concerning multiplicative characters. For instance, we denote now by $y^\alpha := |y|$ the adelic absolute value of $y \in \mathbb{A}^\times$.

5.2 Siegel domains

For convergence issues, we assume basic familiarity with Siegel domains (see e.g. [27, §4] or [2,6] or [2, §12]); the reader may alternatively trust that the general analytic issues concerning convergence are not qualitatively different from those in the model example of Sect. 3. In particular, denote by $\mathrm{ht} : \mathbf{X} \to \mathbb{R}_{>0}$ the function $\mathrm{ht}(g) := \max_{\gamma \in \mathrm{SL}_2(k)} \mathrm{ht}_{\mathbb{A}}(\gamma g)$ where $\mathrm{ht}_{\mathbb{A}}$ is defined with respect to the Iwasawa decomposition by $\mathrm{ht}_{\mathbb{A}}(n(x)t(y)k) := |y|^{1/2}$. Then $\mathrm{ht}(x) \geq c > 0$ for some $c > 0$ depending only upon k; moreover, ht is proper.

5.3 Sobolev norms

We briefly recall the adelic Sobolev norms introduced in [18, §2] which were inspired in turn by [1,31]. For an integer d and a unitary admissible representation V of $\mathrm{Mp}_2(\mathbb{A})$, define the Sobolev norm $\mathcal{S}_d^V$ on V by the formula $\mathcal{S}_d^V(v) := \|\Delta^d v\|$, where Δ denotes the restricted tensor product of the operators defined in Sect. 4.6. This definition applies also to $\mathrm{SL}_2(\mathbb{A})$-modules, which we regard as non-genuine $\mathrm{Mp}_2(\mathbb{A})$-modules. These norms take finite values on smooth vectors and apply in particular when $V = L^2(\mathbf{X})$, but for that space, a finer Sobolev norm $\mathcal{S}_d^{\mathbf{X}}$ is also useful: for $f \in C^\infty(\mathbf{X})$, set $\mathcal{S}_d^{\mathbf{X}}(f) := \|\mathrm{ht}^d \Delta^d f\|_{L^2(\mathbf{X})}$. We omit the superscript, writing $\mathcal{S}_d^V := \mathcal{S}_d$, when V is clear from context. The indices d, d' appearing here and below are implicitlyw restricted to depend only upon $\deg(k)$. We employ as in Sect. 4.6 and [18, §2] the convention of omitting the index when it is implied. Note that $\mathcal{S}_d, \mathcal{S}_{-d}$ are dual; indeed, for $u, v \in V$,

$$\langle u, v \rangle = \langle \Delta^d u, \Delta^{-d} v \rangle. \tag{25}$$

As in [18, §2.6.5], we set

$$C_{\mathrm{Sob}}(V) := \inf_{v \in V, \|v\|=1} \|\Delta v\|. \tag{26}$$

We now record some specialized and annotated forms of the axioms from [20, §2] relevant for Sect. 5:

(S1c) *Sobolev embedding.* Let V be a unitary irreducible admissible representation of $\mathrm{Mp}_2(\mathbb{A})$. Then for each d there exists a d' so that the inclusion of Hilbert spaces $(V, \mathcal{S}_{d'}^V) \to (V, \mathcal{S}_d^V)$ is trace class; moreover (see [18, §2.6.3]), there exists d_0 so that the trace of Δ^{-d_0} is bounded uniformly in V (with the global field k held fixed).

Concretely, this gives implications of the shape

$$\langle u, v \rangle \ll \mathcal{S}_{-d}(u)C(d') \implies \mathcal{S}_d(v) \ll C(d') \tag{27}$$

which read more precisely "given a vector $v \in V$ and a system of scalars $C(d') \geq 0$, if for each d there exists a d' so that the first estimate holds for all $u \in V$, then for each d there exists a d' so that the second estimate holds." Indeed, choosing d_0 so that the sum $\sum_{u \in \mathcal{B}} \mathcal{S}_{-d_0}(u)^2$, taken over u in an orthonormal basis $\mathcal{B}$ of Δ-eigenfunctions in V, is finite, and applying our hypothesis with $d + d_0$ in place of d, we obtain for some d' that

$$\mathcal{S}_d(v)^2 = \sum_{u \in \mathcal{B}} |\langle \Delta^d u, v \rangle|^2 \leq C(d')^2 \sum_{u \in \mathcal{B}} S_{-d_0}(u)^2 \ll C(d')^2,$$

as required. In particular, given an $\mathrm{Mp}_2(\mathbb{A})$-equivariant map $j : W \to V$, we have

$$\langle v, j(w) \rangle \ll \mathcal{S}(v)\mathcal{S}(w) \implies \mathcal{S}_d(j(w)) \ll \mathcal{S}_{d'}(w). \tag{28}$$

Here $v, w \in V, W$; d' depends only upon d. (Precisely, if the estimate on the LHS holds for all v, w and some implied index d_0, then for each d there exists a d' so that the estimate on the RHS holds for all v, w.) Indeed, the hypothesis of (28) is that $\langle v, j(w) \rangle \ll \mathcal{S}_{d_0}(v)\mathcal{S}_{d_0}(w)$ for some d_0. By (25), the estimate

$$\langle v, j(w) \rangle = \langle \Delta^{-d-d_0} v, \Delta^{d+d_0} j(w) \rangle \ll \mathcal{S}_{-d}(v)\mathcal{S}_{d'}(w)$$

holds with $d' := d + 2d_0$. By (27), we obtain the conclusion of (28). The same argument gives for $W = L^2(\mathbf{X})$ that

$$\langle v, j(w) \rangle \ll \mathcal{S}(v)\mathcal{S}^{\mathbf{X}}(w) \implies \mathcal{S}_d(j(w)) \ll \mathcal{S}_{d'}^{\mathbf{X}}(w). \tag{29}$$

(S1d) *Linear functionals can be bounded place-by-place.* Let $\pi = \otimes \pi_\mathfrak{p}$ be a unitary irreducible[1] admissible representation of $\mathrm{Mp}_2(\mathbb{A})$. Let $\ell_\mathfrak{p} : \pi_\mathfrak{p} \to \mathbb{C}$ be functionals indexed by the places $\mathfrak{p}$ of k with the property that for all $\mathfrak{p}$ for which $(k_\mathfrak{p}, \psi_\mathfrak{p})$ is unramified in the sense of Sect. 4.2 and for which there exists a spherical unit vector $v_\mathfrak{p} \in \pi_\mathfrak{p}$, we have $|\ell(v_\mathfrak{p})| \leq 1$; assume also that $\ell_\mathfrak{p}(v_\mathfrak{p}) = 1$ for all $\mathfrak{p}$ outside some finite set. Let $\ell = \prod \ell_\mathfrak{p} : \pi \to \mathbb{C}$ be the restricted product of these functionals. Suppose for some $A > 0$ and $d \in \mathbb{Z}$ that $|\ell_\mathfrak{p}| \leq A\mathcal{S}_d^{\pi_\mathfrak{p}}$ holds for all $\mathfrak{p}$. Then $|\ell| \leq A'\mathcal{S}_{d'}^\pi$, where A', d' depend only upon A, d. In particular, a product of implied constants coming from the finite set of places at which a vector $v \in \pi$ ramifies can always be absorbed into $\mathcal{S}_d^\pi(v)$ if d is large enough.

This axiom applies also to multilinear forms. For instance, if $\ell = \prod \ell_\mathfrak{p} : \pi \otimes \overline{\pi} \to \mathbb{C}$ has the property that $\ell_\mathfrak{p}$ is bounded in magnitude by 1 on spherical unit vectors in the unramified case and satisfies $|\ell_\mathfrak{p}(v_{1,\mathfrak{p}}, v_{2,\mathfrak{p}})| \leq A\mathcal{S}_d(v_{1,\mathfrak{p}})\mathcal{S}_d(v_{2,\mathfrak{p}})$ in general, then $|\ell(v_1, v_2)| \leq A'\mathcal{S}_{d'}(v_1)\mathcal{S}_{d'}(v_2)$ with notation as above; see [18, Remark 2.6.3] and [18, §4.4.1] for details.

(S2a) *Automorphic Sobolev inequality.* There exists d_0 so that $\mathcal{S}_{d_0}^{\mathbf{X}}$ majorizes L^∞-norms.

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[1] Irreducibility is not mentioned explicitly in the hypotheses of (S1d) in [18, §2], but is used in the proof and necessary for the truth of the statement; alternatively, one could restrict to admissible subrepresentations of the space of automorphic forms.

5.4 The basic Weil representation and elementary theta functions

Denote by ω_ψ the Weil representation of $\mathrm{Mp}_2(\mathbb{A})$ on the Schwartz–Bruhat space $\mathcal{S}(\mathbb{A})$ underlying the dual pair $\mathrm{Mp}_2 \times \mathrm{O}_1$ and with respect to the additive character ψ; to dispel any ambiguity, we record that $\omega_\psi(n(b))\phi(x) = \psi(bx^2)\phi(b)$ for $\phi \in \omega$ and $b \in \mathbb{A}$. It is the restricted tensor product of Weil representations of the local metaplectic groups defined in Sect. 4.8. Write $\omega_\psi^2 := \omega_\psi \otimes \overline{\omega_\psi}$.

For $\phi \in \omega_\psi$, the corresponding elementary theta function $\theta_{\psi,\phi} : [\mathrm{Mp}_2] \to \mathbb{C}$ is defined by the absolutely convergent sum

$$\theta_{\psi,\phi}(\sigma) := \sum_{\alpha \in k} \omega_\psi(\sigma)\phi(\alpha).$$

Although we have defined $\theta_{\psi,\phi}$ as a function on $[\mathrm{Mp}_2]$, it is perhaps more natural to regard it here as the restriction of a theta kernel to elements of the form $(\sigma, 1)$ in the product $[\mathrm{Mp}_2] \times O_1(k) \backslash O_1(\mathbb{A})$.

Set $\omega_\psi^{(+)} := \{\phi \in \omega : \phi(x) = \phi(-x) \text{ for all } x \in \mathbb{A}\}$. Its orthogonal complement is known to be the kernel of $\phi \mapsto \theta_{\psi,\phi}$. Except in Sect. 5.12, we consider only one value of ψ; we accordingly abbreviate $\omega := \omega_\psi$, $\omega^2 := \omega_\psi^2$, $\theta_\phi := \theta_{\psi,\phi}$, etc.

By computing Fourier expansions on a Siegel domain, one finds that $|\theta(\sigma)| \ll \mathrm{ht}(\sigma)^{1/4}$, so if φ is a measurable function on $\mathbf{X}$ satisfying

$$\varphi(\sigma) \ll \mathrm{ht}(\sigma)^{1/2-\delta} \text{ for some fixed} \delta > 0, \tag{30}$$

then the integral $\int_\mathbf{X} \theta_{\phi_1} \overline{\theta_{\phi_2}} \varphi$ converges absolutely for all $\phi_1, \phi_2 \in \omega$.

Example. For $k := \mathbb{Q}$, $\psi_\infty(x) = e^{2\pi i x}$ and $\phi = \otimes \phi_v$ with $\phi_\infty(x) := e^{-2\pi x^2}$, $\phi_p := 1_{\mathbb{Z}_p}$, one has for $x, y \in \mathbb{R}$, $\mathbb{R}_+^\times$ with $z := x + iy$ that $\theta_\phi(n(x)t(y^{1/2})) = y^{1/4} \sum_{n \in \mathbb{Z}} \exp(2\pi i n^2 z)$.

5.5 Mellin transforms and Tate integrals

Recall that we have fixed a Haar measure $d^\times y$ on $\mathbb{A}^\times$. It defines a quotient measure, which we also denote by $d^\times y$, on $\mathbb{A}^\times/k^\times$, hence a dual measure $d\chi$ on the space of characters $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^\times$ of given real part $\mathrm{Re}(\chi) = c$ (defined as in Sect. 4.3), so that the Mellin inversion formula $f(1) = \int_{(c)} f^\wedge(\chi) \, d\chi$ holds for all $f \in C_c^\infty(\mathbb{A}^\times/k^\times)$ with the Mellin transform defined by $f^\wedge(\chi) := \int_{y \in \mathbb{A}^\times/k^\times} f(y) y^{-\chi} \, d^\times y$.

We summarize here some standard consequences of the theory of Tate integrals (see [26,30]). For a Schwartz–Bruhat function $\phi \in \mathcal{S}(\mathbb{A})$ and a character χ of $\mathbb{A}^\times/k^\times$ with $\mathrm{Re}(\chi) > 1$, the integral $\int_{y \in \mathbb{A}^\times} y^\chi \phi(y) \, d^\times y$ converges absolutely for $\mathrm{Re}(\chi) > 1$ and extends meromorphically to all χ. We denote by $\int_{y \in \mathbb{A}^\times}^{\mathrm{reg}} y^\chi \phi(y) \, d^\times y$ that meromorphic extension. The possible poles are at $\chi = \alpha$ and $\chi = 0$. One has the global Tate functional equation

$$\int_{y \in \mathbb{A}^\times}^{\mathrm{reg}} y^\chi \phi(y) \, d^\times y = \int_{y \in \mathbb{A}^\times}^{\mathrm{reg}} y^{\alpha-\chi} \phi^\wedge(y) \, d^\times y \tag{31}$$

where $\phi^\wedge$ denotes the Fourier transform with respect to any non-trivial additive character of $\mathbb{A}/k$, such as ψ or ψ_2; it does not matter which.

5.6 Induced representations and Eisenstein series

For each character $\chi : \mathbb{A}^\times \to \mathbb{C}^{(1)}$ whose local components have real part ≥ 0, denote by $\mathcal{I}(\chi)$ its unitary induction to $SL_2(\mathbb{A})$, which consists of smooth functions $f : SL_2(\mathbb{A}) \to \mathbb{C}$ satisfying $f(n(x)t(y)\sigma) = y^{\chi+\alpha} f(\sigma)$ for $x, y, \sigma \in \mathbb{A}, \mathbb{A}^\times, SL_2(\mathbb{A})$; it is the restricted tensor product of the representations defined in Sect. 4.10. When χ is trivial on $k^\times$, so that it defines an automorphic unitary character $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^{(1)}$, denote by $\mathrm{Eis}_\chi : \mathcal{I}(\chi) \to \mathcal{A}(SL_2)$, or simply $\mathrm{Eis} := \mathrm{Eis}_\chi$ when χ is clear from context, the standard Eisenstein intertwiner obtained by averaging over $P(k)\backslash SL_2(k)$ and analytic continuation along holomorphic sections (see e.g. [6]). When χ is unitary, so is $\mathcal{I}(\chi)$, and we equip it with the product of the invariant norms defined in Sect. 4.10.

As in Sect. 4.10, the representation $\mathcal{I}(\chi)$ is reducible when χ is a nontrivial quadratic character; the results of Sect. 5.3 nevertheless apply, either by continuity from the irreducible case or by inspection of the proofs.

The Eisenstein intertwiner Eis_χ has a simple zero for χ the trivial character, so the normalized variant $L(\chi, 1)\mathrm{Eis}_\chi$ makes sense for any unitary χ.

Lemma 9 [18, §2.5.1] *There exists d_0 so that $\int_{(0)} C_{Sob}(\mathcal{I}(\chi))^{-d_0} < \infty$.*

5.7 The residue of the Eisenstein intertwiner

The residue of the association $\chi \mapsto \mathrm{Eis}_\chi : \mathcal{I}(\chi) \to \mathcal{A}(SL_2)$ as χ approaches the character $\alpha = |.|^1$ is given by "integration over $P(\mathbb{A})\backslash SL_2(\mathbb{A})$" in the following sense (see e.g. [6]):[2] Let $f_\chi \in \mathcal{I}(\chi)$ be a holomorphic family defined in a vertical strip containing the character $\chi = \alpha$. Suppose also that f_χ has sufficient decay as $C(\chi) \to \infty$. Then for $\sigma \in SL_2(\mathbb{A})$,

$$\int_{(1+\varepsilon)} \mathrm{Eis}(f_\chi)(\sigma) \, d\chi = \int_{(1-\varepsilon)} \mathrm{Eis}(f_\chi)(\sigma) \, d\chi + \int_{P(\mathbb{A})\backslash SL_2(\mathbb{A})} f_\alpha, \qquad (32)$$

where $\int_{P(\mathbb{A})\backslash SL_2(\mathbb{A})}$ denotes the equivariant functional $\mathcal{I}_\chi(\alpha) \to \mathbb{C}$ compatible with the chosen Haar measures on $P(\mathbb{A})$ and $SL_2(\mathbb{A})$. As a "dimensionality test," note that both $d\chi$ and $\int_{P(\mathbb{A})\backslash SL_2(\mathbb{A})}$ scale inversely with respect to the measure $d^\times y$ on $\mathbb{A}^\times$.

5.8 Bounds for the Eisenstein intertwiner

Lemma 10 *Let χ be a unitary character of $\mathbb{A}^\times/k^\times$. For $f \in \mathcal{I}(\chi)$ and $\varphi \in \mathcal{A}(SL_2)$,*

$$L(\chi, 1)\langle \mathrm{Eis}(f), \varphi \rangle_{L^2(\mathbf{X})} \ll \mathcal{S}(f)\|\varphi\|_\infty.$$

Proof It suffices to estimate the integral of $|L(\chi, 1)\mathrm{Eis}(f)\varphi|$ over a Siegel domain. For χ close to the trivial character (i.e., near the pole of $L(\chi, 1)$), we estimate the constant term of $L(\chi, 1)\mathrm{Eis}(f)$ as in the proof of [18, (4.12)] and its Whittaker function using the argument of [18, (3.23)] and that linear functionals can be bounded place-by-place (S1d), giving

$$L(\chi, 1)\mathrm{Eis}(f)(g) \ll \mathcal{S}(f)\mathrm{ht}(g)^{1/2} \log(\mathrm{ht}(g) + 10). \qquad (33)$$

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2 Strictly speaking, the cited reference discusses Eisenstein series on GL_2. The methods give what we state here for SL_2: it suffices to prove the identity after testing both sides against an incomplete Eisenstein series $\mathrm{Eis}(h)$, $h \in C_c^\infty(N(\mathbb{A})P(k)\backslash SL_2(\mathbb{A}))$, and follows in that case by unfolding the summation defining h and computing the residue of the standard intertwining operator on $\mathcal{I}(\chi)$.

For χ away from the trivial character, we first argue similarly that $\mathrm{Eis}(f)(g) \ll \mathcal{S}(f)\mathrm{ht}(g)^{1/2}$ and then use the coarse bound $L(\chi, 1) \ll C(\chi)^{O(1)} \ll C_{\mathrm{Sob}}(\mathcal{I}(\chi))^{O(1)}$ as in [18, §4.1.8] to absorb the dependence upon χ into the factor $\mathcal{S}(f)$ at the cost of increasing its implied index. Thus (33) holds for any χ, and so

$$L(\chi, 1)\langle \mathrm{Eis}(f), \varphi \rangle_{L^2(\mathbf{X})} \ll \mathcal{S}(f) \int_{\mathbf{X}} \mathrm{ht}(g)^{1/2} \log(\mathrm{ht}(g) + 10) |\varphi|(g).$$

Since $\int_{\mathbf{X}} \mathrm{ht}(g)^{1/2} \log(\mathrm{ht}(g) + 10) < \infty$, the conclusion follows. $\qquad\square$

5.9 Bounds for the Eisenstein projector

For $\varphi \in L^\infty(\mathbf{X})$ there exists, by duality, an element $\Pi_\chi(\varphi) \in \mathcal{I}(\chi)$ so that for all $f \in \mathcal{I}(\chi)$,

$$\langle f, \Pi_\chi(\varphi) \rangle_{\mathcal{I}(\chi)} = \langle \mathrm{Eis}(f), \varphi \rangle_{L^2(\mathbf{X})}. \tag{34}$$

The map Π_χ is linear and equivariant. By continuity and the discussion of Sect. 5.6, we may consider $L(\chi, 1)\Pi_\chi$ even when $\chi = 0$.

Lemma 11 *Let χ be a unitary character of $\mathbb{A}^\times/k^\times$. For any d there exists d' so that*

$$L(\chi, 1)\mathcal{S}_d(\Pi_\chi(\varphi)) \ll \mathcal{S}_{d'}^{\mathbf{X}}(\varphi).$$

Proof By (34), Lemma 10 and the automorphic Sobolev inequality (S2a), we have $\langle f, \Pi_\chi(\varphi) \rangle \ll \mathcal{S}(f)\mathcal{S}^{\mathbf{X}}(\varphi)$. The conclusion follows from Sobolev embedding (S1c) in the form (29). $\qquad\square$

5.10 Change of polarization

Recall that $\omega^2 := \omega \otimes \overline{\omega}$; it descends to a representation of $\mathrm{SL}_2(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}) \otimes \mathcal{S}(\mathbb{A}) \cong \mathcal{S}(\mathbb{A}^2)$. Define the partial Fourier transform $\mathcal{F} : \omega^2 \to \mathcal{S}(\mathbb{A}^2)$ by taking the restricted tensor product of the local maps defined in Sect. 4.11, thus $\mathcal{F}\phi(y_1, y_2) = \int_{t \in \mathbb{A}} \phi(\frac{y_1 + t}{2}, \frac{y_1 - t}{2})\psi(y_2 t)$. By Lemma 6,

$$\mathcal{F}(\omega^2(\sigma)\phi)(x) = \mathcal{F}\phi(x\sigma). \tag{35}$$

For $\phi = \phi_1 \otimes \overline{\phi_2}$, Fourier inversion gives $\mathcal{F}\phi(0) = \int_{x \in \mathbb{A}} \phi_1(x)\overline{\phi_2}(x)$ and $\int_{\mathbb{A}^2} \mathcal{F}\phi = \int_{x \in \mathbb{A}} \phi_1(x)\overline{\phi_2}(-x)$, whence

$$\mathcal{F}\phi(0) + \int_{\mathbb{A}^2} \mathcal{F}\phi = 2\langle \phi_1, \phi_2 \rangle \text{ for } \phi_1, \phi_2 \in \omega^{(+)}. \tag{36}$$

5.11 The regularized global intertwiner

Let $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^\times$ be a nontrivial Hecke character with $\mathrm{Re}(\chi) > -1$. We now define an $\mathrm{Mp}_2(\mathbb{A})$-equivariant map $I_\chi : \omega \otimes \overline{\omega} \to \mathcal{I}(\chi)$. It may be characterized most simply as the unique equivariant map for which

$$I_\chi(\phi_1 \otimes \phi_2)(1) = \int_{y \in \mathbb{A}^\times}^{\mathrm{reg}} y^{-\chi}\phi_1(y)\phi_2(y). \tag{37}$$

By the global Tate functional equation (31) and the identity (35), we may recast this definition in the following form, which is better suited for our purposes:

$$I_\chi(\phi)(\sigma) := \int_{y \in \mathbb{A}^\times}^{\text{reg}} y^{\alpha + \chi} \mathcal{F}\phi(ye_2\sigma) \qquad (38)$$

Equivalently, I_χ is the restricted tensor product of the local maps defined in Sect. 4.12, regularized by the local factors $L(\chi_\mathfrak{p}, 1)$: for pure tensors $\phi = \otimes\phi_\mathfrak{p}$ and a good factorization $d^\times y = \prod d^\times y_\mathfrak{p}$, one has by Lemma 7

$$I_\chi(\phi)(\sigma) = \Lambda(\chi, 1) \prod_\mathfrak{p} \frac{I_{\chi_\mathfrak{p}}(\phi_\mathfrak{p})}{L(\chi_\mathfrak{p}, 1)} = L^{(S)}(\chi, 1) \prod_{\mathfrak{p} \in S} I_{\chi_\mathfrak{p}}(\phi_\mathfrak{p})$$

for S a finite set of places taken large enough in terms of ϕ_1, ϕ_2, σ. In particular, the map $\chi \mapsto L(\chi, 1)^{-1} I_\chi(\phi)$ extends by continuity to all unitary χ; similarly, $\text{Eis}_\chi(I_\chi(\phi))$ makes sense for any unitary χ.

Lemma 12 *For any d, d_0 there exists d' so that for all $\phi = \phi_1 \otimes \overline{\phi_2} \in \omega \otimes \overline{\omega}$ and all nontrivial unitary $\chi : \mathbb{A}^\times \to \mathbb{C}^{(1)}$, one has*

$$L(\chi, 1)^{-1} \mathcal{S}_d(I_\chi(\phi)) \ll \mathcal{S}_{d'}(\phi_1) \mathcal{S}_{d'}(\phi_2) C_{\text{Sob}}(\mathcal{I}(\chi))^{-d_0}.$$

Proof The definition (26) of C_{Sob} implies that $\mathcal{S}_d(I_\chi(\phi)) \ll \mathcal{S}_{d+d_0}(I_\chi(\phi)) C_{\text{Sob}}(\mathcal{I}(\chi))^{-d_0}$, so it suffices to show that for any d there exists d' so that $L(\chi, 1)^{-1} \mathcal{S}_d(I_\chi(\phi)) \ll \mathcal{S}_{d'}(\phi_1) \mathcal{S}_{d'}(\phi_2)$. By Sobolev embedding (S1c) we reduce to showing that for each d there exists d' so that for each factorizable $f = \otimes f_\mathfrak{p} \in \mathcal{I}(\chi)$, one has

$$L(\chi, 1)^{-1} \langle f, I_\chi(\phi) \rangle \ll \mathcal{S}_{-d}(f) \mathcal{S}_{d'}(\phi_1) \mathcal{S}_{d'}(\phi_2).$$

This follows from the fact that linear functionals can be bounded place-by-place (S1d) applied to a suitable multiple of the factorizable functional $\ell = \prod \ell_\mathfrak{p}$ defined by $\ell(\phi) := \langle f, I_\chi(\phi) \rangle / (L(\chi, 1) \mathcal{S}_{-d}(f))$, the required local bounds following from Proposition 8, Lemma 7, the duality (25), and the estimate $L(\chi_\mathfrak{p}, 1) \asymp 1$ for the local factors at finite places $\mathfrak{p}$; compare with the proof of [18, (4.25)]. □

5.12 The anisotropic case

Let $\psi, \psi' : \mathbb{A}/k \to \mathbb{C}^{(1)}$ be nontrivial characters. There exists $a \in k^\times$ so that $\psi'(x) = \psi(ax)$. Assume that $a \notin k^{\times 2}$. Then $\omega_\psi \not\cong \omega_{\psi'}$. Denote by (V, q) the quadratic space k^2 with the form $q(x, y) := x^2 - ay^2$; it is non-split. Then

$$\omega_\psi \otimes \overline{\omega_{\psi'}} \cong \omega_{\psi, V}. \qquad (39)$$

Denote by $\text{SO}(V)$ be the special orthogonal group of V. The quotient

$$[\text{SO}(V)] := \text{SO}(V)(k) \backslash \text{SO}(V)(\mathbb{A})$$

is then a compact abelian group; equip it with the probability Haar. One has a Weil representation $\omega_{\psi, V}$ of $\text{SL}_2(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ as in Sects. 4.8, 5.4. The constructions to follow depend upon ψ, but we omit that dependence from our notation for clarity. For $\phi \in \omega_{\psi, V}$, denote by $\theta_\phi : \mathbf{X} \times [\text{SO}(V)] \to \mathbb{C}$ the theta kernel $\theta_\phi(\sigma, h) := \sum_{\delta \in V} \omega_{\psi, V}(\sigma)\phi(h^{-1}\delta)$. Denote by $A(\text{SO}(V))$ the set of characters τ of $[\text{SO}(V)]$; they are all unitary. For each $\tau \in A(\text{SO}(V))$, denote by $\theta_{\phi, \tau} : \mathbf{X} \to \mathbb{C}$ the theta function $\theta_{\phi, \tau}(\sigma) := \int_{h \in [\text{SO}(V)]} \tau^{-1}(h)\theta_\phi(\sigma, h)$ and

by $\theta(\tau)$ the theta lift $\theta(\tau) := \{\theta_{\phi,\tau} : \phi \in \mathcal{S}(V(\mathbb{A}))\}$; it defines an irreducible automorphic "dihedral" representation of $\mathrm{SL}_2(\mathbb{A})$ which is abstractly unitarizable. Fix the following unitary structure on $\theta(\tau)$:

- If $\tau \neq 1$, then $\theta(\tau)$ is cuspidal; equip it with the unitary structure compatible with its inclusion into $L^2(\mathbf{X})$.
- If $\tau = 1$ is the trivial character of $[\mathrm{SO}(V)]$, then the Siegel–Weil formula (see [32, p182] or [4] and references) implies that $\theta(\mathbf{1}) \subseteq \mathrm{Eis}(\mathcal{I}(\eta))$ where η is the (unitary) quadratic character of $\mathbb{A}^\times/k^\times$ corresponding under class field theory to the quadratic field extension $k(\sqrt{a})/k$. More precisely, one has for each $\phi \in \omega_{\psi,V}$ the identity $\theta_{\phi,\mathbf{1}}(\sigma) = \frac{1}{2}\mathrm{Eis}(J_\eta(\phi))(\sigma)$, where $\mathrm{Eis} : \mathcal{I}(\eta) \to \mathcal{A}(\mathrm{SL}_2)$ is as in Sect. 5.6 and $J_\eta : \omega_{\psi,V} \to \mathcal{I}(\eta)$ is given by $J_\eta(\phi)(\sigma) := \omega_{\psi,V}(\sigma)\phi(0)$. Equip $\theta(\mathbf{1})$ with the unitary structure coming from $\mathcal{I}(\eta)$.

Equip $\theta(\tau)$ with unitary Sobolev norms $\mathcal{S}_d^{\theta(\tau)}$ (Sect. 5.3). As in Sect. 5.6, there are linear maps $\Pi_{\theta(\tau)} : L^\infty(\mathbf{X}) \to \theta(\tau)$ such that $\langle f, \Pi(\varphi)\rangle_{\theta(\tau)} = \langle f, \varphi\rangle_{L^2(\mathbf{X})}$ for all $f \in \theta(\tau), \varphi \in L^\infty(\mathbf{X})$. (The precise choice $L^\infty(\mathbf{X})$ of domain is not particularly important.) For $\phi_1, \phi_2 \in \omega_\psi, \omega_{\psi'}$ we have by the isomorphism (39) and Fourier inversion that

$$\theta_{\phi_1}(\sigma)\overline{\theta_{\phi_2}(\sigma)} = \theta_\phi(\sigma, 1) = \sum_{\tau \in A(\mathrm{SO}(V))} \theta_{\phi,\tau}(\sigma).$$

Lemma 13 *Let $\varphi \in L^\infty(\mathbf{X})$ and notation as above.*

(i) *For $\tau \neq 1$, one has $\mathcal{S}_d^{\theta(\tau)}(\Pi_{\theta(\tau)}(\varphi)) \leq \mathcal{S}_d(\varphi)$.*

(ii) *For each d there exists d' so that $\mathcal{S}_d^{\theta(\mathbf{1})}(\Pi_{\theta(\mathbf{1})}(\varphi)) \ll \mathcal{S}_{d'}^{\mathbf{X}}(\varphi)$.*

(iii) *There exists d_0 so that $\sum_{\tau \in A(SO(V))} C_{Sob}(\theta(\tau))^{-d_0} \ll 1$.*

(iv) *For any d, d_0 there exists d' so that for all τ and all $\phi_1, \phi_2 \in \omega_\psi, \omega_{\psi'}$, one has*

$$\mathcal{S}_d^{\theta(\tau)}(\theta_\tau(\phi_1 \otimes \overline{\phi_2})) \ll \mathcal{S}_{d'}(\phi_1)\mathcal{S}_{d'}(\phi_2)C_{Sob}(\tau)^{-d_0}.$$

The implied constants are allowed to depend upon k, ψ, ψ' and hence upon V.

Proof (i): Immediate from the definitions and normalizations of unitary structures. (ii): Repeat the proof of the case $\chi = \eta$ of Lemma 11. (iii): Follows in a stronger form from [18, §2.5.1]. (iv) When $\tau = 1$, this follows from Lemma 12. A similar proof applies in the case $\tau \neq 1$; we sketch it for completeness. First, reduce formally as in the proofs of Lemmas 12 and 8 to the case $d_0 = d = 0$, then by reduction theory (see Sect. 5.2) to showing for all $x, y \in \mathbb{A}, \mathbb{A}^\times$ for which $|y| \gg 1$ and all k in a fixed compact subset of $\mathrm{SL}_2(\mathbb{A})$ that $\phi := \phi_1 \otimes \overline{\phi_2}$ satisfies $\theta_{\phi,\tau}(n(x)t(y)k) \ll \mathcal{S}(\phi_1)\mathcal{S}(\phi_2)$, then by the distortion property (Sect. 4.6) and equivariance to the case $k = 1$. By definition,

$$\theta_{\phi,\tau}(n(x)t(y)) = \int_{h \in [\mathrm{SO}(V)]} \tau^{-1}(h)|y|\eta(y) \sum_{\delta \in V} \psi(xq(\delta))\phi(h^{-1}y\delta).$$

Since $\tau \neq 1$, the inner sum may be restricted to $\delta \neq 0$. Since $[\mathrm{SO}(V)]$ is compact, we reduce to showing for all $y \in \mathbb{A}^\times$ with $|y| \gg 1$ that

$$|y| \sum_{\delta \in V-\{0\}} |\phi(h^{-1}y\delta)| \ll \mathcal{S}(\phi_1)\mathcal{S}(\phi_2),$$

uniformly for h in a compact subset of $\mathrm{SO}(V)(\mathbb{A})$. We reduce using (S1d) to the case of pure tensors $\phi_i = \otimes \phi_{i,\mathfrak{p}}$ and then by (S4d) to the case that each $\phi_{i,\mathfrak{p}}$ is a $\Delta_\mathfrak{p}$-eigenfunction.

We estimate the size and support of the non-archimedean components and decay of the archimedean components as in the proof of Lemma 4. In the number field case, we then modify y by a suitable element of $k^\times$, using the compactness of $\mathbb{A}^{(1)}/k^\times$, to reduce to showing: for L a fixed lattice in V, the quantity $\#\{\delta \in hL : \|q(\delta)\| \leq Q\} - 1$ vanishes for Q small enough and is otherwise $O(Q^{O(1)})$ for all h in a fixed compact subset of $\mathrm{SO}(V)(\mathbb{A})$, where $\|q(\delta)\|$ denotes the maximum of the norms coming from the archimedean completions. The function field case is similar. □

5.13 The Ξ-function

The Harish–Chandra function $\Xi : \mathrm{SL}_2(\mathbb{A}) \to \mathbb{R}_{>0}$ is the matrix coefficient of the normalized spherical vector in $\mathcal{I}(0)$. It controls the matrix coefficients of tempered representations in the following sense:

Lemma 14 [18, §2.5.1] *Let π be a tempered unitary representation of $\mathrm{SL}_2(\mathbb{A})$. For f_1, $f_2 \in \pi$ and $\sigma \in \mathrm{SL}_2(\mathbb{A})$, one has $\langle f_1, gf_2\rangle \ll \Xi(\sigma)\mathcal{S}(f_1)\mathcal{S}(f_2)$.*

Lemma 14 applies in particular to $\pi = \mathcal{I}(\chi)$ for any unitary character χ of $\mathbb{A}^\times/k^\times$ or to any of the non-split dihedral theta lifts $\pi = \theta(\tau)$ as in Sect. 5.12. For orientation, we record that Ξ factors as $\Xi(\sigma) = \prod_{\mathfrak{p}} \Xi_{\mathfrak{p}}(\sigma_{\mathfrak{p}})$, is left and right invariant under the standard maximal compact, and is given locally at a finite place $\mathfrak{p}$ with uniformizer ϖ and $|\varpi|^{-1} = q$ for $m \geq 0$ by $\Xi_{\mathfrak{p}}(t(\varpi^m)) = (2m+1)/q^m - 1_{m>0}(2m-1)/q^{m+1} \asymp (2m+1)/q^m$.

6 Proofs of the theorems

Proof of Theorem 2 Let $\phi_1, \phi_2 \in \omega_\psi^{(+)}$. Set $\phi := \phi_1 \otimes \overline{\phi_2} \in \omega^2 := \omega_\psi \otimes \overline{\omega_\psi}$. Let $\sigma \in \mathrm{Mp}_2(\mathbb{A})$. By the definition followed by Poisson summation,

$$\theta_{\psi,\phi_1}(\sigma)\overline{\theta_{\psi,\phi_2}}(\sigma) = \sum_{x \in k^2} \omega^2(\sigma)\phi(x) = \sum_{x \in k^2} \mathcal{F}(\omega^2(\sigma))(x).$$

By (35), the RHS equals $\sum_{x \in k^2} \mathcal{F}\phi(x\sigma)$. Since every nonzero element of k^2 is uniquely of the form $te_2\gamma$ for some $t \in k^\times$, $\gamma \in P(k)\backslash \mathrm{SL}_2(k)$, it follows that

$$\theta_{\psi,\phi_1}(\sigma)\overline{\theta_{\psi,\phi_2}}(\sigma) = \mathcal{F}\phi(0) + \mathrm{Eis}(f)(\sigma)$$

where $f(\sigma) := \sum_{t \in k^\times} \mathcal{F}\phi(te_2\sigma)$ and $\mathrm{Eis}(f)(\sigma) := \sum_{\gamma \in P(k)\backslash \mathrm{SL}_2(k)} f(\gamma\sigma)$ is an incomplete Eisenstein series. The asymptotics of f with respect to the Iwasawa height (Sect. 5.1) are analogous to those in Sect. 3. Denote by $f_\chi \in \mathcal{I}(\chi)$ the unitarily normalized Mellin transform

$$f_\chi(\sigma) := \int_{y \in \mathbb{A}^\times/k^\times} y^{-\chi-\alpha} f(t(y)\sigma)\, d^\times y = \int_{y \in \mathbb{A}^\times} y^{\chi+\alpha} \mathcal{F}\phi(ye_2\sigma)\, d^\times y = I_\chi(\phi)(\sigma). \tag{40}$$

By partial integration, $f_\chi(\sigma)$ decays rapidly with respect to χ, whence the rapidly-convergent Mellin expansion

$$\mathrm{Eis}(f) = \int_{(2)} \mathrm{Eis}(f_\chi)\, d\chi. \tag{41}$$

The map $\chi \mapsto \mathrm{Eis}(f_\chi) = \mathrm{Eis}(I_\chi(\phi))$ is holomorphic for $\mathrm{Re}(\chi) \geq 0$ except for a simple pole at $\chi = \alpha$. We shift contours to the line $\mathrm{Re}(\chi) = 0$; thanks to (32), (38) and our measure normalizations, the pole contributes

$$\int_{P(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} I_\alpha(\phi) = \int_{U(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \mathcal{F}\phi(e_2\sigma) = \int_{\mathbb{A}^2} \mathcal{F}\phi.$$

In summary, we have shown that

$$\theta_{\psi,\phi_1}(\sigma)\overline{\theta_{\psi,\phi_2}}(\sigma) = \mathcal{F}\phi(0) + \int_{\mathbb{A}^2} \mathcal{F}\phi + \int_{(0)} \mathrm{Eis}(I_\chi(\phi))(\sigma)\,d\chi.$$

We conclude by (36) and reasoning similar to that following (23). □

Proof of Theorem 3 By (34) and Lemma 14 followed by Lemmas 12 and 11,

$$\int \mathrm{Eis}(I_\chi(\phi)) \cdot \sigma\varphi \ll \Xi(\sigma)\mathcal{S}(I_\chi(\phi))\mathcal{S}(\Pi_\chi\varphi) \ll \Xi(\sigma)\mathcal{S}(\phi)C_{\mathrm{Sob}}(\mathcal{I}(\chi))^{-d_0}\mathcal{S}^{\mathbf{X}}(\varphi).$$

We deduce the $\psi = \psi'$ case by integrating over χ and applying Lemma 9 and Theorem 2. The $\psi \neq \psi'$ case is proved similarly using Lemma 13. □

Remark 6 A slightly lengthier argument gives a stronger (but more complicated, and not obviously more useful) estimate with $\mathcal{S}_d^{\mathbf{X}}(\varphi_0)$ replaced by $\|\mathrm{ht}^{1/2+\varepsilon}\Delta_S^d\varphi_0\|$ for S the finite set of places $\mathfrak{p}$ for which $\sigma_{\mathfrak{p}}$ is not in the maximal compact and Δ_S the product of local Laplacians (Sect. 4.6) at those places. One can also specify more precisely the dependence upon ϕ_1, ϕ_2.

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