Rational curves on fibered varieties

Fabrizio Anella^{1,2}

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Abstract

Let X be a complex projective variety with log terminal singularities and vanishing augmented irregularity. In this paper, we prove that if X admits a relatively minimal genus-one fibration, then it contains a subvariety of codimension one covered by rational curves contracted by the fibration. We then focus on the case of varieties with numerically trivial canonical bundle and we discuss several consequences of this result.

Keywords Genus-one fibrations · Elliptic fiber spaces · Calabi–Yau varieties · Rational curves

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Introduction

The existence of rational curves on algebraic varieties is an interesting and often challenging problem. In the case of Calabi–Yau varieties, for instance, their existence is fully proved only in dimension two by Bogomolov–Mumford [28, Appendix], while in higher dimensions we have only partial results.

On K3 surfaces, any ample linear series contains rational curves; this result justifies the definition of the Beauville–Voisin class as the zero-cycle class of a point on a rational curve [3]. Defining this class in higher dimensions is much more difficult, because it is hard to find an ample divisor $H \xrightarrow{i} X$ with $i_*(\mathrm{CH}_0(H)) = \mathbb{Z}$, and we do not expect in general that there exists a divisor such that $\mathrm{CH}_0(H) = \mathbb{Z}$, e.g. H rational.

The experience with the minimal model program suggests that, even when one is mainly interested in smooth varieties, the natural setting is to allow, at least, log terminal singularities. The aim of this paper is to extend the results proven in [5] to the singular setting typical of the

✉ Fabrizio Anella
fabrizio.anella2@uniroma3.it; anella@math.uni-bonn.de

¹ Department of Mathematics and Physics, Roma Tre University, Largo San Leonardo Murialdo 1, 00146 Rome, Italy

² Present Address: Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

minimal model program. Some of the techniques we use lead us to prove some new results also in the smooth case.

The main result of this paper is the following.

Theorem 1 *Let X be a complex normal projective variety of dimension n with log terminal singularities and vanishing augmented irregularity, i.e. the irregularity of any quasi-étale cover of X is zero. Suppose that there exists a surjective morphism ϕ from X to a variety B of dimension $n - 1$. If there exists a Cartier divisor L on B such that $\phi^*L \sim K_X$, then there exists a subvariety of codimension one in X that is covered by rational curves contracted by ϕ .*

If the variety X has numerically trivial canonical bundle, as a corollary of this theorem we have the following result.

Corollary 1 *Let X be a Calabi–Yau variety of dimension n as in Definition 5. Suppose that there exists a surjective morphism ϕ from X to a variety B of dimension $n - 1$. Then there exists a subvariety of codimension one in X that is covered by rational curves contracted by ϕ .*

We study the implications of this result on the Beauville–Bogomolov decomposition for varieties with trivial canonical bundle. Finally, in Sect. 2, we study the case of a fibration onto a curve and we prove the following result.

Theorem 2 *Let X be a Calabi–Yau variety. Suppose there exists a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor D with numerical dimension one such that $c_2(X) \cdot D = 0$ in $N^3(X)$. Then X contains a rational curve.*

1 Genus-one fibrations

1.1 Definitions and notation

In this paper, every variety will be defined over the complex numbers. We will denote by X a normal projective variety of dimension $n \geq 2$. If X has a morphism to a variety Y , then X_Z denotes the base change for $Z \rightarrow Y$. The notation and standard properties about singularities that are used in this paper can be found, for instance, in [23]. By $\mathbb{Q}$ -divisor we always mean a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. For the reader's convenience, we recall some definitions.

Definition 1 A morphism $f : Z \rightarrow Y$ between normal varieties is called *quasi-étale* if f is quasi-finite and étale in codimension one.

Remark 1 A quasi-étale morphism to a smooth variety is globally étale by purity of the branch locus.

Definition 2 The *irregularity* of a projective variety Y is the non-negative integer $q(Y) := h^1(Y, \mathcal{O}_Y)$. The *augmented irregularity* of Y is the following supremum

$$\tilde{q}(Y) := \sup\{q(Z) \mid Z \rightarrow Y \text{ is a finite quasi-étale cover}\}.$$

The inequality $\tilde{q}(Y) \geq q(Y)$ holds for any variety while, in general, the equality does not hold. Moreover, it may happen that this supremum, and thus the augmented irregularity, is infinite.

Definition 3 A *fibration* is a morphism between normal varieties with connected fibers. A *genus-one fibration* is a fibration such that the general fiber is a smooth genus-one curve. An *elliptic fibration* is a genus-one fibration with a fixed section.

Definition 4 Let $f : X \rightarrow Y$ be a surjective projective morphism between normal quasi-projective varieties. The set of *singular values* of f is the following subset of Y

$$\text{Sv}(f) = \{y \in Y \mid \dim(f^{-1}(y)) > \dim(X) - \dim(Y) \text{ or } f^{-1}(y) \text{ is singular}\}.$$

It is possible to associate to any elliptic curve a complex number, called its j -invariant. This association is modular, which means that an elliptic fibration $f : Y \rightarrow B$ comes with a rational map $j : B \dashrightarrow \mathbb{P}^1$ called the j -map, which is at least defined over the smooth values of f . Good references for standard facts about the j -map of an elliptic fibration are [17, Chap. 4, Sect. 26] and [20].

Remark 2 Consider the following two different definitions of isotriviality for a flat family. One can ask that two general fibers are isomorphic, or that the smooth fibers are isomorphic. In the general setting, the first definition is strictly more general than the second one. An example of this situation is given by a degeneration of an Hirzebruch surface F_n into an F_m with $m > n$, [29, See Example 1.2.11(iii)].

However, for elliptic fibrations, these two definitions coincide. Indeed, a smooth degeneration of an elliptic curve is again elliptic by Kodaira's table [2, Section V.7]. Since the j -invariant is constant on a dense subset of the base, then it is constant. We thus conclude that every smooth fiber is a smooth elliptic curve with the same j -invariant, so the smooth fibers are isomorphic. Since a smooth projective morphism étale-locally admits a section, the same statement holds for a smooth genus-one fibration.

1.2 Fischer–Grauert theorem

A well-known theorem proved by Fischer and Grauert [8] states that a proper holomorphic submersion with isomorphic fibers is locally a product in the complex topology. This means that, given a proper holomorphic submersion $f : X \rightarrow B$ between complex manifolds such that for any $t, s \in B$ the fibers X_t and X_s are isomorphic, for any $p \in B$ there exists a neighborhood $U_p \subset B$, open in the complex topology, such that the family $X_{U_p} \simeq X_p \times U_p$ splits in a product over the base. The same statement does not hold in the Zariski topology, as we can see in the following examples.

Example 1 Let $f : Y' \rightarrow Y$ be any finite unramified (hence étale) morphism between varieties of degree $d > 1$. For any $p \in Y$, the fiber over p is a scheme given by d distinct reduced points and thus all fibers are isomorphic. However, for any $U \subseteq Y$ open in the Zariski topology, the preimage $U' := f^{-1}(U)$ is a non-empty Zariski open subset of Y' . It follows that U' is connected, while the product between d points and U that has d connected components, thus they cannot be isomorphic.

The phenomenon shown in the above example can appear also if fibers are connected.

Example 2 Let $S \xrightarrow{f} \mathbb{P}^1$ be the family of elliptic curves E_t given locally by $y^2 = x^3 - t$ inside the $\mathbb{P}^2$ -bundle $\mathbb{P}(\mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O})$ over $\mathbb{P}^1$. It is possible to check, with a suitable change of coordinates, that over $\mathbb{C}^* \subset \mathbb{P}^1$ the fibers are all isomorphic to $y^2 = x^3 - 1$. Hence the restriction of f over $\mathbb{C}^*$ is a proper holomorphic submersion with isomorphic fibers. Moreover, the total space S is rational, as it can be seen projecting from a point outside S .

However, we claim that it cannot be Zariski locally trivial, and we prove it by contradiction. Suppose that there exists a non-empty Zariski open subset $U \subseteq \mathbb{C}^*$ which trivializes f , i.e. $f|_{S_U} : S_U \simeq U \times E \rightarrow U$ is the projection to the first component. Since S is rational, the composition with the other projection would give a dominant map $\mathbb{P}^2 \dashrightarrow E$, that does not exist.

Motivated by the general philosophy about the relation between complex topology and étale topology, one may expect that the same statement of Fischer–Grauert Theorem holds for the étale topology.

Since we were unable to find a neat reference on this subject, for the reader's convenience, we prove the following proposition that will be useful for what follows.

Proposition 1 *Let $Y \rightarrow B$ be a smooth projective morphism between smooth varieties such that for any $t \in B$ the variety Y_t is isomorphic to a fixed curve of genus 1, i.e. a smooth isotrivial genus-one fibration. Then there exists a finite étale morphism $\tilde{B} \rightarrow B$ such that the pullback $Y_{\tilde{B}} \simeq Y_t \times \tilde{B}$ is a product.*

To prove this proposition we need two preliminary results.

Lemma 1 *If $f : Y \rightarrow B$ is a smooth isotrivial elliptic fibration, then there exists a finite étale map $\tilde{B} \rightarrow B$ such that the pullback family $Y \times_B \tilde{B}$ is isomorphic to the trivial family.*

This result is [17, Corollary 26.5]. The big difference between Proposition 1 and Lemma 1 is that the family of genus-one curves in the lemma has a section.

Lemma 2 *Let $f : Y \rightarrow B$ be a projective morphism between normal varieties. Assume that B is smooth and f is étale locally trivial and the general fiber F has numerically trivial canonical bundle. Then there is a finite étale cover $B' \rightarrow B$ such that the pull back $Y_{B'} \simeq F \times B'$ is globally trivial.*

This lemma is stated and proved in [22, Lemma 17]. Combining Lemma 1 and Lemma 2, we can prove Proposition 1.

Proof of Proposition 1 We have to prove that f is étale locally trivial, i.e. for any $p \in B$ there exists an étale neighborhood U of p such that $Y_U \simeq U \times Y_p$. Choose a point $p \in B$. The morphism is smooth and projective so, locally around p , there exists a multi-section Σ of f that is étale at p .

Indeed, the local structure of smooth morphism can be described in the following way: for any point $y \in Y$ and $t = f(y)$, there exist open neighborhoods V_t and U_y with $U_y \subset f^{-1}(V_t)$ such that $f|_{U_y}$ factorizes as an étale morphism $g : U_y \rightarrow \mathbb{A}_{V_t}^d$ followed by the canonical projection $\mathbb{A}_{V_t}^d \rightarrow V_t$.

Consider a section s of $\mathbb{A}_{V_t}^d \rightarrow V_t$ and the associated fiber product $U \times_{\mathbb{A}_{V_t}^d} s(V_t)$. The image of this fiber product in U_y is the desired étale multi-section. Shrinking Σ we can suppose that the fiber product $Y_\Sigma \rightarrow \Sigma$ is a family of smooth elliptic curves and the fibers are pairwise isomorphic, so by Lemma 1 $Y_\Sigma \simeq \Sigma \times Y_p$. This proves that f is étale locally trivial. And thus the proof follows from Lemma 2. $\square$

1.3 Proof of Theorem 1

To prove Theorem 1, we start with a lemma.

Lemma 3 *Let $X \xrightarrow{\pi} B$ be a genus-one fibration. If the subvariety of singular values $Z := \text{Sv}(\pi)$ has codimension at least two, then the family π is isotrivial.*

Proof Since B is normal, it is smooth in codimension one and the subvariety $Z \cup B_{\text{sing}}$ has codimension at least two. We denote $B_0 := Z^c \cap B_{\text{reg}}$. The j -map $B \dashrightarrow \mathbb{P}^1$ is well-defined on B_0 . Moreover, the image of B_0 under this map is contained in $\mathbb{A}_{\mathbb{C}}^1$. Since B is normal and $(B_0)^c$ has codimension at least two, this map extends to a holomorphic function $j : B \rightarrow \mathbb{C}$. This function must be constant because B is projective and thus it follows that π is isotrivial. $\square$

For the reader's convenience, we state again the theorem that we are going to prove.

Theorem 1 *Let X be a normal projective variety with log terminal singularities such that $\tilde{q}(X) = 0$. Suppose that there exists a surjective morphism $\phi : X \rightarrow B$ to a variety of dimension $n - 1$. If there exists a Cartier divisor L on B such that $\phi^*L \sim K_X$, then there exists a subvariety of codimension one in X that is covered by rational curves contracted by ϕ .*

Proof The proof is divided into several steps, some of which might be already known to experts. In particular Step 1, Step 3 and Step 5 adapt arguments used in [5].

Step 1 The morphism $\phi : X \rightarrow B$ is a genus-one fibration. We can suppose, by taking the normalization of B and passing to Stein factorization, that the morphism ϕ has connected fibers and the base B is normal.

For dimensional reasons, the general fiber is a curve. Since X is normal $X_{\text{sing}} \subset X$ has codimension at least two, it follows that $\phi(X_{\text{sing}}) \subset B$ has positive codimension. Since the restriction to the regular part of X is a morphism from a smooth variety, there is a non-empty open subset $U \subset B$ where the morphism $\phi : X \cap \phi^{-1}(U) \rightarrow U$ is smooth [16, III.10.7]. Let $Z \subset B$ be the union of the singular locus of B and the singular values of ϕ , i.e. $Z := B_{\text{sing}} \cup \text{Sv}(\phi)$. Notice that $B_0 := Z^c$ and $B_0 \cap \phi(X_{\text{sing}})^c$ are non-empty open sets and the morphism $\phi_0 : X_0 := \phi^{-1}(B_0) \rightarrow B_0$ is a smooth proper surjective morphism.

Taking the determinant of the relative cotangent bundle sequence restricted to a fiber that is in the regular part of X , we get the isomorphism $K_E \sim K_{X_0}|_E$. It follows that $K_E \sim K_{X_0}|_E \sim \phi^*L|_E \sim \mathcal{O}_E$. A smooth curve with trivial canonical bundle is a genus-one curve and a smooth degeneration of a genus-one curve has again genus one [2, Section V.7], so every fiber of $\phi_0 : X_0 \rightarrow B_0$ is a smooth genus-one curve.

Step 2 Reduction to the case where the subvariety Z has codimension one in B . Suppose that every irreducible component of Z has codimension at least two. By Lemma 3 the family ϕ is isotrivial, so by Proposition 1 there exists a variety C_0 and a finite étale cover

$C_0 \xrightarrow{\tau} B_0$ such that the pullback $C_0 \times_{B_0} X_0$ is globally trivial, i.e. $C_0 \times_{B_0} X_0 \cong C_0 \times E$.

The morphism induced by the following diagram

$$\begin{array}{ccccc} C_0 \times E & \xrightarrow{\psi^{-1}} & X_0 \times_{B_0} C_0 & \xrightarrow{\tau'} & X_0 \\ & & \downarrow & & \downarrow \\ & & C_0 & \xrightarrow{\tau} & B_0 \end{array}$$

given by the composition $\alpha_0 := \psi^{-1} \circ \tau' : C_0 \times E \rightarrow X_0$ is finite and étale because τ' is the pullback of a finite étale morphism. In particular, the composition of the morphisms $C_0 \times E \xrightarrow{\alpha_0} X_0 \xrightarrow{i} X$ is quasi-finite and étale. By Zariski's Main Theorem [15, Théorème (4.4.3)], a quasi-finite morphism is always the composition of an open immersion and a finite

morphism, so there is a commutative diagram

$$\begin{array}{ccc} C_0 \times E & \xrightarrow{\alpha_0} & X_0 \\ \downarrow i' & & \downarrow i \\ Y & \xrightarrow{\alpha} & X \end{array}$$

where i' is an open immersion and α is a finite morphism. The exceptional locus of ϕ is by definition $\text{Exc}(\phi) = \{x \in X \mid \dim(\phi^{-1}(\phi(x))) > 1\}$. The variety $X_0^c = X_Z$ can be splitted as the disjoint union

$$X_Z = (X_Z \cap \text{Exc}(\phi)) \sqcup (X_Z \cap \text{Exc}(\phi)^c).$$

Since the subvariety $X_Z \cap \text{Exc}(\phi)^c$ has dimension at most $\dim(Z) + 1$ and we are assuming that $\text{cod}_B(Z) \geq 2$, the dimension of X_Z is bounded by $\dim(X_Z \cap \text{Exc}(\phi)^c) \leq n - 2$.

Since $K_X \sim \phi^*(L)$, the anticanonical bundle $-K_X$ is ϕ -nef, hence by [19, Theorem 2] the ϕ -exceptional locus is covered by rational curves contracted by ϕ (even though Kawamata does not say explicitly that the rational curves are contracted by ϕ , this is clear from the proof). This implies that if the exceptional locus of ϕ has codimension one in X , then it is a uniruled subvariety of codimension one of X . This result allows us to assume that $\text{cod}_X(X_Z) \geq 2$.

Since α is finite, then also $i'(C_0 \times E)^c$ has codimension at least two in Y . In particular, since α is étale in $i'(C_0 \times E) \subset Y$, this proves that α is a finite quasi-étale cover of X and $H^1(Y, \mathcal{O}_Y) = 0$. It follows by [23, Proposition 5.20] that Y has klt singularities.

As proved in [14, Proposition 6.9], there is an isomorphism

$$\overline{H^0(Y, \Omega_Y^{[1]})} \simeq H^1(Y, \mathcal{O}_Y).$$

By definition $\Omega_Y^{[1]}$ is a reflexive sheaf, so it is isomorphic to the sheaf of one-forms on the regular part. The variety C_0 is smooth because it is a finite étale cover of B_0 , so $\Omega_Y^{[1]} = i'_* \Omega_{C_0 \times E}^1$.

We obtain a contradiction, since

$$\begin{aligned} 0 &= \overline{H^1(Y, \mathcal{O}_Y)} \simeq H^0(Y, \Omega_Y^{[1]}) \simeq H^0(C_0 \times E, \Omega_{C_0 \times E}^1) \\ &= H^0(C_0, \Omega_{C_0}^1) \oplus H^0(E, \Omega_E^1) \neq 0; \end{aligned}$$

thus we can conclude that if there are no divisors on X covered by vertical rational curves, then Z has codimension one in B .

Step 3 Restriction to a fibration onto a curve with some singular values. Let H be a very ample divisor on B such that $(n - 2)H + L$ is globally generated. The pullback ϕ^*H is a globally generated Cartier divisor and, moreover, since ϕ has connected fibers, there is an isomorphism

$$H^0(X, \phi^*H) \simeq H^0(B, \phi_*(\phi^*H)) \simeq H^0(B, H).$$

This isomorphism implies that general elements in $|H|$ are general also in $|\phi^*(H)|$. So we can choose $n - 2$ general divisors $D_1, \dots, D_{n-2} \in |H|$ such that $C := D_1 \cap \dots \cap D_{n-2}$ is a smooth irreducible curve in B_{reg} not contained in Z and $S := \phi^{-1}(D_1) \cap \dots \cap \phi^{-1}(D_{n-2})$ is a normal surface.

With a little abuse of notation, we denote by ϕ the morphism $\phi|_S$. Since Z has codimension one, it must intersect C . Indeed $Z \cdot C = Z \cdot D_1 \cdot \dots \cdot D_{n-2} = Z \cdot H^{n-2} > 0$ because H is ample in B . This means that ϕ must have some singular fibers.

To prove the existence of a uniruled divisor in X it is sufficient to find a vertical rational curve in the general fiber over Z . Indeed, this means that the codimension one subset $\phi^{-1}(Z) \subset X$ has an irreducible component V such that the general fiber of $\phi|_V : V \rightarrow Z$ contains a rational curve, which is the thesis of the theorem.

Step 4 The case where $\phi^{-1}(p_i) \cap \text{sing}(S) \neq \emptyset$. Let $\bar{S}$ be a minimal resolution of S

$$\begin{array}{ccc} \bar{S} & & \\ \downarrow \nu & \searrow \beta & \\ S & \xrightarrow{\phi} & C. \end{array}$$

We can assume β is relatively minimal. Indeed, if there is a (-1) -curve on $\bar{S}$ contracted by β , the image of such curve is again a rational curve in S because it cannot be contracted to a point by minimality of the resolution. If there are (-1) -curves in the general surface S constructed above, then the union of such rational curves covers a divisor of X .

Let $p_1, \dots, p_k$ be the points of $C \cap Z$. The singular curves $\phi^{-1}(p_i) \subset S$ are exactly $\phi^{-1}(p_i) = \nu(\beta^{-1}(p_i))$. Since β is a minimal genus-one fibration, by Kodaira's table [2, Section V.7], a fiber of β can be a smooth genus-one curve, a sum of (possibly non-reduced) rational curves or a non-reduced genus-one curve.

If $\phi^{-1}(p_i)$ contains some singular point of S , then $\beta^{-1}(p_i) = \nu^{-1}(\phi^{-1}(p_i))$ contains an exceptional divisor of ν , and thus $\beta^{-1}(p_i)$ must be a sum of rational curves. Since not every rational curve of $\beta^{-1}(p_i)$ can be contracted in S , the curve $\phi^{-1}(p_i) = \nu(\beta^{-1}(p_i))$ must be a sum of rational curves in S .

Step 5 The case where $\phi^{-1}(p_i) \subset S_{\text{reg}}$. The curve $\phi^{-1}(p_i)$ is the central fiber of a family $S_0 \xrightarrow{\phi} \Delta$ of genus-one curves. If ϕ is not relatively minimal we can conclude as in the previous step. Since $\phi^{-1}(p_i)$ is not smooth, by Kodaira's table we can conclude that it is either a rational curve or a non-reduced irreducible genus-one curve. We need to exclude the second possibility.

By adjunction formula, the canonical bundle of S_{reg} is base point free. Indeed, the canonical bundle is the restriction of the pullback of a base point free divisor:

$$K_{S_{\text{reg}}} \sim (K_X + (n-2)\phi^*H)|_{S_{\text{reg}}} \sim \phi^*(L + (n-2)H)|_{S_{\text{reg}}}.$$

By [2, V.12.3], the canonical bundle of S_{reg} can be computed using the formula

$$K_{S_{\text{reg}}} \sim \phi^*D + \sum (m_i - 1)F_i$$

for some divisor D on the base, where the sum runs over all the multiple fiber F_i with multiplicity m_i . The restriction of the canonical bundle of S_{reg} to F_i is base point free because $K_{S_{\text{reg}}}$ is base point free. By the above formula it follows that for any i the canonical bundle restricted to F_i is

$$K_{S_{\text{reg}}}|_{F_i} \sim (\phi^*D + \sum (m_i - 1)F_i)|_{F_i} = \mathcal{O}_{F_i}((m_i - 1)F_i).$$

In particular, we can conclude that it has some sections, since it is the restriction of a base point free line bundle.

The line bundle $\mathcal{O}_{F_i}(F_i)$ is torsion of order m_i by [2, Lemma III.8.3]. A non-trivial torsion line bundle has no sections, so for any i the line bundle $\mathcal{O}_{F_i}((m_i - 1)F_i)$ must be trivial, hence the multiplicity of the fiber m_i is one for all i and this is a contradiction. $\square$

Theorem 1 is inspired by [5, Theorem 1.1]; in this paper, a similar result is proven under the additional hypothesis that X is a smooth projective manifold with finite fundamental group.

Remark 3 Let Y be a smooth projective variety with finite fundamental group, then the augmented irregularity is trivial.

Indeed, a quasi-étale cover is an étale cover for purity of the branch locus. The fundamental group of an étale cover $\tilde{Y}$ of Y is a subgroup of the fundamental group of Y , and so it has to be finite. Since the first Betti number of a variety with finite fundamental group is zero, we can conclude, by Hodge theory, that also $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$, and hence $\tilde{q}(Y) = 0$.

A consequence of the above remark is that Theorem 1 is stronger than [5, Theorem 1.1], also in the case of smooth varieties.

An interesting application of Theorem 1 is the following corollary, which was already observed in their context in [5, Corollary 1.5].

Corollary 2 *Let X be a variety with klt singularities, such that $\tilde{q}(X) = 0$, $\kappa(X) = n - 1$ and K_X semiample of exponent one, e.g. mK_X is free $\forall m \gg 0$, then X does contain a uniruled divisor.*

Proof Recall that, fixed a semiample line bundle L , the semiample exponent of L is

$$f(L) := \text{g.c.d.}\{m \in \mathbb{N} : mL \text{ is free}\}.$$

If L itself is free or mL is free $\forall m \gg 0$, then this hypothesis is satisfied. Under our hypothesis, by [24, Theorem 2.1.27], there exists a positive integer m such that $\phi_{mK_X} : X \rightarrow Y_m$ is an algebraic fiber space. Furthermore there exists an ample line bundle A on Y_m such that

$$K_X \sim \phi_{mK_X}^* A.$$

Since the Kodaira dimension of X is $n - 1$, the conclusion follows applying Theorem 1 to the Iitaka fibration of X . $\square$

We conclude this section with an application of Theorem 1.

Example 3 Fix two integer numbers $r \geq 1$ and $d \geq 2$. Consider a smooth hypersurface $X_{3,r} \subset \mathbb{P}^2 \times \mathbb{P}^d$ given as the zero locus of a bihomogeneous polynomial of bidegree $(3, r)$. Consider the natural projection $\pi : X_{3,r} \rightarrow \mathbb{P}^d$.

By Lefschetz Hyperplane Theorem $X_{3,r}$ is simply connected, and thus the augmented irregularity is 0.

By Grothendieck–Lefschetz Theorem, the Picard group of $X_{3,r}$ is isomorphic to $\text{Pic}(\mathbb{P}^2 \times \mathbb{P}^d)$ and, thus, applying adjunction formula, the canonical bundle is

$$K_{X_{3,r}} \sim \mathcal{O}_{X_{3,r}}(0, r - d - 1) \sim \pi^* \mathcal{O}_{\mathbb{P}^d}(r - d - 1).$$

By Theorem 1 it follows that this kind of families of genus-one curves can't be everywhere smooth, and has to degenerate over a divisor of the base in (singular) rational curves.

1.4 Trivial canonical bundle

In this paper, we use the following definition of Calabi–Yau variety.

Definition 5 A Calabi–Yau variety X is a normal projective variety with log terminal singularities such that $K_X \equiv 0$ and $\tilde{q}(X) = 0$.

For Calabi–Yau varieties, we can prove a stronger version of Theorem 1.

Corollary 1 *Let X be a Calabi–Yau variety. Suppose that there exists a morphism $\phi : X \rightarrow B$ whose general fiber is a curve. Then there exists a subvariety of codimension one in X that is covered by rational curves contracted by ϕ .*

Proof By global index one theorem [10, Proposition 2.18], there is a variety X' with canonical singularities and a finite quasi-étale morphism $\alpha : X' \rightarrow X$ such that $K_{X'} \sim 0$. Since a finite quasi-étale cover $Y \rightarrow X'$ is also (after the composition with α) a finite quasi-étale cover of X , we can conclude that $\tilde{q}(X') \leq \tilde{q}(X)$ and thus $\tilde{q}(X') = 0$.

Moreover, notice that:

1. if there is a subvariety $V \subset X'$ of dimension $n - 1$ that is covered by rational curves, then also the variety $\alpha(V) \subset X$ is covered by rational curves;
2. the canonical bundle of X' is the pullback of the trivial line bundle since it is linearly equivalent to the trivial line bundle.

The result follows applying Theorem 1. $\square$

In the singular setting, the dichotomy between irreducible symplectic varieties and Calabi–Yau varieties given by the Beauville–Bogomolov decomposition is studied in [7, 10, 18] and related papers. In particular, in [18, Theorem 1.5], it is proven that there exists a version of the Beauville–Bogomolov decomposition for varieties with klt singularities. As a consequence of some conditions on the reflexive exterior algebra of forms required in the definition of Calabi–Yau varieties and irreducible symplectic varieties used in this paper, it follows that such varieties must have vanishing augmented irregularity.

Let X be a Calabi–Yau variety as in Definition 5; then the abelian factor of the Beauville–Bogomolov decomposition of X is trivial. As a consequence, Corollary 1 applies to any product of such Calabi–Yau varieties and irreducible symplectic varieties.

Let X be a variety with log terminal singularities, and suppose moreover $K_X \equiv 0$ and that there is a surjective morphism $\phi : X \rightarrow B$ to a variety of dimension $n - 1$. By [10, Theorem B], there is a quasi-étale map $f : A \times Y \rightarrow X$ with A an abelian variety of dimension $\tilde{q}(X)$, and $\tilde{q}(Y) = 0$. Passing through the Stein factorization of $\phi \circ f$, we get a genus-one fibration $\alpha : A \times Y \rightarrow \tilde{B}$. If the restriction of α to $\{t\} \times Y$ for general t is a family of curves, then we can apply Corollary 1 and find a uniruled divisor on $\{t\} \times Y$. This implies that there is also a uniruled divisor on $A \times Y$ and hence its image under f is a uniruled subvariety of codimension one in X .

Remark 4 It is not always possible to apply Corollary 1 to the restriction of the fibration to $\{t\} \times \tilde{X}$: it may happen that α is a projection, i.e. $X = E \times Y \rightarrow Y$ for some genus-one curve E .

Corollary 1 is a generalization of [5, Corollary 1.2]. Since the result in [5] requires a Calabi–Yau variety to have finite fundamental group, it may seem that Corollary 1 is more general than [5, Corollary 1.2] also for smooth varieties. In fact, as pointed out in Remark 3, such finiteness condition is a priori stronger than the vanishing of the augmented irregularity.

However, it is a consequence of the Beauville–Bogomolov decomposition for smooth varieties that these two conditions are equivalent: a smooth projective variety with numerically trivial canonical bundle and vanishing augmented irregularity has finite fundamental group. It is conjectured that the same holds in the singular case, at least for varieties with mild singularities.

2 Fibration over curves

In this section, we study the dual of genus-one fibrations: surjective morphisms $\pi : X \rightarrow C$ to a curve. Passing through the Stein factorization, we can assume that π has connected fibers and, since X is normal, that C is smooth. Thus we study only the geometry of a morphism with connected fibers onto a smooth curve. A fiber of a morphism onto a curve is a semiample divisor with numerical dimension one. Thus, a priori, it is more general to work only with nef divisors with numerical dimension one than with fibrations onto a curve.

For the reader's convenience, we recall the definition of numerical dimension of a nef divisor.

Definition 6 Let Y be a normal projective variety. The *numerical dimension* of a nef class $x \in N^1(Y)$ is the maximum integer k such that $x^k \neq 0$ as element in $N^k(Y)$.

The aim of this section is to prove Theorem 2.

Theorem 2 *Let X be a Calabi–Yau variety. Suppose that there exists a nef $\mathbb{Q}$ -divisor D with numerical dimension one such that $c_2(X) \cdot D = 0$ in $N^3(X)$. Then X contains a rational curve.*

Remark 5 Theorem 2 is a generalization of [5, Theorem 1.6], also under the additional hypothesis that the variety is smooth. Indeed, let X be a smooth variety with trivial canonical bundle and a fibration onto a curve with general fiber an abelian variety F , then the class of F in $N^1(X)$ has numerical dimension one and intersect in zero the second Chern class of X , i.e. $F \cdot c_2(X) = 0$ [5, Section 3]. Moreover, a divisor with numerical dimension one and such that the intersection with the second Chern class of X is 0 is only conjecturally semiample.

If the divisor is also semiample, the geometric meaning of Theorem 2 is clear. In this case, the Itaka fibration associated with D is a fibration onto a curve. A general fiber F of such a morphism intersects $c_2(X)$ trivially, i.e. $c_2(F) = F \cdot c_2(X) = 0$. If F is contained in the regular part of X , then by adjunction formula F has automatically trivial canonical bundle. In particular, this implies that there is an abelian variety with a finite quasi-étale cover to F .

2.1 Chern classes for singular varieties

The Todd class and the Chern classes of an arbitrary algebraic scheme are defined in [9, Section 18.3].

Remark 6 Let $\pi : \tilde{X} \rightarrow X$ be a proper birational morphism that is an isomorphism outside $Z \subset X$. Then we have

$$\mathrm{Td}(X) = \pi_* \mathrm{Td}(\tilde{X}) + \alpha \in A_*(X)_{\mathbb{Q}},$$

where α is a class supported in Z . As a consequence, if X is a variety smooth in codimension two, defining $c_2(X) := \pi_* c_2(\tilde{X})$ for some minimal resolution $\pi : \tilde{X} \rightarrow X$, agrees with the definition in [9]. We want to prove that these two definitions agree also for varieties with rational singularities.

Remark 7 Using the same definitions of [9], the Hirzebruch–Riemann–Roch Theorem [9, Corollary 18.3.1] holds for any complete scheme. Let X be a projective variety with rational singularities, e.g. with dlt singularities, and $\pi : \tilde{X} \rightarrow X$ a resolution of singularities. By definition of rational singularities, for any line bundle L on X we have $\chi(X, L) = \chi(\tilde{X}, \pi^* L)$.

Applying Hirzebruch-Riemann-Roch to X and $\tilde{X}$ we obtain

$$\int_X \mathrm{ch}(L) \cdot \mathrm{Td}(X) = \int_{\tilde{X}} \mathrm{ch}(\pi^*L) \cdot \mathrm{Td}(\tilde{X}) = \int_X \mathrm{ch}(L) \cdot \pi_* \mathrm{Td}(\tilde{X})$$

where the last equality follows from the projection formula [9, Proposition 2.5 (c)]. Since this equality holds for any line bundle L , it follows that $c_2(X) = \pi_* c_2(\tilde{X})$ as elements in $N^2(X)$. In particular, for a variety Y with klt singularities we can take as definition $c_2(Y) = \pi_* c_2(\tilde{Y})$ for some resolution of the singularities of Y .

Remark 8 Some alternative definitions of second Chern classes are discussed in [11]. Our definition of second Chern class is the one they define as *birational second Chern class*.

The pseudo-effectiveness of the second Chern class proved by Miyaoka holds also in our setting.

Lemma 4 *Let X be a normal projective variety with canonical singularities and $K_X \equiv 0$. For any nef divisors $D_1, \dots, D_{n-2}$ on X we have $c_2(X) \cdot D_1 \cdots D_{n-2} \geq 0$.*

Proof Let $\nu: \tilde{X} \rightarrow X$ be a terminalization of X . The canonical bundle of $\tilde{X}$ is numerically trivial and $\tilde{X}$ is smooth in codimension two. The divisors $\nu^*D_1, \dots, \nu^*D_{n-2}$ are nef, so applying [27, Theorem 6.6] and the projection formula we have $c_2(\tilde{X}) \cdot D_1 \cdots D_{n-2} \geq 0$. The conclusion follows applying another time the projection formula to ν . $\square$

In our setting, to prove that the second Chern class of X is trivial, it is sufficient to show that $c_2(X) \cdot H^{n-2} = 0$ for some ample divisor H .

Lemma 5 *Let X be a normal projective variety with canonical singularities and $K_X \equiv 0$. Then $c_2(X) = 0$ in $N^2(X)$ if and only if there exist $H_1, \dots, H_{n-2}$ ample line bundles on X such that $c_2(X) \cdot H_1 \cdots H_{n-2} = 0$.*

In particular, if $c_2(X) \neq 0$ in $N^2(X)$, then for any ample line bundle H we have $c_2(X) \cdot H^{n-2} > 0$.

Proof The proof of this lemma is similar to the one of [12, Proposition 4.8]. The first step is the following: let $H_1, \dots, H_{n-2}$ be ample line bundles on X such that $c_2(X) \cdot H_1 \cdots H_{n-2} = 0$ and let $L_1, \dots, L_{n-2}$ be line bundles on X ; we want to show that $c_2(X) \cdot L_1 \cdots L_{n-2} = 0$. Since the ample cone is open in $N^1(X)$ and the intersection product is multilinear, it is enough to prove that the intersection is trivial for L_i ample line bundle. Moreover, up to considering large multiples of the divisors H_i , we can assume that $H_i \pm L_i$ are ample divisors on X .

We prove by induction on k that

$$c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_{n-2} = 0.$$

For $k = 0$ this is the hypothesis. Suppose now that it is true for k , we have

$$\begin{aligned} 0 &= c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_{n-2} \\ &= c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot (H_{k+1} \pm L_{k+1}) \cdots H_{n-2}. \end{aligned}$$

By Lemma 4, both the summands are non-negative, thus they must be zero. For $k = n - 2$ we get

$$0 = \sum c_2(X) \cdot A_1 \cdots A_{n-2}$$

where $A_i \in \{H_i, L_i\}$. Since A_i is nef, this is a zero sum of non-negative numbers whose sum is zero, so every summand must be zero. In particular, we get $c_2(X) \cdot L_1 \cdots L_{n-2} = 0$.

To conclude, we have to prove that, if $c_2(X)$ is non-zero, then for any ample divisor H the number $c_2(X) \cdot H^{n-2}$ is positive. This is immediate since it is a non-zero number by the above argument, and it is non-negative by Lemma 4. $\square$

Remark 9 The second Chern class of a Calabi–Yau variety X is non-zero. This is a well-known fact under the further assumption that X is smooth in codimension two and it is proven for X with canonical singularities and $\mathbb{Q}$ -factorial in [13, Theorem 1.4]. In [26, Theorem 1.2 and Remark 1.5], it is proven that a normal projective variety with klt singularities, trivial canonical bundle and trivial second Chern class is a quasi-étale quotient of an abelian variety, and, as a consequence, the augmented irregularity is equal to the dimension.

Notice that the converse does not hold because $c_2((E \times E)/\pm) = 24$.

2.2 Preliminary results

In this subsection, we present some preliminary results needed to prove Theorem 2.

The well-known statement for $\mathbb{Q}$ -divisors that a nef divisor is big if and only if it has positive top self-intersection [24, Theorem 2.2.13] holds also for $\mathbb{R}$ -divisors.

Lemma 6 *Let Y be a normal projective variety of dimension n and let $D \in N^1(Y)$ be a nef $\mathbb{R}$ -divisor. Then D is big if and only if $D^n > 0$.*

This proof of this lemma is certainly well-known to experts and it is an exercise for others. This lemma can be easily deduced by [6, Theorem 0.5], but to invoke this result for this statement is certainly unnecessary. A detailed direct proof of this fact can be found in [25].

An interesting consequence of the above lemma is the following proposition.

Proposition 2 *Let X be a normal projective variety with log terminal singularities, numerically trivial canonical bundle and no rational curves. Then the ample cone and the big cone coincide.*

Proof Let D be an effective $\mathbb{Q}$ -divisor. For small positive and rational ε , the pair $(X, \varepsilon D)$ is klt. Since there are no rational curves in X , the Cone Theorem [23, Theorem 3.7] tells us that εD is also nef. It follows that the effective cone is contained in the nef cone. Passing to the interior of such cones, we get the thesis. $\square$

Following [24, Def. 1.5.21 and 1.5.26], we define two cones that help us studying nef divisors that are not ample. Notice that these two cones are not convex cones.

Definition 7 The null cone $\mathcal{N}_X \subset N^1(X)$ is the set of classes of divisors D such that $D^n = 0$. The boundary cone $\mathcal{B}_X \subset N^1(X)$ is the boundary of the nef cone.

The following corollary, already well-known by experts, explains which is the relation between these two cones under our hypotheses.

Proposition 3 *Let X be a normal projective variety with log terminal singularities, numerically trivial canonical bundle and no rational curves. The boundary of the ample cone is contained in the null cone, i.e. $\mathcal{B}_X \subset \mathcal{N}_X$.*

Proof In the boundary of the ample cone, there are nef $\mathbb{R}$ -divisors that, by Proposition 2, are not big $\mathbb{R}$ -divisors. These $\mathbb{R}$ -divisors have trivial top self-intersection and thus they are contained in the null cone. $\square$

In Proposition 4, we show how to use Proposition 3 to find (many) divisors with numerical dimension $n - 1$.

Proposition 4 *Let X be a normal projective variety with log terminal singularities, numerically trivial canonical bundle and without rational curves. Let H and D be two divisors on X that are respectively ample and nef of numerical dimension one. There is a (unique) rational number t_0 such that the $\mathbb{Q}$ -divisor $\overline{N}(D, H) = H - t_0 \cdot D$ is nef and has numerical dimension $n - 1$.*

Proof Let $t \in \mathbb{R}$ and let $N_t \subset N^1(X)$ be the line

$$N_t = H + t \cdot D.$$

The intersection of N_t with the null cone gives us an interesting divisor.

This line is parallel to the extremal ray of the nef cone generated by $[D]$. Since the divisor D is nef, the line N_t is contained in the nef cone for $t \geq \frac{-H^n}{nH^{n-1} \cdot D}$ and intersects the null cone when the equality holds.

The divisor in the intersection is

$$\overline{N} = H - \frac{H^n}{nH^{n-1} \cdot D} D.$$

We have that the nef divisor $\overline{N}$ satisfies the following properties:

1. it is a $\mathbb{Q}$ -divisor;
2. it has numerical dimension $n - 1$;
3. it is not big.

The first property holds because H and D are $\mathbb{Q}$ -divisors and $\frac{H^n}{nH^{n-1} \cdot D} \in \mathbb{Q}$; the second one holds because $\overline{N}^{n-1} \cdot D = H^{n-1} \cdot D \neq 0$; the last one holds because by construction $\overline{N}^n = 0$. $\square$

As a consequence, we can prove the following corollary.

Corollary 3 *Let X be a normal projective variety with canonical singularities, numerically trivial canonical bundle and with no rational curves. If $c_2(X) \neq 0$ as element in $N^2(X)$ but $c_2(X) \cdot D = 0$ in $N^3(X)$ for some nef $\mathbb{Q}$ -divisor D with $v(D) = 1$, then there exists an ample $\mathbb{Q}$ -divisor H such that the $\mathbb{Q}$ -divisor $\overline{N}(D, H)$ constructed in Proposition 4 satisfies $c_2(X) \cdot \overline{N}(D, H)^{n-2} > 0$.*

Proof By Proposition 4, X contains a $\mathbb{Q}$ -divisor N of numerical dimension $n - 1$. By Lemma 4 we know that the intersection of $c_2(X)$ with $n - 2$ nef divisors is non-negative. By Lemma 5, for any ample divisor H we have $H^{n-2} \cdot c_2(X) > 0$. Since we assumed $c_2(X) \cdot D = 0$, it follows

$$c_2(X) \cdot (\overline{N}(D, H))^{n-2} = c_2(X) \cdot \left(H - \frac{H^n}{nH^{n-1} \cdot D} D \right)^{n-2} = c_2(X) \cdot H^{n-2} > 0.$$

$\square$

2.3 Proof of Theorem 2

Using the results proven in the previous subsection and Corollary 1, we can finally prove Theorem 2. The idea of Theorem 2 is to find a nef divisor D in X with Iitaka dimension $n - 1$.

Proof of Theorem 2 If the singularities of X are not canonical, then X is uniruled by [22, Theorem 11], thus we can suppose that X has canonical singularities. Moreover, as pointed out in Remark 9, we know that $c_2(X) \neq 0$. Suppose by contradiction that there are no rational curves in X . It follows from Corollary 3 that we can find a nef $\mathbb{Q}$ -divisor $\bar{N}$ such that

$$0 < c_2(X) \cdot \bar{N}^{n-2} = 12 \operatorname{Td}_2(X) \cdot \bar{N}^{n-2}.$$

It follows from [21, Theorem 10] that the divisor $\bar{N}$ induces a genus-one fibration $X \rightarrow B$. Now, we can apply Corollary 1 to find rational curves in X , which is a contradiction. $\square$

In the proof of Theorem 2, we explained that, in our setting, it is sufficient to find a nef $\mathbb{Q}$ -divisor D with numerical dimension $n - 1$ such that $D^{n-2} \cdot c_2(X) > 0$. A careful discussion on the existence of a divisor with these properties in dimension three can be found in [4]. Even if they work with smooth varieties, their proofs of the various results work verbatim also for Calabi–Yau varieties as in Definition 5.

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